On closeness of two discrete weighted sums

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Abstract   The effect that weighted summands have on each other in approximations of $S = w_1 S_1 + w_2 S_2 + \cdots + w_N S_N$ is investigated. Here, S_i's are sums of integer-valued random variables, and w_i denote weights, $i = 1, \ldots, N$. Two cases are considered: the general case of independent random variables when their closeness is ensured by the matching of factorial moments and the case when the S_i has the Markov Binomial distribution. The Kolmogorov metric is used to estimate the accuracy of approximation.

Keywords   Characteristic function, concentration function, factorial moments, Kolmogorov metric, Markov Binomial distribution, weighted random variables

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1   Introduction

Let us consider a typical cluster sampling design: the entire population consists of different clusters, and the probability for each cluster to be selected into a sample is known. The sum of sample elements is then equal to $S = w_1 S_1 + w_2 S_2 + \cdots + w_N S_N$. Here, S_i is the sum of independent identically distributed (iid) random variables (rvs) from the i-th cluster. A similar situation arises in actuarial mathematics when the sum S models the discounted amount of the total net loss of a company, see, for example,

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[24]. Note that then S_i may be the sum of dependent rvs. Of course, in actuarial models, w_i are also typically random, which makes our research just a first step in this direction. In many papers, the limiting behavior of weighted sums is investigated with the emphasis on weights or tails of distributions, see, for example, [6, 16–18, 23, 25–30], and references therein. We, however, concentrate on the impact of $S - w_i S_i$ on $w_i S_i$. Our research is motivated by the following simple example. Let us assume that S_i is in some sense close to Z_i, $i = 1, 2$. Then a natural approximation to $w_1 S_1 + w_2 S_2$ is $w_1 Z_1 + w_2 Z_2$. Suppose that we want to estimate the closeness of both sums in some metric $d(\cdot, \cdot)$. The standard approach which works for the majority of metrics then gives

$$d(w_1 S_1 + w_2 S_2, w_1 Z_1 + w_2 Z_2) \leqslant d(w_1 S_1, w_1 Z_1) + d(w_2 S_2, w_2 Z_2). \quad (1)$$

The triangle inequality (1) is not always useful. For example, let S_1 and Z_1 have the same Poisson distribution with parameter n and let S_2 and Z_2 be Bernoulli variables with probabilities 1/3 and 1/4, respectively. Then (1) ensures the trivial order of approximation $O(1)$ only. Meanwhile, both S and Z can be treated as small (albeit different) perturbations to the same Poisson variable and, therefore, one can expect closeness of their distributions at least for large n. The 'smoothing' effect that other sums have on the approximation of $w_i S_i$ is already observed in [7] (see also references therein). For some general results involving the concentration functions, see, for example, [10, 20].

To make our goals more explicit, we need additional notation. Let $\mathbb{Z}$ denote the set of all integers. Let $\mathcal{F}$ (resp. $\mathcal{F}_Z$, resp. $\mathcal{M}$) denote the set of probability distributions (resp. distributions concentrated on integers, resp. finite signed measures) on $\mathbb{R}$. Let I_a denote the distribution concentrated at real a and set $I = I_0$. Henceforth, the products and powers of measures are understood in the convolution sense. Further, for a measure M, we set $M^0 = I$ and $\exp\{M\} = \sum_{k=0}^{\infty} M^k / k!$. We denote by $\widehat{M}(t)$ the Fourier–Stieltjes transform of M. The real part of $\widehat{M}(t)$ is denoted by $Re\widehat{M}(t)$. Observe also that $\widehat{\exp\{M(t)\}} = \exp\{\widehat{M}(t)\}$. We also use $\mathcal{L}(\xi)$ to denote the distribution of ξ.

The Kolmogorov (uniform) norm $|M|_K$ and the total variation norm $\|M\|$ of M are defined by

$$|M|_K = \sup_{x \in \mathbb{R}} \big| M\big((-\infty, x]\big) \big|, \qquad \|M\| = M^+\{\mathbb{R}\} + M^-\{\mathbb{R}\},$$

respectively. Here $M = M^+ - M^-$ is the Jordan–Hahn decomposition of M. Also, for any two measures M and V, $|M|_K \leqslant \|M\|$, $|MV|_K \leqslant \|M\| \cdot |V|_K$, $|\widehat{M}(t)| \leqslant \|M\|$, $\|\exp\{M\}\| \leqslant \exp\{\|M\|\}$. If $F \in \mathcal{F}$, then $|F|_K = \|F\| = \|\exp\{F - I\}\| = 1$. Observe also that, if M is concentrated on integers, then

$$M = \sum_{k=-\infty}^{\infty} M\{k\} I_k, \qquad \widehat{M}(t) = \sum_{k=-\infty}^{\infty} e^{itk} M\{k\}, \qquad \|M\| = \sum_{k=-\infty}^{\infty} \big| M\{k\} \big|.$$

For $F \in \mathcal{F}$, $h \geqslant 0$, Lévy's concentration function is defined by

$$Q(F, h) = \sup_x F\big\{[x, x + h]\big\}.$$

All absolute positive constants are denoted by the same symbol C. Sometimes to avoid possible ambiguities, the constants C are supplied with indices. Also, the constants depending on parameter N are denoted by $C(N)$. We also assume usual conventions $\sum_{j=a}^{b} = 0$ and $\prod_{j=a}^{b} = 1$, if $b < a$. The notation Θ is used for any signed measure satisfying $\|\Theta\| \leqslant 1$. The notation θ is used for any real or complex number satisfying $|\theta| \leqslant 1$.

2   Sums of independent rvs

The results of this section are partially inspired by a comprehensive analytic research of probability generating functions in [12] and the papers on *mod*-Poisson convergence, see [2, 13, 14], and references therein. Assumptions in the above-mentioned papers are made about the behavior of characteristic or probability generating functions. The inversion inequalities are then used to translate their differences to the differences of distributions. In principle, *mod*-Poisson convergence means that if an initial rv is a perturbation of some Poisson rv, then their distributions must be close. Formally, it is required for $\exp\{-\tilde{\lambda}_n(e^{it} - 1)\} f_n(t)$ to have a limit for some sequence of Poisson parameters $\tilde{\lambda}_n$, as $n \to \infty$. Here, $f_n(t)$ is a characteristic function of an investigated rv. Division by a certain Poisson characteristic function is one of the crucial steps in the proof of Theorem 2.1 below, which makes it applicable to rvs satisfying the *mod*-Poisson convergence definition, provided they can be expressed as sums of independent rvs. Though we use factorial moments, similar to Section 7.1 in [2], our work is much more closer in spirit to [21], where general lemmas about the closeness of lattice measures are proved.

