Modern Stochastics: Theory and Applications 5 (3) (2018) 317–336
https://doi.org/10.15559/18-VMSTA107

Cliquet option pricing in a jump-diffusion Lévy model

Markus Hess

Independent

Markus-Hess@gmx.net (M. Hess)

Received: 5 April 2018, Revised: 14 June 2018, Accepted: 23 June 2018,
Published online: 20 July 2018

Abstract We investigate the pricing of cliquet options in a jump-diffusion model. The considered option is of monthly sum cap style while the underlying stock price model is driven by a drifted Lévy process entailing a Brownian diffusion component as well as compound Poisson jumps. We also derive representations for the density and distribution function of the emerging Lévy process. In this setting, we infer semi-analytic expressions for the cliquet option price by two different approaches. The first one involves the probability distribution function of the driving Lévy process whereas the second draws upon Fourier transform techniques. With view on sensitivity analysis and hedging purposes, we eventually deduce representations for several Greeks while putting emphasis on the Vega.

Keywords Cliquet option pricing, path-dependent exotic option, equity indexed annuity, structured product, sensitivity analysis, Greeks, jump-diffusion model, Lévy process, stochastic differential equation, compound Poisson process, Fourier transform, distribution function

2010 MSC Primary 60G10, 60G51, 60H10; Secondary 91B30, 91B70

1 Introduction

During the last decades, cliquet option based contracts became a very popular and frequently sold investment product in the insurance industry. These contracts can be considered as a customized subclass of equity indexed annuities which combine savings and insurance benefits [3]. The underlying options usually are of monthly sum cap style paying a credited yield based on the sum of monthly-capped rates associated with some reference (stock) index. More precisely, the investor pays a contractually specified amount to the issuer of the option prior to its maturity date and, in turn, receives at maturity a payoff depending on the performance of some designated reference index. In this regard, cliquet type investments belong to the class of path-dependent exotic options. The most popular choice in the insurance branch are cliquet contracts with globally-floored and locally-capped payoffs. These products can be utilized to protect against downside risk while yielding significant upside potential, yet avoiding extreme payoffs due to their local capping (cf. [3, 10, 16]). In [16] cliquet options are regarded as "the height of fashion in the world of equity derivatives".

In the literature, there are different pricing approaches for cliquet options involving e.g. partial differential equations (see [16]), Monte Carlo techniques (see [1, 2, 5, 9]), numerical recursive algorithms related to inverse Laplace transforms (see [10]) and analytical computation methods (see [1, 3, 5, 8, 9, 11]). In [3] the authors provide semi-analytic pricing formulas for path-dependent equity-linked contracts. They distinguish between cliquet options and monthly sum cap contracts and derive expressions for various Greeks. In their approach, it is crucial to know the probability distribution of the returns of the underlying reference index. In [1] the author uses a Lévy process specification to model the evolution of the underlying reference portfolio and investigates the valuation of life insurance policies providing interest rate guarantees. The driving Lévy process is of jump-diffusion type with normally distributed jump amplitudes while a special focus in [1] is laid on valuation under different risk-neutral pricing measures. In [11] the valuation of insurance contracts is discussed while emphasis is put on the impact of different Lévy process model specifications. It is shown that changing the underlying asset model implies a significant change in the prices of guarantees, indicating a substantial model risk. In [8] the pricing of cliquet options in a geometric Meixner model is investigated. The considered option is of monthly sum cap style while the underlying stock price is driven by a pure-jump Meixner–Lévy process yielding Meixner distributed log-returns. The paper [8] provides a specific application of the results derived in the present article. In [10] cliquet option prices are computed numerically by a recursive algorithm involving inverse Laplace transforms. This method is applied to a lognormal and a jump-diffusion model with deterministic volatility as well as to the Heston stochastic volatility model. In addition, a sensitivity analysis in each model is presented. Moreover, in [7] cliquet option pricing in a jump-diffusion model with time-dependent coefficients is examined. The jumps in the stock price trajectory are interspersed by an increasing standard Poisson process and the time-dependent coefficients are approximated by piecewise constant functions. In [7] there are solely cliquet options with a single resetting time discussed. In [16] the author investigates cliquet option pricing with partial differential equations (PDEs) while putting a special focus on the important role of volatility surface modeling. Recently, there have been extensions beyond Lévy settings to regime switching Lévy models (see e.g. [5, 9]). In [5] the authors investigate the pricing of equity-linked annuities with cliquet-style guarantees in regime-switching stochastic volatility models with jumps. They propose a transform-based pricing method involving density projections and continuous-time Markov chain approximations. The considered models include exponential and regime-switching Lévy processes as well as stochastic volatility models with general jump size distributions. In [9] the valuation of equity-linked life insurance contracts in a regime switching Lévy model is studied. The model parameters depend on a continuous-time finite-state Markov chain, and closed-form pricing formulas based on Fourier transform techniques are derived.

