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# Double barrier reflected BSDEs with stochastic Lipschitz coefficient

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**Abstract** This paper proves the existence and uniqueness of a solution to doubly reflected backward stochastic differential equations where the coefficient is stochastic Lipschitz, by means of the penalization method.

**Keywords** BSDE and reflected BSDE, Stochastic Lipschitz coefficient
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## 1 Introduction

Backward Stochastic Differential Equations (BSDEs) were introduced (in the nonlinear case) by Pardoux and Peng [21]. Precisely, given a data $(\xi, f)$ of a square integrable random variable $\xi$ and a progressively measurable function $f$, a solution to BSDE associated with data $(\xi, f)$ is a pair of $\mathcal{F}_t$-adapted processes $(Y, Z)$ satisfying

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s, \quad 0 \le t \le T. \tag{1}$$

These equations have attracted great interest due to their connections with mathematical finance [9, 10], stochastic control and stochastic games [3, 17] and partial differential equations [20, 22].

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In their seminal paper [21], Pardoux and Peng generalized such equations to the Lipschitz condition and proved existence and uniqueness results in a Brownian framework. Moreover, many efforts have been made to relax the Lipschitz condition on the coefficient. In this context, Bender and Kohlmann [2] considered the so-called stochastic Lipschitz condition introduced by El Karoui and Huang [8].

Further, El Karoui et al. [11] have introduced the notion of reflected BSDEs (RBSDEs in short), which is a BSDE but the solution is forced to stay above a lower barrier. In detail, a solution to such equations is a triple of processes $(Y, Z, K)$ satisfying

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + K_T - K_t - \int_t^T Z_s dB_s, \quad Y_t \geq L_t \; 0 \leq t \leq T, \tag{2}$$

where $L$, the so-called barrier, is a given stochastic process. The role of the continuous increasing process $K$ is to push the state process upward with the minimal energy, in order to keep it above $L$; in this sense, it satisfies $\int_0^T (Y_t - L_t)dK_t = 0$. The authors have proved that equation (2) has a unique solution under square integrability of the terminal condition $\xi$ and the barrier $L$, and the Lipschitz property of the coefficient $f$.

RBSDEs have been proven to be powerful tools in mathematical finance [10], mixed game problems [6], providing a probabilistic formula for the viscosity solution to an obstacle problem for a class of parabolic partial differential equations [11].

Later, Cvitanic and Karatzas [6] studied doubly reflected BSDEs (DRBSDEs in short). A solution to such an equation related to a generator $f$, a terminal condition $\xi$ and two barriers $L$ and $U$ is a quadruple of $(Y, Z, K^+, K^-)$ which satisfies

$$\begin{cases} Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \left(K_T^+ - K_t^+\right) - \left(K_T^- - K_t^-\right) - \int_t^T Z_s dB_s \\ L_t \leq Y_t \leq U_t, \; \forall t \leq T \text{ and } \int_0^T (Y_t - L_t)dK_t^+ = \int_0^T (U_t - Y_t)dK_t^- = 0. \end{cases} \tag{3}$$

In this case, a solution $Y$ has to remain between the lower barrier $L$ and upper barrier $U$. This is achieved by the cumulative action of two continuous, increasing reflecting processes $K^{\pm}$. The authors proved the existence and uniqueness of the solution when $f(t, \omega, y, z)$ is Lipschitz on $(y, z)$ uniformly in $(t, \omega)$. At the same time, one of the barriers $L$ or $U$ is regular or they satisfy the so-called Mokobodski condition, which turns out into the existence of a difference of a non-negative supermartingales between $L$ and $U$. In addition, many efforts have been made to relax the conditions on $f$, $L$ and $U$ [1, 15, 16, 18, 19, 27, 29] or to deal with other issues [5, 12–14, 24].

Let us have a look at the pricing problem of an American game option driven by Black–Scholes market model which is given by the following system of stochastic differential equations

$$\begin{cases} dS_t^0 = r(t)S_t^0 dt, & S_0^0 > 0; \\ dS_t = S_t\big(\big(r(t) + \theta(t)\sigma(t)\big)dt + \sigma(t)dB_t\big), & S_0 > 0, \end{cases}$$

where $r(t)$ is the interest rate process, $\theta(t)$ is the risk premium process, $\sigma(t)$ is the volatility process of the market. The fair price of the American game option is defined by

$$Y_t = \inf_{\tau \in \Im_{[0,T]}} \sup_{\nu \in \Im_{[0,T]}} \mathbb{E}\big[e^{-r(t)\sigma(t)\wedge\theta(t)} J(\tau, \nu)|\mathcal{F}_t\big],$$

where $\Im_{[0,T]}$ is the collection of all stopping times $\tau$ with values between 0 and $T$, and $J$ is a *Payoff* given by

$$J(\tau, \nu) = U_\nu \mathbb{1}_{\{\nu<\tau\}} + L_\tau \mathbb{1}_{\{\tau\leq\nu\}} + \xi \mathbb{1}_{\{\nu\wedge\tau=T\}}.$$

Here $r(t)$, $\sigma(t)$ and $\theta(t)$ are stochastic, moreover they are not bounded in general. So the existence results of Cvitanic and Karatzas [6], Li and Shi [19] with completely separated barriers cannot be applied.

Motivated by the above works, the purpose of the present paper is to consider a class of DRBSDEs driven by a Brownian motion with stochastic Lipschitz coefficient. We try to get the existence and uniqueness of solutions to those DRBSDEs by means of the penalization method and the fixed point theorem. Furthermore, the comparison theorem for the solutions to DRBSDEs will be established.

The paper is organized as follows: in Section 2, we give some notations and assumptions needed in this paper. In Section 3, we establish the a priori estimates of solutions to DRBSDEs. In Section 4, we prove the existence and uniqueness of solutions to DRBSDEs via penalization method when one barrier is regular, in the first subsection, then we study the case when the barriers are completely separated, in the second subsection. In Section 5, we give the comparison theorem for the solutions to DRBSDEs. Finally, an Appendix is devoted to the special case of RBSDEs with lower barrier when the generator only depends on $y$; furthermore, the corresponding comparison theorem will be established under the stochastic Lipschitz coefficient.

## 2 Notations

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\leq T}, \mathbb{P})$ be a filtered probability space. Let $(B_t)_{t\leq T}$ be a $d$-dimensional Brownian motion. We assume that $(\mathcal{F}_t)_{t\leq T}$ is the standard filtration generated by the Brownian motion $(B_t)_{t\leq T}$.

We will denote by $|.|$ the Euclidian norm on $\mathbb{R}^d$.

Let's introduce some spaces:

- $\mathcal{L}^2$ is the space of $\mathbb{R}$-valued and $\mathcal{F}_T$-measurable random variables $\xi$ such that
$$\|\xi\|^2 = \mathbb{E}\big[|\xi|^2\big] < +\infty.$$
- $\mathcal{S}^2$ is the space of $\mathbb{R}$-valued and $\mathcal{F}_t$-progressively measurable processes $(K_t)_{t\leq T}$ such that
$$\|K\|^2 = \mathbb{E}\Big[\sup_{0\leq t\leq T} |K_t|^2\Big] < +\infty.$$

Let $\beta > 0$ and $(a_t)_{t\leq T}$ be a non-negative $\mathcal{F}_t$-adapted process. We define the increasing continuous process $A(t) = \int_0^t a^2(s)ds$, for all $t \leq T$, and introduce the following spaces:

- $\mathcal{L}^2(\beta, a)$ is the space of $\mathbb{R}$-valued and $\mathcal{F}_T$-measurable random variables $\xi$ such that
$$\|\xi\|_\beta^2 = \mathbb{E}\big[e^{\beta A(T)}|\xi|^2\big] < +\infty.$$

- $\mathcal{S}^2(\beta, a)$ is the space of $\mathbb{R}$-valued and $\mathcal{F}_t$-adapted continuous processes $(Y_t)_{t\leq T}$ such that
$$\|Y\|_\beta^2 = \mathbb{E}\Big[\sup_{0\leq t\leq T} e^{\beta A(t)}|Y_t|^2\Big] < +\infty.$$

- $\mathcal{S}^{2,a}(\beta, a)$ is the space of $\mathbb{R}$-valued and $\mathcal{F}_t$-adapted processes $(Y_t)_{t\leq T}$ such that
$$\|aY\|_\beta^2 = \mathbb{E}\left[\int_0^T e^{\beta A(t)}\big|a(t)Y_t\big|^2 dt\right] < +\infty.$$

- $\mathcal{H}^2(\beta, a)$ is the space of $\mathbb{R}^d$-valued and $\mathcal{F}_t$-progressively measurable processes $(Z_t)_{t\leq T}$ such that
$$\|Z\|_\beta^2 = \mathbb{E}\left[\int_0^T e^{\beta A(t)}|Z_t|^2 dt\right] < +\infty.$$

- $\mathfrak{B}^2$ is the Banach space of the processes $(Y, Z) \in (\mathcal{S}^2(\beta, a) \cap \mathcal{S}^{2,a}(\beta, a)) \times \mathcal{H}^2(\beta, a)$ with the norm
$$\big\|(Y, Z)\big\|_\beta = \sqrt{\|aY\|_\beta^2 + \|Z\|_\beta^2}.$$

We consider the following conditions:

$(H1)$ The terminal condition $\xi \in \mathcal{L}^2(\beta, a)$.

The coefficient $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}$ satisfies

$(H2)$ $\forall t \in [0, T]$ $\forall (y, z, y', z') \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$, there are two non-negative $\mathcal{F}_t$-adapted processes $\mu$ and $\gamma$ such that
$$\big|f(t, y, z) - f\big(t, y', z'\big)\big| \leq \mu(t)\big|y - y'\big| + \gamma(t)\big|z - z'\big|.$$

$(H3)$ There exists $\epsilon > 0$ such that $a^2(t) := \mu(t) + \gamma^2(t) \geq \epsilon$.

$(H4)$ For all $(y, z) \in \mathbb{R} \times \mathbb{R}^d$, the process $(f(t, y, z))_t$ is progressively measurable and such that
$$\frac{f(., 0, 0)}{a} \in \mathcal{H}^2(\beta, a).$$

The two reflecting barriers $L$ and $U$ are two $\mathcal{F}_t$-adapted and continuous real-valued processes which satisfy

$(H5)$
$$\mathbb{E}\Big[\sup_{0\leq t\leq T} e^{2\beta A(t)}\big|L_t^+\big|^2\Big] + \mathbb{E}\Big[\sup_{0\leq t\leq T} e^{2\beta A(t)}\big|U_t^-\big|^2\Big] < +\infty,$$
where $L^+$ and $U^-$ are the positive and negative parts of $L$ and $U$, respectively.

$(H6)$ $U$ is regular: i.e., there exists a sequence of $(U^n)_{n\geq 0}$ such that

$(i)$ $\forall t \leq T, U_t^n \leq U_t^{n+1}$ and $\lim_{n\to+\infty} U_t^n = U_t$ $\mathbb{P}$-a.s

$(ii)$ $\forall n \geq 0, \forall t \leq T$,

$$U_t^n = U_0^n + \int_0^t u_n(s)ds + \int_0^t v_n(s)dB_s$$

where the processes $u_n$ and $v_n$ are $\mathcal{F}_t$-adapted such that

$$\sup_{n\geq 0}\sup_{0\leq t\leq T}\big(u_n(t)\big)^+ \leq C \quad \text{and} \quad \mathbb{E}\bigg[\int_0^T \big|v_n(s)\big|^2 ds\bigg]^{\frac{1}{2}} < +\infty.$$

**Definition 1.** Let $\beta > 0$ and $a$ be a non-negative $\mathcal{F}_t$-adapted process. A solution to DRBSDE is a quadruple $(Y, Z, K^+, K^-)$ satisfying (3) such that

- $(Y, Z) \in (\mathcal{S}^2(\beta, a) \cap \mathcal{S}^{2,a}(\beta, a)) \times \mathcal{H}^2(\beta, a)$,
- $K^{\pm} \in \mathcal{S}^2$ are two continuous and increasing processes with $K_0^{\pm} = 0$.