In this section, we consider a general case of independent non-identically distributed rvs, forming a triangular array (a scheme of series). Let $S_i = X_{i1} + X_{i2} + \cdots + X_{in_i}$, $Z_i = Z_{i1} + Z_{i2} + \cdots + Z_{in_i}$, $i = 1, 2, \ldots, N$. We assume that all the X_{ij}, Z_{ij} are mutually independent and integer-valued. Observe that, in general, $S = \sum_{i=1}^{N} w_i S_i$ and $Z = \sum_{i=1}^{N} w_i Z_i$ are *not* integer-valued and, therefore, the standard methods of estimation of lattice rvs do not apply. Note also that, since any infinitely divisible distribution can be expressed as a sum of rvs, Poisson, compound Poisson and negative binomial rvs can be used as Z_i.

The distribution of X_{ij} (resp. Z_{ij}) is denoted by F_{ij} (resp. G_{ij}). The closeness of characteristic functions will be determined by the closeness of corresponding factorial moments. Though it is proposed in [2] to use standard factorial moments even for rvs taking negative values, we think that right-hand side and left-hand side factorial moments, already used in [21], are more natural characteristics. Let, for $k = 1, 2, \ldots$, and any $F \in \mathcal{F}_Z$,

$$v_k^+(F_{ij}) = \sum_{m=k}^{\infty} m(m-1) \cdots (m-k+1) F_{ij}\{m\},$$

$$v_k^-(F_{ij}) = \sum_{m=k}^{\infty} m(m-1) \cdots (m-k+1) F_{ij}\{-m\}.$$

For the estimation of the remainder terms we also need the following notation: $\beta_k^{\pm}(F_{ij}, G_{ij}) = v_k^{\pm}(F_{ij}) + v_k^{\pm}(G_{ij})$, $\sigma_{ij}^2 = \max(\mathrm{Var}(X_{ij}), \mathrm{Var}(Z_{ij}))$, and

$$u_{ij} = \min\left\{1 - \frac{1}{2}\|F_{ij}(I_1 - I)\|; \; 1 - \frac{1}{2}\|G_{ij}(I_1 - I)\|\right\}$$

$$= \min\left\{\sum_{k=-\infty}^{\infty} \min\big(F_{ij}\{k\}, F_{ij}\{k-1\}\big); \; \sum_{k=-\infty}^{\infty} \min\big(G_{ij}\{k\}, G_{ij}\{k-1\}\big)\right\}.$$

For the last equality, see (1.9) and (5.15) in [5]. Next we formulate our assumptions. For some fixed integer $s \geqslant 1$, $i = 1, \ldots, N$, $j = 1, \ldots, n_i$,

$$u_{ij} > 0, \qquad \sum_{j=1}^{n_i} u_{ij} \geqslant 1, \qquad n_i \geqslant 1, \qquad w_i > 0, \tag{2}$$

$$v_k^+(F_{ij}) = v_k^+(G_{ij}), \qquad v_k^-(F_{ij}) = v_k^-(G_{ij}), \quad k = 1, 2, \ldots, s \tag{3}$$

$$\beta_{s+1}^+(F_{ij}, G_{ij}) + \beta_{s+1}^-(F_{ij}, G_{ij}) < \infty. \tag{4}$$

Now we are in position to formulate the main result of this section.

Theorem 2.1. *Let assumptions (2)–(4) hold. Then*

$$\big|\mathcal{L}(S) - \mathcal{L}(Z)\big|_K \leqslant C(N, s)\frac{\max_j w_j}{\min_j w_j}\left(\sum_{i=1}^{N}\sum_{l=1}^{n_i} u_{il}\right)^{-1/2}\prod_{l=1}^{N}\left(1 + \sum_{k=1}^{n_l}\sigma_{lk}^2 \Big/ \sum_{k=1}^{n_l} u_{lk}\right)$$

$$\times \sum_{i=1}^{N}\sum_{j=1}^{n_i}\big[\beta_{s+1}^+(F_{ij}, G_{ij}) + \beta_{s+1}^-(F_{ij}, G_{ij})\big]\left(\sum_{k=1}^{n_i} u_{ik}\right)^{-s/2}. \tag{5}$$

If, in addition, s is even and $\beta_{s+2}^+(F_{ij}, G_{ij}) + \beta_{s+2}^-(F_{ij}, G_{ij}) < \infty$, then

$$\big|\mathcal{L}(S) - \mathcal{L}(Z)\big|_K \leqslant C(N, s)\frac{\max_j w_j}{\min_j w_j}\left(\sum_{i=1}^{N}\sum_{l=1}^{n_i} u_{il}\right)^{-1/2}\prod_{l=1}^{N}\left(1 + \sum_{k=1}^{n_l}\sigma_{lk}^2 \Big/ \sum_{k=1}^{n_l} u_{lk}\right)$$

$$\times \sum_{i=1}^{N}\sum_{j=1}^{n_i}\left(\sum_{k=1}^{n_i} u_{ik}\right)^{-s/2}\left(\big|\beta_{s+1}^+(F_{ij}, G_{ij}) - \beta_{s+1}^-(F_{ij}, G_{ij})\big|\right.$$

$$+ \big[\beta_{s+2}^+(F_{ij}, G_{ij}) + \beta_{s+2}^-(F_{ij}, G_{ij})$$

$$\left.+ \beta_{s+1}^-(F_{ij}, G_{ij})\big]\left(\sum_{k=1}^{n_i} u_{ik}\right)^{-1/2}\right). \tag{6}$$

The factor $\left(\sum_{i=1}^{n}\sum_{j=1}^{n_i} u_{ij}\right)^{-1/2}$ estimates the impact of S on approximation of $w_i S_i$. The estimate (6) takes care of a possible symmetry of distributions.

If, in each sum S_i and Z_i, all the rvs are identically distributed, then we can get rid of the factor containing variances. We say that condition (*ID*) is satisfied if, for each $i = 1, 2, \ldots, N$, all rvs X_{ij} and Z_{ij} ($j = 1, \ldots, n_i$) are iid with distributions F_i and G_i, respectively. Observe, that if condition (*ID*) is satisfied, then the characteristic functions of S and Z are respectively equal to

$$\prod_{i=1}^{N} \widehat{F}_i^{n_i}(w_i t), \qquad \prod_{i=1}^{N} \widehat{G}_i^{n_i}(w_i t).$$

We also use notation u_i instead of u_{ij}, since now $u_{i1} = u_{i2} = \cdots = u_{in_i}$.

Theorem 2.2. *Let the assumptions (2)–(4) and the condition (ID) hold. Then*

$$\left|\mathcal{L}(S) - \mathcal{L}(Z)\right|_K \leqslant C(N, s)\frac{\max_j w_j}{\min_j w_j}\left(\sum_{i=1}^{N} n_i u_i\right)^{-1/2}$$

$$\times \sum_{i=1}^{N} \frac{\beta_{s+1}^{+}(F_i, G_i) + \beta_{s+1}^{-}(F_i, G_i)}{n_i^{s/2-1} u_i^{s/2}}. \tag{7}$$

How does Theorem 2.1 compare to the known results? In [4], compound Poisson-type approximations to non-negative iid rvs in each sum were considered under the additional Franken-type condition:

$$v_1^{+}(F_j) - \left(v_1^{+}(F_j)\right)^2 - v_2^{+}(F_j) > 0, \tag{8}$$

see [8]. Similar assumptions were used in [7, 21]. Observe that Franken's condition requires almost all probabilistic mass to be concentrated at 0 and 1. Indeed, then $v_1^{+}(F_j) < 1$ and $F_j\{1\} \geqslant \sum_{k=3}^{\infty} k(k-2)F_j\{k\}$. Meanwhile, Theorems 2.1 and 2.2 hold under much milder assumptions and, as demonstrated in the example below, can be useful even if (8) is not satisfied. Therefore, even for the case of one sum when $N = 1$, our results are new.

