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The risk model with stochastic premiums, dependence and a threshold dividend strategy

Olena Ragulina

Taras Shevchenko National University of Kyiv
Department of Probability Theory, Statistics and Actuarial Mathematics
64 Volodymyrska Str., 01601 Kyiv, Ukraine

ragulina.olena@gmail.com (O. Ragulina)

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Abstract The paper deals with a generalization of the risk model with stochastic premiums where dependence structures between claim sizes and inter-claim times as well as premium sizes and inter-premium times are modeled by Farlie–Gumbel–Morgenstern copulas. In addition, dividends are paid to its shareholders according to a threshold dividend strategy. We derive integral and integro-differential equations for the Gerber–Shiu function and the expected discounted dividend payments until ruin. Next, we concentrate on the detailed investigation of the model in the case of exponentially distributed claim and premium sizes. In particular, we find explicit formulas for the ruin probability in the model without either dividend payments or dependence as well as for the expected discounted dividend payments in the model without dependence. Finally, numerical illustrations are presented.

Keywords Risk model with stochastic premiums, Farlie–Gumbel–Morgenstern copula, threshold strategy, Gerber–Shiu function, expected discounted dividend payments, ruin probability, integro-differential equation
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1 Introduction

In the actuarial literature, a lot of attention is paid to the investigation of the ruin measures such as the ruin probability, the surplus prior to ruin and the deficit at ruin (see, e.g., [6, 34, 39] and references therein). A unified approach to the study of these risk measures together by combining them into one function was proposed by

Gerber and Shiu [22], who introduced the expected discounted penalty function for the classical risk model. The so-called Gerber–Shiu function has been investigated further by many authors (see, e.g., [13, 23, 29, 42, 44]) in more general risk models. In those risk models, claim sizes and inter-claim times are assumed to be mutually independent, which simplifies the investigation of the ruin measures. Nevertheless, this assumption has been proved to be very restrictive in some real applications. For instance, in modelling damages caused by natural catastrophic events, the intensity of the catastrophe and the time elapsed since the last catastrophe are expected to be dependent [10, 38]. That is why more and more authors have concentrated on the investigation of risk models with dependence between claim sizes and inter-claim times recently.

Albrecher and Boxma [1] consider a generalization of the classical risk model, where the distribution of the time between two claims depends on the previous claim size (see also [2] for an extension). Albrecher and Teugels [3] apply the random walk approach and allow the inter-claim time and its subsequent claim size to be dependent according to an arbitrary structure. Boudreault, Cossette, Landriault and Marceau [11] consider a particular dependence structure between the inter-claim time and the subsequent claim size and derive the defective renewal equation satisfied by the expected discounted penalty function. In [33], the authors study the ruin probability in a model where the time between two claim occurrences determines the distribution of the next claim size. The ruin probability in a model with independent but not necessarily identically distributed claim sizes and inter-claim times is investigated in [5] (see also references therein).

Cossette, Marceau and Marri [15, 16] deal with an extension of the classical compound Poisson risk model where a dependence structure between the claim size and the inter-claim time is introduced through a Farlie–Gumbel–Morgenstern copula and its generalization. They derive the integro-differential equation and the Laplace transform of the Gerber–Shiu discounted penalty function and concentrate on exponentially distributed claim sizes. Zhang and Yang [47] extend these results to the compound Poisson risk model perturbed by a Brownian motion. In [12, 24, 46], the authors deal with the Sparre Andersen risk model where the inter-claim times follow the Erlang distribution and extend results obtained in [15, 16].

In all these papers, the dependence structure between the claim sizes and the inter-claim times is described by the Farlie–Gumbel–Morgenstern copula. This copula is often used in applications to introduce dependence structures due to its tractability and simplicity. It allows positive and negative dependence as well as independence. Nevertheless, the Farlie–Gumbel–Morgenstern copula has been shown to be somewhat limited since it does not allow the modeling of high dependencies. Indeed, its dependence parameter is $\theta \in [-1, 1]$, so its Spearman's rho and Kendall's tau are $\rho_\theta = \theta/3 \in [-1/3, 1/3]$ and $\tau_\theta = 2\theta/9 \in [-2/9, 2/9]$, respectively (see, e.g., [7, 8, 37] and references therein). This limited range of dependence restricts the usefulness of this copula for modeling. Note that the dependence parameter θ can be easily estimated from real data due to the simple relations between it and the measures of association ρ_θ and τ_θ. For more information on the Farlie–Gumbel–Morgenstern copula, we refer to, e.g., [21, 37] (see also [16] and references therein for applications of this copula). Despite the popularity of the Farlie–Gumbel–Morgenstern copula, other copulas have been used in risk theory, for instance, an Archimedean copula [4], a Gaussian copula [19] and a Spearman copula [25].

Risk models where an insurance company pays dividends to its shareholders are of great interest in risk theory. Dividend strategies for insurance risk models were first proposed by De Finetti [20], who dealt with a binomial model. Barrier strategies for the classical risk model and its different generalizations have been studied in a number of papers (see, e.g., [14, 27, 28, 30, 31, 43, 45]). For optimal dividend problems in insurance risk models, see the monograph by Schmidli [40] and references therein.

Cossette, Marceau and Marri [17, 18] consider the classical risk process with a constant dividend barrier and a dependence structure between claim sizes and inter-claim times introduced through the Farlie–Gumbel–Morgenstern copula. They analyze the Gerber–Shiu function and the expected discounted dividend payments and then concentrate on exponentially distributed claim sizes investigating the impact of the dependence on ruin quantities. The same model is studied in [32], where, in particular, the authors show that the solution to the integro-differential equation for the Gerber–Shiu function is a linear combination of the Gerber–Shiu function with no barrier and the solution to the associated homogeneous integro-differential equation. For some earlier results in this direction, see also [26].

Shi, Liu and Zhang [41] consider the compound Poisson risk model with a threshold dividend strategy and a dependence structure modeled by the Farlie–Gumbel–Morgenstern copula. They derive integro-differential equations for the Gerber–Shiu function and the expected discounted dividend payments paid until ruin as well as renewal equations for these functions, which are used to obtain explicit formulas for them.

The present paper deals with a generalization of the risk model with stochastic premiums introduced and studied in [9] (see also [34]). In contrast to the classical compound Poisson risk model, where premiums arrive with constant intensity and are not random, in this risk model premiums also form a compound Poisson process, i.e. they arrive at random times and their sizes are also random (see also [35, 36] for a generalization of the classical risk model where an insurance company gets additional funds whenever a claim arrives). In [9], claim sizes and inter-claim times are assumed to be mutually independent, and the same assumption is made concerning premium arrivals. In this paper, we suppose that the dependence structures between claim sizes and inter-claim times as well as premium sizes and inter-premium times are modeled by the Farlie–Gumbel–Morgenstern copulas, which allows positive and negative dependence as well as independence. In addition, we suppose that the insurance company pays dividends to its shareholders according to a threshold dividend strategy. To be more precise, this implies that when the surplus is below some fixed threshold, no dividends are paid, and when the surplus exceeds or equals the threshold, dividends are paid continuously at some constant rate. Our subjects of investigation are the Gerber–Shiu function, a special case of which is the ruin probability, and the expected discounted dividend payments until ruin.

The rest of the paper is organized as follows. In Section 2, we describe the risk model we deal with. In Section 3, we derive integral and integro-differential equations for the Gerber–Shiu function. In Section 4, we obtain corresponding equations for the expected discounted dividend payments until ruin. Section 5 deals with exponentially distributed claim and premium sizes in some special cases of the model. Namely, we consider the ruin probability in the model without either dividend payments or dependence, and the expected discounted dividend payments in the model without dependence. In these simpler models, we reduce the integral and integro-differential equations derived in Sections 2 and 3 to linear differential equations and find explicit solutions to these equations. Section 6 provides some numerical illustrations.

2 Description of the model

Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a probability space satisfying the usual conditions, and let all the stochastic objects we use below be defined on it.

In the risk model with stochastic premiums introduced in [9] (see also [34]), claim sizes form a sequence $(Y_i)_{i\ge1}$ of non-negative independent and identically distributed (i.i.d.) random variables (r.v.'s) with cumulative distribution function (c.d.f.) $F_Y(y) = \mathbb{P}[Y_i \le y]$. The number of claims on the time interval $[0, t]$ is a Poisson process $(N_t)_{t\ge0}$ with constant intensity $\lambda > 0$. Next, premium sizes form a sequence $(\bar{Y}_i)_{i\ge1}$ of non-negative i.i.d. r.v.'s with c.d.f. $\bar{F}_{\bar{Y}}(y) = \mathbb{P}[\bar{Y}_i \le y]$. The number of premiums on the time interval $[0, t]$ is a Poisson process $(\bar{N}_t)_{t\ge0}$ with constant intensity $\bar{\lambda} > 0$. Thus, the total claims and premiums on $[0, t]$ equal $\sum_{i=1}^{N_t} Y_i$ and $\sum_{i=1}^{\bar{N}_t} \bar{Y}_i$, respectively. We set $\sum_{i=1}^{0} Y_i = 0$ if $N_t = 0$, and $\sum_{i=1}^{0} \bar{Y}_i = 0$ if $\bar{N}_t = 0$. In what follows, we also assume that the r.v.'s $(Y_i)_{i\ge1}$ have a probability density function (p.d.f.) $f_Y(y)$ and a finite expectation $\mu > 0$, and the r.v.'s $(\bar{Y}_i)_{i\ge1}$ have a probability density function (p.d.f.) $f_{\bar{Y}}(y)$ and a finite expectation $\bar{\mu} > 0$.

We denote a non-negative initial surplus of the insurance company by x. Let $X_t(x)$ be its surplus at time t provided that the initial surplus is x. Then the surplus process $(X_t(x))_{t\ge0}$ follows the equation

$$X_t(x) = x + \sum_{i=1}^{\bar{N}_t} \bar{Y}_i - \sum_{i=1}^{N_t} Y_i, \quad t \ge 0. \tag{1}$$

In [9], the r.v.'s $(Y_i)_{i\ge1}$ and $(\bar{Y}_i)_{i\ge1}$, and the processes $(N_t)_{t\ge0}$ and $(\bar{N}_t)_{t\ge0}$ are assumed to be mutually independent. In this paper, we suppose that the claim sizes $(Y_i)_{i\ge1}$ and the inter-claim times are not independent but with a dependence structure modeled by a Farlie–Gumbel–Morgenstern copula, and we make the same assumption concerning premium arrivals.

To be more precise, let $(T_i)_{i\ge1}$ be a sequence of inter-arrival times of $(N_t)_{t\ge0}$. In particular, T_1 is the time of the first claim. Thus, $(T_i)_{i\ge1}$ are i.i.d. r.v.'s with p.d.f. $f_T(t) = \lambda e^{-\lambda t}$. We assume that $(Y_i, T_i)_{i\ge1}$ are i.i.d. random vectors and for every fixed $i \ge 1$, the dependence structure between Y_i and T_i is modeled by a Farlie–Gumbel–Morgenstern copula with parameter $\theta \in [-1, 1]$, i.e.

$$C_\theta^{\mathrm{FGM}}(u_1, u_2) = u_1u_2 + \theta u_1u_2(1 - u_1)(1 - u_2), \quad u_1, u_2 \in [0, 1]$$

(see, e.g., [21, 37] for more information on copulas). In other words, a claim size depends on the time elapsed from the previous claim. Therefore, the bivariate c.d.f. of $(Y_i, T_i)_{i\ge1}$ is defined by

$$\begin{aligned} F_{Y,T}(y,t) &= C_\theta^{\mathrm{FGM}}\big(F_Y(y), F_T(t)\big) \\ &= F_Y(y)F_T(t) + \theta F_Y(y)F_T(t)\big(1-F_Y(y)\big)\big(1-F_T(t)\big), \quad y \ge 0,\ t \ge 0. \end{aligned}$$

The corresponding bivariate p.d.f. of $(Y_i, T_i)_{i\ge1}$ is given by

$$\begin{aligned} f_{Y,T}(y,t) &= f_Y(y)f_T(t) + \theta f_Y(y)f_T(t)\big(1-2F_Y(y)\big)\big(1-2F_T(t)\big) \\ &= \lambda e^{-\lambda t} f_Y(y) + \theta h_Y(y)\big(2\lambda e^{-2\lambda t} - \lambda e^{-\lambda t}\big), \quad y \ge 0,\ t \ge 0, \end{aligned} \tag{2}$$

where $h_Y(y) = f_Y(y)(1-2F_Y(y))$, $y \ge 0$. Note that the case $\theta = 0$ corresponds to the situation where the claim sizes and the inter-claim times are independent.

Next, let $(\bar{T}_i)_{i\ge1}$ be a sequence of inter-arrival times of $(\bar{N}_t)_{t\ge0}$. In particular, $\bar{T}_1$ is the time of the first premium. Therefore, $(\bar{T}_i)_{i\ge1}$ are i.i.d. r.v.'s with p.d.f. $f_{\bar{T}}(t) = \bar{\lambda} e^{-\bar{\lambda} t}$. We also suppose that $(\bar{Y}_i, \bar{T}_i)_{i\ge1}$ are i.i.d. random vectors and for every fixed $i \ge 1$, the dependence structure between $\bar{Y}_i$ and $\bar{T}_i$ is modeled by a Farlie–Gumbel–Morgenstern copula with parameter $\bar{\theta} \in [-1, 1]$. So the bivariate p.d.f. of $(\bar{Y}_i, \bar{T}_i)_{i\ge1}$ is given by

$$f_{\bar{Y},\bar{T}}(y,t) = \bar{\lambda} e^{-\bar{\lambda} t} f_{\bar{Y}}(y) + \bar{\theta} h_{\bar{Y}}(y)\big(2\bar{\lambda} e^{-2\bar{\lambda} t} - \bar{\lambda} e^{-\bar{\lambda} t}\big), \quad y \ge 0,\ t \ge 0, \tag{3}$$

where $h_{\bar{Y}}(y) = f_{\bar{Y}}(y)(1-2F_{\bar{Y}}(y))$, $y \ge 0$. The case $\bar{\theta} = 0$ corresponds to the situation where the premium sizes and the inter-premium times are independent. The random vectors $(Y_i, T_i)_{i\ge1}$ and $(\bar{Y}_i, \bar{T}_i)_{i\ge1}$ are assumed to be mutually independent.

From (2) and (3) we obtain the conditional p.d.f.'s of the claim and premium sizes:

$$f_{Y|T}(y \mid t) = \frac{f_{Y,T}(y,t)}{f_T(t)} = f_Y(y) + \theta h_Y(y)\big(2e^{-\lambda t} - 1\big), \quad y \ge 0,\ t \ge 0, \tag{4}$$

and

$$f_{\bar{Y}|\bar{T}}(y \mid t) = \frac{f_{\bar{Y},\bar{T}}(y,t)}{f_{\bar{T}}(t)} = f_{\bar{Y}}(y) + \bar{\theta} h_{\bar{Y}}(y)\big(2e^{-\bar{\lambda} t} - 1\big), \quad y \ge 0,\ t \ge 0. \tag{5}$$

Moreover, we suppose that the insurance company pays dividends to its shareholders according to the following threshold dividend strategy. Let $b > 0$ be a threshold. When the surplus is below b, no dividends are paid. When the surplus exceeds or equals b, dividends are paid continuously at a rate $d > 0$. Let $(X_t^b(x))_{t\ge0}$ denote the modified surplus process under this threshold dividend strategy. Then

$$X_t^b(x) = x + \sum_{i=1}^{\bar{N}_t} \bar{Y}_i - \sum_{i=1}^{N_t} Y_i - d\int_0^t \mathbb{1}\big(X_s^b(x) \ge b\big)\,\mathrm{d}s, \quad t \ge 0, \tag{6}$$

where $\mathbb{1}(\cdot)$ is the indicator function.

Let $(D_t)_{t\ge0}$ denote the dividend distributing process. For the threshold dividend strategy described above, we have

$$\mathrm{d}D_t = \begin{cases} d\,\mathrm{d}t & \text{if } X_t^b(x) \ge b, \\ 0 & \text{if } X_t^b(x) < b. \end{cases}$$

Next, let $\tau_b(x) = \inf\{t \geq 0 \colon X_t^b(x) < 0\}$ be the ruin time for the risk process $(X_t^b(x))_{t\geq 0}$ defined by (6). In what follows, we omit the dependence on x and write τ_b instead of $\tau_b(x)$ when no confusion can arise.

For $\delta_0 \geq 0$, the Gerber–Shiu function is defined by

$$m(x,b) = \mathbb{E}\big[e^{-\delta_0 \tau_b}\, w\big(X_{\tau_b -}^b(x), \big|X_{\tau_b}^b(x)\big|\big)\, \mathbb{1}(\tau_b < \infty) \mid X_0^b(x) = x\big], \quad x \geq 0,$$

where $w(\cdot,\cdot)$ is a bounded non-negative measurable function, $X_{\tau_b -}^b(x)$ is the surplus immediately before ruin and $|X_{\tau_b}^b(x)|$ is a deficit at ruin. Note that if $w(\cdot,\cdot) \equiv 1$ and $\delta_0 = 0$, then $m(x,b)$ becomes the infinite-horizon ruin probability

$$\psi(x) = \mathbb{E}\big[\mathbb{1}(\tau_b < \infty) \mid X_0^b(x) = x\big].$$

For $\delta > 0$, the expected discounted dividend payments until ruin are defined by

$$v(x,b) = \mathbb{E}\bigg[\int_0^{\tau_b} e^{-\delta t}\, \mathrm{d}D_t \mid X_0^b(x) = x\bigg], \quad x \geq 0.$$

For simplicity of notation, we also write $m(x)$ and $v(x)$ instead of $m(x,b)$ and $v(x,b)$, respectively, when no confusion can arise. Moreover, we set

$$m(x,b) = \begin{cases} m_1(x) & \text{if } x \in [0,b], \\ m_2(x) & \text{if } x \in [b,\infty), \end{cases} \tag{7}$$

and

$$v(x,b) = \begin{cases} v_1(x) & \text{if } x \in [0,b], \\ v_2(x) & \text{if } x \in [b,\infty). \end{cases} \tag{8}$$

Thus, the functions $m_1(x)$ and $v_1(x)$ are defined on $[0,b]$, the functions $m_2(x)$ and $v_2(x)$ are defined on $[b,\infty)$ and we have $m_1(b) = m_2(b)$ and $v_1(b) = v_2(b)$.