## 3 A priori estimate

**Lemma 1.** *Let $\beta > 0$ be large enough and assume $(H1)-(H6)$ hold. Let $(Y, Z, K^+, K^-) \in (\mathcal{S}^2(\beta, a) \cap \mathcal{S}^{2,a}(\beta, a)) \times \mathcal{H}^2(\beta, a) \times \mathcal{S}^2 \times \mathcal{S}^2$ be a solution to DRBSDE with data $(\xi, f, L, U)$. Then there exists a constant $C_\beta$ depending only on $\beta$ such that*

$$\begin{aligned}
&\mathbb{E}\bigg[\sup_{0\leq t\leq T} e^{\beta A(t)}|Y_t|^2 + \int_0^T e^{\beta A(t)}\big(a^2(t)|Y_t|^2 + |Z_t|^2\big)dt + \big|K_T^+\big|^2 + \big|K_T^-\big|^2\bigg] \\
&\quad \leq C_\beta \mathbb{E}\bigg[e^{\beta A(T)}|\xi|^2 + \int_0^T e^{\beta A(t)}\frac{|f(t,0,0)|^2}{a^2(t)}dt \\
&\qquad + \sup_{0\leq t\leq T} e^{2\beta A(t)}\big(\big|L_t^+\big|^2 + \big|U_t^-\big|^2\big)\bigg]. \qquad (4)
\end{aligned}$$

**Proof.** Applying Itô's formula and Young's inequality, combined with the stochastic Lipschitz assumption $(H2)$ we can write

$$\begin{aligned}
&e^{\beta A(t)}|Y_t|^2 + \int_t^T \beta e^{\beta A(s)}a^2(s)|Y_s|^2 ds + \int_t^T e^{\beta A(s)}|Z_s|^2 ds \\
&\quad \leq e^{\beta A(T)}|\xi|^2 + \frac{\beta}{2}\int_t^T e^{\beta A(s)}a^2(s)|Y_s|^2 ds + \frac{2}{\beta}\int_t^T e^{\beta A(s)}\frac{|f(s,Y_s,Z_s)|^2}{a^2(s)}ds \\
&\qquad + 2\int_t^T e^{\beta A(s)}Y_s dK_s^+ - 2\int_t^T e^{\beta A(s)}Y_s dK_s^- - 2\int_t^T e^{\beta A(s)}Y_s Z_s dB_s \\
&\quad \leq e^{\beta A(T)}|\xi|^2 + \frac{\beta}{2}\int_t^T e^{\beta A(s)}a^2(s)|Y_s|^2 ds + \frac{6}{\beta}\int_t^T e^{\beta A(s)}a^2(s)|Y_s|^2 ds \\
&\qquad + \frac{6}{\beta}\int_t^T e^{\beta A(s)}|Z_s|^2 ds + \frac{6}{\beta}\int_t^T e^{\beta A(s)}\frac{|f(s,0,0)|^2}{a^2(s)}ds
\end{aligned}$$

$$+2\int_t^T e^{\beta A(s)} Y_s dK_s^+ - 2\int_t^T e^{\beta A(s)} Y_s dK_s^- - 2\int_t^T e^{\beta A(s)} Y_s Z_s dB_s.$$

Using the fact that $dK_s^+ = \mathbb{1}_{\{Y_s = L_s\}} dK_s^+$ and $dK_s^- = \mathbb{1}_{\{Y_s = U_s\}} dK_s^-$, we have

$$\begin{aligned}
&e^{\beta A(t)}|Y_t|^2 + \left(\frac{\beta}{2} - \frac{6}{\beta}\right)\int_t^T e^{\beta A(s)} a^2(s)|Y_s|^2 ds + \left(1 - \frac{6}{\beta}\right)\int_t^T e^{\beta A(s)}|Z_s|^2 ds \\
&\leq e^{\beta A(T)}|\xi|^2 + \frac{6}{\beta}\int_t^T e^{\beta A(s)} \frac{|f(s,0,0)|^2}{a^2(s)} ds + 2\int_t^T e^{\beta A(s)} L_s dK_s^+ \\
&\quad - 2\int_t^T e^{\beta A(s)} U_s dK_s^- - 2\int_t^T e^{\beta A(s)} Y_s Z_s dB_s.
\end{aligned} \tag{5}$$

Taking expectation on both sides above, we get

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T e^{\beta A(s)} a^2(s)|Y_s|^2 ds + \int_0^T e^{\beta A(s)}|Z_s|^2 ds\right] \\
&\quad \leq c_\beta \mathbb{E}\left[e^{\beta A(T)}|\xi|^2 + \int_0^T e^{\beta A(s)} \frac{|f(s,0,0)|^2}{a^2(s)} ds \right.\\
&\qquad \left. + \sup_{0\leq t\leq T} e^{\beta A(t)}\left|L_t^+\right|^2 + \left|K_T^+\right|^2 + \sup_{0\leq t\leq T} e^{\beta A(t)}\left|U_t^-\right|^2 + \left|K_T^-\right|^2\right]
\end{aligned} \tag{6}$$

and by the Burkholder–Davis–Gundy's inequality we obtain

$$\begin{aligned}
&\mathbb{E}\sup_{0\leq t\leq T} e^{\beta A(t)}|Y_t|^2 \\
&\quad \leq \mathcal{C}_\beta \mathbb{E}\left[e^{\beta A(T)}|\xi|^2 + \int_0^T e^{\beta A(s)} \frac{|f(s,0,0)|^2}{a^2(s)} ds \right.\\
&\qquad \left. + 2\int_t^T e^{\beta A(s)} L_s dK_s^+ - 2\int_t^T e^{\beta A(s)} L_s dK_s^-\right]
\end{aligned} \tag{7}$$

$$\begin{aligned}
&\quad \leq \mathcal{C}_\beta \mathbb{E}\left[e^{\beta A(T)}|\xi|^2 + \int_0^T e^{\beta A(s)} \frac{|f(s,0,0)|^2}{a^2(s)} ds \right.\\
&\qquad \left. + \sup_{0\leq t\leq T} e^{2\beta A(t)}\left(\left|L_t^+\right|^2 + \left|U_t^-\right|^2\right) + \left|K_T^+\right|^2 + \left|K_T^-\right|^2\right].
\end{aligned} \tag{8}$$

To conclude, we now give an estimate of ${K_T^+}^2$ and ${K_T^-}^2$. From the equation

$$K_T^+ - K_T^- = Y_0 - \xi - \int_0^T f(s, Y_s, Z_s) ds + \int_0^T Z_s dB_s$$

and the stochastic Lipschitz property $(H2)$, we have

$$\begin{aligned}
&\mathbb{E}\left[\left|K_T^+ - K_T^-\right|^2\right] \\
&\quad \leq 4\mathbb{E}\left[\sup_{0\leq t\leq T} e^{\beta A(t)}|Y_t|^2 + |\xi|^2 + \left(1 + \frac{3}{\beta}\right)\int_0^T e^{\beta A(s)}|Z_s|^2 ds\right.
\end{aligned}$$

$$+\frac{3}{\beta}\int_0^T e^{\beta A(s)}a^2(s)|Y_s|^2ds+\frac{3}{\beta}\int_0^T e^{\beta A(s)}\frac{|f(s,0,0)|^2}{a^2(s)}ds\Bigg].$$

Combining this with (7), we derive that

$$\begin{aligned}\mathbb{E}\big|K_T^+\big|^2+\mathbb{E}\big|K_T^-\big|^2 &\leq \mathfrak{C}_\beta\mathbb{E}\Bigg[e^{\beta A(T)}|\xi|^2+\int_0^T e^{\beta A(s)}\frac{|f(s,0,0)|^2}{a^2(s)}ds\\ &\quad+\sup_{0\leq t\leq T}e^{2\beta A(t)}\big(\big|L_t^+\big|^2+\big|U_t^-\big|^2\big)\Bigg]+\frac{1}{2}\mathbb{E}\big|K_T^+\big|^2+\frac{1}{2}\mathbb{E}\big|K_T^-\big|^2.\end{aligned} \tag{9}$$

The desired result is obtained by estimates (6), (8) and (9). □

## 4 Existence and uniqueness of solution

### 4.1 The obstacle $U$ is regular

In this part, we apply the penalization method and the fixed point theorem to give the existence of the solution to the DRBSDE (3). We first consider the special case when the generator does not depend on $(y,z)$:

$$f(t,y,z)=g(t).$$

**Theorem 1.** *Assume that $\frac{g}{a}\in\mathcal{H}^2(\beta,a)$ and $(H1)$–$(H6)$ hold. Then, the doubly reflected BSDE (3) with data $(\xi,g,L,U)$ has a unique solution $(Y,Z,K^+,K^-)$ that belongs to $(\mathcal{S}^2(\beta,a)\cap\mathcal{S}^{2,a}(\beta,a))\times\mathcal{H}^2(\beta,a)\times\mathcal{S}^2\times\mathcal{S}^2$.*

For all $n\in\mathbb{N}$, let $(Y^n,Z^n,K^{n+})$ be the $\mathcal{F}_t$-adapted process with values in $(\mathcal{S}^2(\beta,a)\cap\mathcal{S}^{2,a}(\beta,a))\times\mathcal{H}^2(\beta,a)\times\mathcal{S}^2$ being a solution to the reflected BSDE with data $(\xi,g(t)-n(y-U_t)^+,L)$. That is

$$\begin{cases}Y_t^n=\xi+\int_t^T g(s)ds-n\int_t^T\big(Y_s^n-U_s\big)^+ds+K_T^{n+}-K_t^{n+}-\int_t^T Z_s^n dB_s\\ Y_t^n\geq L_t,\ \forall t\leq T \text{ and } \int_0^T\big(Y_t^n-L_t\big)dK_t^{n+}=0.\end{cases} \tag{10}$$

We denote $K_t^{n-}:=n\int_0^t(Y_s^n-U_s)^+ds$ and $g^n(s,y):=g(s)-n(y-U_s)^+$.

We have divided the proof of Theorem 1 into sequence of lemmas.