Example. Let $N = 2$, $w_1 = 1$, $w_2 = \sqrt{2}$, and F_j and G_j be defined by $F_j\{0\} = 0.375$, $F_j\{1\} = 0.5$, $F_j\{4\} = 0.125$, $G_j\{0\} = 0.45$, $G_j\{1\} = 0.25$, $G_j\{2\} = 0.25$, $G_j\{5\} = 0.05$, $(j = 1, 2)$. We assume that $n_2 = n$ and $n_1 = \lceil\sqrt{n}\rceil$ is the smallest integer greater or equal to $\sqrt{n}$. Then $v_k^{+}(F_j) = v_k^{+}(G_j)$, $k = 1, 2, 3$, $\beta_4^{+}(F_j, G_j) = 9$, $u_j = 3/8$, $(j = 1, 2)$. Therefore, by Theorem 2.2

$$\left|\mathcal{L}(S) - \mathcal{L}(Z)\right|_K \leqslant \frac{C}{\sqrt{n_1 + n_2}}\left(\frac{1}{n_1} + \frac{1}{n_2}\right) = O\left(n^{-1}\right).$$

In this case, Franken's condition (8) is not satisfied, since $v_1^{+}(F_j) - v_2^{+}(F_j) - (v_1^{+}(F_j))^2 < 0$.

Next we apply Theorem 2.2 to the negative binomial distribution. For real $r > 0$ and $0 < \tilde{p} < 1$, let $\xi \sim \mathrm{NB}(r, \tilde{p})$ denote the distribution with

$$\mathrm{P}(\xi = k) = \binom{r + k - 1}{k}\tilde{p}^r\tilde{q}^k, \quad k = 0, 1, \ldots.$$

Here $\tilde{q} = 1 - \tilde{p}$. Note that r is not necessarily an integer.

Let X_{1j} be concentrated on non-negative integers $(v_k^{-}(F_j) = 0)$. We approximate S_i by $Z_i \sim \mathrm{NB}(r_i, p_i)$ with

$$r_i = \frac{(\mathrm{E}S_i)^2}{\mathrm{Var}\,S_i - \mathrm{E}S_i}, \qquad \tilde{p}_i = \frac{\mathrm{E}S_i}{\mathrm{Var}\,S_i},$$

so that $\mathrm{E}S_i = r_i\tilde{q}_i/\tilde{p}_i$ and $\mathrm{Var}\,S_i = r_i\tilde{q}_i/\tilde{p}_i^2$. Observe that

$$\widehat{G}_j(t) = \left(\frac{\tilde{p}_j}{1 - \tilde{q}_j\mathrm{e}^{\mathrm{i}t}}\right)^{r_j/n_j}. \tag{9}$$

Corollary 2.1. *Let assumptions of Theorem 2.2 hold with X_{1j} concentrated on non-negative integers and let $\mathrm{E}X_{1j}^3 < \infty$, $(j = 1, \ldots, N)$. Let G_j be defined by (9). Then*

$$\big|\mathcal{L}(S) - \mathcal{L}(Z)\big|_K \leqslant C \frac{\max_j w_j}{\min_j w_j} \left(\sum_{i=1}^{N} n_i \tilde{u}_i \right)^{-1/2}$$

$$\times \sum_{k=1}^{N} \left[v_3^+(F_k) + v_1^+(F_k) v_2^+(F_k) + \left(v_1^+(F_k) \right)^3 \right.$$

$$\left. + \frac{(v_2^+(F_k) - (v_1^+(F_k))^2)^2}{v_1^+(F_k)} \right] \tilde{u}_k^{-1}. \tag{10}$$

Here

$$\tilde{u}_k = 1 - \frac{1}{2} \max\left(\|(I_1 - I)F_k\|, \left(r_k \ln \frac{1}{\tilde{p}_k} \right)^{-1/2} \right).$$

Remark 2.1. *(i) Note that*

$$r_k \ln \frac{1}{\tilde{p}_k} = \frac{(v_1^+(F_k))^2}{v_2^+(F_k) - (v_1^+(F_k))^2} \ln \frac{v_2^+(F_k) - (v_1^+(F_k))^2 + v_1^+(F_k)}{v_1^+(F_k)}.$$

(ii) Let $v_k^+(F_j) \asymp C$, $w_j \asymp C$. Then the accuracy of approximation in (10) is of the order $O((n_1 + \cdots + n_N)^{-1/2})$.

3 Sums of Markov Binomial rvs

We already mentioned that it is not always natural to assume independence of rvs. In this section, we still assume that $S = w_1 S_1 + w_2 S_2 + \cdots + w_N S_N$ with mutually independent S_i. On the other hand, we assume that each S_i has a Markov Binomial (MB) distribution, that is, S_i is a sum of Markov dependent Bernoulli variables. Such a sum S has a slightly more realistic interpretation in actuarial mathematics. Assume, for example, that we have N insurance policy holders, i-th of whom can get ill during an insurance period and be paid a claim w_i. The health of the policy holder depends on the state of her/his health in the previous period. Therefore, we have a natural two state (healthy, ill) Markov chain. Then S_i is an aggregate claim for ith insurance policy holder after n_i periods, meanwhile S is an aggregate claim of all holders. Limit behavior of the MB distribution is a popular topic among mathematicians, discussed in numerous papers, see, for example, [3, 9, 11], and references therein.

Let $0, \xi_{i1}, \ldots, \xi_{in_i}, \ldots$, $(i = 1, 2, \ldots, N)$ be a Markov chain with the transition probabilities

$$\mathrm{P}(\xi_{ik} = 1 \,|\, \xi_{i,k-1} = 1) = p_i, \qquad \mathrm{P}(\xi_{ik} = 0 \,|\, \xi_{i,k-1} = 1) = q_i,$$
$$\mathrm{P}(\xi_{i,k} = 1 \,|\, \xi_{i,k-1} = 0) = \overline{q}_i, \qquad \mathrm{P}(\xi_{ik} = 0 \,|\, \xi_{i,k-1} = 0) = \overline{p}_i,$$
$$p_i + q_i = \overline{q}_i + \overline{p}_i = 1, \qquad p_i, \overline{q}_i \in (0, 1), \quad k \in \mathbb{N}.$$

