Modern Stochastics: Theory and Applications 1 (2014) 109–116
DOI: 10.15559/vmsta-2014.10

On the distribution of integral functionals of a homogeneous diffusion process

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Received: 21 July 2014, Revised: 18 August 2014, Accepted: 19 August 2014,
Published online: 29 August 2014

Abstract In this article, we study homogeneous transient diffusion processes. We provide the basic distributions of their local times. It helps to get exact formulas and upper bounds for the moments, exponential moments, and potentials of integral functionals of transient diffusion processes. Some of the results generalize the corresponding results of Salminen and Yor for the Brownian motion with drift.

Keywords Homogeneous transient diffusion process, distribution of local time, integral functional, moments, exponential moments and potentials

2010 MSC Primary: 60J60, secondary: 60J55

1 Introduction

We consider a family $\{X_t^x, t \ge 0, x \in \mathbb{R}\}$ of one-dimensional homogeneous diffusion processes defined on a complete filtered probability space $\{\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}, \mathsf{P}\}$ by a stochastic differential equation

$$dX_t^x = b\big(X_t^x\big)dt + a\big(X_t^x\big)dW_t, \quad t \ge 0, \tag{1}$$

with initial condition $X_0^x = x \in \mathbb{R}$, where $\{W_t, t \ge 0\}$ is a standard $\mathcal{F}_t$-Wiener process. If the initial condition is not important, we will denote the process in question

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by X. Let the coefficients a, b of equation (1) be continuous and satisfy any conditions of the existence of a nonexplosive weak solution on $\mathbb{R}$. Assume also that $a(x) \neq 0$ for $x \in \mathbb{R}$. We further introduce several objects related to the family $\{X_t^x, t \geq 0, x \in \mathbb{R}\}$.

The generator of a diffusion process X is defined for $f \in C^2(\mathbb{R})$ as

$$\mathcal{L}f(x) = \frac{a(x)^2}{2} f''(x) + b(x) f'(x).$$

Define the functions

$$\varphi(x_0, x) = \exp\left\{-2\int_{x_0}^{x} \frac{b(u)}{a(u)^2} du\right\}, \qquad \Phi(x_0, x) = \int_{x_0}^{x} \varphi(x_0, z) dz,$$

$x_0, x \in \mathbb{R} \cup \{-\infty, +\infty\}$. It is easy to see that, for a fixed $x_0 \in \mathbb{R}$, the function $\Phi(x_0, \cdot)$ solves a second-order homogeneous differential equation $\mathcal{L}\Phi(x_0, \cdot) = 0$.

For $x, y \in \mathbb{R}$, let $\tau_y^x = \inf\{t \geq 0, X_t^x = y\}$ be the first moment of hitting the point y. For any $(a, b) \subset \mathbb{R}$ and $x \in (a, b)$, let $\tau_{a,b}^x = \inf\{t \geq 0, X_t^x \notin (a, b)\} = \tau_a^x \wedge \tau_b^x$ be the first moment of exiting the interval (a, b). (We use the convention $\inf \varnothing = +\infty$.)

For any $t > 0$ and $y \in \mathbb{R}$, define the local time of the process X^x at the point y on the interval $[0, t]$ by

$$L_t^x(y) = a(y)^2 \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t \mathbb{I}\{|X_s^x - y| \leq \varepsilon\} ds. \tag{2}$$

(The factor $a^2(y)$ is included to agree with the general Meyer–Tanaka definition of the local time of a semimartingale [7].) The limit in (2) exists almost surely and defines a continuous nondecreasing process $\{L_t^x(y), t \geq 0\}$ for any $x, y \in \mathbb{R}$. The local time on the whole interval $[0, +\infty)$ will be denoted by $L_\infty^x(y) = \lim_{t \to +\infty} L_t^x(y)$.

In this article, we focus on the *transient* diffusion processes, that is, those converging to $+\infty$ or $-\infty$ as $t \to \infty$. We use the explicit distribution of L_∞^x to study integral functionals of the form $J_\infty(f) = \int_0^\infty f(X_s^x) ds$, which can be interpreted as continuous perpetuities in the framework of financial mathematics. We follow the approach of Salminen and Yor [8] to study integral functionals of a Wiener process with positive drift and generalize their results to homogeneous transient diffusion processes. Applying the results of [6], we establish criteria of convergence of almost sure finiteness of the functionals $J_\infty(f)$, calculate their moments and potentials, and bound their exponential moments.