**Lemma 2.** *There exists a positive constant $C$ such that*

$$\sup_{0\leq t\leq T}n\big(Y_t^n-U_t\big)^+\leq C\quad \mathbb{P}\text{-}a.s.$$

**Proof.** For all $n,m\geq 0$, let $(Y^{n,m},Z^{n,m})$ be the solution to the following BSDE

$$Y_t^{n,m}=\xi-\int_t^T\big\{g(s)+m\big(Y_s^{n,m}-L_s\big)^- -n\big(Y_s^{n,m}-U_s\big)^+\big\}ds-\int_t^T Z_s^{n,m}dB_s.$$

We denote $\bar{Y}^{n,m} = Y^{n,m} - U^m$. Then we have

$$\bar{Y}_t^{n,m} = \xi - U_T^m + \int_t^T \big(g(s) + u_m(s)\big)ds - n\int_t^T \big(\bar{Y}_s^{n,m} - \big(U_s - U_s^m\big)\big)^+ ds$$
$$+ m\int_t^T \big(\bar{Y}_s^{n,m} - \big(L_s - U_s^m\big)\big)^- ds - \int_t^T \big(Z_s^{n,m} - v_n(s)\big)dB_s.$$

For $n \geq 0$, let $\mathcal{D}_n$ be the class of $\mathcal{F}_t$-progressively measurable process taking values in $[0, n]$. For $\nu \in \mathcal{D}_n$ and $\lambda \in \mathcal{D}_m$ we denote $R_t = e^{-\int_0^t (\nu(s)+\lambda(s))ds}$. Applying Itô's formula to $R_t \bar{Y}_t^{n,m}$ and using the same arguments as on page 2042 of [6], one can show that

$$\bar{Y}_t^{n,m} \leq \operatorname*{ess\,sup}_{\lambda\in\mathcal{D}_m} \operatorname*{ess\,inf}_{\nu\in\mathcal{D}_n} \mathbb{E}\bigg[\int_t^T e^{-\int_t^s (\nu(r)+\lambda(r))dr}\big|u_m(s)\big|ds|\mathcal{F}_t\bigg].$$

From the assumption $(H6)(ii)$, we can write $\bar{Y}_t^{n,m} \vee 0 \leq \frac{C}{n}$. It follows that

$$\forall t \leq T, \qquad n\big(\bar{Y}_t^{n,m} \vee 0\big) \xrightarrow[m\to+\infty]{} n\big(Y_t^n - U_t\big)^+ \leq C \quad \mathbb{P}\text{-}a.s. \qquad \square$$

**Lemma 3.** *There exists a positive constant $C'_\beta$ depending only on $\beta$ such that for all $n \geq 0$*

$$\mathbb{E}\bigg[\sup_{0\leq t\leq T} e^{\beta A(t)}\big|Y_t^n\big|^2 + \int_0^T e^{\beta A(t)}a^2(t)\big|Y_t^n\big|^2dt + \int_0^T e^{\beta A(t)}\big|Z_t^n\big|^2dt + \big|K_T^{n+}\big|^2\bigg]$$
$$\leq C'_\beta\mathbb{E}\bigg[e^{\beta A(T)}|\xi|^2 + \int_0^T e^{\beta A(t)}\bigg|\frac{g(t)}{a(t)}\bigg|^2dt$$
$$+ \sup_{0\leq t\leq T} e^{2\beta A(t)}\big|U_t^-\big|^2 + \sup_{0\leq t\leq T} e^{2\beta A(t)}\big|L_t^+\big|^2\bigg].$$

**Proof.** Itô's formula implies for $t \leq T$:

$$\beta\mathbb{E}\int_t^T e^{\beta A(s)}a^2(s)\big|Y_s^n\big|^2ds + \mathbb{E}\int_t^T e^{\beta A(s)}\big|Z_s^n\big|^2ds$$
$$\leq \mathbb{E}e^{\beta A(T)}|\xi|^2 + \frac{\beta}{2}\mathbb{E}\int_t^T e^{2\beta A(s)}a^2(s)\big|Y_s^n\big|^2ds + \frac{2}{\beta}\mathbb{E}\int_t^T e^{\beta A(s)}\frac{|g(s)|^2}{a^2(s)}ds$$
$$+ 2\mathbb{E}\bigg[\sup_{n\geq 0}\sup_{0\leq t\leq T} n\big(Y_t^n - U_t\big)^+\int_t^T e^{\beta A(s)}U_s^-ds\bigg] + 2\mathbb{E}\bigg[\int_t^T e^{\beta A(s)}L_s dK_s^{n+}\bigg].$$

Here we used the fact that $-nY_s^n(Y_s^n - U_s)^+ \leq nU^-(Y_s^n - U_s)^+$ and $dK_s^{n+} = \mathbb{1}_{\{Y_s^n = L_s\}}dK_s^{n+}$. We conclude, by the Burkholder–Davis–Gundy's inequality, that

$$\mathbb{E}\sup_{0\leq t\leq T} e^{\beta A(t)}\big|Y_t^n\big|^2 + \mathbb{E}\int_0^T e^{\beta A(s)}a^2(s)\big|Y_s^n\big|^2ds + \mathbb{E}\int_0^T e^{\beta A(s)}\big|Z_s^n\big|^2ds$$
$$\leq c'_p\mathbb{E}\bigg[e^{\beta A(T)}|\xi|^2 + \int_0^T e^{\beta A(s)}\frac{|g(s)|^2}{a^2(s)}ds$$

$$+\sup_{0\le t\le T} e^{2\beta A(t)}\big|U_t^-\big|^2+\sup_{0\le t\le T} e^{2\beta A(t)}\big|L_t^+\big|^2+\big|K_T^{n+}\big|^2\Big].$$

In the same way as (9), we can prove that

$$\mathbb{E}\big|K_T^{n+}\big|^2\le C'_p\mathbb{E}\Big[e^{\beta A(T)}|\xi|^2+\int_0^T e^{\beta A(s)}\frac{|g(s)|^2}{a^2(s)}ds$$
$$+\sup_{0\le t\le T} e^{2\beta A(t)}\big|U_t^-\big|^2+\sup_{0\le t\le T} e^{2\beta A(t)}\big|L_t^+\big|^2\Big].$$

We obtain the desired result. □

**Lemma 4.** *There exist two $\mathcal{F}_t$-adapted processes $(Y_t)_{t\le T}$ and $(K_t^+)_{t\le T}$ such that $Y^n\searrow Y$, $K^{n+}\nearrow K^+$ and*

$$\mathbb{E}\Big[\sup_{0\le t\le T}\big|K_t^{n+}-K_t^+\big|^2\Big]\xrightarrow[n\to+\infty]{}0.$$

**Proof.** The comparison Theorem 5 (below) shows that $Y_t^0\ge Y_t^n\ge Y_t^{n+1}$ and $K_t^{n+}\le K_t^{(n+1)+}$ for all $t\le T$. Therefore, there exist processes $Y$ and $K^+$ such that, as $n\to+\infty$, for all $t\le T$, $Y_t^n\searrow Y_t$ and $K_t^{n+}\nearrow K_t^+$. Since the process $K^+$ is continuous, it follows by Dini's theorem that

$$\mathbb{E}\Big[\sup_{0\le t\le T}\big|K_t^{n+}-K_t^+\big|^2\Big]\xrightarrow[n\to+\infty]{}0.$$ □

**Lemma 5.**

$$\mathbb{E}\Big[\sup_{0\le t\le T} e^{\beta A(t)}\big|\big(Y_t^n-U_t\big)^+\big|^2\Big]\xrightarrow[n\to+\infty]{}0.$$

**Proof.** Since $Y_t\le Y_t^n\le Y_t^0$, we can replace $U_t$ by $U_t\vee Y^0$; that is, we may assume that $\mathbb{E}\sup_{0\le t\le T} e^{\beta A(t)}|U_t|^2<+\infty$.

Let $(\widetilde{Y}^n,\widetilde{Z}^n,\widetilde{K}^n)$ be the solution to the following Reflected BSDE associated with $(\xi, g-n(y-U), L)$:

$$\begin{cases}\widetilde{Y}_t^n=\xi+\displaystyle\int_t^T\big(g(s)-n\big(\widetilde{Y}_s^n-U_s\big)\big)ds+\widetilde{K}_T^n-\widetilde{K}_t^n-\int_t^T\widetilde{Z}_s^n dB_s\\ \widetilde{Y}_t^n\ge L_t,\ \forall t\le T \text{ and } \displaystyle\int_0^T\big(\widetilde{Y}_t^n-L_t\big)d\widetilde{K}_t^n=0.\end{cases}\tag{11}$$

The comparison Theorem 5 shows that $Y^n\le\widetilde{Y}^n$ and $d\widetilde{K}^n\le dK^{n+}\le dK^+$. Let $\tau\le T$ be a stopping time. Then we can write

$$\widetilde{Y}_\tau^n=\mathbb{E}\Big[e^{-n(T-\tau)}\xi+\int_\tau^T e^{-n(s-\tau)}\big(g(s)+nU_s\big)ds+\int_\tau^T e^{-n(s-\tau)}d\widetilde{K}_s^n|\mathcal{F}_\tau\Big].$$

Since $\mathbb{E}\sup_{0\le t\le T} e^{\beta A(t)}U_t^2<+\infty$, we obtain

$$e^{-n(T-\tau)}\xi+n\int_\tau^T e^{-n(s-\tau)}U_s ds\xrightarrow[n\to+\infty]{}\xi\mathbb{1}_{\tau=T}+U_\tau\mathbb{1}_{\tau<T}\quad\mathbb{P}\text{-a.s. in }\mathcal{L}^2$$

and the conditional expectation converges also in $\mathcal{L}^2$. Moreover,

$$\left|\int_\tau^T e^{-n(s-\tau)} g(s)ds\right|^2 \leq \int_\tau^T e^{\beta A(s)} \left|\frac{g(s)}{a(s)}\right|^2 ds \int_\tau^T e^{-2n(s-\tau)} e^{-\beta A(s)} a^2(s)ds.$$

Then

$$\int_\tau^T e^{-n(s-\tau)} g(s)ds \xrightarrow[n\to+\infty]{} 0 \quad \mathbb{P}\text{-a.s. in } \mathcal{L}^2.$$

In addition,

$$0 \leq \int_\tau^T e^{-n(s-\tau)} d\widetilde{K}_s^n \leq \int_\tau^T e^{-n(s-\tau)} dK_s^+ \xrightarrow[n\to+\infty]{} 0 \text{ in } \mathcal{L}^1.$$

Consequently,

$$\widetilde{Y}_\tau^n \xrightarrow[n\to+\infty]{} \xi \mathbb{1}_{\tau=T} + U_\tau \mathbb{1}_{\tau<T} \quad \mathbb{P}\text{-a.s. in } \mathcal{L}^1.$$

Therefore, $Y_\tau \leq U_\tau$ $\mathbb{P}$-a.s. We deduce, from Theorem 86 page 220 in Dellacherie and Meyer [7], that $Y_t \leq U_t$ for all $t \leq T$ $\mathbb{P}$-a.s and then $e^{\beta A(t)}(Y_t^n - U_t)^+ \searrow 0$ for all $t \leq T$ $\mathbb{P}$-a.s. By Dini's theorem, we have $\sup_{0\leq t\leq T} e^{\beta A(t)}(Y_t^n - U_t)^+ \searrow 0$ $\mathbb{P}$-a.s. and the result follows from the Lebesgue's dominated convergence theorem. □

**Lemma 6.** *There exist two processes $(Z_t)_{t\leq T}$ and $(K_t^-)_{t\leq T}$ such that*

$$\mathbb{E}\int_0^T e^{\beta A(t)} a^2(t) \left|Y_t^n - Y_t\right|^2 dt + \mathbb{E}\int_0^T e^{\beta A(t)} \left|Z_t^n - Z_t\right|^2 dt \xrightarrow[n\to+\infty]{} 0.$$

*Moreover,*

$$\mathbb{E} \sup_{0\leq t\leq T} e^{\beta A(t)} \left|Y_t^n - Y_t\right|^2 + \mathbb{E} \sup_{0\leq t\leq T} \left|K_t^{n-} - K_t^-\right|^2 \xrightarrow[n\to+\infty]{} 0.$$

**Proof.** For all $n \geq p \geq 0$ and $t \leq T$, applying Itô's formula and taking expectation yields that

$$\begin{aligned}
&\mathbb{E}\left[e^{\beta A(t)}\left|Y_t^n - Y_t^p\right|^2 + \beta\int_t^T e^{\beta A(s)} a^2(s)\left|Y_s^n - Y_s^p\right|^2 ds + \int_t^T e^{\beta A(s)}\left|Z_s^n - Z_s^p\right|^2 ds\right] \\
&\quad \leq 2\mathbb{E}\left[\int_t^T e^{\beta A(s)} \left(Y_s^p - U_s\right)^+ n\left(Y_s^n - U_s\right)^+ ds\right] \\
&\qquad + 2\mathbb{E}\left[\int_t^T e^{\beta A(s)} \left(Y_s^n - U_s\right)^+ p\left(Y_s^p - U_s\right)^+ ds\right] \\
&\quad \leq \mathbb{E}\left[\sup_{0\leq t\leq T} \left(e^{\beta A(t)}\left(Y_t^p - U_t\right)^+\right)^2\right]^{\frac{1}{2}} \mathbb{E}\left[\left(\int_t^T n\left(Y_s^n - U_s\right)^+ ds\right)^2\right]^{\frac{1}{2}} \\
&\qquad + \mathbb{E}\left[\sup_{0\leq t\leq T} \left(e^{\beta A(t)}\left(Y_t^n - U_t\right)^+\right)^2\right]^{\frac{1}{2}} \mathbb{E}\left[\left(\int_t^T p\left(Y_s^p - U_s\right)^+ ds\right)^2\right]^{\frac{1}{2}}
\end{aligned}$$

since $(Y_s^n - Y_s^p)d(K_s^{n+} - K_s^{p+}) \leq 0$. Therefore, using Lemmas 2 and 5, we obtain

$$\mathbb{E}\int_0^T e^{\beta A(s)} a^2(s)\left|Y_s^n - Y_s^p\right|^2 ds + \mathbb{E}\int_0^T e^{\beta A(s)}\left|Z_s^n - Z_s^p\right|^2 ds \xrightarrow[n,p\to+\infty]{} 0.$$