3 Equations for the Gerber–Shiu function

Theorem 1. *Let the surplus process $(X_t^b(x))_{t\geq 0}$ follow (6) under the above assumptions with $\theta \neq 0$ and $\bar{\theta} \neq 0$. Moreover, let the p.d.f.'s $f_Y(y)$ and $f_{\bar{Y}}(y)$ have the derivatives $f_Y'(y)$ and $f_{\bar{Y}}'(y)$ on $\mathbb{R}_+$, which are continuous and bounded on $\mathbb{R}_+$, and let $w(u_1,u_2)$ have the second derivatives $w''_{u_1u_1}(u_1,u_2)$, $w''_{u_1u_2}(u_1,u_2)$ and $w''_{u_2u_2}(u_1,u_2)$ on $\mathbb{R}_+^2$, which are continuous and bounded on $\mathbb{R}_+^2$ as functions of two variables. Then the Gerber–Shiu function $m(x)$ satisfies the equations*

$$\begin{aligned}
&(\lambda + \bar{\lambda} + \delta_0) m_1(x) \\
&\quad = \lambda\bigg(\int_0^x m_1(x-y) f_Y(y)\,\mathrm{d}y + \int_x^\infty w(x, y-x) f_Y(y)\,\mathrm{d}y\bigg) \\
&\qquad + \frac{\lambda\theta(\bar{\lambda}+\delta_0)}{2\lambda + \bar{\lambda} + \delta_0}\bigg(\int_0^x m_1(x-y) h_Y(y)\,\mathrm{d}y + \int_x^\infty w(x, y-x) h_Y(y)\,\mathrm{d}y\bigg)
\end{aligned}$$

$$
\begin{aligned}
&+\bar{\lambda}\bigg(\int_0^{b-x} m_1(x+y) f_{\bar{Y}}(y)\,\mathrm{d}y+\int_{b-x}^{\infty} m_2(x+y) f_{\bar{Y}}(y)\,\mathrm{d}y\bigg)\\
&+\frac{\bar{\lambda}\bar{\theta}(\lambda+\delta_0)}{\lambda+2\bar{\lambda}+\delta_0}\bigg(\int_0^{b-x} m_1(x+y) h_{\bar{Y}}(y)\,\mathrm{d}y+\int_{b-x}^{\infty} m_2(x+y) h_{\bar{Y}}(y)\,\mathrm{d}y\bigg),\\
&\qquad\qquad x\in[0,b],
\end{aligned}\tag{9}
$$

and

$$
\begin{aligned}
&d^3 m_2'''(x)+(4\lambda+4\bar{\lambda}+3\delta_0)d^2 m_2''(x)\\
&\quad+\big((\lambda+2\bar{\lambda}+\delta_0)(3\lambda+2\bar{\lambda}+2\delta_0)+(2\lambda+\bar{\lambda}+\delta_0)(\lambda+\bar{\lambda}+\delta_0)\big)d m_2'(x)\\
&\quad+(\lambda+2\bar{\lambda}+\delta_0)(2\lambda+\bar{\lambda}+\delta_0)(\lambda+\bar{\lambda}+\delta_0)m_2(x)\\
&=(\lambda+2\bar{\lambda}+\delta_0)(2\lambda+\bar{\lambda}+\delta_0)\beta_1(x)+(3\lambda+3\bar{\lambda}+2\delta_0)d\beta_1'(x)\\
&\quad+d^2\beta_1''(x)-2(\lambda+2\bar{\lambda}+\delta_0)\beta_2(x)-2d\beta_2'(x)\\
&\quad+2\bar{\lambda}^2\bar{\theta}(\bar{\lambda}-\lambda)\int_0^{\infty} m_2(x+y)h_{\bar{Y}}(y)\,\mathrm{d}y,\quad x\in[b,\infty),
\end{aligned}\tag{10}
$$

where

$$
\begin{aligned}
\beta_1(x)=\lambda&\int_{x-b}^{x} m_1(x-y)\big(f_Y(y)+\theta h_Y(y)\big)\,\mathrm{d}y\\
&+\lambda\int_0^{x-b} m_2(x-y)\big(f_Y(y)+\theta h_Y(y)\big)\,\mathrm{d}y\\
&+\lambda\int_x^{\infty} w(x,y-x)\big(f_Y(y)+\theta h_Y(y)\big)\,\mathrm{d}y\\
&+\bar{\lambda}\int_0^{\infty} m_2(x+y)\big(f_{\bar{Y}}(y)+\bar{\theta}h_{\bar{Y}}(y)\big)\,\mathrm{d}y,\quad x\in[b,\infty),
\end{aligned}
$$

and

$$
\begin{aligned}
\beta_2(x)=\lambda^2\theta&\bigg(\int_{x-b}^{x} m_1(x-y)h_Y(y)\,\mathrm{d}y+\int_0^{x-b} m_2(x-y)h_Y(y)\,\mathrm{d}y\\
&+\int_x^{\infty} w(x,y-x)h_Y(y)\,\mathrm{d}y\bigg)+\bar{\lambda}^2\bar{\theta}\int_0^{\infty} m_2(x+y)h_{\bar{Y}}(y)\,\mathrm{d}y,\quad x\in[b,\infty).
\end{aligned}
$$

Proof. It is easily seen that the time of the first jump of $(X_t^b(x))_{t\geq 0}$, $T_1\wedge\bar{T}_1$, is exponentially distributed with mean $1/(\lambda+\bar{\lambda})$. Furthermore, $\mathbb{P}[T_1\wedge\bar{T}_1=T_1]=\lambda/(\lambda+\bar{\lambda})$ and $\mathbb{P}[T_1\wedge\bar{T}_1=\bar{T}_1]=\bar{\lambda}/(\lambda+\bar{\lambda})$.

We first deal with the case $x\in[0,b]$. Considering the time and the size of the first jump of $(X_t^b(x))_{t\geq 0}$ and applying the law of total probability we obtain

$$
\begin{aligned}
m(x)=\int_0^{\infty} e^{-(\lambda+\bar{\lambda})t}\bigg(&\lambda\int_0^{x} e^{-\delta_0 t} m(x-y) f_{Y|T}(y\mid t)\,\mathrm{d}y\\
&+\lambda\int_x^{\infty} e^{-\delta_0 t} w(x,y-x) f_{Y|T}(y\mid t)\,\mathrm{d}y
\end{aligned}\tag{11}
$$

$$+\,\bar\lambda\int_0^\infty e^{-\delta_0 t} m(x+y)\, f_{\bar Y|\bar T}(y\,|\,t)\,\mathrm{d}y\Big)\mathrm{d}t, \quad x\in[0,b].$$

Substituting (4) and (5) into (11) and taking into account (7) give

$$\begin{aligned} m_1(x) = \int_0^\infty e^{-(\lambda+\bar\lambda+\delta_0)t}\Big(\lambda\int_0^x m_1(x-y)\big(f_Y(y)+\theta h_Y(y)\big(2e^{-\lambda t}-1\big)\big)\,\mathrm{d}y \\ +\lambda\int_x^\infty w(x,y-x)\big(f_Y(y)+\theta h_Y(y)\big(2e^{-\lambda t}-1\big)\big)\,\mathrm{d}y \\ +\bar\lambda\int_0^{b-x} m_1(x+y)\big(f_{\bar Y}(y)+\bar\theta h_{\bar Y}(y)\big(2e^{-\bar\lambda t}-1\big)\big)\,\mathrm{d}y \\ +\bar\lambda\int_{b-x}^\infty m_2(x+y)\big(f_{\bar Y}(y)+\bar\theta h_{\bar Y}(y)\big(2e^{-\bar\lambda t}-1\big)\big)\,\mathrm{d}y\Big)\mathrm{d}t, \quad x\in[0,b]. \end{aligned} \tag{12}$$

Separating the integrals on the right-hand side of (12) into integrals w.r.t. either t or y yields

$$\begin{aligned} m_1(x) = \lambda\int_0^\infty e^{-(\lambda+\bar\lambda+\delta_0)t}\,\mathrm{d}t\,\Big(\int_0^x m_1(x-y)f_Y(y)\,\mathrm{d}y \\ +\int_x^\infty w(x,y-x)f_Y(y)\,\mathrm{d}y\Big) \\ +\lambda\theta\int_0^\infty e^{-(\lambda+\bar\lambda+\delta_0)t}\,\big(2e^{-\lambda t}-1\big)\,\mathrm{d}t\,\Big(\int_0^x m_1(x-y)h_Y(y)\,\mathrm{d}y \\ +\int_x^\infty w(x,y-x)h_Y(y)\,\mathrm{d}y\Big) \\ +\bar\lambda\int_0^\infty e^{-(\lambda+\bar\lambda+\delta_0)t}\,\mathrm{d}t\,\Big(\int_0^{b-x} m_1(x+y)f_{\bar Y}(y)\,\mathrm{d}y \\ +\int_{b-x}^\infty m_2(x+y)f_{\bar Y}(y)\,\mathrm{d}y\Big) \\ +\bar\lambda\bar\theta\int_0^\infty e^{-(\lambda+\bar\lambda+\delta_0)t}\,\big(2e^{-\bar\lambda t}-1\big)\,\mathrm{d}t\,\Big(\int_0^{b-x} m_1(x+y)h_{\bar Y}(y)\,\mathrm{d}y \\ +\int_{b-x}^\infty m_2(x+y)h_{\bar Y}(y)\,\mathrm{d}y\Big), \quad x\in[0,b]. \end{aligned} \tag{13}$$

Taking the integrals w.r.t. t on the right-hand side of (13) we get

$$\begin{aligned} m_1(x) = \frac{\lambda}{\lambda+\bar\lambda+\delta_0}\Big(\int_0^x m_1(x-y)f_Y(y)\,\mathrm{d}y+\int_x^\infty w(x,y-x)f_Y(y)\,\mathrm{d}y\Big) \\ +\frac{\lambda\theta(\bar\lambda+\delta_0)}{(2\lambda+\bar\lambda+\delta_0)(\lambda+\bar\lambda+\delta_0)}\Big(\int_0^x m_1(x-y)h_Y(y)\,\mathrm{d}y \\ +\int_x^\infty w(x,y-x)h_Y(y)\,\mathrm{d}y\Big) \end{aligned}$$

$$+\frac{\bar{\lambda}}{\lambda+\bar{\lambda}+\delta_0}\left(\int_0^{b-x} m_1(x+y) f_{\bar{Y}}(y)\,\mathrm{d}y+\int_{b-x}^{\infty} m_2(x+y) f_{\bar{Y}}(y)\,\mathrm{d}y\right)$$
$$+\frac{\bar{\lambda}\bar{\theta}(\lambda+\delta_0)}{(\lambda+2\bar{\lambda}+\delta_0)(\lambda+\bar{\lambda}+\delta_0)}\left(\int_0^{b-x} m_1(x+y) h_{\bar{Y}}(y)\,\mathrm{d}y\right.$$
$$\left.+\int_{b-x}^{\infty} m_2(x+y) h_{\bar{Y}}(y)\,\mathrm{d}y\right),\quad x\in[0,b],$$

which yields (9).

Let now $x\in[b,\infty)$. Considering the time and the size of the first jump of $(X_t^b(x))_{t\geq 0}$ and applying the law of total probability we have

$$\begin{aligned}
m(x)=&\int_0^{(x-b)/d} e^{-(\lambda+\bar{\lambda})t}\left(\lambda\int_0^{x-dt} e^{-\delta_0 t} m(x-dt-y) f_{Y|T}(y\,|\,t)\,\mathrm{d}y\right.\\
&+\lambda\int_{x-dt}^{\infty} e^{-\delta_0 t} w(x-dt, y-x-dt) f_{Y|T}(y\,|\,t)\,\mathrm{d}y\\
&\left.+\bar{\lambda}\int_0^{\infty} e^{-\delta_0 t} m(x-dt+y) f_{\bar{Y}|\bar{T}}(y\,|\,t)\,\mathrm{d}y\right)\mathrm{d}t\\
&+\int_{(x-b)/d}^{\infty} e^{-(\lambda+\bar{\lambda})t}\left(\lambda\int_0^{b} e^{-\delta_0 t} m(b-y) f_{Y|T}(y\,|\,t)\,\mathrm{d}y\right.\\
&+\lambda\int_b^{\infty} e^{-\delta_0 t} w(b, y-b) f_{Y|T}(y\,|\,t)\,\mathrm{d}y\\
&\left.+\bar{\lambda}\int_0^{\infty} e^{-\delta_0 t} m(b+y) f_{\bar{Y}|\bar{T}}(y\,|\,t)\,\mathrm{d}y\right)\mathrm{d}t,\quad x\in[b,\infty).
\end{aligned}\tag{14}$$

Substituting (4) and (5) into (14) and taking into account (7) give

$$m_2(x)=I_{1,2,3}(x)+I_{4,5,6}(x),\quad x\in[b,\infty),\tag{15}$$

where

$$\begin{aligned}
I_{1,2,3}(x)=&\int_0^{(x-b)/d} e^{-(\lambda+\bar{\lambda}+\delta_0)t}\\
&\times\left(\lambda\int_0^{x-dt-b} m_2(x-dt-y)\big(f_Y(y)+\theta h_Y(y)\big(2e^{-\lambda t}-1\big)\big)\,\mathrm{d}y\right.\\
&+\lambda\int_{x-dt-b}^{x-dt} m_1(x-dt-y)\big(f_Y(y)+\theta h_Y(y)\big(2e^{-\lambda t}-1\big)\big)\,\mathrm{d}y\\
&+\lambda\int_{x-dt}^{\infty} w(x-dt, y-x+dt)\big(f_Y(y)+\theta h_Y(y)\big(2e^{-\lambda t}-1\big)\big)\,\mathrm{d}y\\
&\left.+\bar{\lambda}\int_0^{\infty} m_2(x-dt+y)\big(f_{\bar{Y}}(y)+\bar{\theta} h_{\bar{Y}}(y)\big(2e^{-\bar{\lambda}t}-1\big)\big)\,\mathrm{d}y\right)\mathrm{d}t
\end{aligned}$$

and

$$I_{4,5,6}(x)=\int_{(x-b)/d}^{\infty} e^{-(\lambda+\bar{\lambda}+\delta_0)t}$$

$$\times \bigg(\lambda \int_0^b m_1(b-y)\big(f_Y(y)+\theta h_Y(y)\big(2e^{-\lambda t}-1\big)\big)\,\mathrm{d}y$$
$$+\lambda \int_b^\infty w(b,y-b)\big(f_Y(y)+\theta h_Y(y)\big(2e^{-\lambda t}-1\big)\big)\,\mathrm{d}y$$
$$+\bar\lambda \int_0^\infty m_2(b+y)\big(f_{\bar Y}(y)+\bar\theta h_{\bar Y}(y)\big(2e^{-\bar\lambda t}-1\big)\big)\,\mathrm{d}y\bigg)\mathrm{d}t.$$

Changing the variable $x-dt=s$ in the outer integral of the expression for $I_{1,2,3}(x)$ yields

$$\begin{aligned}
I_{1,2,3}(x) &= \frac{1}{d}\int_b^x e^{-(\lambda+\bar\lambda+\delta_0)(x-s)/d} \\
&\quad\times \bigg(\lambda \int_0^{s-b} m_2(s-y)\big(f_Y(y)+\theta h_Y(y)\big(2e^{-\lambda(x-s)/d}-1\big)\big)\,\mathrm{d}y \\
&\quad+\lambda \int_{s-b}^{s} m_1(s-y)\big(f_Y(y)+\theta h_Y(y)\big(2e^{-\lambda(x-s)/d}-1\big)\big)\,\mathrm{d}y \\
&\quad+\lambda \int_{s}^{\infty} w(s,y-s)\big(f_Y(y)+\theta h_Y(y)\big(2e^{-\lambda(x-s)/d}-1\big)\big)\,\mathrm{d}y \\
&\quad+\bar\lambda \int_0^\infty m_2(s+y)\big(f_{\bar Y}(y)+\bar\theta h_{\bar Y}(y)\big(2e^{-\bar\lambda(x-s)/d}-1\big)\big)\,\mathrm{d}y\bigg)\mathrm{d}s \\
&= \frac{1}{d}e^{-(\lambda+\bar\lambda+\delta_0)x/d} I_1(x)+\frac{2}{d}e^{-(2\lambda+\bar\lambda+\delta_0)x/d} I_2(x) \\
&\quad+\frac{2}{d}e^{-(\lambda+2\bar\lambda+\delta_0)x/d} I_3(x), \quad x\in[b,\infty),
\end{aligned} \tag{16}$$

where

$$\begin{aligned}
I_1(x) &= \int_b^x e^{(\lambda+\bar\lambda+\delta_0)s/d}\bigg(\lambda\int_0^{s-b} m_2(s-y)\big(f_Y(y)-\theta h_Y(y)\big)\,\mathrm{d}y \\
&\quad+\lambda\int_{s-b}^{s} m_1(s-y)\big(f_Y(y)-\theta h_Y(y)\big)\,\mathrm{d}y \\
&\quad+\lambda\int_{s}^{\infty} w(s,y-s)\big(f_Y(y)-\theta h_Y(y)\big)\,\mathrm{d}y \\
&\quad+\bar\lambda\int_0^\infty m_2(s+y)\big(f_{\bar Y}(y)-\bar\theta h_{\bar Y}(y)\big)\,\mathrm{d}y\bigg)\mathrm{d}s, \\
I_2(x) &= \lambda\theta\int_b^x e^{(2\lambda+\bar\lambda+\delta_0)s/d}\bigg(\int_0^{s-b} m_2(s-y)h_Y(y)\,\mathrm{d}y \\
&\quad+\int_{s-b}^{s} m_1(s-y)h_Y(y)\,\mathrm{d}y+\int_s^\infty w(s,y-s)h_Y(y)\,\mathrm{d}y\bigg)\mathrm{d}s
\end{aligned}$$

and

$$I_3(x) = \bar\lambda\bar\theta\int_b^x e^{(\lambda+2\bar\lambda+\delta_0)s/d}\bigg(\int_0^\infty m_2(s+y)h_{\bar Y}(y)\,\mathrm{d}y\bigg)\mathrm{d}s.$$