2 The distribution of the local time of a transient diffusion process

In this section, we concentrate on the explicit distribution of $L_\infty^x(y)$. According to the classical results (see, e.g., [4]), in the case where $\Phi(x, +\infty) = -\Phi(x, -\infty) = +\infty$ for some (equivalently, for all) $x \in \mathbb{R}$, the diffusion process X is recurrent, that is,

$$\mathsf{P}\Big(\limsup_{t \to +\infty} X_t^x = +\infty, \liminf_{t \to +\infty} X_t^x = -\infty\Big) = 1.$$

Therefore, $L_\infty^x(y) = +\infty$ for all $x, y \in \mathbb{R}$ a.s.

In what follows, we will consider only the case of a transient process X, where at least one of the integrals $\Phi(x,+\infty)$ and $\Phi(x,-\infty)$ is finite. We formulate the following statement concerning the distribution of $L^x_\infty(y)$ that can be easily deduced from the results of [3, 1].

Proposition 1. *1. In each of the cases $x=y$, $x<y$ and $-\Phi(0,-\infty)=+\infty$, and $x>y$ and $\Phi(0,+\infty)=+\infty$, the local time $L^x_\infty(y)$ is exponentially distributed with parameter $\psi_y(0)$ given by (5).*

2. If $x<y$ and $-\Phi(0,-\infty)<+\infty$, then the local time $L^x_\infty(y)$ is distributed as $\kappa\xi$, where ξ is exponentially distributed with parameter $\psi_y(0)$, and κ is an independent of ξ Bernoulli random variable with

$$\mathsf{P}(\kappa=0)=1-\mathsf{P}(\kappa=1)=\frac{\Phi(y,x)}{\Phi(y,-\infty)}.$$

3. If $x>y$ and $\Phi(0,+\infty)<+\infty$, then the local time $L^x_\infty(y)$ is distributed as $\kappa\xi$, where ξ is exponentially distributed with parameter $\psi_y(0)$, and κ is an independent of ξ Bernoulli random variable with

$$\mathsf{P}(\kappa=0)=1-\mathsf{P}(\kappa=1)=\frac{\Phi(y,x)}{\Phi(y,+\infty)}.$$

Proof. By the strong Markov property of the process X, for any $l\ge 0$ and $x,y\in\mathbb{R}$,

$$\mathsf{P}\big(L^x_\infty(y)>l\big)=\mathsf{P}\big(L^y_\infty(y)>l\big)\mathsf{P}\big(\tau^x_y<+\infty\big).$$

The probability $\mathsf{P}(\tau^x_y<+\infty)=1-\mathsf{P}(\tau^x_y=+\infty)$ can be found with the help of the well-known formula (see, e.g., [3, Section VIII.6, (18)]): for $x\in(a,b)$,

$$\mathsf{P}\big(X^x_{\tau_{a,b}}=b\big)=\frac{\Phi(a,x)}{\Phi(a,b)}.$$

Then the value of probability in question depends on x, y and on the integrals $\Phi(x,+\infty)$, $\Phi(x,-\infty)$. Specifically, if $x>y$, then

$$\mathsf{P}\big(\tau^x_y=\infty\big)=\lim_{a\to+\infty}\mathsf{P}\big(X^x_{\tau_{y,a}}=a\big)=\lim_{a\to+\infty}\frac{\Phi(y,x)}{\Phi(y,a)},$$

whence

$$\mathsf{P}\big(\tau^x_y=+\infty\big)=\begin{cases}\frac{\Phi(y,x)}{\Phi(y,+\infty)}, & \Phi(x,+\infty)<+\infty,\\ 0, & \Phi(x,+\infty)=+\infty.\end{cases}\tag{3}$$