It follows that $(Z^n)_{n\geq 0}$ is a Cauchy sequence in complete space $\mathcal{H}^2(\beta, a)$. Then there exists an $\mathcal{F}_t$-progressively measurable process $(Z_t)_{t\leq T}$ such that the sequence $(Z^n)_{n\geq 0}$ tends toward $Z$ in $\mathcal{H}^2(\beta, a)$. On the other hand, by the Burkholder–Davis–Gundy's inequality, one can derive that

$$\begin{aligned}
&\mathbb{E}\sup_{0\leq t\leq T} e^{\beta A(t)}\left|Y_t^n - Y_t^p\right|^2 \\
&\quad\leq \mathbb{E}\Big[\sup_{0\leq t\leq T}\big(e^{\beta A(t)}\big(Y_t^p - U_t\big)^+\big)^2\Big]^{\frac{1}{2}}\mathbb{E}\bigg[\bigg(\int_t^T n\big(Y_s^n - U_s\big)^+ ds\bigg)^2\bigg]^{\frac{1}{2}} \\
&\qquad+ \mathbb{E}\Big[\sup_{0\leq t\leq T}\big(e^{\beta A(t)}\big(Y_t^n - U_t\big)^+\big)^2\Big]^{\frac{1}{2}}\mathbb{E}\bigg[\bigg(\int_t^T p\big(Y_s^p - U_s\big)^+ ds\bigg)^2\bigg]^{\frac{1}{2}} \\
&\qquad+ \frac{1}{2}\mathbb{E}\sup_{0\leq t\leq T} e^{\beta A(t)}\left|Y_t^n - Y_t^p\right|^2 + 2c^2\mathbb{E}\int_t^T e^{\beta A(s)}\left|Z_s^n - Z_s^p\right|^2 ds
\end{aligned}$$

where $c$ is a universal non-negative constant. It follows that

$$\mathbb{E}\sup_{0\leq t\leq T} e^{\beta A(t)}\left|Y_t^n - Y_t^p\right|^2 \xrightarrow[n,p\to+\infty]{} 0$$

and then

$$\mathbb{E}\bigg[\sup_{0\leq t\leq T} e^{\beta A(t)}\left|Y_t^n - Y_t\right|^2 + \int_0^T e^{\beta A(t)} a^2(t)\left|Y_t^n - Y_t\right|^2 dt\bigg] \xrightarrow[n\to+\infty]{} 0.$$

Now, we set

$$K_t^- = Y_t - Y_0 + \int_0^t g(s)ds + K_t^+ - K_0^+ - \int_0^t Z_s dB_s.$$

One can show, at least for a subsequence (which we still index by $n$), that

$$\mathbb{E}\sup_{0\leq t\leq T}\left|K_t^{n-} - K_t^-\right|^2 \xrightarrow[n\to+\infty]{} 0.$$

The proof is completed. ☐

**Proof of Theorem 1.** Obviously, the process $(Y_t, Z_t, K_t^+, K_t^-)_{t\leq T}$ satisfies, for all $t \leq T$,

$$Y_t = \xi + \int_t^T g(s)ds + \big(K_T^+ - K_t^+\big) - \big(K_T^- - K_t^-\big) - \int_t^T Z_s dB_s.$$

Since $Y_t^n \geq L_t$ and from Lemma 5 we have $L_t \leq Y_t \leq U_t$.

In the following, we want to show that

$$\int_0^T (Y_t - L_t) dK_t^+ = \int_0^T (U_t - Y_t) dK_t^- = 0 \quad \mathbb{P}\text{-a.s.}$$

Note that

$$\int_0^T (Y_t - L_t) dK_t^+ = \int_0^T \big(Y_t - Y_t^n\big) dK_t^+ + \int_0^T \big(Y_t^n - L_t\big)\big(dK_t^+ - dK_t^{n+}\big).$$

Let $\omega \in \Omega$ be fixed. It follows from Lemma 4 that, for any $\varepsilon > 0$, there exists $n(\omega)$ such that $\forall n \geq n(\omega)$, $Y_t(\omega) \leq Y_t^n(\omega) + \varepsilon$. Hence

$$\int_0^T \big(Y_t(\omega) - Y_t^n(\omega)\big) dK_t^+(\omega) \leq \varepsilon K_T^+(\omega). \tag{12}$$

On the other hand, since the function $(Y_t(\omega) - L_t(\omega))_{t \leq T}$ is continuous, then there exists a sequence of non-negative step functions $(f^m(\omega))_{m \geq 0}$ which converges uniformly on $[0, T]$ to $Y_t(\omega) - L_t(\omega)$. That is

$$\big|Y_t(\omega) - L_t(\omega) - f_t^m(\omega)\big| < \varepsilon.$$

It follows that

$$\begin{aligned} &\int_0^T \big(Y_t(\omega) - L_t(\omega)\big) d\big(K_t^+(\omega) - K_t^{n+}(\omega)\big) \\ &\quad \leq \varepsilon\big(K_T^+(\omega) + K_T^{n+}(\omega)\big) + \int_0^T f_t^m(\omega) d\big(K_t^+(\omega) - K_t^{n+}(\omega)\big). \end{aligned}$$

Further,

$$\varepsilon\big(K_T^+(\omega) + K_T^{n+}(\omega)\big) \xrightarrow[n \to +\infty]{} 2\varepsilon K_T^+(\omega)$$

and, since $(f^m(\omega))_{m \geq 0}$ is a step function,

$$\int_0^T f_t^m(\omega) d\big(K_t^+(\omega) - K_t^{n+}(\omega)\big) \xrightarrow[m \to +\infty]{} 0.$$

Therefore, we have

$$\limsup_{n \to +\infty} \int_0^T \big(Y_t^n - L_t\big) d\big(K_t^+ - K_t^{n+}\big) \leq 2\varepsilon K_T^+(\omega).$$

From (12) we deduce that

$$\int_0^T (Y_t - L_t) dK_t^+ \leq 3\varepsilon K_T^+(\omega).$$

The arbitrariness of $\varepsilon$ and $Y \geq L$, show that $\int_0^T (Y_t - L_t) dK_t^+ = 0$. Further, by Lemma 4 and the result treated on p. 465 of Saisho [25] we can write

$$\int_0^T \big(U_s - Y_s^n\big) n\big(Y_s^n - U_s\big) ds \xrightarrow[n \to +\infty]{} \int_0^T (U_s - Y_s) dK_s^-. \tag{13}$$

Since $\int_0^T (U_s - Y_s^n)n(Y_s^n - U_s)ds = \int_0^T (U_s - Y_s^n)dK_s^{n-} \le 0$ for each $n \ge 0$ $\mathbb{P}$-a.s. and for each $n, m \ge 0, n \ne m$,

$$\mathbb{E}\Big[\Big|\int_0^T \big(Y_s^n - Y_s^m\big)dK_s^{m-}\Big|\Big] \le \mathbb{E}\Big[\sup_{0\le t\le T} e^{\beta A(t)}\big|Y_t^n - Y_t^m\big|K_T^{m-}\Big] \xrightarrow[n,m\to+\infty]{} 0.$$

Then we have

$$\limsup_{n\to+\infty} \int_0^T \big(U_s - Y_s^n\big)dK_t^{n-} \le 0 \quad \mathbb{P}\text{-a.s.} \tag{14}$$

Combining (13) and (14), we get $\int_0^T (U_s - Y_s)dK_s^- \le 0$ $\mathbb{P}$-a.s. Noting that $Y \le U$, we conclude that $\int_0^T (U_s - Y_s)dK_s^- = 0$. Consequently, $(Y_t, Z_t, K_t^+, K_t^-)$ is the solution to (3) associated to the data $(\xi, g, L, U)$. $\square$

We can now state the main result:

**Theorem 2.** *Assume $(H1)$–$(H6)$ hold for a sufficient large $\beta$. Then DRBSDE (3) has a unique solution $(Y, Z, K^+, K^-)$ that belongs to $(\mathcal{S}^2(\beta, a) \cap \mathcal{S}^{2,a}(\beta, a)) \times \mathcal{H}^2(\beta, a) \times \mathcal{S}^2 \times \mathcal{S}^2$.*

**Proof.** Given $(\phi, \psi) \in \mathfrak{B}^2$, consider the following DRBSDE :

$$\begin{cases} Y_t = \xi + \displaystyle\int_t^T f(s, \phi_s, \psi_s)ds + (K_T^+ - K_t^+) - (K_T^- - K_t^-) - \int_t^T Z_s dB_s \quad t \le T \\ L_t \le Y_t \le U_t, \ \forall t \le T \text{ and } \displaystyle\int_0^T (Y_t - L_t)dK_t^+ = \int_0^T (U_t - Y_t)dK_t^- = 0. \end{cases} \tag{15}$$

From $(H2)$ and $(H3)$, we have

$$\big|f(t, \phi_t, \psi_t)\big|^2 \le 3\big(a(t)^4|\phi_t|^2 + a(t)^2|\psi_t|^2 + \big|f(t, 0, 0)\big|^2\big).$$

It follows from $(H4)$ that $\frac{f}{a} \in \mathcal{H}^2(\beta, a)$ and then (15) has a unique solution $(Y, Z, K^+, K^-)$.

We define a mapping

$$\begin{array}{cccc} \varphi : & \mathfrak{B}^2 & \longrightarrow & \mathfrak{B}^2 \\ & (\phi, \psi) & \longmapsto & (Y, Z) \end{array}$$

Let $\varphi(\phi, \psi) = (Y, Z)$ and $\varphi(\phi', \psi') = (Y', Z')$ where $(Y, Z, K^+, K^-)$ (resp. $(Y', Z', K^{+'}, K^{-'})$) is the unique solution to the DRBSDE associated with data $(\xi, f(., \phi, \psi), L, U)$ (resp. $(\xi, f(., \phi', \psi'), L, U)$). Denote $\Delta\Gamma = \Gamma - \Gamma'$ for $\Gamma = Y, Z, K^+, K^-, \phi, \psi$ and $\Delta f_t = f(t, {\phi'}_t, {\psi'}_t) - f(t, \phi_t, \psi_t)$. Applying Itô's formula to $e^{\beta A(t)}|\Delta Y_t|^2$ and taking expectation we have

$$\begin{aligned} &\mathbb{E}e^{\beta A(t)}|\Delta Y_t|^2 + \beta\mathbb{E}\int_t^T e^{\beta A(s)}a^2(s)|\Delta Y_s|^2 ds + \mathbb{E}\int_t^T e^{\beta A(s)}|\Delta Z_s|^2 ds \\ &\quad \le 2\mathbb{E}\int_t^T e^{\beta A(s)}\Delta Y_s \Delta f_s ds \end{aligned}$$

$$\leq \alpha\beta\mathbb{E}\int_t^T e^{\beta A(s)}a^2(s)|\Delta Y_s|^2 ds + \frac{2}{\alpha\beta}\mathbb{E}\int_t^T e^{\beta A(s)}\big(a^2(s)|\Delta\phi_s|^2 + |\Delta\psi_s|^2\big)ds.$$