Separating the integrals in the expression for $I_{4,5,6}(x)$ into integrals w.r.t. either t or y we get

$$
\begin{aligned}
I_{4,5,6}(x) = \lambda \int_{(x-b)/d}^{\infty} & e^{-(\lambda+\bar{\lambda}+\delta_0)t}\,\mathrm{d}t \left(\int_0^b m_1(b-y) f_Y(y)\,\mathrm{d}y \right.\\
&\left. + \int_b^{\infty} w(b, y-b) f_Y(y)\,\mathrm{d}y\right)\\
& + \lambda\theta \int_{(x-b)/d}^{\infty} e^{-(\lambda+\bar{\lambda}+\delta_0)t} \left(2e^{-\lambda t}-1\right)\mathrm{d}t \left(\int_0^b m_1(b-y) h_Y(y)\,\mathrm{d}y \right.\\
&\left. + \int_b^{\infty} w(b, y-b) h_Y(y)\,\mathrm{d}y\right)\\
& + \bar{\lambda} \int_{(x-b)/d}^{\infty} e^{-(\lambda+\bar{\lambda}+\delta_0)t}\,\mathrm{d}t \int_0^{\infty} m_2(x+y) f_{\bar{Y}}(y)\,\mathrm{d}y\\
& + \bar{\lambda}\bar{\theta} \int_{(x-b)/d}^{\infty} e^{-(\lambda+\bar{\lambda}+\delta_0)t} \left(2e^{-\bar{\lambda} t}-1\right)\mathrm{d}t \int_0^{\infty} m_2(b+y) h_{\bar{Y}}(y)\,\mathrm{d}y.
\end{aligned}
\tag{17}
$$

Taking the integrals w.r.t. t on the right-hand side of (17) we obtain

$$
I_{4,5,6}(x) = I_4(x) + I_5(x) + I_6(x), \quad x \in [b, \infty), \tag{18}
$$

where

$$
\begin{aligned}
I_4(x) = {} & \frac{e^{-(\lambda+\bar{\lambda}+\delta_0)(x-b)/d}}{\lambda+\bar{\lambda}+\delta_0}\\
& \times \left(\lambda \int_0^b m_1(b-y) f_Y(y)\,\mathrm{d}y + \lambda \int_b^{\infty} w(b, y-b) f_Y(y)\,\mathrm{d}y \right.\\
& - \lambda\theta \int_0^b m_1(b-y) h_Y(y)\,\mathrm{d}y - \lambda\theta \int_b^{\infty} w(b, y-b) h_Y(y)\,\mathrm{d}y\\
& \left. + \bar{\lambda} \int_0^{\infty} m_2(b+y) f_{\bar{Y}}(y)\,\mathrm{d}y - \bar{\lambda}\bar{\theta} \int_0^{\infty} m_2(b+y) h_{\bar{Y}}(y)\,\mathrm{d}y\right),\\
I_5(x) = {} & \frac{2\lambda\theta e^{-(2\lambda+\bar{\lambda}+\delta_0)(x-b)/d}}{2\lambda+\bar{\lambda}+\delta_0}\\
& \times \left(\int_0^b m_1(b-y) h_Y(y)\,\mathrm{d}y + \int_b^{\infty} w(b, y-b) h_Y(y)\,\mathrm{d}y\right)
\end{aligned}
$$

and

$$
I_6(x) = \frac{2\bar{\lambda}\bar{\theta} e^{-(\lambda+2\bar{\lambda}+\delta_0)(x-b)/d}}{\lambda+2\bar{\lambda}+\delta_0} \int_0^{\infty} m_2(b+y) h_{\bar{Y}}(y)\,\mathrm{d}y.
$$

Thus, substituting (16) and (18) into (15) we have

$$
\begin{aligned}
m_2(x) = {} & \frac{1}{d} e^{-(\lambda+\bar{\lambda}+\delta_0)x/d} I_1(x) + \frac{2}{d} e^{-(2\lambda+\bar{\lambda}+\delta_0)x/d} I_2(x)\\
& + \frac{2}{d} e^{-(\lambda+2\bar{\lambda}+\delta_0)x/d} I_3(x) + I_4(x) + I_5(x) + I_6(x), \quad x \in [b, \infty).
\end{aligned}
\tag{19}
$$

It is easily seen from (9) that $m_1(x)$ is continuous on $[0, b]$, and from (19) we conclude that $m_2(x)$ is continuous on $[b, \infty)$. Indeed, the right-hand sides of (9) and (19) are continuous on $[0, b]$ and $[b, \infty)$, respectively, and so are the left-hand sides. Therefore, from (19) we deduce that $m_2(x)$ is differentiable on $[b, \infty)$. Differentiating (19) gives

$$\begin{aligned} m_2'(x) = &-\frac{\lambda + \bar{\lambda} + \delta_0}{d^2} e^{-(\lambda+\bar{\lambda}+\delta_0)x/d} I_1(x) \\ &- \frac{2(2\lambda + \bar{\lambda} + \delta_0)}{d^2} e^{-(2\lambda+\bar{\lambda}+\delta_0)x/d} I_2(x) \\ &- \frac{2(\lambda + 2\bar{\lambda} + \delta_0)}{d^2} e^{-(\lambda+2\bar{\lambda}+\delta_0)x/d} I_3(x) \\ &- \frac{\lambda + \bar{\lambda} + \delta_0}{d} I_4(x) - \frac{2\lambda + \bar{\lambda} + \delta_0}{d} I_5(x) \\ &- \frac{\lambda + 2\bar{\lambda} + \delta_0}{d} I_6(x) + \frac{1}{d} \beta_1(x), \quad x \in [b, \infty), \end{aligned} \tag{20}$$

where the function $\beta_1(x)$ is defined in the assertion of the theorem.

Multiplying (19) by $(\lambda + \bar{\lambda} + \delta_0)/d$ and adding (20) we get

$$\begin{aligned} dm_2'(x) + (\lambda + \bar{\lambda} + \delta_0)m_2(x) = &-\frac{2\lambda}{d} e^{-(2\lambda+\bar{\lambda}+\delta_0)x/d} I_2(x) \\ - \frac{2\bar{\lambda}}{d} e^{-(\lambda+2\bar{\lambda}+\delta_0)x/d} I_3(x) &- \lambda I_5(x) - \bar{\lambda} I_6(x) + \beta_1(x), \quad x \in [b, \infty). \end{aligned} \tag{21}$$

Since $m_1(x)$ and $m_2(x)$ are continuous and bounded on $[0, b]$ and $[b, \infty)$, respectively, and $w(u_1, u_2)$ is continuous and bounded on $\mathbb{R}_+^2$ as a function of two variables, from (21) we conclude that so is $m_2'(x)$ on $[b, \infty)$. Taking into account that $f_Y'(y)$ and $f_{\bar{Y}}'(y)$ are continuous and bounded on $\mathbb{R}_+$, and $w_{u_1}'(u_1, u_2)$ and $w_{u_2}'(u_1, u_2)$ are continuous and bounded on $\mathbb{R}_+^2$, from (9) we conclude that so is $m_1'(x)$ on $[0, b]$. Hence, $\beta_1(x)$ is differentiable on $[b, \infty)$. From this and (21) it follows that $m_2(x)$ is twice differentiable on $[b, \infty)$. Differentiating (21) gives

$$\begin{aligned} dm_2''(x) + (\lambda + \bar{\lambda} + \delta_0)m_2'(x) = &\frac{2\lambda(2\lambda + \bar{\lambda} + \delta_0)}{d^2} e^{-(2\lambda+\bar{\lambda}+\delta_0)x/d} I_2(x) \\ + \frac{2\bar{\lambda}(\lambda + 2\bar{\lambda} + \delta_0)}{d^2} e^{-(\lambda+2\bar{\lambda}+\delta_0)x/d} I_3(x) &+ \frac{\lambda(2\lambda + \bar{\lambda} + \delta_0)}{d} I_5(x) \\ + \frac{\bar{\lambda}(\lambda + 2\bar{\lambda} + \delta_0)}{d} I_6(x) + \beta_1'(x) &- \frac{2}{d} \beta_2(x), \quad x \in [b, \infty), \end{aligned} \tag{22}$$

where

$$\begin{aligned} \beta_1'(x) = \lambda &\int_{x-b}^{x} m_1'(x-y)\big(f_Y(y) + \theta h_Y(y)\big)\,\mathrm{d}y \\ &+ \lambda \int_0^{x-b} m_2'(x-y)\big(f_Y(y) + \theta h_Y(y)\big)\,\mathrm{d}y \end{aligned}$$

$$
\begin{aligned}
&+\lambda\int_x^\infty \big(w'_{u_1}(x,y-x)-w'_{u_2}(x,y-x)\big)\big(f_Y(y)+\theta h_Y(y)\big)\,\mathrm{d}y\\
&+\bar\lambda\int_0^\infty m'_2(x+y)\big(f_{\bar Y}(y)+\bar\theta h_{\bar Y}(y)\big)\,\mathrm{d}y\\
&+\lambda\big(m_1(0)-w(x,0)\big)\big(f_Y(x)+\theta h_Y(x)\big),\quad x\in[b,\infty),
\end{aligned}
$$

which is continuous and bounded on $[b,\infty)$, and the function $\beta_2(x)$ is defined in the assertion of the theorem. Here $w'_{u_1}(\cdot,\cdot)$ and $w'_{u_2}(\cdot,\cdot)$ stand for the partial derivatives of $w(u_1,u_2)$ w.r.t. the first and the second variables, respectively.

Multiplying (21) by $(2\lambda+\bar\lambda+\delta_0)/d$ and adding (22) we obtain

$$
\begin{aligned}
&d^2m''_2(x)+(3\lambda+2\bar\lambda+2\delta_0)dm'_2(x)+(2\lambda+\bar\lambda+\delta_0)(\lambda+\bar\lambda+\delta_0)m_2(x)\\
&\quad=\frac{2\bar\lambda(\bar\lambda-\lambda)}{d}e^{-(\lambda+2\bar\lambda+\delta_0)x/d}I_3(x)+\bar\lambda(\bar\lambda-\lambda)I_6(x)\\
&\qquad+(2\lambda+\bar\lambda+\delta_0)\beta_1(x)+d\beta'_1(x)-2\beta_2(x),\quad x\in[b,\infty).
\end{aligned}\tag{23}
$$

It is easily seen from (23) that $m''_2(x)$ is continuous and bounded on $[b,\infty)$. Taking into account that $f'_Y(y)$ and $f'_{\bar Y}(y)$ are continuous and bounded on $\mathbb{R}_+$, and $w''_{u_1u_1}(u_1,u_2)$, $w''_{u_1u_2}(u_1,u_2)$ and $w''_{u_2u_2}(u_1,u_2)$ are continuous and bounded on $\mathbb{R}^2_+$, from (9) we conclude that so is $m''_1(x)$ on $[0,b]$. Hence, $\beta_1(x)$ is twice differentiable on $[b,\infty)$. Moreover, applying similar arguments shows that $\beta_2(x)$ is differentiable on $[b,\infty)$. From this and (23) it follows that $m_2(x)$ has the third derivative on $[b,\infty)$. Differentiating (23) gives

$$
\begin{aligned}
&d^2m'''_2(x)+(3\lambda+2\bar\lambda+2\delta_0)dm''_2(x)+(2\lambda+\bar\lambda+\delta_0)(\lambda+\bar\lambda+\delta_0)m'_2(x)\\
&\quad=-\frac{2\bar\lambda(\bar\lambda-\lambda)(\lambda+2\bar\lambda+\delta_0)}{d^2}e^{-(\lambda+2\bar\lambda+\delta_0)x/d}I_3(x)\\
&\qquad-\frac{\bar\lambda(\bar\lambda-\lambda)(\lambda+2\bar\lambda+\delta_0)}{d}I_6(x)+(2\lambda+\bar\lambda+\delta_0)\beta'_1(x)+d\beta''_1(x)\\
&\qquad-2\beta'_2(x)+\frac{2\bar\lambda^2\bar\theta(\bar\lambda-\lambda)}{d}\int_0^\infty m_2(x+y)h_{\bar Y}(y)\,\mathrm{d}y,\quad x\in[b,\infty).
\end{aligned}\tag{24}
$$

Multiplying (23) by $(\lambda+2\bar\lambda+\delta_0)/d$ and adding (24) yield (10), which completes the proof. □

Remark 1. *To solve equations* (9) *and* (10)*, we need some boundary conditions. The first one is* $m_1(b)=m_2(b)$. *Next, using standard considerations (see, e.g., [34, 36, 39]) we can show that* $\lim_{x\to\infty}m_2(x)=0$ *provided that the net profit condition holds. Finally, we can substitute* $x=b$ *into the intermediate equations (e.g., equation* (21)*) to get additional boundary conditions involving derivatives of* $m_2(x)$. *Furthermore, equations* (9) *and* (10) *are not solvable analytically in the general case, so we can use, for instance, numerical methods. Nevertheless, we can give explicit expressions for* $m(x)$ *in some particular cases (see Section 5). The uniqueness of the required solutions to these equations should be justified in each case.*

Remark 2. *The corresponding model without dividend payments is obtained by $b \to \infty$. In this case, the Gerber–Shiu function $m(x)$ satisfies the integral equation*

$$\begin{aligned}(\lambda+\bar{\lambda}+\delta_0)m(x)\\ &= \lambda\biggl(\int_0^x m(u) f_Y(x-u)\,\mathrm{d}u + \int_0^\infty w(x,u) f_Y(x+u)\,\mathrm{d}u\biggr)\\ &\quad + \frac{\lambda\theta(\bar{\lambda}+\delta_0)}{2\lambda+\bar{\lambda}+\delta_0}\biggl(\int_0^x m(u)h_Y(x-u)\,\mathrm{d}u + \int_0^\infty w(x,u)h_Y(x+u)\,\mathrm{d}u\biggr)\\ &\quad + \bar{\lambda}\int_x^\infty m(u) f_{\bar{Y}}(u-x)\,\mathrm{d}u + \frac{\bar{\lambda}\bar{\theta}(\lambda+\delta_0)}{\lambda+2\bar{\lambda}+\delta_0}\int_x^\infty m(u)h_{\bar{Y}}(u-x)\,\mathrm{d}u,\\ &\qquad x\in[0,\infty).\end{aligned} \tag{25}$$

Note that equation (25) *for the ruin probability coincides with the equation derived in [9] (see also [34]) if $\theta = 0$ and $\bar{\theta} = 0$.*

Remark 3. *In Theorem 1, we assume that $\theta \neq 0$ and $\bar{\theta} \neq 0$. Otherwise, we do not need to differentiate* (19) *three times and can obtain equations not involving the third derivative of $m_2(x)$ instead of* (10).

Thus, if $\theta = 0$ and $\bar{\theta} = 0$, from (21) *we have*

$$dm_2'(x) + (\lambda+\bar{\lambda}+\delta_0)m_2(x) = \beta_1(x), \quad x\in[b,\infty). \tag{26}$$

Next, if $\theta \neq 0$ and $\bar{\theta} = 0$, from (23) *we have*

$$\begin{aligned}&d^2m_2''(x) + (3\lambda+2\bar{\lambda}+2\delta_0)dm_2'(x) + (2\lambda+\bar{\lambda}+\delta_0)(\lambda+\bar{\lambda}+\delta_0)m_2(x)\\ &\quad = (2\lambda+\bar{\lambda}+\delta_0)\beta_1(x) + d\beta_1'(x) - 2\beta_2(x), \quad x\in[b,\infty).\end{aligned} \tag{27}$$

Finally, if $\theta = 0$ and $\bar{\theta} \neq 0$, multiplying (21) *by $(\lambda+2\bar{\lambda}+\delta_0)/d$ and adding* (22) *we get*

$$\begin{aligned}&d^2m_2''(x) + (2\lambda+3\bar{\lambda}+2\delta_0)dm_2'(x) + (\lambda+2\bar{\lambda}+\delta_0)(\lambda+\bar{\lambda}+\delta_0)m_2(x)\\ &\quad = (\lambda+2\bar{\lambda}+\delta_0)\beta_1(x) + d\beta_1'(x) - 2\beta_2(x), \quad x\in[b,\infty).\end{aligned} \tag{28}$$

Note that to obtain (26)*–*(28)*, it is enough to have weaker smoothness assumptions on $f_Y(y)$, $f_{\bar{Y}}(y)$ and $w(u_1,u_2)$.*

Equation (9) *holds in all possible cases. Furthermore,* (9) *involves no derivatives and holds under weaker assumptions than* (10). *To be more precise, we do not need the differentiability of $f_Y(y)$, $f_{\bar{Y}}(y)$ and $w(u_1,u_2)$ to get* (9).