For $x<y$,

$$\begin{aligned}\mathsf{P}\big(\tau^x_y=\infty\big)&=\lim_{a\to-\infty}\big(1-\mathsf{P}\big(X^x_{\tau_{a,y}}=y\big)\big)=\lim_{a\to-\infty}\frac{\Phi(a,y)-\Phi(a,x)}{\Phi(a,y)}\\ &=\lim_{a\to-\infty}\frac{\varphi(a,x)\Phi(x,y)}{-\varphi(a,y)\Phi(y,a)}=\lim_{a\to-\infty}\frac{-\varphi(a,x)\varphi(x,y)\Phi(y,x)}{-\varphi(a,y)\Phi(y,a)}\\ &=\lim_{a\to-\infty}\frac{\Phi(y,x)}{\Phi(y,a)};\end{aligned}$$

therefore,

$$\mathsf{P}\big(\tau_y^x = +\infty\big) = \begin{cases} \frac{\Phi(y,x)}{\Phi(y,-\infty)}, & -\Phi(x,-\infty) < +\infty, \\ 0, & -\Phi(x,-\infty) = +\infty. \end{cases} \tag{4}$$

Thus, it is sufficient to determine the distribution of variables $L_\infty^x(x)$. But it was proved in [1, II.13, II.27] that $\mathsf{P}(L_\infty^x(x) > l) = \exp(-l\psi_x(0))$, where

$$\psi_x(0) = \frac{1}{2}\left(\frac{1}{\Phi(x,+\infty)} - \frac{1}{\Phi(x,-\infty)}\right) \tag{5}$$

with $\frac{1}{\infty} := 0$. Hence, the proof follows. □

Consider two examples where the parameters of the distribution of local time can be calculated explicitly.

Example 1. Let $a(x) \equiv a \neq 0$ and $b(x) \equiv b$ be constant. Then

$$\varphi(x, y) = e^{-2b(y-x)/a^2}$$

and

$$\Phi(x, y) = \frac{a^2}{2b}\big(1 - e^{-2b(y-x)/a^2}\big) \quad \text{for } b \neq 0; \qquad \Phi(x, y) = y - x \quad \text{for } b = 0.$$

In this case, the diffusion process X is transient if and only if $b \neq 0$; moreover, $-\Phi(0,-\infty) = +\infty$ and $\Phi(0,+\infty) < +\infty$ for $b > 0$, and $-\Phi(0,-\infty) < +\infty$ and $\Phi(0,+\infty) = +\infty$ for $b < 0$. The cases are symmetric; therefore, we consider only the case $b > 0$.

Now $\psi_x(0) = \frac{b}{a^2}$. Thus, for $x \le y$, the local time $L_\infty^x(y)$ is exponentially distributed with parameter $\frac{b}{a^2}$. For $x > y$, the local time is distributed as $\kappa\xi$, where ξ has an exponential distribution with parameter $\frac{b}{a^2}$, and κ is a Bernoulli random variable independent of ξ and distributed as $\mathsf{P}(\kappa = 1) = 1 - \mathsf{P}(\kappa = 0) = e^{-2b(x-y)/a^2}$. Using the properties of exponential distribution, we see that these cases can be combined: $L_\infty^x(y) \stackrel{d}{=} (\xi - 2(x-y)_+)_+$, where $a_+ = a \vee 0$.

Example 2. Let $a(x) = \sqrt{x^2+1}$ and $b(x) = x$. Then $\varphi(x, y) = \frac{x^2+1}{y^2+1}$ and $\Phi(x, y) = (1+x^2)(\arctan y - \arctan x)$. We see that the process is transient and $-\Phi(0,-\infty) = \Phi(0,\infty) = \frac{\pi}{2} < \infty$.