We have used the fact that $\Delta Y_s d(\Delta K_s^+ - \Delta K_s^-) \leq 0$. Choosing $\alpha\beta = 4$ and $\beta > 5$, we can write

$$\|\varphi(\phi,\psi)\|_\beta^2 \leq \frac{1}{2}\|(\phi,\psi)\|_\beta^2.$$

It follows that $\varphi$ is a strict contraction mapping on $\mathfrak{B}^2$ and then $\varphi$ has a unique fixed point which is the solution to the DRBSDE (3). □

**Remark 1.** *If we consider $U = +\infty$, we obtain the BSDE with one continuous reflecting barrier $L$, then we proved the existence and uniqueness of the solution to RBSDE (2) by means of a penalization method. Before this work, Wen Lü [26] showed the existence and uniqueness result for this class of equations via the Snell envelope notion.*

### 4.2 Completely separated barriers

In this section we will prove the existence of solution to (3) when the barriers are completely separated, i.e., $L_t < U_t$, $\forall t \leq T$. Then

($\mathcal{H}7$) there exists a continuous semimartingale

$$H_t = H_0 + \int_0^t h_s dB_s - V_t^+ + V_t^-, \quad H_T = \xi$$

with $h \in \mathcal{H}^2(0,a)$ and $V^\pm \in \mathcal{S}^2$ ($V_0^\pm = 0$) are two nondecreasing continuous processes, such that

$$L_t \leq H_t \leq U_t \qquad 0 \leq t \leq T. \tag{16}$$

We will show the existence by the general penalization method. We first consider the special case when the generator does not depend on $(y,z)$:

$$f(t,y,z) = f(t).$$

Let $(Y^n, Z^n) \in (\mathcal{S}^2(\beta,a)\cap\mathcal{S}^{2,a}(\beta,a))\times\mathcal{H}^2(\beta,a)$ be solution to the following BSDE

$$\begin{aligned} Y_t^n = \xi + \int_t^T f(s)ds - n\int_t^T \big(Y_s^n - U_s\big)^+ ds + n\int_t^T \big(Y_s^n - L_s\big)^- ds \\ - \int_t^T Z_s^n dB_s. \end{aligned} \tag{17}$$

We denote $K_t^{n+} := n\int_0^t (Y_s^n - L_s)^- ds$, $K_t^{n-} := n\int_0^t (Y_s^n - U_s)^+ ds$, $K_t^n = K_t^{n+} - K_t^{n-}$ and $f^n(s,y) = f(s) - n(y - U_s)^+ + n(y - L_s)^-$.

Now let us derive the uniform a priori estimates of $(Y^n, Z^n, K^{n+}, K^{n-})$.

**Lemma 7.** *There exists a positive constant $\kappa$ independent of $n$ such that, $\forall n \geq 0$,*

$$\mathbb{E}\Big[\sup_{0\leq t\leq T} e^{\beta A(t)}\big|Y_t^n\big|^2 + \int_0^T e^{\beta A(t)} a^2(t)\big|Y_t^n\big|^2 dt + \int_0^T e^{\beta A(t)}\big|Z_t^n\big|^2 dt + \big|K_T^{n+}\big|^2 + \big|K_T^{n-}\big|^2\Big] \leq \kappa.$$

**Proof.** Consider the RBSDE with data $(\xi, f, L)$. That is,

$$\begin{cases} \overline{Y}_t = \xi + \int_t^T f(s)ds + \overline{K}_T - \overline{K}_t - \int_t^T \overline{Z}_s dB_s \\ \overline{Y}_t \geq L_t, \ \forall t \leq T \text{ and } \int_0^T (\overline{Y}_t - L_t)d\overline{K}_t = 0. \end{cases} \tag{18}$$

From Appendix A there exists a unique triplet of processes $(\overline{Y}, \overline{Z}, \overline{K}) \in (\mathcal{S}^2(\beta, a) \cap \mathcal{S}^{2,a}(\beta, a)) \times \mathcal{H}^2(\beta, a) \times \mathcal{S}^2$ being the solution to RBSDE (18). We consider the penalization equation associated with the RBSDE (18), for $n \in \mathbb{N}$,

$$\overline{Y}_t^n = \xi + \int_t^T f(s)ds + n\int_t^T \big(\overline{Y}_s^n - L_s\big)^- ds - \int_t^T \overline{Z}_s^n dB_s.$$

The Remark 2 implies that $\overline{Y}_t^0 \leq \overline{Y}_t^n \leq \overline{Y}^{n+1}$ and $Y_t^n \leq \overline{Y}_t^n$ for all $t \leq T$. Therefore, as $n \longrightarrow +\infty$ for all $t \leq T$, $\overline{Y}_t^n \nearrow \overline{Y}_t$. Hence $Y_t^n \leq \overline{Y}_t$.

Similarly, we consider the RBSDE with data $(\xi, f, U)$. There exists a unique triplet of processes $(\underline{Y}, \underline{Z}, \underline{K}) \in (\mathcal{S}^2(\beta, a) \cap \mathcal{S}^{2,a}(\beta, a)) \times \mathcal{H}^2(\beta, a) \times \mathcal{S}^2$, which satisfies

$$\begin{cases} \underline{Y}_t = \xi + \int_t^T f(s)ds - (\underline{K}_T - \underline{K}_t) - \int_t^T \underline{Z}_s dB_s \\ \underline{Y}_t \leq U_t, \ \forall t \leq T \text{ and } \int_0^T (U_t - \underline{Y}_t)d\underline{K}_t = 0. \end{cases} \tag{19}$$

By the penalization equation associated with the RBSDE (19)

$$\underline{Y}_t^n = \xi + \int_t^T f(s)ds - n\int_t^T \big(\underline{Y}_s^n - U_s\big)^+ ds - \int_t^T \underline{Z}_s^n dB_s$$

and the Remark 2, we deduce that $Y_t^n \geq \underline{Y}_t$ for all $t \leq T$. Then we can write

$$\mathbb{E}\sup_{0\leq t\leq T} e^{\beta A(t)}\big|Y_t^n\big|^2 \leq \max\Big\{\mathbb{E}\sup_{0\leq t\leq T} e^{\beta A(t)}|\overline{Y}_t|^2, \mathbb{E}\sup_{0\leq t\leq T} e^{\beta A(t)}|\underline{Y}_t|^2\Big\} \leq \kappa. \tag{20}$$

On the other hand, using Itô's formula and taking expectation implies for $t \leq T$:

$$\begin{aligned} &\beta\mathbb{E}\int_t^T e^{\beta A(s)} a^2(s)\big|Y_s^n\big|^2 ds + \mathbb{E}\int_t^T e^{\beta A(s)}\big|Z_s^n\big|^2 ds \\ &\quad\leq \mathbb{E}e^{\beta A(T)}|\xi|^2 + 2\mathbb{E}\int_t^T e^{\beta A(s)} Y_s^n f(s)ds \end{aligned}$$

$$
\begin{aligned}
& -2n\mathbb{E}\int_t^T e^{\beta A(s)} Y_s^n \big(Y_s^n - U_s\big)^+ ds + 2n\mathbb{E}\int_t^T e^{\beta A(s)} Y_s^n \big(Y_s^n - L_s\big)^- ds \\
& \leq \mathbb{E}e^{\beta A(T)}|\xi|^2 + \frac{\beta}{2}\mathbb{E}\int_t^T e^{\beta A(s)} a^2(s)\big|Y_s^n\big|^2 ds + \frac{2}{\beta}\mathbb{E}\int_t^T e^{\beta A(s)} \frac{|f(s)|^2}{a^2(s)} ds \\
& + 2n\mathbb{E}\int_t^T e^{\beta A(s)} U_s^- \big(Y_s^n - U_s\big)^+ ds + 2n\mathbb{E}\int_t^T e^{\beta A(s)} L_s^+ \big(Y_s^n - L_s\big)^- ds.
\end{aligned}
$$

Hence

$$
\begin{aligned}
& \frac{\beta}{2}\mathbb{E}\int_t^T e^{\beta A(s)} a^2(s)\big|Y_s^n\big|^2 ds + \mathbb{E}\int_t^T e^{\beta A(s)}\big|Z_s^n\big|^2 ds \\
& \leq \mathbb{E}e^{\beta A(T)}|\xi|^2 + \frac{2}{\beta}\mathbb{E}\int_t^T e^{\beta A(s)}\left|\frac{f(s)}{a(s)}\right|^2 ds + \frac{1}{\alpha}\mathbb{E}\sup_{0\leq t\leq T} e^{2\beta A(t)}\big(\big|L_t^+\big|^2 + \big|U_t^-\big|^2\big) \\
& + \alpha\mathbb{E}\left[\int_t^T n\big(Y_s^n - U_s\big)^+ ds\right]^2 + \alpha\mathbb{E}\left[\int_t^T n\big(Y_s^n - L_s\big)^- ds\right]^2. \qquad (21)
\end{aligned}
$$

Now we need to estimate $\mathbb{E}[\int_t^T n(Y_s^n - U_s)^+ ds]^2 + \mathbb{E}[\int_t^T n(Y_s^n - L_s)^- ds]^2$. For this, let us consider the following stopping times

$$
\begin{cases}
\tau_0 = 0, \\
\tau_{2l+1} = \inf\big\{t > \tau_{2l} \ \mid \ Y_t^n \leq L_t\big\} \wedge T, & l \geq 0 \\
\tau_{2l+2} = \inf\big\{t > \tau_{2l+1} \ \mid \ Y_t^n \geq U_t\big\} \wedge T, & l \geq 0.
\end{cases}
$$

Since $Y$, $L$ and $U$ are continuous processes and $L < U$, $\tau_l < \tau_{l+1}$ on the set $\{\tau_{l+1} < T\}$. In addition the sequence $(\tau_l)_{l\geq 0}$ is of stationary type (i.e. $\forall\omega \in \Omega$, there exists $l_0(\omega)$ such that $\tau_{l_0}(\omega) = T$). Indeed, let us set $G = \{\omega \in \Omega, \tau_l(\omega) < T, \ l \geq 0\}$, and we will show that $\mathbb{P}(G) = 0$. We assume that $\mathbb{P}(G) > 0$, therefore for $\omega \in G$, we have $Y_{\tau_{2l+1}} \leq L_{\tau_{2l+1}}$ and $Y_{\tau_{2l}} \geq U_{\tau_{2l}}$. Since $(\tau_l)_{l\geq 0}$ is nondecreasing sequence then $\tau_l \nearrow \tau$, hence $U_\tau \leq Y_\tau \leq L_\tau$ which is contradiction since $L < U$. We deduce that $\mathbb{P}(G) = 0$. Obviously $Y^n \geq L$ on the interval $[\tau_{2l}, \tau_{2l+1}]$, then the BSDE (17) becomes

$$
Y_{\tau_{2l}}^n = Y_{\tau_{2l+1}}^n + \int_{\tau_{2l}}^{\tau_{2l+1}} f(s)ds - n\int_{\tau_{2l}}^{\tau_{2l+1}} \big(Y_s^n - U_s\big)^+ ds - \int_{\tau_{2l}}^{\tau_{2l+1}} Z_s^n dB_s. \qquad (22)
$$

On the other hand, using the assumption $(\mathcal{H}7)$, we get

$$
\begin{aligned}
Y_{\tau_{2l}}^n \geq H_{\tau_{2l}} \text{ on } \{\tau_{2l} < T\} \quad & \text{and} \quad Y_{\tau_{2l}}^n = H_{\tau_{2l}} = \xi \text{ on } \{\tau_{2l} = T\}, \\
Y_{\tau_{2l+1}}^n \leq H_{\tau_{2l+1}} \text{ on } \{\tau_{2l+1} < T\} \quad & \text{and} \quad Y_{\tau_{2l+1}}^n = H_{\tau_{2l+1}} = \xi \text{ on } \{\tau_{2l+1} = T\}.
\end{aligned}
$$