4 Equations for the expected discounted dividend payments until ruin

Theorem 2. *Let the surplus process $(X_t^b(x))_{t\ge 0}$ follow* (6) *under the above assumptions with $\theta \neq 0$ and $\bar{\theta} \neq 0$. Moreover, let the p.d.f.'s $f_Y(y)$ and $f_{\bar{Y}}(y)$ have the derivatives $f_Y'(y)$ and $f_{\bar{Y}}'(y)$ on $\mathbb{R}_+$, which are continuous and bounded on $\mathbb{R}_+$. Then the expected discounted dividend payments until ruin $v(x)$ satisfy the equations*

$$
\begin{aligned}
&(\lambda+\bar{\lambda}+\delta)v_1(x)\\
&\quad=\lambda\int_0^x v_1(x-y)f_Y(y)\,\mathrm{d}y+\frac{\lambda\theta(\bar{\lambda}+\delta)}{2\lambda+\bar{\lambda}+\delta}\int_0^x v_1(x-y)h_Y(y)\,\mathrm{d}y\\
&\qquad+\bar{\lambda}\biggl(\int_0^{b-x} v_1(x+y)f_{\bar{Y}}(y)\,\mathrm{d}y+\int_{b-x}^{\infty} v_2(x+y)f_{\bar{Y}}(y)\,\mathrm{d}y\biggr)\\
&\qquad+\frac{\bar{\lambda}\bar{\theta}(\lambda+\delta)}{\lambda+2\bar{\lambda}+\delta}\biggl(\int_0^{b-x} v_1(x+y)h_{\bar{Y}}(y)\,\mathrm{d}y+\int_{b-x}^{\infty} v_2(x+y)h_{\bar{Y}}(y)\,\mathrm{d}y\biggr),\\
&\qquad\qquad x\in[0,b],
\end{aligned}
\tag{29}
$$

and

$$
\begin{aligned}
&d^3v_2'''(x)+(4\lambda+4\bar{\lambda}+3\delta)d^2v_2''(x)\\
&\quad+\bigl((\lambda+2\bar{\lambda}+\delta)(3\lambda+2\bar{\lambda}+2\delta)+(2\lambda+\bar{\lambda}+\delta)(\lambda+\bar{\lambda}+\delta)\bigr)dv_2'(x)\\
&\quad+(\lambda+2\bar{\lambda}+\delta)(2\lambda+\bar{\lambda}+\delta)(\lambda+\bar{\lambda}+\delta)v_2(x)\\
&\quad=(\lambda+2\bar{\lambda}+\delta)(2\lambda+\bar{\lambda}+\delta)\beta_3(x)+(3\lambda+3\bar{\lambda}+2\delta)d\beta_3'(x)+d^2\beta_3''(x)\\
&\qquad-2(\lambda+2\bar{\lambda}+\delta)\beta_4(x)-2d\beta_4'(x)+(\lambda+2\bar{\lambda}+\delta)(2\lambda+\bar{\lambda}+\delta)d\\
&\qquad+2\bar{\lambda}^2\bar{\theta}(\bar{\lambda}-\lambda)\int_0^{\infty} v_2(x+y)h_{\bar{Y}}(y)\,\mathrm{d}y,\quad x\in[b,\infty),
\end{aligned}
\tag{30}
$$

where

$$
\begin{aligned}
\beta_3(x)=\lambda&\int_{x-b}^{x} v_1(x-y)\bigl(f_Y(y)+\theta h_Y(y)\bigr)\,\mathrm{d}y\\
&+\lambda\int_0^{x-b} v_2(x-y)\bigl(f_Y(y)+\theta h_Y(y)\bigr)\,\mathrm{d}y\\
&+\bar{\lambda}\int_0^{\infty} v_2(x+y)\bigl(f_{\bar{Y}}(y)+\bar{\theta}h_{\bar{Y}}(y)\bigr)\,\mathrm{d}y,\quad x\in[b,\infty),
\end{aligned}
$$

and

$$
\begin{aligned}
\beta_4(x)=\lambda^2\theta&\biggl(\int_{x-b}^{x} v_1(x-y)h_Y(y)\,\mathrm{d}y+\int_0^{x-b} v_2(x-y)h_Y(y)\,\mathrm{d}y\biggr)\\
&+\bar{\lambda}^2\bar{\theta}\int_x^{\infty} v_2(u)h_{\bar{Y}}(u-x)\,\mathrm{d}u,\quad x\in[b,\infty).
\end{aligned}
$$

Proof. The proof is similar to the proof of Theorem 1, so we omit detailed considerations.

Let $x\in[0,b]$. Considering the time and the size of the first jump of $(X_t^b(x))_{t\ge0}$ and applying the law of total probability, we obtain

$$
\begin{aligned}
v(x)=\int_0^{\infty} e^{-(\lambda+\bar{\lambda})t}\biggl(&\lambda\int_0^x e^{-\delta t}v(x-y)f_{Y|T}(y\,|\,t)\,\mathrm{d}y\\
&+\bar{\lambda}\int_0^{\infty} e^{-\delta t}v(x+y)f_{\bar{Y}|\bar{T}}(y\,|\,t)\,\mathrm{d}y\biggr)\mathrm{d}t,\quad x\in[0,b].
\end{aligned}
\tag{31}
$$

Comparing (31) with (11) and applying arguments similar to those in the proof of Theorem 1 yield (29).

Let now $x \in [b, \infty)$. By the law of total probability, we have

$$\begin{aligned} v(x) = \int_0^{(x-b)/d} & e^{-(\lambda+\bar{\lambda})t}\bigg((\lambda+\bar{\lambda})\int_0^t de^{-\delta s}\,\mathrm{d}s \\ & + \lambda\int_0^{x-dt} e^{-\delta t} v(x-dt-y) f_{Y\,|\,T}(y\,|\,t)\,\mathrm{d}y \\ & + \bar{\lambda}\int_0^{\infty} e^{-\delta t} v(x-dt+y) f_{\bar{Y}\,|\,\bar{T}}(y\,|\,t)\,\mathrm{d}y\bigg)\mathrm{d}t \\ + \int_{(x-b)/d}^{\infty} & e^{-(\lambda+\bar{\lambda})t}\bigg((\lambda+\bar{\lambda})\int_0^{(x-b)/d} de^{-\delta s}\,\mathrm{d}s \\ & + \lambda\int_0^{b} e^{-\delta t} v(b-y) f_{Y\,|\,T}(y\,|\,t)\,\mathrm{d}y \\ & + \bar{\lambda}\int_0^{\infty} e^{-\delta t} v(b+y) f_{\bar{Y}\,|\,\bar{T}}(y\,|\,t)\,\mathrm{d}y\bigg)\mathrm{d}t, \quad x \in [b, \infty). \end{aligned} \tag{32}$$

Taking into account that

$$\begin{aligned} & \int_0^{(x-b)/d} (\lambda+\bar{\lambda}) e^{-(\lambda+\bar{\lambda})t} \int_0^t de^{-\delta s}\,\mathrm{d}s\,\mathrm{d}t \\ & \quad = \frac{(\lambda+\bar{\lambda})d}{\delta}\int_0^{(x-b)/d} e^{-(\lambda+\bar{\lambda})t}\big(1-e^{-\delta t}\big)\,\mathrm{d}t \\ & \quad = \frac{d}{\delta}\bigg(\frac{\delta}{\lambda+\bar{\lambda}+\delta} - e^{-(\lambda+\bar{\lambda})(x-b)/d} + \frac{\lambda+\bar{\lambda}}{\lambda+\bar{\lambda}+\delta}\, e^{-(\lambda+\bar{\lambda}+\delta)(x-b)/d}\bigg) \end{aligned}$$

and

$$\begin{aligned} & \int_{(x-b)/d}^{\infty} (\lambda+\bar{\lambda}) e^{-(\lambda+\bar{\lambda})t} \int_0^{(x-b)/d} de^{-\delta s}\,\mathrm{d}s\,\mathrm{d}t \\ & \quad = \frac{(\lambda+\bar{\lambda})d}{\delta}\big(1-e^{-\delta(x-b)/d}\big)\int_{(x-b)/d}^{\infty} e^{-(\lambda+\bar{\lambda})t}\,\mathrm{d}t \\ & \quad = \frac{d}{\delta}\big(e^{-(\lambda+\bar{\lambda})(x-b)/d} - e^{-(\lambda+\bar{\lambda}+\delta)(x-b)/d}\big), \end{aligned}$$

substituting (4) and (5) into (32) and using (8) give

$$\begin{aligned} v_2(x) &= I_{7,8,9}(x) + I_{10,11,12}(x) \\ & \quad + \frac{d}{\lambda+\bar{\lambda}+\delta}\big(1-e^{-(\lambda+\bar{\lambda}+\delta)(x-b)/d}\big), \quad x \in [b, \infty), \end{aligned} \tag{33}$$

where

$$\begin{aligned} I_{7,8,9}(x) = \int_0^{(x-b)/d} & e^{-(\lambda+\bar{\lambda}+\delta)t} \\ & \times \bigg(\lambda\int_0^{x-dt-b} v_2(x-dt-y)\big(f_Y(y)+\theta h_Y(y)\big(2e^{-\lambda t}-1\big)\big)\,\mathrm{d}y \end{aligned}$$

$$+ \lambda \int_{x-dt-b}^{x-dt} v_1(x-dt-y)\big(f_Y(y)+\theta h_Y(y)\big(2e^{-\lambda t}-1\big)\big)\,\mathrm{d}y$$
$$+ \bar{\lambda} \int_0^{\infty} v_2(x-dt+y)\big(f_{\bar{Y}}(y)+\bar{\theta} h_{\bar{Y}}(y)\big(2e^{-\bar{\lambda} t}-1\big)\big)\,\mathrm{d}y\bigg)\mathrm{d}t$$

and

$$I_{10,11,12}(x) = \int_{(x-b)/d}^{\infty} e^{-(\lambda+\bar{\lambda}+\delta)t}$$
$$\times \bigg(\lambda \int_0^{b} v_1(b-y)\big(f_Y(y)+\theta h_Y(y)\big(2e^{-\lambda t}-1\big)\big)\,\mathrm{d}y$$
$$+ \bar{\lambda} \int_0^{\infty} v_2(b+y)\big(f_{\bar{Y}}(y)+\bar{\theta} h_{\bar{Y}}(y)\big(2e^{-\bar{\lambda} t}-1\big)\big)\,\mathrm{d}y\bigg)\mathrm{d}t.$$

Changing the variable $x - dt = s$ in the outer integral of the expression for $I_{7,8,9}(x)$ yields

$$I_{7,8,9}(x) = \frac{1}{d}\, e^{-(\lambda+\bar{\lambda}+\delta)x/d}\, I_7(x) + \frac{2}{d}\, e^{-(2\lambda+\bar{\lambda}+\delta)x/d}\, I_8(x)$$
$$+ \frac{2}{d}\, e^{-(\lambda+2\bar{\lambda}+\delta)x/d}\, I_9(x), \quad x \in [b,\infty), \tag{34}$$

where

$$I_7(x) = \int_b^{x} e^{(\lambda+\bar{\lambda}+\delta)s/d}\bigg(\lambda \int_0^{s-b} v_2(s-y)\big(f_Y(y)-\theta h_Y(y)\big)\,\mathrm{d}y$$
$$+ \lambda \int_{s-b}^{s} v_1(s-y)\big(f_Y(y)-\theta h_Y(y)\big)\,\mathrm{d}y$$
$$+ \bar{\lambda} \int_0^{\infty} v_2(s+y)\big(f_{\bar{Y}}(y)-\bar{\theta} h_{\bar{Y}}(y)\big)\,\mathrm{d}y\bigg)\mathrm{d}s,$$
$$I_8(x) = \lambda\theta \int_b^{x} e^{(2\lambda+\bar{\lambda}+\delta)s/d}\bigg(\int_0^{s-b} v_2(s-y)h_Y(y)\,\mathrm{d}y$$
$$+ \int_{s-b}^{s} v_1(s-y)h_Y(y)\,\mathrm{d}y\bigg)\mathrm{d}s$$

and

$$I_9(x) = \bar{\lambda}\bar{\theta} \int_b^{x} e^{(\lambda+2\bar{\lambda}+\delta)s/d}\bigg(\int_0^{\infty} v_2(s+y)h_{\bar{Y}}(y)\,\mathrm{d}y\bigg)\mathrm{d}s.$$

Separating the integrals in the expression for $I_{10,11,12}(x)$ into integrals w.r.t. either t or y and then taking the integrals w.r.t. t we obtain

$$I_{10,11,12}(x) = I_{10}(x) + I_{11}(x) + I_{12}(x), \quad x \in [b,\infty), \tag{35}$$

where

$$I_{10}(x) = \frac{e^{-(\lambda+\bar{\lambda}+\delta)(x-b)/d}}{\lambda+\bar{\lambda}+\delta}$$

$$\times\left(\lambda\int_0^b v_1(b-y)f_Y(y)\,\mathrm{d}y-\lambda\theta\int_0^b v_1(b-y)h_Y(y)\,\mathrm{d}y\right.$$
$$\left.+\bar\lambda\int_0^\infty v_2(b+y)f_{\bar Y}(y)\,\mathrm{d}y-\bar\lambda\bar\theta\int_0^\infty v_2(b+y)h_{\bar Y}(y)\,\mathrm{d}y\right),$$
$$I_{11}(x)=\frac{2\lambda\theta e^{-(2\lambda+\bar\lambda+\delta)(x-b)/d}}{2\lambda+\bar\lambda+\delta}\int_0^b v_1(b-y)h_Y(y)\,\mathrm{d}y$$

and

$$I_{12}(x)=\frac{2\bar\lambda\bar\theta e^{-(\lambda+2\bar\lambda+\delta)(x-b)/d}}{\lambda+2\bar\lambda+\delta}\int_0^\infty v_2(b+y)h_{\bar Y}(y)\,\mathrm{d}y.$$

Thus, substituting (34) and (35) into (33) we have

$$\begin{aligned}v_2(x)&=\frac{1}{d}e^{-(\lambda+\bar\lambda+\delta)x/d}I_7(x)+\frac{2}{d}e^{-(2\lambda+\bar\lambda+\delta)x/d}I_8(x)\\&\quad+\frac{2}{d}e^{-(\lambda+2\bar\lambda+\delta)x/d}I_9(x)+I_{10}(x)+I_{11}(x)+I_{12}(x)\\&\quad+\frac{d}{\lambda+\bar\lambda+\delta}\left(1-e^{-(\lambda+\bar\lambda+\delta)(x-b)/d}\right),\quad x\in[b,\infty).\end{aligned}\tag{36}$$

It is easily seen from (29) that $v_1(x)$ is continuous on $[0,b]$, and from (36) we conclude that $v_2(x)$ is continuous on $[b,\infty)$. Hence, from (36) we deduce that $v_2(x)$ is differentiable on $[b,\infty)$. Differentiating (36) gives

$$\begin{aligned}v_2'(x)&=-\frac{\lambda+\bar\lambda+\delta}{d^2}e^{-(\lambda+\bar\lambda+\delta)x/d}I_7(x)\\&\quad-\frac{2(2\lambda+\bar\lambda+\delta)}{d^2}e^{-(2\lambda+\bar\lambda+\delta)x/d}I_8(x)\\&\quad-\frac{2(\lambda+2\bar\lambda+\delta)}{d^2}e^{-(\lambda+2\bar\lambda+\delta)x/d}I_9(x)\\&\quad-\frac{\lambda+\bar\lambda+\delta}{d}I_{10}(x)-\frac{2\lambda+\bar\lambda+\delta}{d}I_{11}(x)-\frac{\lambda+2\bar\lambda+\delta}{d}I_{12}(x)\\&\quad+\frac{1}{d}\beta_3(x)+e^{-(\lambda+\bar\lambda+\delta)(x-b)/d},\quad x\in[b,\infty),\end{aligned}\tag{37}$$

where the function $\beta_3(x)$ is defined in the assertion of the theorem.

Multiplying (36) by $(\lambda+\bar\lambda+\delta)/d$ and adding (37) we get

$$\begin{aligned}dv_2'(x)+(\lambda+\bar\lambda+\delta)v_2(x)&=-\frac{2\lambda}{d}e^{-(2\lambda+\bar\lambda+\delta)x/d}I_8(x)\\&\quad-\frac{2\bar\lambda}{d}e^{-(\lambda+2\bar\lambda+\delta)x/d}I_9(x)-\lambda I_{11}(x)-\bar\lambda I_{12}(x)\\&\quad+\beta_3(x)+d,\quad x\in[b,\infty).\end{aligned}\tag{38}$$

Since $v_1(x)$ and $v_2(x)$ are continuous and bounded on $[0,b]$ and $[b,\infty)$, respectively, from (38) we conclude that so is $v_2'(x)$ on $[b,\infty)$. Taking into account that

$f_Y'(y)$ and $f_{\bar{Y}}'(y)$ are continuous and bounded on $\mathbb{R}_+$, from (29) we conclude that so is $v_1'(x)$ on $[0, b]$. Hence, $\beta_3(x)$ is differentiable on $[b, \infty)$. From this and (38) it follows that $v_2(x)$ is twice differentiable on $[b, \infty)$. Differentiating (38) gives

$$\begin{aligned} dv_2''(x) + (\lambda + \bar{\lambda} + \delta)v_2'(x) &= \frac{2\lambda(2\lambda + \bar{\lambda} + \delta)}{d^2} e^{-(2\lambda+\bar{\lambda}+\delta)x/d} I_8(x) \\ &\quad + \frac{2\bar{\lambda}(\lambda + 2\bar{\lambda} + \delta)}{d^2} e^{-(\lambda+2\bar{\lambda}+\delta)x/d} I_9(x) + \frac{\lambda(2\lambda + \bar{\lambda} + \delta)}{d} I_{11}(x) \\ &\quad + \frac{\bar{\lambda}(\lambda + 2\bar{\lambda} + \delta)}{d} I_{12}(x) + \beta_3'(x) - \frac{2}{d}\beta_4(x), \quad x \in [b, \infty), \end{aligned} \tag{39}$$

where

$$\begin{aligned} \beta_3'(x) &= \lambda \int_{x-b}^{x} v_1'(x-y)\big(f_Y(y) + \theta h_Y(y)\big)\,\mathrm{d}y \\ &\quad + \lambda \int_0^{x-b} v_2'(x-y)\big(f_Y(y) + \theta h_Y(y)\big)\,\mathrm{d}y \\ &\quad + \bar{\lambda} \int_0^{\infty} v_2'(x+y)\big(f_{\bar{Y}}(y) + \bar{\theta} h_{\bar{Y}}(y)\big)\,\mathrm{d}y \\ &\quad + \lambda v_1(0)\big(f_Y(x) + \theta h_Y(x)\big), \quad x \in [b, \infty), \end{aligned}$$

which is continuous and bounded on $[b, \infty)$, and the function $\beta_4(x)$ is defined in the assertion of the theorem.