The distribution of $S_i = \xi_{i1} + \cdots + \xi_{in_i}$ $(n_i \in \mathbb{N})$ is called the Markov binomial distribution with parameters $p_i, q_i, \overline{p}_i, \overline{q}_i, n_i$. The definition of a MB rv slightly differs

from paper to paper. We use the one from [3]. Note that the Markov chain, considered above, is not necessarily stationary. Furthermore, the distribution of $w_i S_i$ is denoted by $H_{in} = \mathcal{L}(w_i S_i)$. For approximation of H_{in} we use the signed compound Poisson (CP) measure with matching mean and variance. Such signed CP approximations usually outperform both the normal and CP approximations, see, for example, [1, 3, 20]. Let

$$\gamma_i = \frac{q_i \overline{q}_i}{q_i + \overline{q}_i}, \qquad \widehat{Y}_i(t) = \frac{q_i \mathrm{e}^{\mathrm{i} w_i t}}{1 - p_i \mathrm{e}^{\mathrm{i} w_i t}} - 1.$$

Observe that $\widehat{Y}_i(t) + 1$ is the characteristic function of the geometric distribution. Let Y_i be a measure corresponding to $\widehat{Y}_i(t)$. For approximation of H_{in} we use the signed CP measure D_{in}

$$D_{in} = \exp\left\{\left(\frac{\gamma_i(\overline{q}_i - p_i)}{q_i + \overline{q}_i} + n_i \gamma_i\right) Y_i \right.$$
$$\left. - n_i \left(\frac{q_i \overline{q}_i^2}{(q_i + \overline{q}_i)^2}\left(p_i + \frac{q_i}{q_i + \overline{q}_i}\right) + \frac{\gamma_i^2}{2}\right) Y_i^2\right\}. \tag{11}$$

The CP limit occurs when $n\overline{q}_i \to \tilde{\lambda}$, see, for example, [3]. Therefore, we assume $\overline{q}_i$ to be small, though not necessarily vanishing. Let, for some fixed integer $k_0 \geqslant 2$,

$$\overline{q}_i \geqslant \frac{1}{n^{k_0}}, \qquad 0 < p_i \leqslant \frac{1}{2}, \qquad \overline{q}_i \leqslant \frac{1}{30}, \qquad w_i > 0, \qquad n_i \geqslant 1, \quad i = 1, \ldots, N. \tag{12}$$

In principle, the first assumption in (12) can be dropped, but then exponentially vanishing remainder terms appear in all results, making them very complicated.

Theorem 3.1. *Let $H_{in} = \mathcal{L}(w_i S_i)$ and let D_{in} be defined by (11), $i = 1, \ldots, N$. Let the conditions stated in (12) be satisfied. Then*

$$\left|\prod_{i=1}^{N} H_{in} - \prod_{i=1}^{N} D_{in}\right|_K \leqslant C(N, k_0)\frac{\max w_i}{\min w_i} \cdot \frac{\sum_{i=1}^{N} \overline{q}_i(p_i + \overline{q}_i)}{\sqrt{\sum_{k=1}^{N} \max(n_k \overline{q}_k, 1)}}. \tag{13}$$

Remark 3.1. *Let all $\overline{q}_i \geqslant C$, $i = 1, \ldots, N$. Then, obviously, the right-hand side of (13) is majorized by*

$$C(N, k_0)\frac{\max w_i}{\min w_i} \cdot \frac{1}{\sqrt{\max n_k}}.$$

Therefore, even in this case, the result is comparable with the Berry–Esseen theorem.

4 Auxiliary results

Lemma 4.1. *Let $h > 0$, $W \in \mathcal{M}$, $W\{\mathbb{R}\} = 0$, $U \in \mathcal{F}$ and $|\widehat{U}(t)| \leqslant C\widehat{V}(t)$, for $|t| \leqslant 1/h$ and some symmetric distribution V having non-negative characteristic function. Then*

$$|WU|_K \leqslant C \int_{|t| \leqslant 1/h} \left|\frac{\widehat{W}(t)\widehat{U}(t)}{t}\right| \mathrm{d}t + C\|W\|Q(U, h)$$

$$\leqslant C \left(\sup_{|t| \leqslant 1/h} \frac{|\widehat{W}(t)|}{|t|} \cdot \frac{1}{h} + \|W\| \right) Q(V, h).$$

Lemma 4.1 is a version of Le Cam's smoothing inequality, see Lemma 9.3 in [5] and Lemma 3 on p. 402 in [15].

Lemma 4.2. *Let $F \in \mathcal{F}$, $h > 0$ and $a > 0$. Then*

$$Q(F, h) \leqslant \left(\frac{96}{95} \right)^2 h \int_{|t| \leqslant 1/h} |\widehat{F}(t)| \, dt, \tag{14}$$

$$Q(F, h) \leqslant \left(1 + \left(\frac{h}{a} \right) \right) Q(F, a), \tag{15}$$

$$Q(\exp\{a(F - I)\}, h) \leqslant \frac{C}{\sqrt{a F\{|x| > h\}}}. \tag{16}$$

If, in addition, $\widehat{F}(t) \geqslant 0$, then

$$h \int_{|t| \leqslant 1/h} |\widehat{F}(t)| \, dt \leqslant C Q(F, h). \tag{17}$$

Lemma 4.2 contains well-known properties of Levy's concentration function, see, for example, Chapter 1 in [19] or Section 1.5 in [5].

Expansion in left-hand and right-hand factorial moments for Fourier–Stieltjes transforms is given in [21]. Here we need its analogue for distributions.

Lemma 4.3. *Let $F \in \mathcal{F}_Z$ and, for some $s \geqslant 1$, $v_{s+1}^+(F) + v_{s+1}^-(F) < \infty$. Then*

$$F = I + \sum_{m=1}^{s} \frac{v_m^+(F)}{m!} (I_1 - I)^m + \sum_{m=1}^{s} \frac{v_m^-(F)}{m!} (I_{-1} - I)^m$$
$$+ \frac{v_{s+1}^+(F) + v_{s+1}^-(F)}{(s+1)!} (I_1 - I)^{s+1} \Theta. \tag{18}$$

Proof. For measures, concentrated on non-negative integers, (18) is given in [5], Lemma 2.1. Observe that distribution F can be expressed as a mixture $F = p^+ F^+ + p^- F^-$ of distributions F^+, F^- concentrated on non-negative and negative integers, respectively. Then Lemma 2.1 from [5] can be applied in turn to F^+ and to F^- (with I_{-1}). The remainder terms can be combined, since $(I_{-1} - I) = I_{-1}(I - I_1) = (I_1 - I)\Theta$. □

Lemma 4.4. *Let $F, G \in \mathcal{F}_Z$ and, for some $s \geqslant 1$, $v_j^+(F) = v_j^+(G)$, $v_j^-(F) = v_j^-(G)$, $(j = 1, 2, \ldots, s)$. If $\beta_{s+1}^+(F, G) + \beta_{s+1}^-(F, G) < \infty$, then*

$$F - G = \frac{\beta_{s+1}^+(F, G) + \beta_{s+1}^-(F, G)}{(s+1)!} (I_1 - I)^{s+1} \Theta.$$