The aim of the present paper is to provide analytical pricing formulas for globally-floored locally-capped cliquet options with multiple resetting times where the underlying reference stock index is driven by a drifted time-homogeneous Lévy process with Brownian diffusion component and compound Poisson jumps. In our framework, jumps represent rare events such as crashes, large drawdowns or upward movements. The dates of e.g. market crashes are modeled as arrival times of a standard Poisson process while the jump amplitudes can be both positive and negative. With reference to Section 4.1.1 in [4], we state that jump-diffusion models are easy to simulate and efficient Monte Carlo methods for option pricing are available. Jump-diffusion models also perform well when it comes to implied volatility smile interpolation (see Section 13 in [4]). In our setup, we derive cliquet option price formulas under two different approaches: once by using the distribution function of the driving Lévy process and once by applying Fourier transform techniques. In the context of sensitivity analysis, we eventually provide expressions for several Greeks related to our model.

The paper is organized as follows: In Section 2 we introduce the jump-diffusion stock price model and derive representations for the probability density and distribution function of the driving Lévy process. In Section 3 we are concerned with cliquet option pricing under both a distribution function and a Fourier transform approach. Section 4 is dedicated to sensitivity analysis and the computation of different Greeks. In Section 5 we draw the conclusions and briefly mention some future research topics.

2 A Lévy stock price model and its distributional properties

Let $(\Omega, \mathbb{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{Q})$ be a filtered probability space satisfying the usual hypotheses, i.e. $\mathcal{F}_t = \mathcal{F}_{t+} := \cap_{s>t}\mathcal{F}_s$ constitutes a right-continuous filtration and $\mathbb{F}$ denotes the sigma-algebra augmented by all $\mathbb{Q}$-null sets (cf. p. 3 in [13]). Here, $\mathbb{Q}$ is a risk-neutral probability measure and $0 < T < \infty$ denotes a finite time horizon. In the sequel, we introduce a stochastic model for the stock price process S_t. Let $t \in [0, T]$ and consider the stochastic differential equation (SDE)

$$dS_t = \eta(t, S_t)\, dt + \sigma(t, S_t)\, dW_t + \int_{\mathbb{R}} \theta(t, z, S_{t-})\, dN(t, z)$$

where η, σ and θ are deterministic functions, W constitutes an $\mathcal{F}$-adapted standard Brownian motion under $\mathbb{Q}$ and N is a Poisson random measure (PRM). We further introduce the $\mathbb{Q}$-compensated PRM

$$d\widetilde{N}(s, z) := dN(s, z) - d\nu(z)\, ds \tag{2.1}$$

which constitutes an $(\mathcal{F}, \mathbb{Q})$-martingale integrator on $[0, T] \times \mathbb{R}$ with positive and finite Lévy measure ν satisfying $\nu(\{0\}) = 0$ and

$$\int_{\mathbb{R}} \left(1 \wedge z^2\right) d\nu(z) < \infty$$

(cf. Eq. (3.14) in [4]). In the above setup, we refer to η, σ and θ as the drift, volatility and jump function, respectively. We assume that W and N are $\mathbb{Q}$-independent and set

$$\mathcal{F}_t := \sigma\{S_u : 0 \le u \le t\}$$

for all $t \in [0, T]$. In the next step, we specify the emerging coefficients as follows

$$\eta(t, S_t) := \eta(t)S_t, \qquad \sigma(t, S_t) := \sigma(t)S_t, \qquad \theta(t, z, S_{t-}) := \theta(t, z)S_{t-}$$

while assuming that $\theta(t, z) > -1$ for all $(t, z) \in [0, T] \times \mathbb{R}$ and that

$$\mathbb{E}_{\mathbb{Q}}\bigg[\int_0^T \bigg(\big|\eta(t)\big| + \sigma^2(t) + \int_{\mathbb{R}} \theta^2(t, z)\, d\nu(z)\bigg)\, dt\bigg] < \infty$$

(cf. Section 9.1 in [6]). Consequently, we obtain the geometric SDE

$$\frac{dS_t}{S_{t-}} = \eta(t)\, dt + \sigma(t)\, dW_t + \int_{\mathbb{R}} \theta(t, z)\, dN(t, z)$$

which possesses the discontinuous Doléans-Dade solution

$$S_t = S_0 \exp\bigg\{\int_0^t \bigg(\eta(s) - \frac{1}{2}\sigma^2(s)\bigg)\, ds + \int_0^t \sigma(s)\, dW_s + \int_0^t \int_{\mathbb{R}} \ln\big(1 + \theta(s, z)\big)\, dN(s, z)\bigg\}$$

for all $t \in [0, T]$. From now on, we set $\eta(t) \equiv \eta$, $\sigma(t) \equiv \sigma > 0$ and $\theta(t, z) := e^z - 1$ in order to obtain a time-homogeneous Lévy process specification. If we do so, the latter equation can be written as