Multiplying (38) by $(2\lambda + \bar{\lambda} + \delta)/d$ and adding (39) we obtain

$$\begin{aligned} &d^2 v_2''(x) + (3\lambda + 2\bar{\lambda} + 2\delta)dv_2'(x) + (2\lambda + \bar{\lambda} + \delta)(\lambda + \bar{\lambda} + \delta)v_2(x) \\ &\quad = \frac{2\bar{\lambda}(\bar{\lambda} - \lambda)}{d} e^{-(\lambda+2\bar{\lambda}+\delta)x/d} I_9(x) + \bar{\lambda}(\bar{\lambda} - \lambda)I_{12}(x) \\ &\qquad + (2\lambda + \bar{\lambda} + \delta)\beta_3(x) + d\beta_3'(x) - 2\beta_4(x) \\ &\qquad + (2\lambda + \bar{\lambda} + \delta)d, \quad x \in [b, \infty). \end{aligned} \tag{40}$$

It is easily seen from (40) that $v_2''(x)$ is continuous and bounded on $[b, \infty)$. Taking into account that $f_Y'(y)$ and $f_{\bar{Y}}'(y)$ are continuous and bounded on $\mathbb{R}_+$, from (29) we conclude that so is $v_1''(x)$ on $[0, b]$. Hence, $\beta_3(x)$ is twice differentiable on $[b, \infty)$. Moreover, applying similar arguments shows that $\beta_4(x)$ is differentiable on $[b, \infty)$. From this and (40) it follows that $v_2(x)$ has the third derivative on $[b, \infty)$. Differentiating (40) gives

$$\begin{aligned} &d^2 v_2'''(x) + (3\lambda + 2\bar{\lambda} + 2\delta)dv_2''(x) + (2\lambda + \bar{\lambda} + \delta)(\lambda + \bar{\lambda} + \delta)v_2'(x) \\ &\quad = -\frac{2\bar{\lambda}(\bar{\lambda} - \lambda)(\lambda + 2\bar{\lambda} + \delta)}{d^2} e^{-(\lambda+2\bar{\lambda}+\delta)x/d} I_9(x) \\ &\qquad - \frac{\bar{\lambda}(\bar{\lambda} - \lambda)(\lambda + 2\bar{\lambda} + \delta)}{d} I_{12}(x) + (2\lambda + \bar{\lambda} + \delta)\beta_3'(x) + d\beta_3''(x) \\ &\qquad - 2\beta_4'(x) + \frac{2\bar{\lambda}^2\bar{\theta}(\bar{\lambda} - \lambda)}{d} \int_0^{\infty} v_2(x+y)h_{\bar{Y}}(y)\,\mathrm{d}y, \quad x \in [b, \infty). \end{aligned} \tag{41}$$

Multiplying (40) by $(\lambda + 2\bar{\lambda} + \delta_0)/d$ and adding (41) yield (30), which completes the proof. □

Remark 4. *To solve equations (29) and (30), we use the following boundary conditions. First of all, we have $v_1(b) = v_2(b)$. Next, if the net profit condition holds, applying arguments similar to those in [40, p. 70] we can show that $\lim_{x\to\infty} v_2(x) = d/\delta$. Moreover, we can substitute $x = b$ into the intermediate equations (e.g., equation (38)) to get additional boundary conditions involving derivatives of $v_2(x)$. The uniqueness of the required solutions should also be justified. If $\theta = 0$ and $\bar{\theta} = 0$, we can find explicit solutions to the equations (see Section 5).*

Remark 5. *If at least one of the parameters θ and $\bar{\theta}$ is equal to 0, we do not need to differentiate (36) three times and can obtain equations not involving the third derivative of $v_2(x)$ instead of (30).*

Thus, if $\theta = 0$ and $\bar{\theta} = 0$, from (38) we have

$$d v_2'(x) + (\lambda + \bar{\lambda} + \delta_0) v_2(x) = \beta_3(x) + d, \quad x \in [b, \infty). \tag{42}$$

If $\theta \neq 0$ and $\bar{\theta} = 0$, from (40) we get

$$\begin{aligned} &d^2 v_2''(x) + (3\lambda + 2\bar{\lambda} + 2\delta_0) d v_2'(x) + (2\lambda + \bar{\lambda} + \delta)(\lambda + \bar{\lambda} + \delta) v_2(x) \\ &\quad = (2\lambda + \bar{\lambda} + \delta)\beta_3(x) + d\beta_3'(x) - 2\beta_4(x) + (2\lambda + \bar{\lambda} + \delta) d, \quad x \in [b, \infty). \end{aligned} \tag{43}$$

If $\theta = 0$ and $\bar{\theta} \neq 0$, multiplying (38) by $(\lambda + 2\bar{\lambda} + \delta_0)/d$ and adding (39) we have

$$\begin{aligned} &d^2 v_2''(x) + (2\lambda + 3\bar{\lambda} + 2\delta) d v_2'(x) + (\lambda + 2\bar{\lambda} + \delta)(\lambda + \bar{\lambda} + \delta) v_2(x) \\ &\quad = (\lambda + 2\bar{\lambda} + \delta)\beta_3(x) + d\beta_3'(x) - 2\beta_4(x) + (\lambda + 2\bar{\lambda} + \delta) d, \quad x \in [b, \infty). \end{aligned} \tag{44}$$

To obtain (42)–(44), it is enough to have weaker smoothness assumptions on $f_Y(y)$ and $f_{\bar{Y}}(y)$.

Equation (29) is true in all possible cases. Since (29) involves no derivatives, it holds under weaker assumptions than (30). To obtain (29), we do not need the differentiability of $f_Y(y)$ and $f_{\bar{Y}}(y)$.

5 Exponentially distributed claim and premium sizes

In this section, we deal with exponentially distributed claim and premium sizes, i.e.

$$f_Y(y) = \frac{1}{\mu} e^{-y/\mu}, \qquad h_Y(y) = \frac{2}{\mu} e^{-2y/\mu} - \frac{1}{\mu} e^{-y/\mu}, \quad y \geq 0, \tag{45}$$

and

$$f_{\bar{Y}}(y) = \frac{1}{\bar{\mu}} e^{-y/\bar{\mu}}, \qquad h_{\bar{Y}}(y) = \frac{2}{\bar{\mu}} e^{-2y/\bar{\mu}} - \frac{1}{\bar{\mu}} e^{-y/\bar{\mu}}, \quad y \geq 0. \tag{46}$$

5.1 The ruin probability in the model without dividend payments

If no dividends are paid, equation (25) for the ruin probability $\psi(x)$ takes the form

$$
\begin{aligned}
(\lambda+\bar{\lambda})\psi(x) &= \lambda\bigg(\int_0^x \psi(u) f_Y(x-u)\,\mathrm{d}u + \int_0^\infty f_Y(x+u)\,\mathrm{d}u\bigg) \\
&\quad + \frac{\lambda\bar{\lambda}\theta}{2\lambda+\bar{\lambda}}\bigg(\int_0^x \psi(u) h_Y(x-u)\,\mathrm{d}u + \int_0^\infty h_Y(x+u)\,\mathrm{d}u\bigg) \\
&\quad + \bar{\lambda}\int_x^\infty \psi(u) f_{\bar{Y}}(u-x)\,\mathrm{d}u \\
&\quad + \frac{\lambda\bar{\lambda}\bar{\theta}}{\lambda+2\bar{\lambda}}\int_x^\infty \psi(u) h_{\bar{Y}}(u-x)\,\mathrm{d}u, \quad x\in[0,\infty).
\end{aligned} \tag{47}
$$

Substituting (45) and (46) into (47) gives

$$
\begin{aligned}
(\lambda+\bar{\lambda})\psi(x) &= \bigg(\lambda - \frac{\lambda\bar{\lambda}\theta}{2\lambda+\bar{\lambda}}\bigg) I_{13}(x) + \frac{\lambda\bar{\lambda}\theta}{2\lambda+\bar{\lambda}} I_{14}(x) \\
&\quad + \bigg(\bar{\lambda} - \frac{\lambda\bar{\lambda}\bar{\theta}}{\lambda+2\bar{\lambda}}\bigg) I_{15}(x) + \frac{\lambda\bar{\lambda}\bar{\theta}}{\lambda+2\bar{\lambda}} I_{16}(x) \\
&\quad + \bigg(\lambda - \frac{\lambda\bar{\lambda}\theta}{2\lambda+\bar{\lambda}}\bigg) e^{-x/\mu} + \frac{\lambda\bar{\lambda}\theta}{2\lambda+\bar{\lambda}} e^{-2x/\mu}, \quad x\in[0,\infty),
\end{aligned} \tag{48}
$$

where

$$
\begin{aligned}
I_{13}(x) &= \frac{1}{\mu} e^{-x/\mu}\int_0^x \psi(u) e^{u/\mu}\,\mathrm{d}u, & I_{14}(x) &= \frac{2}{\mu} e^{-2x/\mu}\int_0^x \psi(u) e^{2u/\mu}\,\mathrm{d}u, \\
I_{15}(x) &= \frac{1}{\bar{\mu}} e^{x/\bar{\mu}}\int_x^\infty \psi(u) e^{-u/\bar{\mu}}\,\mathrm{d}u, & I_{16}(x) &= \frac{2}{\bar{\mu}} e^{2x/\bar{\mu}}\int_x^\infty \psi(u) e^{-2u/\bar{\mu}}\,\mathrm{d}u.
\end{aligned}
$$

We now show that if either $\theta=0$ or $\bar{\theta}=0$, integro-differential equation (48) can be reduced to a third-order linear differential equation with constant coefficients.

Lemma 1. *Let the surplus process $(X_t(x))_{t\geq 0}$ follow (1) under the above assumptions, and let claim and premium sizes be exponentially distributed with means μ and $\bar{\mu}$, respectively.*

If $\theta\neq 0$ and $\bar{\theta}=0$, then $\psi(x)$ is a solution to the differential equation

$$
\begin{aligned}
&\mu^2\bar{\mu}(\lambda+\bar{\lambda})(2\lambda+\bar{\lambda})\psi'''(x) \\
&\quad + \big(\mu\bar{\mu}(2\lambda+3\bar{\lambda})(2\lambda+\bar{\lambda}) - \lambda\mu^2(2\lambda+\bar{\lambda}) + \lambda\bar{\lambda}\mu\bar{\mu}\theta\big)\psi''(x) \\
&\quad + \big(2(\bar{\lambda}\bar{\mu}-\lambda\mu)(2\lambda+\bar{\lambda}) + \lambda\bar{\lambda}\mu\theta\big)\psi'(x) = 0, \quad x\in[0,\infty).
\end{aligned} \tag{49}
$$

If $\theta=0$ and $\bar{\theta}\neq 0$, then $\psi(x)$ is a solution to the differential equation

$$
\begin{aligned}
&\mu\bar{\mu}^2(\lambda+\bar{\lambda})(\lambda+2\bar{\lambda})\psi'''(x) \\
&\quad + \big(-\mu\bar{\mu}(3\lambda+2\bar{\lambda})(\lambda+2\bar{\lambda}) + \bar{\lambda}\bar{\mu}^2(\lambda+2\bar{\lambda}) + \lambda\bar{\lambda}\mu\bar{\mu}\bar{\theta}\big)\psi''(x) \\
&\quad + \big(2(\lambda\mu-\bar{\lambda}\bar{\mu})(\lambda+2\bar{\lambda}) + \lambda\bar{\lambda}\bar{\mu}\bar{\theta}\big)\psi'(x) = 0, \quad x\in[0,\infty).
\end{aligned} \tag{50}
$$

Proof. First of all, note that

$$I'_{13}(x) = -\frac{1}{\mu} I_{13}(x) + \frac{1}{\mu}\psi(x), \qquad I'_{14}(x) = -\frac{2}{\mu} I_{14}(x) + \frac{2}{\mu}\psi(x),$$
$$I'_{15}(x) = \frac{1}{\bar{\mu}} I_{15}(x) - \frac{1}{\bar{\mu}}\psi(x), \qquad I'_{16}(x) = \frac{2}{\bar{\mu}} I_{16}(x) - \frac{2}{\bar{\mu}}\psi(x).$$

From (48), it is easily seen that $\psi(x)$ is differentiable on $[0, \infty)$. Therefore, differentiating (48) yields

$$\begin{aligned}
&(\lambda + \bar{\lambda})\psi'(x) \\
&\quad = -\frac{1}{\mu}\left(\lambda - \frac{\lambda\bar{\lambda}\theta}{2\lambda + \bar{\lambda}}\right) I_{13}(x) - \frac{2}{\mu}\frac{\lambda\bar{\lambda}\theta}{2\lambda + \bar{\lambda}} I_{14}(x) \\
&\qquad + \frac{1}{\bar{\mu}}\left(\bar{\lambda} - \frac{\lambda\bar{\lambda}\bar{\theta}}{\lambda + 2\bar{\lambda}}\right) I_{15}(x) + \frac{2}{\bar{\mu}}\frac{\lambda\bar{\lambda}\bar{\theta}}{\lambda + 2\bar{\lambda}} I_{16}(x) + \left(\frac{1}{\mu}\left(\lambda - \frac{\lambda\bar{\lambda}\theta}{2\lambda + \bar{\lambda}}\right)\right. \\
&\qquad \left. + \frac{2}{\mu}\frac{\lambda\bar{\lambda}\theta}{2\lambda + \bar{\lambda}} - \frac{1}{\bar{\mu}}\left(\bar{\lambda} - \frac{\lambda\bar{\lambda}\bar{\theta}}{\lambda + 2\bar{\lambda}}\right) - \frac{2}{\bar{\mu}}\frac{\lambda\bar{\lambda}\bar{\theta}}{\lambda + 2\bar{\lambda}}\right)\psi(x) \\
&\qquad - \frac{1}{\mu}\left(\lambda - \frac{\lambda\bar{\lambda}\theta}{2\lambda + \bar{\lambda}}\right) e^{-x/\mu} - \frac{2}{\mu}\frac{\lambda\bar{\lambda}\theta}{2\lambda + \bar{\lambda}} e^{-2x/\mu}, \quad x \in [0, \infty).
\end{aligned} \tag{51}$$

Multiplying (51) by μ and adding (48) we obtain

$$\begin{aligned}
&\mu(\lambda + \bar{\lambda})\psi'(x) + \bar{\lambda}\psi(x) \\
&\quad = -\frac{\lambda\bar{\lambda}\theta}{2\lambda + \bar{\lambda}} I_{14}(x) + \left(1 + \frac{\mu}{\bar{\mu}}\right)\left(\bar{\lambda} - \frac{\lambda\bar{\lambda}\bar{\theta}}{\lambda + 2\bar{\lambda}}\right) I_{15}(x) + \left(1 + \frac{2\mu}{\bar{\mu}}\right)\frac{\lambda\bar{\lambda}\bar{\theta}}{\lambda + 2\bar{\lambda}} I_{16}(x) \\
&\qquad + \left(\frac{\lambda\bar{\lambda}\theta}{2\lambda + \bar{\lambda}} - \frac{\mu}{\bar{\mu}}\left(\bar{\lambda} + \frac{\lambda\bar{\lambda}\bar{\theta}}{\lambda + 2\bar{\lambda}}\right)\right)\psi(x) - \frac{\lambda\bar{\lambda}\theta}{2\lambda + \bar{\lambda}} e^{-2x/\mu}, \quad x \in [0, \infty).
\end{aligned} \tag{52}$$

From (52), it follows that $\psi(x)$ is twice differentiable on $[0, \infty)$. Differentiating (52) gives

$$\begin{aligned}
&\mu(\lambda + \bar{\lambda})\psi''(x) + \bar{\lambda}\psi'(x) \\
&\quad = \frac{2}{\mu}\frac{\lambda\bar{\lambda}\theta}{2\lambda + \bar{\lambda}} I_{14}(x) + \frac{1}{\bar{\mu}}\left(1 + \frac{\mu}{\bar{\mu}}\right)\left(\bar{\lambda} - \frac{\lambda\bar{\lambda}\bar{\theta}}{\lambda + 2\bar{\lambda}}\right) I_{15}(x) \\
&\qquad + \frac{2}{\bar{\mu}}\left(1 + \frac{2\mu}{\bar{\mu}}\right)\frac{\lambda\bar{\lambda}\bar{\theta}}{\lambda + 2\bar{\lambda}} I_{16}(x) + \left(\frac{\lambda\bar{\lambda}\theta}{2\lambda + \bar{\lambda}} - \frac{\mu}{\bar{\mu}}\left(\bar{\lambda} + \frac{\lambda\bar{\lambda}\bar{\theta}}{\lambda + 2\bar{\lambda}}\right)\right)\psi'(x) \\
&\qquad - \left(\frac{2}{\mu}\frac{\lambda\bar{\lambda}\theta}{2\lambda + \bar{\lambda}} + \frac{1}{\bar{\mu}}\left(1 + \frac{\mu}{\bar{\mu}}\right)\left(\bar{\lambda} - \frac{\lambda\bar{\lambda}\bar{\theta}}{\lambda + 2\bar{\lambda}}\right) + \frac{2}{\bar{\mu}}\left(1 + \frac{2\mu}{\bar{\mu}}\right)\frac{\lambda\bar{\lambda}\bar{\theta}}{\lambda + 2\bar{\lambda}}\right)\psi(x) \\
&\qquad + \frac{2}{\mu}\frac{\lambda\bar{\lambda}\theta}{2\lambda + \bar{\lambda}} e^{-2x/\mu}, \quad x \in [0, \infty).
\end{aligned} \tag{53}$$

Multiplying (53) by $(-\bar{\mu})$ and adding (52) we get

$$\begin{aligned}
&-\mu\bar{\mu}(\lambda+\bar{\lambda})\psi''(x)+(\lambda\mu-\bar{\lambda}\bar{\mu})\psi'(x)\\
&\quad=-\left(1+\frac{2\bar{\mu}}{\mu}\right)\frac{\lambda\bar{\lambda}\theta}{2\lambda+\bar{\lambda}}\,I_{14}(x)-\left(1+\frac{2\mu}{\bar{\mu}}\right)\frac{\lambda\bar{\lambda}\bar{\theta}}{\lambda+2\bar{\lambda}}\,I_{16}(x)\\
&\qquad+\left(-\frac{\lambda\bar{\lambda}\bar{\mu}\theta}{2\lambda+\bar{\lambda}}+\frac{\lambda\bar{\lambda}\mu\bar{\theta}}{\lambda+2\bar{\lambda}}\right)\psi'(x)+\left(\left(1+\frac{2\bar{\mu}}{\mu}\right)\frac{\lambda\bar{\lambda}\theta}{2\lambda+\bar{\lambda}}\right.\\
&\qquad\left.+\left(1+\frac{2\mu}{\bar{\mu}}\right)\frac{\lambda\bar{\lambda}\bar{\theta}}{\lambda+2\bar{\lambda}}\right)\psi(x)-\left(1+\frac{2\bar{\mu}}{\mu}\right)\frac{\lambda\bar{\lambda}\theta}{2\lambda+\bar{\lambda}}\,e^{-2x/\mu},\quad x\in[0,\infty).
\end{aligned}\tag{54}$$