If, in addition, $\beta_{s+2}^+(F, G) + \beta_{s+2}^-(F, G) < \infty$ and s is even, then

$$F - G = \frac{\beta_{s+1}^+(F, G) - \beta_{s+1}^-(F, G)}{(s+1)!} (I_1 - I)^{s+1}$$
$$+ \left[\beta_{s+2}^+(F, G) + \beta_{s+2}^-(F, G) + \beta_{s+1}^-(F, G) \right] (I_1 - I)^{s+2} \Theta C(s).$$

Proof. Observe that

$$(I_1 - I)^{s+1} + (I_{-1} - I)^{s+1} = (I_1 - I)^{s+1} - (I_{-1})^{s+1}(I_1 - I)^{s+1}$$

$$= (I_1 - I)^{s+1} I_{-1}(I_1 - I) \sum_{j=1}^{s+1} (I_{-1})^{s+1-j}$$

$$= (I_1 - I)^{s+2} \Theta(s + 1).$$

The lemma now follows from (18). □

Lemma 4.5. *Let* $F \in \mathcal{F}_Z$ *with mean* $\mu(F)$ *and variance* $\sigma^2(F)$, *both finite. Then, for all* $|t| \leqslant \pi$,

$$\left| \widehat{F}(t) \right| \leqslant 1 - \frac{(1 - \|(I_1 - I)F\|/2)t^2}{4\pi}$$

$$\leqslant \exp\left\{ -\frac{(1 - \|(I_1 - I)F\|/2)}{\pi} \sin^2 \frac{t}{2} \right\}, \tag{19}$$

$$\left| \left(\widehat{F}(t) \mathrm{e}^{-\mathrm{i}t\mu(F)} \right)' \right| \leqslant \pi^2 \sigma^2(F) |\sin(t/2)|. \tag{20}$$

The first estimate in (19) is given in [2] p. 884, the second estimate in (19) is trivial. For the proof of (20), see p. 81 in [5].

Lemma 4.6. *Let* $M \in \mathcal{M}$ *be concentrated on* $\mathbb{Z}$, $\sum_{k\in\mathbb{Z}} |k||M\{k\}| < \infty$. *Then, for any* $a \in \mathbb{R}$, $b > 0$ *the following inequality holds*

$$\|M\| \leqslant (1 + b\pi)^{1/2} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\left| \widehat{M}(t) \right|^2 + \frac{1}{b^2} \left| \left(\mathrm{e}^{-\mathrm{i}ta} \widehat{M}(t) \right)' \right|^2 \right) \mathrm{d}t \right)^{1/2}.$$

Lemma 4.6 is a well-known inversion inequality for lattice distributions. Its proof can be found, for example, in [5], Lemma 5.1.

Lemma 4.7. *Let* $H_{in} = \mathcal{L}(w_i S_i)$ *and let* D_{in} *be defined by* (11), $i = 1, \ldots, N$. *Let conditions* (12) *hold. Then, for* $i = 1, 2, \ldots, N$,

$$H_{in} - D_{in} = \overline{q}_i (p_i + \overline{q}_i) Y_i \exp\{n_i \gamma_i Y_i / 60\} \Theta C + (p_i + \overline{q}_i)(I_{w_i} - I) \Theta C \mathrm{e}^{-C_i n_i},$$

$$H_{in} = \exp\{n_i \gamma_i Y_i / 30\} \Theta C + (p_i + \overline{q}_i)(I_{w_i} - I) \Theta C \mathrm{e}^{-C_i n_i},$$

$$D_{in} = \exp\{n_i \gamma_i Y_i / 30\} \Theta C, \qquad \mathrm{e}^{-C_i n_i} \leqslant \frac{C(k_0)\overline{q}_i}{\sqrt{\max(n_i \overline{q}_i, 1)}},$$

$$\left| \widehat{Y}_i(t) \right| \leqslant 4 |\sin(t w_i/2)|, \qquad \mathrm{Re}\,\widehat{Y}_i(t) \geqslant -\frac{4}{3} \sin^2(t w_i/2), \qquad \frac{\overline{q}_i}{2} \leqslant \gamma_i \leqslant \overline{q}_i.$$

Proof. The statements follow from Lemma 5.4, Lemma 5.1 and the relations given on pp. 1131–1132 in [3]. The estimate for $\mathrm{e}^{-C_i n_i}$ follows from the first assumption in (12) and the following simple estimate

$$\mathrm{e}^{-C_i n_i} \leqslant \mathrm{e}^{-C_i n_i/2} \mathrm{e}^{-C_i n_i \overline{q}_i/2} \leqslant \frac{C(k_0)}{n_i^{k_0}} \frac{2}{1 + C_i n_i \overline{q}_1}$$

$$\leqslant \frac{C(k_0)\overline{q}_i}{\min(1, C_i)(1 + n_i \overline{q}_i)} \leqslant \frac{C(k_0)\overline{q}_i}{\min(1, C_i) \max(n_i \overline{q}_i, 1)}. \qquad \square$$

5   Proofs for sums of independent rvs

Proof of Theorem 2.1. Let $F_{ij,w}$ (resp. $G_{ij,w}$) denote the distribution of $w_i X_{ij}$ (resp. $w_i Z_{ij}$). Note that $\widehat{F}_{ij,w}(t) = \widehat{F}_{ij}(w_i t)$. By the triangle inequality

$$
\begin{aligned}
\left|\mathcal{L}(S) - \mathcal{L}(Z)\right|_K &= \left|\prod_{i=1}^{N} \mathcal{L}(w_i S_i) - \prod_{i=1}^{N} \mathcal{L}(w_i Z_i)\right|_K \\
&\leqslant \sum_{i=1}^{N} \left|\left(\mathcal{L}(w_i S_i) - \mathcal{L}(w_i Z_i)\right) \prod_{l=1}^{i-1} \mathcal{L}(w_l S_l) \prod_{l=i+1}^{N} \mathcal{L}(w_l Z_l)\right|_K.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\mathcal{L}(w_i S_i) - \mathcal{L}(w_i Z_i) &= \prod_{j=1}^{n_i} F_{ij,w} - \prod_{j=1}^{n_i} G_{ij,w} \\
&= \sum_{j=1}^{n_i} (F_{ij,w} - G_{ij,w}) \prod_{k=1}^{j-1} F_{ik,w} \prod_{k=j+1}^{n_i} G_{ik,w}.
\end{aligned}
$$

For the sake of brevity, let

$$
E_{ij} := \prod_{k=1}^{j-1} F_{ik,w} \prod_{k=j+1}^{n_i} G_{ik,w},
$$

$$
T_i := \prod_{l=1}^{i-1} \mathcal{L}(w_l S_l) \prod_{l=i+1}^{N} \mathcal{L}(w_l Z_l) = \prod_{l=1}^{i-1} \prod_{m=1}^{n_l} F_{lm,w} \prod_{l=i+1}^{N} \prod_{m=1}^{n_l} G_{lm,w}.
$$