From (22) and the definition of process $H$ we obtain

$$
\begin{aligned}
n\int_{\tau_{2l}}^{\tau_{2l+1}} \big(Y_s^n - U_s\big)^+ ds & \leq H_{\tau_{2l+1}} - H_{\tau_{2l}} + \int_{\tau_{2l}}^{\tau_{2l+1}} f(s)ds - \int_{\tau_{2l}}^{\tau_{2l+1}} Z_s^n dB_s \\
& \leq \int_{\tau_{2l}}^{\tau_{2l+1}} \big(h_s - Z_s^n\big)dB_s + \int_{\tau_{2l}}^{\tau_{2l+1}} \big|f(s)\big|ds + V_{\tau_{2l+1}}^- - V_{\tau_{2l}}^-.
\end{aligned}
$$

By summing in $l$, using the fact that $Y^n \leq U$ on the interval $[\tau_{2l+1}, \tau_{2l+2}]$, we can write for $t \leq T$

$$\mathbb{E}\Big[n\int_t^T (Y_s^n - U_s)^+ ds\Big]^2 \leq 4\Big(\mathbb{E}\int_t^T |h_s|^2 ds + \mathbb{E}\int_t^T e^{\beta A_s}\big|Z_s^n\big|^2 ds + \frac{T}{\beta}\mathbb{E}\int_t^T e^{\beta A_s}\frac{|f(s)|^2}{a^2(s)}ds + \mathbb{E}\big|V_T^-\big|^2\Big). \tag{23}$$

In the same way, we obtain

$$\mathbb{E}\Big[n\int_t^T (Y_s^n - L_s)^- ds\Big]^2 \leq 4\Big(\mathbb{E}\int_t^T |h_s|^2 ds + \mathbb{E}\int_t^T e^{\beta A_s}\big|Z_s^n\big|^2 ds + \frac{T}{\beta}\mathbb{E}\int_t^T e^{\beta A_s}\frac{|f(s)|^2}{a^2(s)}ds + \mathbb{E}\big|V_T^+\big|^2\Big). \tag{24}$$

Combining (23), (24) with (21), we obtain the desired result. □

**Lemma 8.**

1. $\mathbb{E}\sup\limits_{0\leq t\leq T} e^{\beta A(t)}|(Y_t^n - U_t)^+|^2 \xrightarrow[n\to+\infty]{} 0.$
2. $\mathbb{E}\sup\limits_{0\leq t\leq T} e^{\beta A(t)}|(Y_t^n - L_t)^-|^2 \xrightarrow[n\to+\infty]{} 0.$

**Proof.** Consider the following BSDE for each $n \in \mathbb{N}$

$$\begin{aligned}\widehat{Y}_t^n &= \xi + \int_t^T f(s)ds + n\int_t^T \big(L_s - \widehat{Y}_s^n\big)ds - \int_t^T \widehat{Z}_s^n dB_s \\ &= \xi + \int_t^T f(s)ds + n\int_t^T \big(\widehat{Y}_s^n - L_s\big)^- ds - n\int_t^T \big(L_s - \widehat{Y}_s^n\big)^- ds - \int_t^T \widehat{Z}_s^n dB_s.\end{aligned}$$

By the Remark 2, we have $Y_t^n \geq \widehat{Y}_t^n$ for all $t \leq T$. Let $\nu$ be a stopping time such that $\nu \leq T$. Then

$$\widehat{Y}_\nu^n = \mathbb{E}\Big[e^{-n(T-\nu)}\xi + \int_\nu^T e^{-n(s-\nu)}f(s)ds + n\int_\nu^T e^{-n(s-\nu)}L_s ds|\mathcal{F}_\nu\Big]. \tag{25}$$

It is easily seen that

$$e^{-n(T-\nu)}\xi + n\int_\nu^T e^{-n(s-\nu)}L_s ds \xrightarrow[n\to+\infty]{} \xi\mathbb{1}_{\nu=T} + L_\nu\mathbb{1}_{\nu<T} \qquad \mathbb{P}\text{-a.s. in } \mathcal{L}^2.$$

Moreover, the conditional expectation converges also in $\mathcal{L}^2$. In addition, by the Hölder inequality, we have

$$\Big|\int_\nu^T e^{-n(s-\nu)}f(s)ds\Big|^2$$

$$\leq \left(\int_{\nu}^{T} e^{\beta A(s)}\left|\frac{f(s)}{a(s)}\right|^2 ds\right)\left(\int_{\nu}^{T} e^{-2n(s-\nu)-\beta A(s)} a^2(s)ds\right) \xrightarrow[n\to+\infty]{} 0.$$

Thus $\int_{\nu}^{T} e^{-n(s-\nu)} f(s)ds \xrightarrow[n\to+\infty]{} 0$ $\mathbb{P}$-a.s. in $\mathcal{L}^2$.

Now, we denote

$$\widehat{y}_t^n := e^{-n(T-t)}\xi + \int_t^T e^{-n(s-t)}\big(f(s)+nL_s\big)ds,$$

$$\tilde{y}_t^n := e^{-n(T-t)}L_T + \int_t^T e^{-n(s-t)}\big(f(s)+nL_s\big)ds$$

and

$$X_t^n := e^{-n(T-t)}L_T + n\int_t^T e^{-n(s-t)}L_s ds - L_t.$$

By the fact that $L$ is uniformly continuous on $[0,T]$, it can be shown that the sequence $(X_t^n)_{n\geq 1}$ uniformly converges in $t$, and the same for $(X_t^{n-})_{n\geq 1}$. Lebesgue's dominated convergence theorem implies that

$$\lim_{n\to+\infty}\mathbb{E}\sup_{0\leq t\leq T} e^{\beta A(t)}\big|\big(\widehat{y}_t^n - L_t\big)^-\big|^2 = \lim_{n\to+\infty}\mathbb{E}\sup_{0\leq t\leq T} e^{\beta A(t)}\big|\big(\tilde{y}_t^n - L_t\big)^-\big|^2$$

$$\leq 2\lim_{n\to+\infty}\mathbb{E}\left[\sup_{0\leq t\leq T} e^{\beta A(t)}\big|X_t^{n-}\big|^2 + \sup_{0\leq t\leq T} e^{\beta A(t)}\left|\int_t^T e^{-n(s-t)} f(s)ds\right|^2\right] = 0.$$

So, from (25), Jensen's inequality and Doob's maximal quadratic inequality (see Theorem 20, p. 11 in [23]), we have

$$\mathbb{E}\sup_{0\leq t\leq T} e^{\beta A(t)}\big|\big(\widehat{Y}_t^n - L_t\big)^-\big|^2 \leq \mathbb{E}\sup_{0\leq t\leq T} e^{\beta A(t)}\big|\mathbb{E}\big[\big(\widehat{y}_t^n - L_t\big)^- |\mathcal{F}_t\big]\big|^2$$

$$\leq 4\mathbb{E}\sup_{0\leq t\leq T} e^{\beta A(t)}\big|\big(\widehat{y}_t^n - L_t\big)^-\big|^2 \xrightarrow[n\to+\infty]{} 0.$$

From the fact that $Y_t^n \geq \widehat{Y}_t^n$ for all $t\leq T$ we deduce that

$$\lim_{n\to+\infty}\mathbb{E}\sup_{0\leq t\leq T} e^{\beta A(t)}\big|\big(Y_t^n - L_t\big)^-\big|^2 = 0.$$

Similarly to proof of the Lemma 5, we can obtain

$$\lim_{n\to+\infty}\mathbb{E}\sup_{0\leq t\leq T} e^{\beta A(t)}\big|\big(Y_t^n - U_t\big)^+\big|^2 = 0. \qquad \square$$

**Lemma 9.** *For each $n\geq p\geq 0$, we have*

$$\mathbb{E}\left[\sup_{0\leq t\leq T} e^{\beta A(t)}\big|Y_t^n - Y_t^p\big|^2 + \int_0^T e^{\beta A(t)}a^2(t)\big|Y_t^n - Y_t^p\big|^2 dt + \int_0^T e^{\beta A(t)}\big|Z_t^n - Z_t^p\big|^2 dt + \sup_{0\leq t\leq T}\big|K_t^n - K_t^p\big|^2\right] \xrightarrow[n,p\to+\infty]{} 0.$$

**Proof.** Itô's formula implies that

$$\begin{aligned}
&\mathbb{E}e^{\beta A(t)}\left|Y_t^n-Y_t^p\right|^2+\mathbb{E}\int_t^T e^{\beta A(s)}\left(\beta a^2(s)\left|Y_s^n-Y_s^p\right|^2+\left|Z_s^n-Z_s^p\right|^2\right)ds\\
&\quad\le 2\mathbb{E}\int_t^T e^{\beta A(s)}\left(Y_s^n-Y_s^p\right)\left(dK_s^{n+}-dK_s^{p+}\right)\\
&\qquad-2\mathbb{E}\int_t^T e^{\beta A(s)}\left(Y_s^n-Y_s^p\right)\left(dK_s^{n-}-dK_s^{p-}\right)\\
&\quad\le 2\mathbb{E}\int_t^T e^{\beta A(s)}\left(Y_s^n-L_s\right)^-dK_s^{p+}+2\mathbb{E}\int_t^T e^{\beta A(s)}\left(Y_s^p-L_s\right)^-dK_s^{n+}\\
&\qquad+2\mathbb{E}\int_t^T e^{\beta A(s)}\left(Y_s^n-U_s\right)^+dK_s^{p-}+2\mathbb{E}\int_t^T e^{\beta A(s)}\left(Y_s^p-U_s\right)^+dK_s^{n-}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\beta\mathbb{E}\int_t^T e^{\beta A(s)}a^2(s)\left|Y_s^n-Y_s^p\right|^2ds+\mathbb{E}\int_t^T e^{\beta A(s)}\left|Z_s^n-Z_s^p\right|^2ds\\
&\quad\le 2\mathbb{E}\sup_{0\le t\le T}e^{\beta A(t)}\left(Y_t^n-L_t\right)^-K_T^{p+}+2\mathbb{E}\sup_{0\le t\le T}e^{\beta A(t)}\left(Y_t^p-L_t\right)^-K_T^{n+}\\
&\qquad+2\mathbb{E}\sup_{0\le t\le T}e^{\beta A(t)}\left(Y_t^n-U_t\right)^+K_T^{p-}+2\mathbb{E}\sup_{0\le t\le T}e^{\beta A(t)}\left(Y_t^p-U_t\right)^+K_T^{n-}.
\end{aligned}$$

Lemma 8 implies that

$$\mathbb{E}\int_t^T e^{\beta A(s)}a^2(s)\left|Y_s^n-Y_s^p\right|^2ds+\mathbb{E}\int_t^T e^{\beta A(s)}\left|Z_s^n-Z_s^p\right|^2)ds\xrightarrow[n,p\to+\infty]{}0. \tag{26}$$

On the other hand, by the Burkholder–Davis–Gundy's inequality, we get

$$\mathbb{E}\sup_{0\le t\le T}e^{\beta A(t)}\left|Y_t^n-Y_t^p\right|^2\xrightarrow[n,p\to+\infty]{}0. \tag{27}$$

From the equation

$$K_t^n=Y_0^n-Y_t^n-\int_0^t f(s)ds+\int_0^t Z_s^n dB_s\quad 0\le t\le T, \tag{28}$$

we can conclude that

$$\mathbb{E}\sup_{0\le t\le T}\left|K_t^n-K_t^p\right|^2\xrightarrow[n,p\to+\infty]{}0. \tag{29}$$

The proof is completed. □

The main result of this section is the following:

**Theorem 3.** *Assume that $L<U$. Then the DRBSDE (3) has a unique solution $(Y,Z,K^+,K^-)$ that belongs to $(\mathcal{S}^2(\beta,a)\cap\mathcal{S}^{2,a}(\beta,a))\times\mathcal{H}^2(\beta,a)\times\mathcal{S}^2\times\mathcal{S}^2$.*

**Proof.** From Lemma 9, we obtain that there exists an adapted process $(Y, Z, K) \in (\mathcal{S}^2(\beta, a) \cap \mathcal{S}^{2,a}(\beta, a)) \times \mathcal{H}^2(\beta, a) \times \mathcal{S}^2$ such that

$$\mathbb{E}\Big[\sup_{0\le t\le T} e^{\beta A(t)}\big|Y_t^n - Y_t\big|^2 + \int_0^T e^{\beta A(t)} a^2(t)\big|Y_t^n - Y_t\big|^2 dt \\ + \int_0^T e^{\beta A(t)}\big|Z_t^n - Z_t\big|^2 dt + \sup_{0\le t\le T}\big|K_t^n - K_t\big|^2\Big] \xrightarrow[n\to+\infty]{} 0. \tag{30}$$

Then, passing to the limit as $n \to +\infty$ in the equation

$$Y_t^n = \xi + \int_t^T f(s)ds + K_T^n - K_t^n - \int_t^T Z_s^n dB_s,$$

we obtain

$$Y_t = \xi + \int_t^T f(s)ds + K_T - K_t - \int_t^T Z_s dB_s.$$

Let $\tau \le T$ be a stopping time, by Lemma 7 we obtain that the sequences $K_\tau^{n\pm}$ are bounded in $\mathcal{L}^2$, consequently, there exist $\mathcal{F}_\tau$-measurable random variables $K_\tau^\pm$ in $\mathcal{L}^2$, such that there exist the subsequences of $K_\tau^{n\pm}$ weakly converging in $K_\tau^\pm$.