Let now $\theta\neq 0$ and $\bar{\theta}=0$. Then (54) takes the form

$$\begin{aligned}
&-\mu\bar{\mu}(\lambda+\bar{\lambda})\psi''(x)+(\lambda\mu-\bar{\lambda}\bar{\mu})\psi'(x)\\
&\quad=-\left(1+\frac{2\bar{\mu}}{\mu}\right)\frac{\lambda\bar{\lambda}\theta}{2\lambda+\bar{\lambda}}\,I_{14}(x)-\frac{\lambda\bar{\lambda}\bar{\mu}\theta}{2\lambda+\bar{\lambda}}\,\psi'(x)+\left(1+\frac{2\bar{\mu}}{\mu}\right)\frac{\lambda\bar{\lambda}\theta}{2\lambda+\bar{\lambda}}\,\psi(x)\\
&\qquad-\left(1+\frac{2\bar{\mu}}{\mu}\right)\frac{\lambda\bar{\lambda}\theta}{2\lambda+\bar{\lambda}}\,e^{-2x/\mu},\quad x\in[0,\infty).
\end{aligned}\tag{55}$$

From (55), it follows that $\psi(x)$ has the third derivative on $[0,\infty)$. Differentiating (55) yields

$$\begin{aligned}
&-\mu\bar{\mu}(\lambda+\bar{\lambda})\psi'''(x)+(\lambda\mu-\bar{\lambda}\bar{\mu})\psi''(x)\\
&\quad=\frac{2}{\mu}\left(1+\frac{2\bar{\mu}}{\mu}\right)\frac{\lambda\bar{\lambda}\theta}{2\lambda+\bar{\lambda}}\,I_{14}(x)-\frac{\lambda\bar{\lambda}\bar{\mu}\theta}{2\lambda+\bar{\lambda}}\,\psi''(x)+\left(1+\frac{2\bar{\mu}}{\mu}\right)\frac{\lambda\bar{\lambda}\theta}{2\lambda+\bar{\lambda}}\,\psi'(x)\\
&\qquad-\frac{2}{\mu}\left(1+\frac{2\bar{\mu}}{\mu}\right)\frac{\lambda\bar{\lambda}\theta}{2\lambda+\bar{\lambda}}\,\psi(x)+\frac{2}{\mu}\left(1+\frac{2\bar{\mu}}{\mu}\right)\frac{\lambda\bar{\lambda}\theta}{2\lambda+\bar{\lambda}}\,e^{-2x/\mu},\quad x\in[0,\infty).
\end{aligned}\tag{56}$$

Multiplying (56) by μ and adding (55) multiplied by 2 we obtain

$$\begin{aligned}
&-\mu^2\bar{\mu}(\lambda+\bar{\lambda})\psi'''(x)+\big(\mu(\lambda\mu-\bar{\lambda}\bar{\mu})-2\mu\bar{\mu}(\lambda+\bar{\lambda})\big)\psi''(x)+2(\lambda\mu-\bar{\lambda}\bar{\mu})\psi'(x)\\
&\quad=\frac{\lambda\bar{\lambda}\mu\bar{\mu}\theta}{2\lambda+\bar{\lambda}}\,\psi''(x)+\frac{\lambda\bar{\lambda}\mu\theta}{2\lambda+\bar{\lambda}}\,\psi'(x),\quad x\in[0,\infty),
\end{aligned}$$

from which (49) follows.

If $\theta=0$ and $\bar{\theta}\neq 0$, then (54) takes the form

$$\begin{aligned}
&-\mu\bar{\mu}(\lambda+\bar{\lambda})\psi''(x)+(\lambda\mu-\bar{\lambda}\bar{\mu})\psi'(x)\\
&\quad=-\left(1+\frac{2\mu}{\bar{\mu}}\right)\frac{\lambda\bar{\lambda}\bar{\theta}}{\lambda+2\bar{\lambda}}\,I_{16}(x)+\frac{\lambda\bar{\lambda}\mu\bar{\theta}}{\lambda+2\bar{\lambda}}\psi'(x)\\
&\qquad+\left(1+\frac{2\mu}{\bar{\mu}}\right)\frac{\lambda\bar{\lambda}\bar{\theta}}{\lambda+2\bar{\lambda}}\,\psi(x),\quad x\in[0,\infty).
\end{aligned}\tag{57}$$

From (57), it follows that $\psi(x)$ has the third derivative on $[0,\infty)$. Differentiating (57) gives

$$
\begin{aligned}
&-\mu\bar{\mu}(\lambda+\bar{\lambda})\psi'''(x)+(\lambda\mu-\bar{\lambda}\bar{\mu})\psi''(x)\\
&\quad=-\frac{2}{\bar{\mu}}\left(1+\frac{2\mu}{\bar{\mu}}\right)\frac{\lambda\bar{\lambda}\bar{\theta}}{\lambda+2\bar{\lambda}}\,I_{16}(x)+\frac{\lambda\bar{\lambda}\mu\bar{\theta}}{\lambda+2\bar{\lambda}}\psi''(x)+\left(1+\frac{2\mu}{\bar{\mu}}\right)\frac{\lambda\bar{\lambda}\bar{\theta}}{\lambda+2\bar{\lambda}}\,\psi'(x)\\
&\qquad+\frac{2}{\bar{\mu}}\left(1+\frac{2\mu}{\bar{\mu}}\right)\frac{\lambda\bar{\lambda}\bar{\theta}}{\lambda+2\bar{\lambda}}\,\psi(x),\quad x\in[0,\infty).
\end{aligned}\tag{58}
$$

Multiplying (58) by $(-\bar{\mu})$ and adding (57) multiplied by 2 we get

$$
\begin{aligned}
&\mu\bar{\mu}^2(\lambda+\bar{\lambda})\psi'''(x)-\big(\bar{\mu}(\lambda\mu-\bar{\lambda}\bar{\mu})+2\mu\bar{\mu}(\lambda+\bar{\lambda})\big)\psi''(x)+2(\lambda\mu-\bar{\lambda}\bar{\mu})\psi'(x)\\
&\quad=-\frac{\lambda\bar{\lambda}\mu\bar{\mu}\bar{\theta}}{\lambda+2\bar{\lambda}}\,\psi''(x)-\frac{\lambda\bar{\lambda}\bar{\mu}\bar{\theta}}{\lambda+2\bar{\lambda}}\,\psi'(x),\quad x\in[0,\infty),
\end{aligned}
$$

from which (50) follows. □

To formulate the next theorem, we define the following constants:

$$
\begin{aligned}
\mathrm{D}_1&=\big(\mu\bar{\mu}(2\lambda+3\bar{\lambda})(2\lambda+\bar{\lambda})-\lambda\mu^2(2\lambda+\bar{\lambda})+\lambda\bar{\lambda}\mu\bar{\mu}\theta\big)^2\\
&\quad-4\mu^2\bar{\mu}(\lambda+\bar{\lambda})(2\lambda+\bar{\lambda})\big(2(\bar{\lambda}\bar{\mu}-\lambda\mu)(2\lambda+\bar{\lambda})+\lambda\bar{\lambda}\mu\theta\big),\\
z_2&=\frac{-(\mu\bar{\mu}(2\lambda+3\bar{\lambda})(2\lambda+\bar{\lambda})-\lambda\mu^2(2\lambda+\bar{\lambda})+\lambda\bar{\lambda}\mu\bar{\mu}\theta)+\sqrt{\mathrm{D}_1}}{2\mu^2\bar{\mu}(\lambda+\bar{\lambda})(2\lambda+\bar{\lambda})},\\
z_3&=\frac{-(\mu\bar{\mu}(2\lambda+3\bar{\lambda})(2\lambda+\bar{\lambda})-\lambda\mu^2(2\lambda+\bar{\lambda})+\lambda\bar{\lambda}\mu\bar{\mu}\theta)-\sqrt{\mathrm{D}_1}}{2\mu^2\bar{\mu}(\lambda+\bar{\lambda})(2\lambda+\bar{\lambda})}
\end{aligned}
$$

and

$$
\begin{aligned}
\Delta_1&=\left(\lambda+\bar{\lambda}-\frac{\bar{\lambda}}{1-\bar{\mu}z_2}\right)\left(\mu(\lambda+\bar{\lambda})z_3-\left(\bar{\lambda}+\frac{\bar{\lambda}\mu}{\bar{\mu}}\right)\frac{\bar{\mu}z_3}{1-\bar{\mu}z_3}-\frac{\lambda\bar{\lambda}\theta}{2\lambda+\bar{\lambda}}\right)\\
&\quad-\left(\lambda+\bar{\lambda}-\frac{\bar{\lambda}}{1-\bar{\mu}z_3}\right)\left(\mu(\lambda+\bar{\lambda})z_2-\left(\bar{\lambda}+\frac{\bar{\lambda}\mu}{\bar{\mu}}\right)\frac{\bar{\mu}z_2}{1-\bar{\mu}z_2}-\frac{\lambda\bar{\lambda}\theta}{2\lambda+\bar{\lambda}}\right).
\end{aligned}
$$

Theorem 3. *Let the surplus process $(X_t(x))_{t\ge 0}$ follow (1) under the above assumptions with $\theta\neq 0$ and $\bar{\theta}=0$. Moreover, let claim and premium sizes be exponentially distributed with means μ and $\bar{\mu}$, respectively, and let $\bar{\lambda}\bar{\mu}>\lambda\mu$.*

If $2(\bar{\lambda}\bar{\mu}-\lambda\mu)(2\lambda+\bar{\lambda})+\lambda\bar{\lambda}\mu\theta\le 0$, then

$$
\psi(x)=\frac{\lambda(1-\bar{\mu}z_3)}{\lambda(1-\bar{\mu}z_3)-\lambda\mu z_3}\,e^{z_3x},\quad x\in[0,\infty).\tag{59}
$$

If $2(\bar{\lambda}\bar{\mu}-\lambda\mu)(2\lambda+\bar{\lambda})+\lambda\bar{\lambda}\mu\theta>0$, then

$$
\psi(x)=C_2e^{z_2x}+C_3e^{z_3x},\quad x\in[0,\infty),\tag{60}
$$

where the constants C_2 and C_3 are determined from the system of linear equations

$$
\left(\lambda+\bar{\lambda}-\frac{\bar{\lambda}}{1-\bar{\mu}z_2}\right)C_2+\left(\lambda+\bar{\lambda}-\frac{\bar{\lambda}}{1-\bar{\mu}z_3}\right)C_3=\lambda\tag{61}
$$

and

$$\begin{aligned}&\left(\mu(\lambda+\bar{\lambda})z_2-\left(\bar{\lambda}+\frac{\bar{\lambda}\mu}{\bar{\mu}}\right)\frac{\bar{\mu}z_2}{1-\bar{\mu}z_2}-\frac{\lambda\bar{\lambda}\theta}{2\lambda+\bar{\lambda}}\right)C_2\\&\quad+\left(\mu(\lambda+\bar{\lambda})z_3-\left(\bar{\lambda}+\frac{\bar{\lambda}\mu}{\bar{\mu}}\right)\frac{\bar{\mu}z_3}{1-\bar{\mu}z_3}-\frac{\lambda\bar{\lambda}\theta}{2\lambda+\bar{\lambda}}\right)C_3=-\frac{\lambda\bar{\lambda}\theta}{2\lambda+\bar{\lambda}}\end{aligned}\tag{62}$$

provided that $\Delta_1\neq 0$.

Proof. By Lemma 1, $\psi(x)$ is a solution to (49). We now find the general solution to (49). Its characteristic equation is

$$\begin{aligned}&\mu^2\bar{\mu}(\lambda+\bar{\lambda})(2\lambda+\bar{\lambda})z^3+\big(\mu\bar{\mu}(2\lambda+3\bar{\lambda})(2\lambda+\bar{\lambda})-\lambda\mu^2(2\lambda+\bar{\lambda})+\lambda\bar{\lambda}\mu\bar{\mu}\theta\big)z^2\\&\quad+\big(2(\bar{\lambda}\bar{\mu}-\lambda\mu)(2\lambda+\bar{\lambda})+\lambda\bar{\lambda}\mu\theta\big)z=0.\end{aligned}\tag{63}$$

It is evident that $z_1=0$ is a solution to (63). Next, we prove that the equation

$$\begin{aligned}&\mu^2\bar{\mu}(\lambda+\bar{\lambda})(2\lambda+\bar{\lambda})z^2+\big(\mu\bar{\mu}(2\lambda+3\bar{\lambda})(2\lambda+\bar{\lambda})-\lambda\mu^2(2\lambda+\bar{\lambda})+\lambda\bar{\lambda}\mu\bar{\mu}\theta\big)z\\&\quad+\big(2(\bar{\lambda}\bar{\mu}-\lambda\mu)(2\lambda+\bar{\lambda})+\lambda\bar{\lambda}\mu\theta\big)=0\end{aligned}\tag{64}$$

has two real roots. To this end, we show that its discriminant D_1 defined before the assertion of the theorem is positive. We have

$$\begin{aligned}\mathrm{D}_1/\mu^2&=\big((2\lambda\bar{\mu}+3\bar{\lambda}\bar{\mu}-\lambda\mu)(2\lambda+\bar{\lambda})+\lambda\bar{\lambda}\bar{\mu}\theta\big)^2\\&\quad-8\bar{\mu}(\lambda+\bar{\lambda})(\bar{\lambda}\bar{\mu}-\lambda\mu)(2\lambda+\bar{\lambda})^2-4\lambda\bar{\lambda}\mu\bar{\mu}\theta(\lambda+\bar{\lambda})(2\lambda+\bar{\lambda})\\&=(2\lambda+\bar{\lambda})^2\big((2\lambda\bar{\mu}+3\bar{\lambda}\bar{\mu}-\lambda\mu)^2-8\bar{\mu}(\lambda+\bar{\lambda})(\bar{\lambda}\bar{\mu}-\lambda\mu)\big)\\&\quad+2\lambda\bar{\lambda}\bar{\mu}\theta(2\lambda+\bar{\lambda})\big(2\lambda\bar{\mu}+3\bar{\lambda}\bar{\mu}-\lambda\mu-2\mu(\lambda+\bar{\lambda})\big)+(\lambda\bar{\lambda}\bar{\mu}\theta)^2\\&=(2\lambda+\bar{\lambda})^2\big(2\lambda(\mu+\bar{\mu})+(\bar{\lambda}\bar{\mu}-\lambda\mu)\big)^2\\&\quad+2\lambda\bar{\lambda}\bar{\mu}\theta(2\lambda+\bar{\lambda})\big(2(\bar{\mu}-\mu)(\lambda+\bar{\lambda})+(\bar{\lambda}\bar{\mu}+\lambda\mu)\big)+(\lambda\bar{\lambda}\bar{\mu}\theta)^2\\&=(2\lambda+\bar{\lambda})^2(\bar{\lambda}\bar{\mu}-\lambda\mu)^2+2\lambda\bar{\lambda}\bar{\mu}\theta(2\lambda+\bar{\lambda})(\bar{\lambda}\bar{\mu}-\lambda\mu)+(\lambda\bar{\lambda}\bar{\mu}\theta)^2\\&\quad+(2\lambda+\bar{\lambda})^2\big(4\lambda^2(\mu+\bar{\mu})^2+4\lambda(\mu+\bar{\mu})(\bar{\lambda}\bar{\mu}-\lambda\mu)\big)\\&\quad+4\lambda\bar{\lambda}\bar{\mu}\theta(\bar{\mu}-\mu)(\lambda+\bar{\lambda})(2\lambda+\bar{\lambda})\\&=(2\lambda+\bar{\lambda})^2(\bar{\lambda}\bar{\mu}-\lambda\mu)^2+2\lambda\bar{\lambda}\bar{\mu}\theta(2\lambda+\bar{\lambda})(\bar{\lambda}\bar{\mu}-\lambda\mu)+(\lambda\bar{\lambda}\bar{\mu}\theta)^2\\&\quad+4\lambda(2\lambda+\bar{\lambda})\big(\lambda^2(\mu+\bar{\mu})^2+\lambda(\mu+\bar{\mu})(\bar{\lambda}\bar{\mu}-\lambda\mu)\\&\qquad+(\lambda+\bar{\lambda})\big(\lambda(\mu+\bar{\mu})^2+(\mu+\bar{\mu})(\bar{\lambda}\bar{\mu}-\lambda\mu)+\bar{\lambda}\bar{\mu}\theta(\bar{\mu}-\mu)\big)\big).\end{aligned}$$

Since $\bar{\lambda}\bar{\mu}>\lambda\mu$, it suffices to show that

$$\lambda(\mu+\bar{\mu})^2+(\mu+\bar{\mu})(\bar{\lambda}\bar{\mu}-\lambda\mu)+\bar{\lambda}\bar{\mu}\theta(\bar{\mu}-\mu))>0.$$

It is obvious that the minimal value of the expression on the left-hand side of the above inequality is attained when either $\theta=1$ or $\theta=-1$ and equals either $\lambda\bar{\mu}(\mu+\bar{\mu})+2\lambda\bar{\mu}^2$ or $\lambda\bar{\mu}(\mu+\bar{\mu})+2\bar{\lambda}\mu\bar{\mu}$, respectively. Both these expressions are positive. Thus, $D_1 > 0$ and (64) has two real roots z_2 and z_3 defined before the assertion of the theorem.

Next, it is easily seen that

$$\mu\bar{\mu}(2\lambda + 3\bar{\lambda})(2\lambda + \bar{\lambda}) - \lambda\mu^2(2\lambda + \bar{\lambda}) + \lambda\bar{\lambda}\mu\bar{\mu}\theta > 0.$$

Indeed, since $\bar{\lambda}\bar{\mu} > \lambda\mu$, we have

$$\begin{aligned}
&\mu\bar{\mu}(2\lambda + 3\bar{\lambda})(2\lambda + \bar{\lambda}) - \lambda\mu^2(2\lambda + \bar{\lambda}) + \lambda\bar{\lambda}\mu\bar{\mu}\theta \\
&\quad = \mu(2\lambda + \bar{\lambda})(2\lambda\bar{\mu} + 3\bar{\lambda}\bar{\mu} - \lambda\mu) + \lambda\bar{\lambda}\mu\bar{\mu}\theta \\
&\quad > 2\mu\bar{\mu}(2\lambda + \bar{\lambda})(\lambda + \bar{\lambda}) - \lambda\bar{\lambda}\mu\bar{\mu} > 3\lambda\bar{\lambda}\mu\bar{\mu} > 0.
\end{aligned}$$

Therefore, by Vieta's theorem applied to (64), $z_3 < 0$. Moreover, $z_2 \geq 0$ if $2(\bar{\lambda}\bar{\mu} - \lambda\mu)(2\lambda + \bar{\lambda}) + \lambda\bar{\lambda}\mu\theta \leq 0$, and $z_2 < 0$ if $2(\bar{\lambda}\bar{\mu} - \lambda\mu)(2\lambda + \bar{\lambda}) + \lambda\bar{\lambda}\mu\theta > 0$.