Then, combining both equations given above with Lemma 4.4 , we get

$$
\begin{aligned}
\left|\mathcal{L}(S) - \mathcal{L}(Z)\right|_K \leqslant C(s) \sum_{i=1}^{N} \sum_{j=1}^{n_i} \big[\beta_{s+1}^{+}(F_{ij}, G_{ij}) \\
+ \beta_{s+1}^{-}(F_{ij}, G_{ij}) \big] \left|(I_{w_i} - I)^{s+1} E_{ij} T_i\right|_K.
\end{aligned} \tag{21}
$$

Let $|t| \leqslant \pi / \max_i w_i$. Then it follows from (19) that

$$
\left|\widehat{E}_{ij}(t)\widehat{T}_i(t)\right| \leqslant e^{u_{ij} \sin^2(tw_i/2)/\pi} \exp\left\{-\frac{1}{\pi} \sum_{l=1}^{N} \sum_{m=1}^{n_l} u_{lm} \sin^2 \frac{t w_l}{2}\right\}. \tag{22}
$$

Observe that $e^{u_{ij} \sin^2(tw_i/2)/\pi} \leqslant e^{1/\pi} = C$. Next, let

$$
L := \frac{1}{8\pi} \sum_{l=1}^{N} \sum_{m=1}^{n_l} u_{lm} \big[(I_{w_l} - I) + (I_{-w_l} - I)\big]. \tag{23}
$$

It is not difficult to check, that $\exp\{L\}$ is a CP distribution with non-negative characteristic function. Also, by the definition of exponential measure, $\exp\{-L\}$, which can be called *the inverse* to $\exp\{L\}$, is a signed measure with finite variation. We have

$$
|(I_{w_i} - I)^{s+1} E_{ij} T_i|_K = |(I_{w_i} - I)^{s+1} E_{ij} T_i \exp\{-L\} \exp\{L\}|_K. \tag{24}
$$

Next step is similar to the definition of *mod*-Poisson convergence. We apply Lemma 4.1 with $h = \max w_i/\pi$ and $U_1 = \exp\{L\}$ and $W_1 = (I_{w_i} - I)^{s+1} E_{ij} T_i \exp\{-L\}$. By Lemma 4.2,

$$Q(\exp\{L\}, h) \leqslant C \frac{\max w_i}{\min w_i} \cdot Q(\exp\{L\}, \min w_i/2)$$

$$\leqslant C \frac{\max w_i}{\min w_i} \left(\sum_{l=1}^{N} \sum_{m=1}^{n_l} u_{lm} \right)^{-1/2}. \tag{25}$$

From (22) and (23), it follows that

$$\left| \frac{\widehat{W}_1(t)}{t} \right| \cdot \frac{1}{h} \leqslant C(s) \frac{|\sin(t w_i/2)|^{s+1}}{h|t|} \exp\left\{ -\frac{1}{2\pi} \sum_{l=1}^{N} \sum_{m=1}^{n_l} u_{lm} \sin^2 \frac{t w_l}{2} \right\}$$

$$\leqslant C(s) \frac{w_i}{h} |\sin(t w_i/2)|^s \exp\left\{ -\frac{1}{2\pi} \sum_{m=1}^{n_i} u_{im} \sin^2(t w_i/2) \right\}$$

$$\leqslant C(s) \left(\sum_{m=1}^{n_i} u_{im} \right)^{-s/2}. \tag{26}$$

It remains to estimate $\|W_1\|$. Let

$$\Phi_{lm,w} := F_{lm,w} \exp\left\{ \frac{1}{8\pi} u_{lm} \left[(I_{w_l} - I) + (I_{-w_l} - I) \right] \right\},$$

$$\Psi_{lm,w} := G_{lm,w} \exp\left\{ \frac{1}{8\pi} u_{lm} \left[(I_{w_l} - I) + (I_{-w_l} - I) \right] \right\}$$

Then by the properties of the total variation norm,

$$\|W_1\| \leqslant \left\| \exp\left\{ \frac{1}{8} u_{ij} \left[(I_{w_i} - I) + (I_{-w_i} - I) \right] \right\} \right\|$$

$$\times \left\| (I_{w_i} - I)^{s+1} \prod_{k=1}^{j-1} \Phi_{ik,w} \prod_{k=j+1}^{n_i} \Psi_{ik,w} \right\|$$

$$\times \prod_{l=1}^{i-1} \left\| \prod_{m=1}^{n_l} \Phi_{lm,w} \right\| \prod_{l=i+1}^{N} \left\| \prod_{m=1}^{n_l} \Psi_{lm,w} \right\|. \tag{27}$$

The first norm in (27) is bounded by $\exp\{\frac{1}{8} u_{ij} [\|I_{w_i} - I\| + \|I_{-w_i} - I\|]\} \leqslant \exp\{1/2\}$. The total variation norm is invariant with respect to scale. Therefore, without loss of generality, we can switch to $w_l = 1$. In this case, we use the notations Φ_{ik}, Ψ_{ik}. Then, again employing the inverse CP measures, we get

$$\left\| (I_{w_i} - I)^{s+1} \prod_{k=1}^{j-1} \Phi_{ik,w} \prod_{k=j+1}^{n_i} \Psi_{ik,w} \right\|$$

$$= \left\| (I_1 - I)^{s+1} \prod_{k=1}^{j-1} \Phi_{ik} \prod_{k=j+1}^{n_i} \Psi_{ik} \right\|$$

$$= \left\| (I_1 - I)^{s+1} \prod_{k=1}^{j-1} \Phi_{ik} \prod_{k=j+1}^{n_i} \Psi_{ik} \exp\{u_{ij}(I_1 - I)\} \exp\{u_{ij}(I - I_1)\} \right\|$$

$$\leqslant e^2 \left\| (I_1 - I)^{s+1} \exp\{u_{ij}(I_1 - I)\} \prod_{k=1}^{j-1} \Phi_{ik} \prod_{k=j+1}^{n_i} \Psi_{ik} \right\|.$$

We apply Lemma 4.6 with $a = u_{ij} + \sum_{k \neq i}^{n_i} \mu_{ik}$, $b = 1$, where $\mu_{ik} = v_1^+(F_{ik}) + v_1^-(F_{ik})$ is the mean of F_{ik} and, due to assumption (3), of G_{ik}. Let

$$\widehat{\Delta}(t) := (e^{it} - 1)^{s+1} \exp\{u_{ij}(e^{it} - 1 - it)\} \prod_{k=1}^{j-1} \widehat{\Phi}_{ik}(t) e^{-it\mu_{ik}} \prod_{k=j+1}^{n_i} \widehat{\Psi}_{ik} e^{-it\mu_{ik}}.$$