Now we set $\mathcal{K}_\tau = K_\tau^+ - K_\tau^-$. By [28] (Mazu's Lemma, p. 120), there exists, for every $n \in \mathbb{N}$, an integer $N \ge n$ and a convex combination $\sum_{j=n}^N \zeta_j^{\tau,n}(K_\tau^\pm)_j$ with $\zeta_j^{\tau,n} \ge 0$ and $\sum_{j=n}^N \zeta_j^{\tau,n} = 1$ such that

$$\mathcal{K}_\tau^{n\pm} := \sum_{j=n}^N \zeta_j^{\tau,n}\big(K_\tau^\pm\big)_j \xrightarrow[n\to+\infty]{} K_\tau^\pm. \tag{31}$$

Denoting $\mathcal{K}_\tau^n = \mathcal{K}_\tau^{n+} - \mathcal{K}_\tau^{n-}$, it follows that

$$\mathbb{E}\big|\mathcal{K}_\tau^n - \mathcal{K}_\tau\big|^2 \xrightarrow[n\to+\infty]{} 0. \tag{32}$$

Thanks to (30), we have $\|K_\tau^n - K_\tau\|_{\mathcal{L}^2} < \varepsilon$ for all $\varepsilon > 0$. Therefore

$$\begin{aligned}\big\|\mathcal{K}_\tau^n - K_\tau\big\|_{\mathcal{L}^2} &= \Big\|\sum_{j=n}^N \zeta_j^{\tau,n}\big(\big(K_\tau^\pm\big)_j - K_\tau\big)\Big\|_{\mathcal{L}^2} \\ &\le \sum_{j=n}^N \zeta_j^{\tau,n}\big\|\big(K_\tau^\pm\big)_j - K_\tau\big\|_{\mathcal{L}^2} < \varepsilon.\end{aligned}$$

Hence

$$\mathbb{E}\big|\mathcal{K}_\tau^n - K_\tau\big|^2 \xrightarrow[n\to+\infty]{} 0. \tag{33}$$

Combining (32) and (33), we obtain $\mathcal{K}_\tau = K_\tau$ a.s. Therefore, from Theorem 86, p. 220 in [7] we have $\mathcal{K}_t = K_t$ for all $t \le T$. On the other hand, (31) implies that, for $\tau = T$, there exists a subsequence of $\mathcal{K}_T^{n+} := \sum_{j=n}^N \zeta_j^{T,n}(K_T^+)_j$ (resp.

$\mathcal{K}_T^{n-} := \sum_{j=n}^{N} \zeta_j^{T,n}(K_T^-)_j)$ converging a.s. to $K_T^+$ (resp. $K_T^-$). Then for $\mathbb{P}$-a.s. $\omega \in \Omega$, the sequence $\mathcal{K}_T^{n+}(\omega)$ (resp. $\mathcal{K}_T^{n-}(\omega)$) is bounded. Using Theorem 4.3.3, p. 88 in [4], there exists a subsequence of $\mathcal{K}_t^{n+}(\omega)$ (resp. $\mathcal{K}_t^{n-}(\omega)$) tending to $K_t^+(\omega)$ (resp. $K_t^-(\omega)$), weakly.

On the other hand, by the definition of stopping times $(\tau_l)_{l\geq 0}$, we have

$$\begin{cases} Y_t^n > L_t, & \text{on } [\tau_{2l}, \tau_{2l+1}[; \\ Y_t^n < U_t, & \text{on } [\tau_{2l+1}, \tau_{2l+2}[. \end{cases}$$

Then

$$L_t \mathbb{1}_{[\tau_{2i},\tau_{2i+1}]}(t) \leq Y_t^n \leq U_t \mathbb{1}_{[\tau_{2i+1},\tau_{2i+2}]}(t).$$

By summing in $i$, $i = 0, \dots, l$ and passing to limit in $n$, we obtain $L_t \leq Y_t \leq U_t$. Now, we would have to show the Skorokhod's conditions. Indeed, since $\mathcal{K}_t^{n+}(\omega)$ tends to $K_t^+(\omega)$, using the result treated in p. 465 of [25] we can write

$$\int_0^T \big(Y_t^n(\omega) - L_t(\omega)\big) d\mathcal{K}_t^{n+}(\omega) \xrightarrow[n\to+\infty]{} \int_0^T \big(Y_t(\omega) - L_t(\omega)\big) dK_t^+(\omega). \tag{34}$$

Since $\int_0^T (Y_t^n - L_t) dK_t^{n+} \leq 0$, $\forall n \geq 0$ a.s., and $\forall n, m \geq 0$, $n \neq m$,

$$\mathbb{E}\bigg[\bigg|\int_0^T \big(Y_t^n - Y_t^m\big) dK_t^{m+}\bigg|\bigg] \leq \mathbb{E}\Big[\sup_{0\leq t\leq T} e^{\beta A(t)} \big|Y_t^n - Y_t^m\big| K_T^{m+}\Big] \xrightarrow[n,m\to+\infty]{} 0,$$

then by

$$\int_0^T \big(Y_t^n - L_t\big) dK_t^{m+} = \int_0^T \big(Y_t^n - Y_t^m\big) dK_t^{m+} + \int_0^T \big(Y_t^m - L_t\big) dK_t^{m+}$$

we have

$$\limsup_{n\to+\infty} \int_0^T \big(Y_t^n - L_t\big) d\mathcal{K}_t^{n+} \leq 0 \quad \mathbb{P}\text{-a.s.} \tag{35}$$

Combining (34) and (35), we get $\int_0^T (Y_t - L_t) dK_t^+ \leq 0$ $\mathbb{P}$-a.s. Noting that $Y \geq L$, we conclude that $\int_0^T (Y_t - L_t) dK_t^+ = 0$. By a similar consideration, we can prove $\int_0^T (U_t - Y_t) dK_t^- = 0$.

Finally, using the fixed point theorem we construct a strict contraction mapping $\varphi$ on $\mathfrak{B}^2$ and conclude that $(Y_t, Z_t, K_t^+, K_t^-)$ is the unique solution to DRBSDE (3) associated with data $(\xi, f, L, U)$. $\square$

## 5 Comparison theorem

In this section we prove a comparison theorem for the DRBSDE under the stochastic Lipschitz assumptions on generators.

**Theorem 4.** *Let $(Y^1, Z^1, K^{1+}, K^{1-})$ and $(Y^2, Z^2, K^{2+}, K^{2-})$ be respectively the solutions to the DRBSDE with data $(\xi^1, f^1, L^1, U^1)$ and $(\xi^2, f^2, L^2, U^2)$. Assume in addition the following:*

- $\xi^1 \leq \xi^2$ *a.s.*
- $f^1(t, Y^2, Z^2) \leq f^2(t, Y^2, Z^2) \quad \forall t \in [0, T]$ *a.s.*
- $L_t^1 \leq L_t^2$ *and* $U_t^1 \leq U_t^2 \quad \forall t \in [0, T]$ *a.s.*

*Then*

$$\forall t \leq T, \qquad Y_t^1 \leq Y_t^2 \quad a.s.$$

**Proof.** Let $\bar{\Re} = \Re^1 - \Re^2$ for $\Re = Y, Z, K^+, K^+, \xi$ and

- $\zeta_t = \mathbb{1}_{\{\bar{Y}_t \neq 0\}} \dfrac{f^1(t, Y_t^1, Z_t^1) - f^1(t, Y_t^2, Z_t^1)}{\bar{Y}_t}$;
- $\eta_t = \mathbb{1}_{\{\bar{Z}_t \neq 0\}} \dfrac{f^1(t, Y_t^2, Z_t^1) - f^1(t, Y_t^2, Z_t^2)}{\bar{Z}_t}$;
- $\delta_t = f^1(t, Y_t^2, Z_t^2) - f^2(t, Y_t^2, Z_t^2)$.

Applying the Meyer–Itô formula (Theorem 66, p. 210 in [23]), there exists a continuous nondecreasing process $(\mathcal{A}_t)_{t \leq T}$ such that

$$\begin{aligned} \left|\bar{Y}_t^+\right|^2 &= 2\int_t^T \bar{Y}_s^+(\zeta_s \bar{Y}_s + \eta_s \bar{Z}_s + \delta_s)ds - 2\int_t^T \bar{Y}_s^+ \bar{Z}_s dB_s \\ &\quad + 2\int_t^T \bar{Y}_s^+ d\bar{K}_s^+ - 2\int_t^T \bar{Y}_s^+ d\bar{K}_s^- - (\mathcal{A}_T - \mathcal{A}_t). \end{aligned}$$

Suppose in addition that

$$\mathbb{E}\int_0^T \mu_t dt < +\infty \quad \text{and} \quad \mathbb{E}\int_0^T |\gamma_t|^2 dt < +\infty.$$

Let $\{\Gamma_{t,s}, 0 \leq t \leq s \leq T\}$ be the process defined as

$$\Gamma_{t,s} = \exp\left\{\int_t^s \left(\zeta_u - \frac{1}{2}|\eta_u|^2\right)du + \int_t^s \eta_u dB_u\right\} > 0$$

being a solution to the linear stochastic differential equation

$$\Gamma_{t,s} = 1 + \int_t^s \zeta_u \Gamma_{t,u} du + \int_t^s \eta_u \Gamma_{t,u} dB_u.$$

Applying the integration by parts and taking expectation yield

$$\begin{aligned} &\mathbb{E}\left[e^{\beta A(t)}\left|\bar{Y}_t^+\right|^2\right] + \beta\mathbb{E}\int_0^T e^{\beta A(s)}\Gamma_{t,s} a^2(s)\left|\bar{Y}_s^+\right|^2 ds \\ &\quad \leq \mathbb{E}\left[\int_t^T e^{\beta A(s)}\Gamma_{t,s}\zeta_s\left|\bar{Y}_s^+\right|^2 ds\right] + 2\mathbb{E}\left[\int_t^T e^{\beta A(s)}\Gamma_{t,s}\delta_s \bar{Y}_s^+ ds\right] \\ &\qquad + 2\mathbb{E}\int_t^T e^{\beta A(s)}\Gamma_{t,s}\bar{Y}_s^+ dK_s^+ - 2\mathbb{E}\int_t^T e^{\beta A(s)}\Gamma_{t,s}\bar{Y}_s^+ dK_s^-. \end{aligned}$$

Remark that

$$\bar{Y}_s^+ d\bar{K}_s^+ = \big(L_s^1 - Y_s^2\big)\mathbb{1}_{Y_s^1 > Y_s^2} dK_s^{1+} - \big(Y_s^1 - L_s^2\big)\mathbb{1}_{Y_s^1 > Y_s^2} dK_s^{2+} \leq 0$$

and

$$\bar{Y}_s^+ d\bar{K}_s^- = \big(Y_s^1 - U_s^2\big)\mathbb{1}_{Y_s^1 > Y_s^2} dK_s^{2-} - \big(U_s^1 - Y_s^2\big)\mathbb{1}_{Y_s^1 > Y_s^2} dK_s^{1-} \leq 0.$$

Since $\delta_s \leq 0$ and $|\zeta_s| \leq a^2(s)$, one can derive that

$$\mathbb{E}\big[e^{\beta A(t)}\big|\bar{Y}_t^+\big|^2\big] \leq 0.$$

It follows that $\bar{Y}_t^+ = 0$, i.e $Y_t^1 \leq Y_t^2$ for all $t \leq T$ a.s. □

**Remark 2.**

- *If $U^i = +\infty$ for $i = 1, 2$, then $dK^{i-} = 0$ and the comparison holds also for the reflected BSDE (2).*
- *If $U^i = +\infty$ and $L^i = -\infty$ for $i = 1, 2$, then $dK^{i\pm} = 0$ and the comparison holds also for the BSDE (1).*

## A Appendix

In this section, we study a special case of the reflected BSDE when the generator depends only on $y$.