Thus, the general solution to (49) has the form

$$\psi(x) = C_1 + C_2 e^{z_2 x} + C_3 e^{z_3 x}, \quad x \in [0, \infty), \tag{65}$$

where C_1, C_2 and C_3 are some constants. To determine them, we use the following boundary conditions. Firstly, since $\bar{\lambda}\bar{\mu} > \lambda\mu$, using standard considerations (see, e.g., [34, 36, 39]) we can easily show that $\lim_{x\to\infty} \psi(x) = 0$. Consequently, $C_1 = 0$ and $C_2 = 0$ if $2(\bar{\lambda}\bar{\mu} - \lambda\mu)(2\lambda + \bar{\lambda}) + \lambda\bar{\lambda}\mu\theta \leq 0$, and $C_1 = 0$ if $2(\bar{\lambda}\bar{\mu} - \lambda\mu)(2\lambda + \bar{\lambda}) + \lambda\bar{\lambda}\mu\theta > 0$. Secondly, we use intermediate equations to find other constants.

Let now $2(\bar{\lambda}\bar{\mu} - \lambda\mu)(2\lambda + \bar{\lambda}) + \lambda\bar{\lambda}\mu\theta \leq 0$. The constant C_3 is determined by letting $x = 0$ in (48), i.e. from the equation

$$(\lambda + \bar{\lambda})\psi(0) = \frac{\bar{\lambda}}{\bar{\mu}} \int_0^\infty \psi(u) e^{-u/\bar{\mu}}\, \mathrm{d}u + \lambda. \tag{66}$$

Substituting $\psi(x) = C_3 e^{z_3 x}$ into (66) gives

$$(\lambda + \bar{\lambda})C_3 = \frac{\bar{\lambda}}{1 - \bar{\mu} z_3} C_3 + \lambda,$$

from which (59) follows immediately.

If $2(\bar{\lambda}\bar{\mu} - \lambda\mu)(2\lambda + \bar{\lambda}) + \lambda\bar{\lambda}\mu\theta > 0$, then the constants C_2 and C_3 are determined by letting $x = 0$ in (48) and (52), i.e. from (66) and the equation

$$\begin{aligned}
&\mu(\lambda + \bar{\lambda})\psi'(0) + \bar{\lambda}\psi(0) \\
&\quad = \frac{\bar{\lambda}}{\bar{\mu}}\left(1 + \frac{\mu}{\bar{\mu}}\right) \int_0^\infty \psi(u) e^{-u/\bar{\mu}}\, \mathrm{d}u + \left(\frac{\lambda\bar{\lambda}\theta}{2\lambda + \bar{\lambda}} - \frac{\bar{\lambda}\mu}{\bar{\mu}}\right)\psi(0) - \frac{\lambda\bar{\lambda}\theta}{2\lambda + \bar{\lambda}}.
\end{aligned} \tag{67}$$

Substituting (60) into (66) and (67) yields equations (61) and (62), respectively. The system of equations (61) and (62) has a unique solution provided that $\Delta_1 \neq 0$.

Note that letting $x = 0$ in (54) (and in (52) when $C_2 = 0$) gives no additional information about unknown constants. Nevertheless, the equalities must hold for the values of the constants found from (48) (and (52) when $C_2 \neq 0$). Consequently, differential equation (49) has the unique solution given by (59) or (60). Since we have derived (49) from (48) without any additional assumptions, we conclude that the function $\psi(x)$ given by (59) or (60) is a unique solution to (48) satisfying the certain conditions. This guaranties that the solution we have found is the ruin probability and completes the proof. □

The case $\theta = 0$ and $\bar{\theta} \neq 0$ can be considered in a similar way by finding the required solution to equation (50).

5.2 The ruin probability in the model without dependence

Let now $\theta = 0$ and $\bar{\theta} = 0$. We set

$$\psi(x) = \begin{cases} \psi_1(x) & \text{if } x \in [0, b], \\ \psi_2(x) & \text{if } x \in [b, \infty). \end{cases}$$

Then equations (9) and (26) for the ruin probability $\psi(x)$ in the case of exponentially distributed claim and premium sizes take the form

$$\begin{aligned}
&(\lambda + \bar{\lambda})\psi_1(x) \\
&\quad = \frac{\lambda}{\mu} e^{-x/\mu} \int_0^x \psi_1(u) e^{u/\mu}\, \mathrm{d}u + \lambda e^{-x/\mu} + \frac{\bar{\lambda}}{\bar{\mu}} e^{x/\bar{\mu}} \int_x^b \psi_1(u) e^{-u/\bar{\mu}}\, \mathrm{d}u \\
&\qquad + \frac{\bar{\lambda}}{\bar{\mu}} e^{x/\bar{\mu}} \int_b^\infty \psi_2(u) e^{-u/\bar{\mu}}\, \mathrm{d}u, \quad x \in [0, b],
\end{aligned} \tag{68}$$

and

$$\begin{aligned}
&d\psi_2'(x) + (\lambda + \bar{\lambda})\psi_2(x) \\
&\quad = \frac{\lambda}{\mu} e^{-x/\mu} \int_0^b \psi_1(u) e^{u/\mu}\, \mathrm{d}u + \frac{\lambda}{\mu} e^{-x/\mu} \int_b^x \psi_2(u) e^{u/\mu}\, \mathrm{d}u + \lambda e^{-x/\mu} \\
&\qquad + \frac{\bar{\lambda}}{\bar{\mu}} e^{x/\bar{\mu}} \int_x^\infty \psi_2(u) e^{-u/\bar{\mu}}\, \mathrm{d}u, \quad x \in [b, \infty),
\end{aligned} \tag{69}$$

respectively.

We now show that integro-differential equations (68) and (69) can be reduced to linear differential equations with constant coefficients.

Lemma 2. *Let the surplus process $(X_t^b(x))_{t \ge 0}$ follow (6) under the above assumptions with $\theta = 0$ and $\bar{\theta} = 0$, and let claim and premium sizes be exponentially distributed with means μ and $\bar{\mu}$, respectively. Then $\psi_1(x)$ and $\psi_2(x)$ are solutions to the differential equations*

$$\mu\bar{\mu}(\lambda + \bar{\lambda})\psi_1''(x) + (\bar{\lambda}\bar{\mu} - \lambda\mu)\psi_1'(x) = 0, \quad x \in [0, b], \tag{70}$$

and

$$\begin{aligned}
&d\mu\bar{\mu}\psi_2'''(x) + \big(d\bar{\mu} - d\mu + \mu\bar{\mu}(\lambda + \bar{\lambda})\big)\psi_2''(x) \\
&\quad + (\bar{\lambda}\bar{\mu} - \lambda\mu - d)\psi_2'(x) = 0, \quad x \in [b, \infty).
\end{aligned} \tag{71}$$

Proof. From (68), it is easily seen that $\psi(x)$ is differentiable on $[0, b]$. Differentiating (68) yields

$$\begin{aligned}(\lambda+\bar{\lambda})\psi_1'(x)&\\ =-\frac{1}{\mu}\Big(\frac{\lambda}{\mu}\,e^{-x/\mu}\int_0^x\psi_1(u)e^{u/\mu}\,\mathrm{d}u+\lambda e^{-x/\mu}\Big)&\\ +\frac{1}{\bar{\mu}}\Big(\frac{\bar{\lambda}}{\bar{\mu}}\,e^{x/\bar{\mu}}\int_x^b\psi_1(u)e^{-u/\bar{\mu}}\,\mathrm{d}u+\frac{\bar{\lambda}}{\bar{\mu}}\,e^{x/\bar{\mu}}\int_b^\infty\psi_2(u)e^{-u/\bar{\mu}}\,\mathrm{d}u\Big)&\\ +\Big(\frac{\lambda}{\mu}-\frac{\bar{\lambda}}{\bar{\mu}}\Big)\psi_1(x),\quad x\in[0,b].&\end{aligned}\tag{72}$$

Multiplying (72) by $(-\bar{\mu})$ and adding (68) we obtain

$$\begin{aligned}&-\bar{\mu}(\lambda+\bar{\lambda})\psi_1'(x)+\lambda\Big(1+\frac{\bar{\mu}}{\mu}\Big)\psi_1(x)\\ &=\Big(1+\frac{\bar{\mu}}{\mu}\Big)\Big(\frac{\lambda}{\mu}\,e^{-x/\mu}\int_0^x\psi_1(u)e^{u/\mu}\,\mathrm{d}u+\lambda e^{-x/\mu}\Big),\quad x\in[0,b].\end{aligned}\tag{73}$$

From (73), it follows that $\psi(x)$ is twice differentiable on $[0, b]$. Differentiating (73) gives

$$\begin{aligned}&-\bar{\mu}(\lambda+\bar{\lambda})\psi_1''(x)+\lambda\Big(1+\frac{\bar{\mu}}{\mu}\Big)\psi_1'(x)\\ &=-\frac{1}{\mu}\Big(1+\frac{\bar{\mu}}{\mu}\Big)\Big(\frac{\lambda}{\mu}\,e^{-x/\mu}\int_0^x\psi_1(u)e^{u/\mu}\,\mathrm{d}u+\lambda e^{-x/\mu}\Big)\\ &\quad+\frac{\lambda}{\mu}\Big(1+\frac{\bar{\mu}}{\mu}\Big)\psi_1(x),\quad x\in[0,b].\end{aligned}\tag{74}$$

Multiplying (74) by μ and adding (73) we get (70).

From (69), it is easily seen that $\psi(x)$ is twice differentiable on $[b,\infty)$. Differentiating (69) yields

$$\begin{aligned}&d\psi_2''(x)+(\lambda+\bar{\lambda})\psi_2'(x)\\ &=-\frac{1}{\mu}\Big(\frac{\lambda}{\mu}\,e^{-x/\mu}\int_0^b\psi_1(u)e^{u/\mu}\,\mathrm{d}u+\frac{\lambda}{\mu}\,e^{-x/\mu}\int_b^x\psi_2(u)e^{u/\mu}\,\mathrm{d}u+\lambda e^{-x/\mu}\Big)\\ &\quad+\frac{\bar{\lambda}}{\bar{\mu}^2}\,e^{x/\bar{\mu}}\int_x^\infty\psi_2(u)e^{-u/\bar{\mu}}\,\mathrm{d}u+\Big(\frac{\lambda}{\mu}-\frac{\bar{\lambda}}{\bar{\mu}}\Big)\psi_2(x),\quad x\in[b,\infty).\end{aligned}\tag{75}$$

Multiplying (75) by μ and adding (69) we obtain

$$\begin{aligned}&d\mu\psi_2''(x)+\big(d+\mu(\lambda+\bar{\lambda})\big)\psi_2'(x)+\bar{\lambda}\Big(1+\frac{\mu}{\bar{\mu}}\Big)\psi_2(x)\\ &=\frac{\bar{\lambda}}{\bar{\mu}}\Big(1+\frac{\mu}{\bar{\mu}}\Big)e^{x/\bar{\mu}}\int_x^\infty\psi_2(u)e^{-u/\bar{\mu}}\,\mathrm{d}u,\quad x\in[b,\infty).\end{aligned}\tag{76}$$

From (76), it follows that $\psi(x)$ has the third derivative on $[b,\infty)$. Differentiating (76) gives

$$\begin{aligned} & d\mu\psi_2'''(x) + \big(d+\mu(\lambda+\bar{\lambda})\big)\psi_2''(x) + \bar{\lambda}\left(1+\frac{\mu}{\bar{\mu}}\right)\psi_2'(x) \\ & = \frac{\bar{\lambda}}{\bar{\mu}^2}\left(1+\frac{\mu}{\bar{\mu}}\right)e^{x/\bar{\mu}}\int_x^\infty \psi_2(u)e^{-u/\bar{\mu}}\,\mathrm{d}u - \frac{\bar{\lambda}}{\bar{\mu}}\left(1+\frac{\mu}{\bar{\mu}}\right)\psi_2(x),\ \ x\in[b,\infty). \end{aligned} \tag{77}$$

Multiplying (77) by $(-\bar{\mu})$ and adding (76) we get (71). □

To formulate the next theorem, we define the following constants:

$$\begin{aligned} \mathrm{D}_2 &= \big(d\bar{\mu} - d\mu + \mu\bar{\mu}(\lambda+\bar{\lambda})\big)^2 - 4d\mu\bar{\mu}(\bar{\lambda}\bar{\mu} - \lambda\mu - d), \\ z_5 &= \frac{\lambda\mu - \bar{\lambda}\bar{\mu}}{\mu\bar{\mu}(\lambda+\bar{\lambda})}, \\ z_7 &= \frac{-(d\bar{\mu} - d\mu + \mu\bar{\mu}(\lambda+\bar{\lambda})) + \sqrt{\mathrm{D}_2}}{2d\mu\bar{\mu}} \end{aligned}$$

and

$$z_8 = \frac{-(d\bar{\mu} - d\mu + \mu\bar{\mu}(\lambda+\bar{\lambda})) - \sqrt{\mathrm{D}_2}}{2d\mu\bar{\mu}}.$$

Theorem 4. *Let the surplus process $(X_t^b(x))_{t\ge 0}$ follow (6) under the above assumptions with $\theta = 0$ and $\bar{\theta} = 0$, and let claim and premium sizes be exponentially distributed with means μ and $\bar{\mu}$, respectively, and let $\bar{\lambda}\bar{\mu} > \lambda\mu + d$. Then we have*

$$\psi_1(x) = C_4 + C_5 e^{z_5 x}, \quad x\in[0,b], \tag{78}$$

and

$$\psi_2(x) = C_7 e^{z_7 x} + C_8 e^{z_8 x}, \quad x\in[b,\infty), \tag{79}$$

where the constants C_4, C_5, C_7 and C_8 are determined from the system of linear equations (80)–(83):

$$\big(\lambda e^{b/\bar{\mu}} + \bar{\lambda}\big)C_4 + \frac{\lambda+\bar{\lambda}}{\mu+\bar{\mu}}\big(\bar{\mu}e^{b/\bar{\mu}} + \mu e^{z_5 b}\big)C_5 + \frac{\bar{\lambda}e^{z_7 b}}{\bar{\mu}z_7 - 1}C_7 + \frac{\bar{\lambda}e^{z_8 b}}{\bar{\mu}z_8 - 1}C_8 = \lambda e^{b/\bar{\mu}}, \tag{80}$$

$$\lambda\left(1+\frac{\bar{\mu}}{\mu}\right)C_4 + \frac{\bar{\mu}(\lambda+\bar{\lambda})}{\mu}C_5 = \lambda\left(1+\frac{\bar{\mu}}{\mu}\right), \tag{81}$$

$$C_4 + e^{z_5 b}C_5 - e^{z_7 b}C_7 - e^{z_8 b}C_8 = 0 \tag{82}$$

and

$$\begin{aligned} & \lambda\big(e^{-b/\mu} - 1\big)C_4 + \frac{\bar{\mu}(\lambda+\bar{\lambda})}{\mu+\bar{\mu}}\big(e^{-b/\mu} - e^{z_5 b}\big)C_5 \\ & + \left(\lambda+\bar{\lambda}+dz_7 + \frac{\bar{\lambda}}{\bar{\mu}z_7 - 1}\right)e^{z_7 b}C_7 \\ & + \left(\lambda+\bar{\lambda}+dz_8 + \frac{\bar{\lambda}}{\bar{\mu}z_8 - 1}\right)e^{z_8 b}C_8 = \lambda e^{-b/\mu}. \end{aligned} \tag{83}$$

Proof. By Lemma 2, $\psi_1(x)$ and $\psi_2(x)$ are solutions to (70) and (71). We now find the general solutions to these equations.

It is easily seen that the characteristic equation corresponding to (70) has two roots: $z_4 = 0$ and z_5 given before the assertion of the theorem. Hence, (78) is true with some constants C_4 and C_5.

The characteristic equation corresponding to (71) has the form

$$d\mu\bar{\mu}z^3 + \big(d\bar{\mu} - d\mu + \mu\bar{\mu}(\lambda + \bar{\lambda})\big)z^2 + (\bar{\lambda}\bar{\mu} - \lambda\mu - d)z = 0. \tag{84}$$

It is obvious that $z_6 = 0$ is a solution to (84). We now show that the equation

$$d\mu\bar{\mu}z^2 + \big(d\bar{\mu} - d\mu + \mu\bar{\mu}(\lambda + \bar{\lambda})\big)z + (\bar{\lambda}\bar{\mu} - \lambda\mu - d) = 0. \tag{85}$$

has two negative roots. We first notice that its discriminant D_2 defined above is positive. Indeed, we have

$$\begin{aligned}
\mathrm{D}_2 &= d^2(\bar{\mu} - \mu)^2 + \mu^2\bar{\mu}^2(\lambda + \bar{\lambda})^2 + 2d\mu\bar{\mu}(\lambda + \bar{\lambda})(\bar{\mu} - \mu) + 4d\mu\bar{\mu}(d + \lambda\mu - \bar{\lambda}\bar{\mu})\\
&= d^2(\mu + \bar{\mu})^2 + \mu^2\bar{\mu}^2(\lambda + \bar{\lambda})^2 + 2d\mu\bar{\mu}(\lambda - \bar{\lambda})(\mu + \bar{\mu})\\
&= \big(d(\mu + \bar{\mu}) + \mu\bar{\mu}(\lambda - \bar{\lambda})\big)^2 + 4\lambda\bar{\lambda}\mu^2\bar{\mu}^2 > 0.
\end{aligned}$$

Therefore, (85) has two real roots. Next, by the conditions of the theorem, we have

$$\bar{\lambda}\bar{\mu} - \lambda\mu - d > 0$$

and

$$d\bar{\mu} - d\mu + \mu\bar{\mu}(\lambda + \bar{\lambda}) = \mu(\bar{\lambda}\bar{\mu} - \lambda\mu - d) + \lambda\mu^2 + \lambda\mu\bar{\mu} + d\bar{\mu} > 0,$$

which shows that both roots are negative. Consequently, we get

$$\psi_2(x) = C_6 + C_7 e^{z_7 x} + C_8 e^{z_8 x}, \quad x \in [b, \infty),$$

with some constants C_6, C_7 and C_8. Moreover, since $\bar{\lambda}\bar{\mu} > \lambda\mu + d$, using standard considerations (see, e.g., [34, 36, 39]) we can easily show that $\lim_{x\to\infty}\psi(x) = 0$, which yields $C_6 = 0$. Thus, we obtain (79).