It follows from (19) that

$$\left| \Delta(t) \right| \leqslant C(s) \left| \sin(t/2) \right|^{s+1} \exp\left\{ -\frac{1}{2\pi} \sum_{m=1}^{n_i} u_{im} \sin^2(t/2) \right\}$$

$$\leqslant C(s) \left(\sum_{m=1}^{n_i} u_{im} \right)^{-s/2}.$$

For the estimation of $|\Delta'(t)|$, observe that by (19) and (20)

$$\left| \left(\widehat{\Phi}_{ik}(t) e^{-it\mu_{ik}} \right)' \right| \leqslant \left| \widehat{F}_{ik}(t) e^{-it\mu_{ik}} \frac{u_{ik}}{\pi} \sin(t/2) e^{(u_{ik}/2\pi) \sin^2(t/2)} \right|$$

$$+ \left| \left(\widehat{F}_{ik}(t) e^{-it\mu_{ik}} \right)' e^{(u_{ik}/2\pi) \sin^2(t/2)} \right|$$

$$\leqslant C(s) \left(u_{ik} + \sigma_{ik}^2 \right) \left| \sin(t/2) \right|$$

$$\leqslant C(s) \left(u_{ik} + \sigma_{ik}^2 \right) \left| \sin(t/2) \right| \exp\left\{ -\frac{u_{ik}}{\pi} \sin^2(t/2) \right\} e^{1/\pi}.$$

The same bound holds for $|(\widehat{\Psi}_{ik}(t) \exp\{-it\mu_{ik}\})'|$. The direct calculation shows that

$$\left| \left((e^{it} - 1)^{s+1} \exp\{u_{ij}(e^{it} - 1 - it)\} \right)' \right| \leqslant C(s) \left| \sin(t/2) \right|^s \exp\left\{ -\frac{1}{\pi} u_{ij} \sin^2(t/2) \right\}.$$

Taking into account of the previous two estimates, it is not difficult to prove that

$$\left| \Delta'(t) \right| \leqslant C(s) \left| \sin(t/2) \right|^s \exp\left\{ -\frac{1}{\pi} \sum_{k=1}^{n_i} u_{ik} \sin^2(t/2) \right\}$$

$$\times \left(1 + \sin^2(t/2) \sum_{k=1, k \neq j}^{n_i} \left(u_{ik} + \sigma_{ik}^2 \right) \right)$$

$$\leqslant C(s)\Big(\sum_{k=1}^{n_i} u_{ik}\Big)^{-s/2}\Big(1+\sum_{k=1}^{n_i}\sigma_{ik}^2/\sum_{k=1}^{n_i}u_{ik}\Big).$$

From Lemma 4.6, it follows that

$$\Big\|(I_{w_i}-I)^{s+1}\prod_{k=1}^{j-1}\Phi_{ik,w}\prod_{k=j+1}^{n_i}\Psi_{ik,w}\Big\|\leqslant C(s)\Big(\sum_{k=1}^{n_i}u_{ik}\Big)^{-s/2}\Big(1+\sum_{k=1}^{n_i}\sigma_{ik}^2/\sum_{k=1}^{n_i}u_{ik}\Big).$$
(28)

The remaining two norms in (27) can be estimated similarly:

$$\Big\|\prod_{m=1}^{n_l}\Phi_{lm,w}\Big\|,\ \Big\|\prod_{m=1}^{n_l}\Psi_{lm,w}\Big\|\leqslant C\Big(1+\sum_{m=1}^{n_l}\sigma_{lm}^2/\sum_{m=1}^{n_l}u_{lm}\Big).$$
(29)

Substituting (28), (29) into (27), we obtain

$$\|W_1\|\leqslant C(N,s)\Big(\sum_{m=1}^{n_i}u_{im}\Big)^{-s/2}\prod_{l=1}^{N}\Big(1+\sum_{k=1}^{n_l}\sigma_{lk}^2/\sum_{k=1}^{n_l}u_{lk}\Big).$$
(30)

Combining (30) with (25), (26) and (24), we get

$$\big|(I_{w_i}-I)^{s+1}E_{ij}T_i\big|_K\leqslant C(N,s)\frac{\max_j w_j}{\min_j w_j}\Big(\sum_{i=1}^{N}\sum_{k=1}^{n_i}u_{ik}\Big)^{-1/2}$$
$$\times\Big(\sum_{m=1}^{n_i}u_{im}\Big)^{-s/2}\prod_{l=1}^{N}\Big(1+\sum_{k=1}^{n_l}\sigma_{lk}^2/\sum_{k=1}^{n_l}u_{lk}\Big).$$

Substituting the last estimate into (21) we complete the proof of (5). The proof of (6) is very similar and, therefore, omitted. □

Proof of Theorem 2.2. We outline only the differences from the proof of Theorem 2.1. No use of convolution with the inverse Poisson measure is required, since we have powers of $F_i^{n_i}$, which can be used for Levy's concentration function. Let $\lfloor a\rfloor$ denote an integer part of a and let $a(k):=\lfloor(k-1)/2\rfloor$, $b(k):=\lfloor(n_i-k)/2\rfloor$. Then, as in the proof of Theorem 2.1, we obtain

$$\big|\mathcal{L}(S)-\mathcal{L}(Z)\big|_K\leqslant C(s)\sum_{i=1}^{N}\sum_{k=1}^{n_i}\big(\beta_{s+1}^+(F_i,G_i)+\beta_{s+1}^-(F_i,G_i)\big)$$
$$\times\Big|(I_{w_i}-I)^{s+1}F_{iw}^{a(k)}G_{iw}^{b(k)}F_{iw}^{a(k)}G_{iw}^{b(k)}\prod_{j=1}^{i-1}F_{jw}^{n_j}\prod_{j=i+1}^{N}G_{jw}^{n_j}\Big|_K.$$

Here F_{iw} and G_{iw} denote the distributions of w_iX_{ij} and w_iZ_{ij}, respectively. We can apply Lemma 4.1 to the Kolmogorov norm given above, taking $W=(I_{w_i}-I)^{s+1}F_{iw}^{a(k)}G_{iw}^{b(k)}$. The remaining distribution is used in Levy's concentration function. The Fourier–Stieltjes transform $\widehat{W}(t)/t$ is estimated exactly as in the proof of Theorem 2.1. The total variation of any distribution is equal to 1, therefore $\|W\|\leqslant\|I_{w_i}-I\|\leqslant 2$ and we can avoid application of Lemma 4.6. □

Proof of Corollary 2.1. As proved in [1], p. 144,

$$\frac{1}{2}\left\| G_k(I_1 - I) \right\| \leqslant \left(\frac{p_k v_1^+(F_k)}{q_k} \ln \frac{1}{p_k} \right)^{-1/2}.$$

Observe that $v_1^+(F_j) = v_1^+(G_j)$ and $v_2^+(F_j) = v_2^+(G_j)$. It remains to find $v_3^+(G_j)$ and apply Theorem 2.2. $\qquad\square$

6 Proof of Theorem 3.1

The proof is similar to the one given in [22]. Let $A_i = \exp\{n_i \gamma_i Y_i / 30\}$. From Lemma 4.7, it follows that

$$H_{in} = A_i \Theta_i C + e^{-C_i n_i} \Theta_i C, \qquad D_{in} = A_i \Theta_i C, \quad i = 1, 2, \dots, N.$$