We consider the following reflected BSDE

$$\begin{cases} Y_t = \xi + \displaystyle\int_t^T f(s, Y_s)ds + K_T - K_t - \int_t^T Z_s dB_s \\ Y_t \geq L_t \ \forall t \leq T \ \text{ and } \displaystyle\int_0^T (Y_t - L_t)dK_t = 0 \end{cases} \tag{36}$$

where $(\xi, f, L)$ satisfies the following assumptions:

- $\xi \in \mathcal{S}^2(\beta, a)$;
- $f$ is Lipschitz, i.e. there exists a positive constant $\mu$ such that $\forall (t, y, y') \in [0, T] \times \mathbb{R} \times \mathbb{R}$
$$\big|f(t, y) - f\big(t, y'\big)\big| \leq \mu\big|y - y'\big|;$$
- $\dfrac{f(t, 0)}{a} \in \mathcal{H}^2(\beta, a)$;
- $\mathbb{E}[\sup\limits_{0 \leq t \leq T} e^{2\beta A(t)} |L_t^+|^2] < +\infty$.

As in [11], we prove the existence and uniqueness of a solution to (36) by means of the penalization method. Indeed, for each $n \in \mathbb{N}$, we consider the following BSDE:

$$Y_t^n = \xi + \int_t^T f\big(s, Y_s^n\big)ds + n\int_t^T \big(Y_s^n - L_s\big)^- ds - \int_t^T Z_s^n dB_s. \tag{37}$$

We denote $K_t^n := n\int_0^t (Y_s^n - L_s)^- ds$ and $f^n(t,y) = f(t,y) + n(y - L_t)^-$. Remark that $f^n$ is Lipschitz and

$$\mathbb{E}|\xi|^2 + \mathbb{E}\int_0^T \big|f^n(t,0)\big|^2 dt \le \mathbb{E}\big[e^{\beta A(T)}|\xi|^2\big] + \frac{2}{\beta}\mathbb{E}\Big[\int_0^T e^{\beta A(t)}\Big|\frac{f(t,0)}{a(t)}\Big|^2 dt\Big]$$
$$+ 2n^2 T\mathbb{E}\Big[\sup_{0\le t\le T} e^{2\beta A(t)}\big|L_t^+\big|^2\Big].$$

From [21], there exists a unique process $(Y^n, Z^n)$ being a solution to the BSDE (37). The sequence $(Y^n, Z^n, K^n)_n$ satisfies the uniform estimate

$$\mathbb{E}\sup_{0\le t\le T} e^{\beta A(t)}\big|Y_t^n\big|^2 + \mathbb{E}\Big[\int_0^T e^{\beta A(s)}a^2(s)\big|Y_s^n\big|^2 ds + \mathbb{E}\int_0^T e^{\beta A(s)}\big|Z_s^n\big|^2 ds\Big]$$
$$\le C\mathbb{E}\Big[e^{\beta A(T)}|\xi|^2 + \int_0^T e^{\beta A(s)}\frac{|f(s,0)|^2}{a^2(s)}ds + \sup_{0\le t\le T} e^{2\beta A(s)}\big|L_s^+\big|^2\Big].$$

where $C$ is a positive constant depending only on $\beta$, $\mu$ and $\epsilon$.

Now we establish the convergence of sequence $(Y^n, Z^n, K^n)$ to the solution to (36). Obviously $f^n(t,y) \le f^{n+1}(t,y)$ for each $n \in \mathbb{N}$, and it follows from Remark 2 that $Y^n \le Y^{n+1}$. Hence there exists a process $Y$ such that $Y_t^n \nearrow Y_t$ $0 \le t \le T$ a.s. From the a priori estimates and Fatou's lemma, we have

$$\mathbb{E}\Big[\sup_{0\le t\le T} e^{\beta A(t)}|Y_t|^2\Big] \le \liminf_{n\to+\infty}\mathbb{E}\Big[\sup_{0\le t\le T} e^{\beta A(t)}\big|Y_t^n\big|^2\Big] \le C.$$

Then by the dominated convergence, one can derive that

$$\mathbb{E}\Big[\int_0^T e^{\beta A(s)}\big|Y_s^n - Y_s\big|^2 ds\Big] \xrightarrow[n\to+\infty]{} 0.$$

On the other hand, for all $n \ge p \ge 0$ and $t \le T$, we have

$$\mathbb{E}e^{\beta A(t)}\big|Y_t^n - Y_t^p\big|^2 + \Big(\beta - \frac{2\mu}{\epsilon}\Big)\mathbb{E}\int_t^T e^{\beta A(s)}a^2(s)\big|Y_s^n - Y_s^p\big|^2 ds$$
$$+ \mathbb{E}\int_t^T e^{\beta A(s)}\big|Z_s^n - Z_s^p\big|^2 ds$$
$$\le 2\mathbb{E}\int_t^T e^{\beta A(s)}\big(Y_s^n - L_s\big)^- dK_s^p + \mathbb{E}\int_t^T e^{\beta A(s)}\big(Y_s^p - L_s\big)^- dK_s^n.$$

Similarly to Lemma 8, we can easily prove that

$$\mathbb{E}\sup_{0\le t\le T} e^{\beta A(t)}\big|\big(Y_t^n - L_t\big)^-\big|^2 \xrightarrow[n\to+\infty]{} 0. \tag{38}$$

By the above result an the a priori estimates, one can derive that

$$\mathbb{E}\Big[\int_t^T e^{\beta A(s)}\big(Y_s^n - L_s\big)^- dK_s^p + \int_t^T e^{\beta A(s)}\big(Y_s^p - L_s\big)^- dK_s^n\Big] \xrightarrow[n,p\to+\infty]{} 0.$$

Thus

$$\mathbb{E}\Big[\int_t^T e^{\beta A(s)} a^2(s)\big|Y_s^n - Y_s^p\big|^2 ds + \int_t^T e^{\beta A(s)}\big|Z_s^n - Z_s^p\big|^2 ds\Big] \xrightarrow[n,p\to+\infty]{} 0.$$

Moreover, by the Burkholder–Davis–Gundy's inequality, one can derive that

$$\mathbb{E}\Big[\sup_{0\le t\le T} e^{\beta A(t)}\big|Y_t^n - Y_t^p\big|^2\Big] \xrightarrow[n,p\to+\infty]{} 0.$$

Further, from the equation (37), we have also

$$\mathbb{E}\Big[\sup_{0\le t\le T} \big|K_t^n - K_t^p\big|^2\Big] \xrightarrow[n,p\to+\infty]{} 0.$$

Consequently there exists a pair of progressively measurable processes $(Z, K)$ such that

$$\mathbb{E}\int_0^T e^{\beta A(t)}\big|Z_t^n - Z_t\big|^2 dt + \mathbb{E}\sup_{0\le t\le T}\big|K_t^n - K_t\big|^2 \xrightarrow[n\to+\infty]{} 0.$$

Obviously the triplet $(Y, Z, K)$ satisfies (36). It remains to check the Skorokhod condition. We have just seen that the sequence $(Y^n, K^n)$ tends to $(Y, K)$ uniformly in $t$ in probability. Then the measure $dK^n$ tends to $dK$ weakly in probability, hence

$$\int_0^T \big(Y_t^n - L_t\big) dK_t^n \xrightarrow[n\to+\infty]{\mathbb{P}} \int_0^T (Y_t - L_t) dK_t.$$

We deduce from the equation (38) that $\int_0^T (Y_t^n - L_t)dK_t^n \le 0$, $n \in \mathbb{N}$, which implies that $\int_0^T (Y_t - L_t)dK_t \le 0$. On the other hand, since $Y_t \ge L_t$ then $\int_0^T (Y_t - L_t)dK_t \ge 0$. Hence $\int_0^T (Y_t - L_t)dK_t = 0$.

**Remark 3** (Special cases)**.** *The coefficients* $g^n(s, y) = g(s) - n(y - U_s)^+$ *and* $\tilde{g}^n(s, y) = g(s) - n(y - U_s)$ *are Lipschitz and satisfy*

$$\begin{aligned}
&\mathbb{E}\int_0^T e^{\beta A(s)}\left|\frac{g^n(s,0)}{a(s)}\right|^2 ds + \mathbb{E}\int_0^T e^{\beta A(s)}\left|\frac{\tilde{g}^n(s,0)}{a(s)}\right|^2 ds \\
&\le 4\mathbb{E}\int_0^T e^{\beta A(s)}\left|\frac{g(s)}{a(s)}\right|^2 ds + \frac{4n^2T}{\epsilon}\mathbb{E}\Big[\sup_{n\ge 0} e^{2\beta A(t)}\big|U_t^-\big|^2\Big] < +\infty.
\end{aligned}$$

*Then the Reflected BSDEs (10) and (11) have a unique solution.*

**Theorem 5** (Comparison theorem)**.** *Let* $(Y^1, Z^1, K^1)$ *and* $(Y^2, Z^2, K^2)$ *be solutions to the Reflected BSDE (36) with data* $(\xi^1, f^1, L)$ *and* $(\xi^2, f^2, L)$ *respectively. If we have*

- $f^1(t,y) \le f^2(t,y)$ *a.s.* $\forall (t,y)$,
- $\xi^1 \le \xi^2$ *a.s.,*

*then* $Y_t^1 \le Y_t^2$ *and* $K_t^1 \ge K_t^2$ $\forall t \in [0,T]$ *a.s.*

**Proof.** We consider the penalized equations relative to the Reflected BSDE with data $(\xi^i, f^i, L)$ for $i = 1, 2$ and $n \in \mathbb{N}$, as follows

$$Y_t^{n,i} = \xi^i + \int_t^T f^i\big(s, Y_s^{n,i}\big)ds + n\int_t^T \big(Y_s^{n,i} - L_s\big)^- - \int_t^T Z_s^{n,i} dB_s.$$

Let $f_n^i(t,y) := f^i(t,y) + n(y - L_s)^-$. So, by the comparison theorem, we have $Y_t^{n,1} \le Y_t^{n,2}$ for $t \le T$. Since $K_t^{n,i} = n\int_0^t (Y_s^{n,i} - L_s)^- ds$ for $i = 1, 2$, we deduce that $K_t^{n,1} \ge K_t^{n,2}$ for $t \le T$. But $Y_t^{n,i} \nearrow Y_t^i$ and $K_t^{n,i} \longrightarrow K_t^i$ as $n \longrightarrow +\infty$ for $i = 1, 2$, and it follows that $Y_t^1 \le Y_t^2$ and $K_t^1 \ge K_t^2$ for $t \le T$. □

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