Due to Corollary 1, the local time $L_\infty^x(y)$ is distributed as $\kappa\xi$, where ξ has an exponential distribution with parameter

$$\begin{aligned}\psi_x(0) &= \frac{1}{2\Phi(x,+\infty)} - \frac{1}{2\Phi(x,+\infty)} \\ &= \frac{1}{(1+x^2)(\pi - 2\arctan x)} - \frac{1}{(1+x^2)(\pi + 2\arctan x)} \\ &= \frac{4\arctan x}{(1+x^2)(\pi^2 - 4\arctan^2 x)},\end{aligned}$$

and κ is a Bernoulli random variable, which is independent of ξ and distributed as

$$\mathsf{P}(\kappa = 1) = 1 - \mathsf{P}(\kappa = 0) = \begin{cases} \frac{\pi - 2\arctan x}{\pi - 2\arctan y}, & x \geq y, \\ \frac{\pi + 2\arctan x}{\pi + 2\arctan y}, & x < y. \end{cases}$$

3 Integral functionals of a transient diffusion processes

For a measurable function $f : \mathbb{R} \to \mathbb{R}$ such that f and f/a^2 are locally integrable, define the integral functional

$$J_\infty^x(f) = \int_0^\infty f\big(X_s^x\big) ds.$$

We will study the questions of finiteness and existence of moments of $J_\infty^x(f)$. We start with the well-known occupation density formula (see, e.g., [4, 7])

$$J_\infty^x(f) = \int_{\mathbb{R}} \frac{f(y)}{a(y)^2} L_\infty^x(y) dy. \tag{6}$$

If the process X^x is recurrent, then $L_\infty^x(y) = \infty$ a.s. for all $y \in \mathbb{R}$, so $J_\infty^x(f)$ is undefined unless f is identically zero. Therefore, we will require that the process X is transient. We recall that this holds iff $\Phi(0, +\infty)$ of $\Phi(0, -\infty)$ is finite. Moreover, if $\Phi(0, +\infty) < +\infty$ and $\Phi(0, -\infty) = -\infty$, then $X_s^x \to +\infty$ a.s.; if $\Phi(0, +\infty) = +\infty$ and $\Phi(0, -\infty) > -\infty$, then $X_s^x \to -\infty$ a.s.; if $\Phi(0, +\infty) < +\infty$ and $\Phi(0, -\infty) > -\infty$, then $X_s^x \to +\infty$ on a set A_+ of positive probability, and $X_s^x \to -\infty$ on a set $A_- = \Omega \setminus A_+$ of positive probability.

We start with a criterion of almost sure finiteness of $J_\infty^x(f)$. It was obtained in [5] in the case where only one of the integrals $\Phi(0, +\infty)$ and $\Phi(0, -\infty)$ is finite; a complete analysis was made in [6]. Define

$$I_1(f) = \int_0^{+\infty} \frac{|f(y)|}{a(y)^2} \Phi(y, +\infty) dy, \qquad I_2(f) = \int_{-\infty}^0 \frac{|f(y)|}{a(y)^2} \Phi(y, -\infty) dy.$$

Theorem 1 ([6]). *For arbitrary $x \in \mathbb{R}$, the following statements hold.*

- *Let $\Phi(0, +\infty) < +\infty$ and $\Phi(0, -\infty) = -\infty$.*

  If $I_1(f) < +\infty$, then $J_\infty^x(f) \in \mathbb{R}$ a.s.

  If $I_1(f) = \infty$, then $J_\infty^x(f) = \infty$ a.s.

- *Let $\Phi(0, +\infty) = +\infty$ and $\Phi(0, -\infty) > -\infty$.*

  If $I_2(f) < +\infty$, then $J_\infty^x(f) \in \mathbb{R}$ a.s.

  If $I_2(f) = -\infty$, then $J_\infty^x(f) = \infty$ a.s.

- *Let $\Phi(0, +\infty) < +\infty$ and $\Phi(0, -\infty) > -\infty$.*

  If $I_1(f) < +\infty$, then $J_\infty^x(f) \in \mathbb{R}$ a.s. on A_+.

  If $I_1(f) = \infty$, then $J_\infty^x(f) = \infty$ a.s. on A_+.

  If $I_2(f) < +\infty$, then $J_\infty^x(f) \in \mathbb{R}$ a.s. on A_-.

  If $I_2(f) = +\infty$, then $J_\infty^x(f) = \infty$ a.s. on A_-.

In what follows, we consider the case where $\Phi(0,+\infty) < +\infty$ and $\Phi(0,-\infty) = -\infty$, the other cases being similar. The next result is a direct consequence of Proposition 1.