Here we have added index to Θ_i emphasizing that they might be different for different i. As usual, we assume that the convolution $\prod_{k=N+1}^{N} = \prod_{k=1}^{0} = I$. Let also denote by $\sum_i^*$ summation over all indices $\{j_1, j_2, \dots, j_{i-1} \in \{0, 1\}\}$. Taking into account Lemma 4.7 and the properties of the Kolmogorov and total variation norms given in the Introduction, we get

$$\left| \prod_{i=1}^{N} H_{in} - \prod_{i=1}^{N} D_{in} \right|_K$$

$$\leqslant \sum_{i=1}^{N} \left| (H_{in} - D_{in}) \prod_{k=1}^{i-1} H_{kn} \prod_{k=i+1}^{N} D_{kn} \right|_K$$

$$\leqslant \sum_{i=1}^{N} \left| (H_{in} - D_{in}) \sum_i^* \prod_{k=1}^{i-1} A_k^{j_k} \Theta_k C \right.$$

$$\left. \times \prod_{k=i+1}^{N} A_k \Theta_k C \prod_{k=1}^{i-1} e^{-(1-j_k)n_k C_k} \Theta_k C \right|_K$$

$$\leqslant C(N) \sum_{i=1}^{N} \overline{q}_i (p_i + \overline{q}_i) \sum_i^* \left| Y_i \exp\{n_i \gamma_i Y_i / 60\} \prod_{k=1}^{i-1} A_k^{j_k} \prod_{k=i+1}^{N} A_k \right|_K$$

$$\times \prod_{k=1}^{i-1} e^{-(1-j_k)n_k C_k} + C \sum_{i=1}^{N} (p_i + \overline{q}_i) e^{-C_i n_i}$$

$$\times \sum_i^* \left| (I_{w_i} - I) \prod_{k=1}^{i-1} A_k^{j_k} \prod_{k=i+1}^{N} A_k \right|_K \prod_{k=1}^{i-1} e^{-(1-j_k)n_k C_k}. \tag{31}$$

Both summands on the right-hand side of (31) are estimated similarly. Observe that

$$\left| Y_i \exp\{n_i \gamma_i Y_i / 60\} \prod_{k=1}^{i-1} A_k^{j_k} \prod_{k=i+1}^{N} A_k \right|_K$$

$$= \left| Y_i \exp\left\{ \frac{n_i \gamma_i Y_i}{60} + \frac{1}{30} \sum_{k=1}^{i-1} j_k n_k \gamma_k Y_k + \frac{1}{30} \sum_{k=i+1}^{N} n_k \gamma_k Y_k \right\} \right|_K.$$

Next we apply Lemma 4.1 with $W = Y_i$ and $h = \max w_i / \pi$ and V with

$$\widehat{V}(t) = \exp\left\{ -\frac{1}{90} \left[\sum_{k=1}^{i-1} j_k \max(n_k \overline{q}_k, 1) \sin^2(t w_k / 2) \right.\right.$$

$$\left.\left. + \sum_{k=i}^{N} \max(n_k \overline{q}_k, 1) \sin^2(t w_k / 2) \right] \right\}.$$

By Lemma 4.7

$$\frac{|\widehat{Y}_i(t)|}{t} \frac{1}{h} + \|Y_i\| \leqslant C.$$

Observe that

$$\left| \exp\left\{ \frac{n_i \gamma_i}{60} \widehat{Y}_i(t) + \frac{1}{30} \sum_{k=1}^{i-1} j_k n_k \gamma_k \widehat{Y}_k(t) + \frac{1}{30} \sum_{k=i+1}^{N} \gamma_k \widehat{Y}_k(t) \right\} \right|$$

$$\leqslant \exp\left\{ -\frac{n_i \gamma_i \sin^2(t w_i / 2)}{45} - \frac{2}{45} \sum_{k=1}^{i-1} j_k n_k \gamma_k \sin^2(t w_k / 2) \right.$$

$$\left. - \frac{2}{45} \sum_{k=i+1}^{N} n_k \gamma_k \sin^2(t w_k / 2) \right\}$$

$$\leqslant \exp\left\{ -\frac{1}{90} \left[\sum_{k=1}^{i-1} j_k n_k \overline{q}_k \sin^2(t w_k / 2) + \sum_{k=i}^{N} n_k \overline{q}_k \sin^2(t w_k / 2) \right] \right\}$$

$$\leqslant e^{N/90} \exp\left\{ -\frac{1}{90} \left[\sum_{k=1}^{i-1} j_k (n_k \overline{q}_k + 1) \sin^2(t w_k / 2) \right.\right.$$

$$\left.\left. + \sum_{k=i}^{N} (n_k \overline{q}_k + 1) \sin^2(t w_k / 2) \right] \right\}$$

$$\leqslant e^{N/90} \exp\left\{ -\frac{1}{90} \left[\sum_{k=1}^{i-1} j_k \max(n_k \overline{q}_k, 1) \sin^2(t w_k / 2) \right.\right.$$

$$\left.\left. + \sum_{k=i}^{N} \max(n_k \overline{q}_k, 1) \sin^2(t w_k / 2) \right] \right\}$$

$$= e^{N/90} \widehat{V}(t).$$

Therefore, using Lemma 4.2, we prove

$$\left| Y_i \exp\{n_i \gamma_i Y_i / 60\} \prod_{k=1}^{i-1} A_k^{j_k} \prod_{k=i+1}^{N} A_k \right|_K$$

$$\leqslant C(N) Q\left(V, \max_i w_i/h\right)$$

$$\leqslant C(N)\left(\frac{\max w_i}{\min w_i}\right) Q(V, \min w_i/2)$$

$$\leqslant C(N)\left(\frac{\max w_i}{\min w_i}\right)\left(\sum_{k=1}^{i-1} j_k \max(n_k \overline{q}_k, 1) + \sum_{k=i+1}^{N} \max(n_k \overline{q}_k, 1)\right)^{-1/2}. \quad (32)$$

Next observe that by Lemma 4.7,

$$\left|\prod_{k=1}^{i-1} e^{-(1-j_k)n_k C_k}\right| = C \exp\left\{-\sum_{k=1}^{i-1}(1-j_k)C_k n_k\right\}$$

$$\leqslant \frac{C(k_0, N)}{\max(1, \sqrt{\sum_{k=1}^{i-1}(1-j_k)\max(n_k \overline{q}_k, 1)})}.$$

The last estimate, (32) and the trivial inequality $1/(ab) < 2/(a+b)$, valid for any $a, b \geqslant 1$, allow us to obtain

$$\sum_{i=1}^{N} \overline{q}_i (p_i + \overline{q}_i) \sum_i^* \left|Y_i \exp\{n_i \gamma_i Y_i/60\} \prod_{k=1}^{i-1} A_k^{j_k} \prod_{k=i+1}^{N} A_k\right|_K \prod_{k=1}^{i-1} e^{-(1-j_k)n_k C_k}$$

$$\leqslant C(k_0, N) \frac{\max w_j}{\min w_j} \cdot \frac{\sum_{i=1}^{N} \overline{q}_i (p_i + \overline{q}_i)}{\sqrt{\sum_{k=1}^{N} \max(n_k \overline{q}_k, 1)}}.$$

The estimation of the second sum in (31) is almost identical and, therefore, omitted. $\square$

Acknowledgement

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