The constants C_4, C_5, C_7 and C_8 are determined by letting $x = 0$ in (68) and (73), taking into account that $\psi_1(b) = \psi_2(b)$ and letting $x = b$ in (69).

Substituting (78) and (79) into (68) and (73) as $x = 0$ and into (69) as $x = b$ and doing some simplifications yield equations (80), (81) and (83), respectively. Substituting (78) and (79) into the equality $\psi_1(b) = \psi_2(b)$ gives (82).

We denote the determinant of the system of equations (80)–(83) by Δ_2. A standard computation shows that

$$\begin{aligned}
\Delta_2 = d(z_7 - z_8)e^{(z_7+z_8)b}&\Bigg(\lambda e^{z_5 b}\left(\frac{\bar{\lambda}\bar{\mu}}{\mu} - \lambda\right)\\
&+ \frac{\bar{\mu}^2 z_7 z_8}{\bar{\mu}^2 z_7 z_8 - \bar{\mu}(z_7 + z_8) + 1}\left((\lambda + \bar{\lambda})\frac{\bar{\lambda}\bar{\mu}}{\mu} - \lambda\bar{\lambda}e^{z_5 b}\left(1 + \frac{\bar{\mu}}{\mu}\right)\right)\Bigg),
\end{aligned}$$

which is positive. Indeed, $z_7 - z_8 > 0$ by definition, $\bar{\lambda}\bar{\mu}/\mu - \lambda > 0$ by the conditions of the theorem and $\bar{\mu}^2 z_7 z_8 - \bar{\mu}(z_7 + z_8) + 1 > 0$ since $z_7 < 0$ and $z_8 < 0$. Moreover, since $z_5 < 0$, we have

$$(\lambda + \bar{\lambda})\frac{\bar{\lambda}\bar{\mu}}{\mu} - \lambda\bar{\lambda}e^{z_5 b}\left(1 + \frac{\bar{\mu}}{\mu}\right) > (\lambda + \bar{\lambda})\frac{\bar{\lambda}\bar{\mu}}{\mu} - \lambda\bar{\lambda}\left(1 + \frac{\bar{\mu}}{\mu}\right) = \bar{\lambda}\left(\frac{\bar{\lambda}\bar{\mu}}{\mu} - \lambda\right) > 0.$$

Thus, since $\Delta_2 \neq 0$, the system of equations (80)–(83) has a unique solution. Furthermore, note that letting $x = b$ in (76) gives no additional information about unknown constants, but the equality in (76) holds for the values of the constants found from the system of equations (80)–(83). Therefore, each of differential equations (70) and (71) has the unique solution given by (78) or (79), respectively. Since we have derived these equations from (68) and (69) without any additional assumptions, we conclude that the functions $\psi_1(x)$ and $\psi_2(x)$ given by (78) and (79) are unique solutions to (68) and (69) satisfying the certain conditions. This guaranties that the functions $\psi_1(x)$ and $\psi_2(x)$ we have found coincide with the ruin probability on the intervals $[0, b]$ and $[b, \infty)$, respectively, which completes the proof. □

5.3 *The expected discounted dividend payments until ruin in the model without dependence*

We now also assume that $\theta = 0$ and $\bar{\theta} = 0$. Then equations (29) and (42) for the expected discounted dividend payments $v(x)$ in the case of exponentially distributed claim and premium sizes take the form

$$\begin{aligned}(\lambda + \bar{\lambda} + \delta)v_1(x) \\ &= \frac{\lambda}{\mu} e^{-x/\mu} \int_0^x v_1(u)e^{u/\mu}\,\mathrm{d}u + \frac{\bar{\lambda}}{\bar{\mu}} e^{x/\bar{\mu}} \int_x^b v_1(u)e^{-u/\bar{\mu}}\,\mathrm{d}u \\ &\quad + \frac{\bar{\lambda}}{\bar{\mu}} e^{x/\bar{\mu}} \int_b^\infty v_2(u)e^{-u/\bar{\mu}}\,\mathrm{d}u, \quad x \in [0, b],\end{aligned} \tag{86}$$

and

$$\begin{aligned}dv_2'(x) + (\lambda + \bar{\lambda} + \delta)v_2(x) \\ &= \frac{\lambda}{\mu} e^{-x/\mu} \int_0^b v_1(u)e^{u/\mu}\,\mathrm{d}u + \frac{\lambda}{\mu} e^{-x/\mu} \int_b^x v_2(u)e^{u/\mu}\,\mathrm{d}u \\ &\quad + \frac{\bar{\lambda}}{\bar{\mu}} e^{x/\bar{\mu}} \int_x^\infty v_2(u)e^{-u/\bar{\mu}}\,\mathrm{d}u + d, \quad x \in [b, \infty),\end{aligned} \tag{87}$$

respectively.

Lemma 3 below shows that integro-differential equations (86) and (87) can be reduced to linear differential equations with constant coefficients.

Lemma 3. *Let the surplus process $(X_t^b(x))_{t\geq 0}$ follow (6) under the above assumptions with $\theta = 0$ and $\bar{\theta} = 0$, and let claim and premium sizes be exponentially distributed with means μ and $\bar{\mu}$, respectively. Then $v_1(x)$ and $v_2(x)$ are solutions to the differential equations*

$$\mu\bar{\mu}(\lambda+\bar{\lambda}+\delta)v_1''(x) + \big(\bar{\mu}(\bar{\lambda}+\delta) - \mu(\lambda+\delta)\big)v_1'(x) - \delta v_1(x) = 0, \quad x \in [0, b], \tag{88}$$

and

$$\begin{aligned} &d\mu\bar{\mu}v_2'''(x) + \big(d(\bar{\mu}-\mu) + \mu\bar{\mu}(\lambda+\bar{\lambda}+\delta)\big)v_2''(x) \\ &\quad + \big(\bar{\mu}(\bar{\lambda}+\delta) - \mu(\lambda+\delta) - d\big)v_2'(x) - \delta v_2(x) = -d, \quad x \in [b,\infty). \end{aligned} \tag{89}$$

The proof of the lemma is similar to the proof of Lemma 2.

To formulate the next theorem, we define the following constants:

$$\begin{aligned} \mathrm{D}_3 &= \big(\bar{\mu}(\bar{\lambda}+\delta) - \mu(\lambda+\delta)\big)^2 + 4\delta\mu\bar{\mu}(\lambda+\bar{\lambda}+\delta), \\ \mathrm{D}_4 &= -18\delta d\mu\bar{\mu}\big(d(\bar{\mu}-\mu) + \mu\bar{\mu}(\lambda+\bar{\lambda}+\delta)\big)\big(\bar{\mu}(\bar{\lambda}+\delta) - \mu(\lambda+\delta) - d\big) \\ &\quad + 4\delta\big(d(\bar{\mu}-\mu) + \mu\bar{\mu}(\lambda+\bar{\lambda}+\delta)\big)^3 \\ &\quad + \big(d(\bar{\mu}-\mu) + \mu\bar{\mu}(\lambda+\bar{\lambda}+\delta)\big)^2\big(\bar{\mu}(\bar{\lambda}+\delta) - \mu(\lambda+\delta) - d\big)^2 \\ &\quad - 4d\mu\bar{\mu}\big(\bar{\mu}(\bar{\lambda}+\delta) - \mu(\lambda+\delta) - d\big)^3 - 27(\delta d\mu\bar{\mu})^2, \\ z_9 &= \frac{-(\bar{\mu}(\bar{\lambda}+\delta) - \mu(\lambda+\delta)) + \sqrt{\mathrm{D}_3}}{2\mu\bar{\mu}(\lambda+\bar{\lambda}+\delta)} \end{aligned}$$

and

$$z_{10} = \frac{-(\bar{\mu}(\bar{\lambda}+\delta) - \mu(\lambda+\delta)) - \sqrt{\mathrm{D}_3}}{2\mu\bar{\mu}(\lambda+\bar{\lambda}+\delta)}.$$

Theorem 5. *Let the surplus process $(X_t^b(x))_{t\ge 0}$ follow (6) under the above assumptions with $\theta = 0$ and $\bar{\theta} = 0$, and let claim and premium sizes be exponentially distributed with means μ and $\bar{\mu}$, respectively, and let $\bar{\lambda}\bar{\mu} > \lambda\mu + d$ and $\mathrm{D}_4 > 0$. Then we have*

$$v_1(x) = C_9 e^{z_9 x} + C_{10} e^{z_{10} x}, \quad x \in [0,b], \tag{90}$$

and

$$v_2(x) = C_{11} e^{z_{11} x} + C_{12} e^{z_{12} x} + d/\delta, \quad x \in [b,\infty), \tag{91}$$

where z_{11} and z_{12} are negative roots of the cubic equation

$$d\mu\bar{\mu}z^3 + \big(d(\bar{\mu}-\mu) + \mu\bar{\mu}(\lambda+\bar{\lambda}+\delta)\big)z^2 + \big(\bar{\mu}(\bar{\lambda}+\delta) - \mu(\lambda+\delta) - d\big)z - \delta = 0 \tag{92}$$

and the constants C_9, C_{10}, C_{11} and C_{12} are determined from the system of linear equations (93)–(96):

$$\begin{aligned} &\left((\lambda+\bar{\lambda}+\delta)e^{b/\bar{\mu}} - \frac{\bar{\lambda}}{\bar{\mu}z_9 - 1}\big(e^{z_9 b} - e^{b/\bar{\mu}}\big)\right)C_9 \\ &\quad + \left((\lambda+\bar{\lambda}+\delta)e^{b/\bar{\mu}} - \frac{\bar{\lambda}}{\bar{\mu}z_{10} - 1}\big(e^{z_{10} b} - e^{b/\bar{\mu}}\big)\right)C_{10} \\ &\quad + \frac{\bar{\lambda}e^{z_{11}b}}{\bar{\mu}z_{11} - 1}C_{11} + \frac{\bar{\lambda}e^{z_{12}b}}{\bar{\mu}z_{12} - 1}C_{12} = \frac{d\bar{\lambda}}{\delta}, \end{aligned} \tag{93}$$

$$\begin{aligned} &\left(\lambda+\delta+\frac{\lambda\bar{\mu}}{\mu} - \bar{\mu}z_9(\lambda+\bar{\lambda}+\delta)\right)C_9 \\ &\quad + \left(\lambda+\delta+\frac{\lambda\bar{\mu}}{\mu} - \bar{\mu}z_{10}(\lambda+\bar{\lambda}+\delta)\right)C_{10} = 0, \end{aligned} \tag{94}$$

$$e^{z_9 b} C_9 + e^{z_{10} b} C_{10} - e^{z_{11} b} C_{11} - e^{z_{12} b} C_{12} = d/\delta \tag{95}$$

and

$$\begin{aligned}
&\frac{\lambda}{\mu z_9 + 1}\big(e^{-b/\mu} - e^{z_9 b}\big) C_9 + \frac{\lambda}{\mu z_{10} + 1}\big(e^{-b/\mu} - e^{z_{10} b}\big) C_{10} \\
&\quad + \left(\lambda + \bar{\lambda} + \delta + d z_{11} + \frac{\bar{\lambda}}{\bar{\mu} z_{11} - 1}\right) e^{z_{11} b} C_{11} \\
&\quad + \left(\lambda + \bar{\lambda} + \delta + d z_{12} + \frac{\bar{\lambda}}{\bar{\mu} z_{12} - 1}\right) e^{z_{12} b} C_{12} = -d\left(1 + \frac{\lambda}{\delta}\right)
\end{aligned} \tag{96}$$

provided that its determinant is not equal to 0.

Proof. The proof is similar to the proof of Theorem 4, so we omit detailed considerations. By Lemma 3, $v_1(x)$ and $v_2(x)$ are solutions to (88) and (89).

It is easily seen that $D_3 > 0$. Hence the characteristic equation corresponding to (88) has two real roots z_9 and z_{10} given before the assertion of the theorem. This yields (90) with some constants C_9 and C_{10}.

The assumption $D_4 > 0$ guarantees that cubic equation (92) has three distinct real roots. Consequently, the general solution to (89) is given by

$$v_2(x) = C_{11} e^{z_{11} x} + C_{12} e^{z_{12} x} + C_{13} e^{z_{13} x}, \quad x \in [b, \infty)$$

with some constants C_{11}, C_{12} and C_{13}.

By Vieta's theorem, we conclude that (92) has either two or no negative roots. Since $\bar{\lambda}\bar{\mu} > \lambda\mu + d$, applying arguments similar to those in [40, p. 70] shows that $\lim_{x\to\infty} v_2(x) = d/\delta$. Therefore, if (92) had no negative roots, the function $v_2(x)$ would be constant, which is impossible. From this we deduce that (92) has two negative roots. We denote them by z_{11} and z_{12}. Since $z_{13} > 0$, we get $C_{13} = 0$, which yields (91).

To determine the constants C_9, C_{10}, C_{11} and C_{12}, we apply considerations similar to those in the proof of Theorem 4 and obtain the system of linear equations (93)–(96), which has a unique solution provided that its determinant is not equal to 0. Finally, applying arguments similar to those in the proof of Theorem 4 guaranties that the functions $v_1(x)$ and $v_2(x)$ we have found coincide with the expected discounted dividend payments on the intervals $[0, b]$ and $[b, \infty)$. □

6 Numerical illustrations

We now present numerical examples for the results obtained in Section 5. The claim and premium sizes are also assumed to be exponentially distributed. Set $\lambda = 0.1$, $\bar{\lambda} = 2.3$, $\mu = 3$ and $\bar{\mu} = 0.2$.

Let now the conditions of Theorem 3 hold. Then applying this theorem we can calculate the ruin probability for $x \in [0, \infty)$ in the model without dividend payments for different values of θ:

- if $\theta = -0.9$, then $\psi(x) \approx 0.929934 e^{-0.022277x} - 0.006234 e^{-0.744001x}$;

Table 1. The ruin probabilities in the model without dividend payments for different values of θ

$x \setminus \theta$	-0.9	-0.5	-0.1	0.1	0.5	0.9
0	0.923700	0.808017	0.694653	0.638820	0.528807	0.420945
1	0.906484	0.766009	0.629915	0.563467	0.433686	0.307949
2	0.888003	0.724172	0.570650	0.497607	0.358674	0.228911
5	0.831762	0.608107	0.423229	0.343831	0.208380	0.100718
7	0.795628	0.540438	0.346536	0.268996	0.146531	0.060380
10	0.744222	0.452611	0.256692	0.186208	0.086812	0.028761
15	0.665780	0.336734	0.155646	0.100887	0.036402	0.008589
20	0.595603	0.250520	0.094376	0.054662	0.015276	0.002591
50	0.305291	0.042479	0.004690	0.001383	0.000083	0.000002
70	0.195532	0.013013	0.000634	0.000119	0.000003	0.000000

- if $\theta = -0.5$, then $\psi(x) \approx 0.817753e^{-0.059151x} - 0.009736e^{-0.712238x}$;
- if $\theta = -0.1$, then $\psi(x) \approx 0.698198e^{-0.100061x} - 0.003545e^{-0.676439x}$;
- if $\theta = 0.1$, then $\psi(x) \approx 0.634275e^{-0.122565x} + 0.004545e^{-0.656490x}$;
- if $\theta = 0.5$, then $\psi(x) \approx 0.492433e^{-0.173655x} + 0.036374e^{-0.610511x}$;
- if $\theta = 0.9$, then $\psi(x) \approx 0.309485e^{-0.239185x} + 0.111461e^{-0.550092x}$.

Table 1 presents the results of calculations for some values of x.

Next, we denote by $\psi_0(x)$ the ruin probability in this model where $\theta = 0$. It is given by

$$\psi_0(x) = \frac{\lambda(\mu + \bar{\mu})}{\bar{\mu}(\lambda + \bar{\lambda})} \exp\left(-\frac{(\bar{\lambda}\bar{\mu} - \lambda\mu)x}{\mu\bar{\mu}(\lambda\bar{\lambda})}\right), \quad x \in [0, \infty)$$

(see [9, 34]). In our example, $\psi_0(x) \approx 0.666667e^{-0.111111x}$. The values of $\psi_0(x)$ for some x are given in Table 2.

Let now the conditions of Theorems 4 and 5 hold. Set additionally $b = 5$, $d = 0.1$ and $\delta = 0.01$. Applying Theorems 4 and 5 we can calculate the ruin probability $\psi(x)$ and the expected discounted dividend payments until ruin $v(x)$:

$$\begin{aligned}
\psi_1(x) &\approx 0.389315 + 0.407125e^{-0.111111x}, \quad x \in [0, 5],\\
\psi_2(x) &\approx 0.809486e^{-0.051863x} - 1.24332 \cdot 10^{39} e^{-19.281470x}, \quad x \in [5, \infty);\\
v_1(x) &\approx 4.555889e^{0.049220x} - 2.296416e^{-0.140506x}, \quad x \in [0, 5],\\
v_2(x) &\approx 10 - 9.149114e^{-0.107684x} + 4.07834 \cdot 10^{40} e^{-19.405407x}, \quad x \in [5, \infty).
\end{aligned}$$

The results of calculations for some values of x are given in Table 2.

The results presented in Tables 1 and 2 show that the positive dependence between the claim sizes and the inter-claim times decreases the ruin probability and the negative dependence increases it. This conclusion seems to be natural. Indeed, in the case of negative dependence, the situation where large claims arrive in short time intervals is more probable, which obviously leads to ruin in the near future. Moreover, it is easily seen from Table 2 that dividend payments substantially increase the ruin probability, which is also an expected conclusion.

Table 2. The ruin probabilities with and without dividend payments and the expected discounted dividend payments in the model without dependence

x	$\psi_0(x)$	$\psi(x)$	$v(x)$
0	0.666667	0.796440	2.259472
1	0.596560	0.753626	2.790339
2	0.533825	0.715315	3.293343
5	0.382502	0.622904	4.689607
7	0.306284	0.563044	5.694612
10	0.219462	0.481915	6.883176
15	0.125917	0.371835	8.180807
20	0.072245	0.286900	8.938193
50	0.002577	0.060536	9.958020
70	0.000279	0.021455	9.995128

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