Lemma 1. *Let $\Phi(0,+\infty) < +\infty$ and $\Phi(0,-\infty) = -\infty$. Then, for any $k \geq 1$,*

$$\mathsf{E}\big[L_\infty^x(y)^k\big] = k!\big(2\Phi(y,+\infty)\big)^k \quad \text{for } x \leq y$$

and

$$\begin{aligned}\mathsf{E}\big[L_\infty^x(y)^k\big] &= 2^k k!\Phi(y,+\infty)^{k-1}\big(\Phi(y,+\infty) - \Phi(y,x)\big) \\ &= 2^k k!\Phi(y,+\infty)^{k-1}\varphi(y,x)\Phi(x,+\infty)\end{aligned}$$

for $x > y$.

Example 3. Let $a = 1$ and $b = \mu > 0$ with some constant μ, so that X is a Brownian motion with constant positive drift. Furthermore, in this case, $\Phi(y,+\infty) = 1/2\mu$, $\varphi(y,x) = \exp\{-2\mu(x-y)\}$, and $\Phi(0,-\infty) = -\infty$. Therefore, the criterion for $J_\infty(f)$ to be finite is $\int_0^\infty |f(x)|dx < \infty$, which coincides with that of [8]. For what concerns the moments of local times, in this case, $\mathsf{E}[L_\infty^x(y)] = 1/\mu$ for $x \leq y$ and $\mathsf{E}[L_\infty^x(y)] = \frac{1}{\mu}\exp\{-2\mu(x-y)\}$ for $x > y$.

We further derive conditions for $\mathsf{E}[J_\infty^x(f)]$ to be finite.

Theorem 2. *Let $\Phi(0,+\infty) < +\infty$, $\Phi(0,-\infty) = -\infty$, and $I_1(f) < +\infty$. Assume additionally that*

$$\int_{-\infty}^x \frac{|f(u)|}{a(u)^2}\varphi(u,x)du < +\infty.$$

Then

$$\mathsf{E}\big[J_\infty^x(f)\big] = 2\int_x^{+\infty} \frac{f(u)}{a(u)^2}\Phi(u,+\infty)du + 2\Phi(x,+\infty)\int_{-\infty}^x \frac{f(u)}{a(u)^2}\varphi(u,x)du.$$

Proof. The statement immediately follows from Lemma 1 and the Fubini theorem. □

Remark 1. For a Brownian motion with positive drift μ, a sufficient condition for $\mathsf{E}[J_\infty^x(f)]$ to be finite is

$$\int_x^{+\infty} \big|f(u)\big|du + e^{-2\mu x}\int_{-\infty}^x \big|f(u)\big|e^{2\mu u}du < \infty;$$

and in that case, we have the equality

$$\mathsf{E}\big[J_\infty^x(f)\big] = \frac{1}{\mu}\int_x^{+\infty} f(u)du + \frac{1}{\mu}e^{-2\mu x}\int_{-\infty}^x f(u)e^{2\mu u}du.$$

Obviously, the requirement $\int_{\mathbb{R}} |f(u)|du < \infty$ is also sufficient, as stated in [8].

Now we continue with the moments of $J_\infty(f)$ of higher order.

Theorem 3. *Let $\Phi(0,+\infty) < +\infty$ and $\Phi(0,-\infty) = -\infty$. The moments of higher order admit the following bound: for any $k > 1$,*

$$\big(\mathsf{E}\big[\big|J_\infty^x(f)\big|^k\big]\big)^{1/k} \le 2(k!)^{1/k}\bigg(\int_x^{+\infty}\frac{|f(u)|}{a(u)^2}\Phi(u,+\infty)du + \Phi(x,+\infty)^{1/k}\int_{-\infty}^x \frac{|f(u)|}{a(u)^2}\Phi(u,+\infty)^{1-1/k}\varphi(u,x)^{1/k}du\bigg). \tag{7}$$

Proof. We use representation (6) and the generalized Minkowski inequality to get the following equalities and bounds:

$$\big(\mathsf{E}\big[\big|J_\infty^x(f)\big|^k\big]\big)^{1/k} = \bigg(\mathsf{E}\bigg[\bigg|\int_{\mathbb{R}}\frac{f(y)}{a(y)^2}L_\infty^x(y)dy\bigg|^k\bigg]\bigg)^{1/k} \le \int_{\mathbb{R}}\frac{|f(y)|}{a(y)^2}\big(\mathsf{E}\big[L_\infty^x(y)^k\big]\big)^{1/k}dy. \tag{8}$$

Now (7) follows immediately from (8) and Lemma 1. □

We conclude with the existence of potential and exponential moments. Some related results were obtained in [5].

Definition 1. The integral functional $J_\infty(f)$ has a bounded potential P if

$$P = \sup_{x\in\mathbb{R}} \mathsf{E}\big[J_\infty^x(f)\big] < \infty.$$

The following result is an immediate corollary of Theorem 2.

Theorem 4. *Let $\Phi(0,+\infty) < +\infty$, $\Phi(0,-\infty) = -\infty$, and*

$$P_0 = 2\sup_{x\in\mathbb{R}}\bigg(\int_x^{+\infty}\frac{|f(u)|}{a(u)^2}\Phi(u,+\infty)du + \Phi(x,+\infty)\int_{-\infty}^x\frac{|f(u)|}{a(u)^2}\varphi(u,x)du\bigg) < \infty.$$

Then the integral functional $J_\infty(f)$ has a bounded potential $P \le P_0$.

Theorem 5. *Let $\Phi(0,+\infty) < +\infty$, $\Phi(0,-\infty) = -\infty$, and $P_0 < \infty$. Then*

$$\mathsf{E}\big[\exp\big(\lambda J_\infty^x(f)\big)\big] \le \frac{1}{1-\lambda P_0}$$

for $\lambda P_0 < 1$.

Proof. We apply the following result of Dellacherie and Meyer [2], see also [8, Lemma 5.2]. Let A be a continuous adapted nondecreasing process starting at zero such that there exists a constant $C > 0$ satisfying $\mathsf{E}[A_\infty - A_t \mid \mathcal{F}_t] \le C$ for any $t \ge 0$. Then

$$\mathsf{E}\big[\exp(\lambda A_\infty)\big] \le \frac{1}{1-\lambda C}$$

for $\lambda < C^{-1}$.

Set $A_t = \int_0^t |f(X_s^x)|ds$. Then it follows from the Markov property of X and Theorems 2 and 4 that $\mathsf{E}[A_\infty - A_t \mid \mathcal{F}_t] \le P_0$ for any $t \ge 0$, whence the proof follows. □

References

[1] Borodin, A.N., Salminen, P.: Handbook of Brownian Motion—Facts and Formulae, 2nd edn. Probability and Its Applications. Birkhäuser Verlag, Basel (2002). MR1912205

[2] Dellacherie, C., Meyer, P.-A.: Probabilities and Potential. C. North-Holland Mathematics Studies, vol. 151. North-Holland Publishing Co., Amsterdam (1988). MR0939365

[3] Gikhman, I.I., Skorokhod, A.V.: Introduction to the Theory of Random Processes. Dover Publications Inc., Mineola, NY (1996). MR1435501

[4] Itô, K., McKean, H.P. Jr.: Diffusion Processes and Their Sample Paths. Springer, Berlin (1974). MR0345224

[5] Khoshnevisan, D., Salminen, P., Yor, M.: A note on a.s. finiteness of perpetual integral functionals of diffusions. Electron. Commun. Probab. **11**, 108–117 (2006). MR2231738

[6] Mijatović, A., Urusov, M.: Convergence of integral functionals of one-dimensional diffusions. Electron. Commun. Probab. **17**, 1–13 (2012), article 61. MR3005734

[7] Pitman, J., Yor, M.: Hitting, occupation and inverse local times of one-dimensional diffusions: martingale and excursion approaches. Bernoulli **9**(1), 1–24 (2003). MR1963670

[8] Salminen, P., Yor, M.: Properties of perpetual integral functionals of Brownian motion with drift. Ann. Inst. Henri Poincaré Probab. Stat. **41**(3), 335–347 (2005). MR2139023