$$S_t = S_0 e^{X_t} \tag{2.2}$$

with a real-valued Lévy process

$$X_t := \gamma t + \sigma W_t + \int_0^t \int_{\mathbb{R}} z\, dN(s, z) \tag{2.3}$$

where $\gamma := \eta - \sigma^2/2$ and $t \in [0, T]$. Note that $X_0 = 0$ $\mathbb{Q}$-a.s. We denote the characteristic triplet of X by (γ, σ, ν). Moreover, the first moment of X is given by

$$\mathbb{E}_{\mathbb{Q}}[X_t] = t\bigg(\gamma + \int_{\mathbb{R}} z\, d\nu(z)\bigg)$$

whereas the characteristic function of X can be computed by the Lévy–Khinchin formula (see e.g. [4, 6, 14, 15]) due to

$$\phi_{X_t}(u) := \mathbb{E}_{\mathbb{Q}}\big[e^{iuX_t}\big] = e^{\psi(u)t} \tag{2.4}$$

with $i^2 = -1$, $u \in \mathbb{R}$, $t \in [0, T]$ and characteristic exponent

$$\psi(u) := iu\gamma - \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}} \big[e^{iuz} - 1\big]\, d\nu(z). \tag{2.5}$$

More details on Lévy processes can be found in e.g. [4, 6, 14, 15]. In the next step, we define the discounted stock price

$$\hat{S}_t := \frac{S_t}{B_t}$$

where S_t is such as defined in (2.2) and $B_t := e^{rt}$ is the value of a bank account with normalized initial capital $B_0 = 1$ and risk-less interest rate $r > 0$. Due to (2.2), we find

$$\hat{S}_t = S_0 e^{X_t - rt}$$

while Itô's formula yields the following geometric SDE under $\mathbb{Q}$

$$\frac{d\hat{S}_t}{\hat{S}_{t-}} = \left(\eta - r + \int_{\mathbb{R}} \left[e^z - 1\right] d\nu(z)\right) dt + \sigma\, dW_t + \int_{\mathbb{R}} \left[e^z - 1\right] d\widetilde{N}(t, z).$$

In accordance to no-arbitrage theory, the discounted stock price process $\hat{S}_t$ must form a martingale under the risk-neutral probability measure $\mathbb{Q}$. For this reason, we have to require the drift restriction

$$\eta = r - \int_{\mathbb{R}} \left[e^z - 1\right] d\nu(z).$$

With this particular choice of the drift coefficient η, we obtain

$$\frac{dS_t}{S_{t-}} = r\, dt + \sigma\, dW_t + \int_{\mathbb{R}} \left[e^z - 1\right] d\widetilde{N}(t, z)$$

under $\mathbb{Q}$. Summing up, if we model the stock price process S_t as in the latter equation, then the discounted stock price $\hat{S}_t$ constitutes a $\mathbb{Q}$-martingale.

Furthermore, let us define the Fourier transform, respectively inverse Fourier transform, of a function $q \in \mathcal{L}^1(\mathbb{R})$ via

$$\hat{q}(y) := \int_{\mathbb{R}} q(x) e^{iyx}\, dx, \qquad q(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{q}(y) e^{-iyx}\, dy.$$

Proposition 2.1 (density function)**.** *Suppose that the Lévy process X_t is such as defined in (2.3). Then for all $t \in [0, T]$ and $x \in \mathbb{R}$ the probability density function $f_{X_t}(x)$ of X_t under $\mathbb{Q}$ can be represented as*

$$f_{X_t}(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \exp\left\{-iux + t\left(iu\gamma - \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}} \left[e^{iuz} - 1\right] d\nu(z)\right)\right\} du. \quad (2.6)$$

Proof. Note that the characteristic function (2.4) is the Fourier transform of the density function $f_{X_t}(\cdot)$, that is,

$$\phi_{X_t}(u) = \int_{\mathbb{R}} e^{iux} f_{X_t}(x)\, dx.$$

We next apply the inverse Fourier transform and hereafter take (2.4) and (2.5) into account which yields the density function (2.6). $\square$

In what follows, we investigate in detail the jump part of the Lévy process X denoted by

$$L_t := \int_0^t \int_{\mathbb{R}} z\, dN(s, z) = \sum_{j=1}^{N_t} Y_j$$

which constitutes a càdlàg, finite activity compound Poisson process (CPP) with finitely many jumps in each time interval. In the latter equation, N_t constitutes a standard Poisson process under $\mathbb{Q}$ with deterministic jump intensity $\lambda > 0$, i.e. $N_t \sim Poi(\lambda t)$, while $Y_1, Y_2, \ldots$ are i.i.d. random variables modeling the jump amplitudes. We put $\beta := \mathbb{E}_{\mathbb{Q}}[Y_1]$. Recall that the compensated compound Poisson process $(L_t - \beta\lambda t)_t$ constitutes an $(\mathcal{F}_t, \mathbb{Q})$-martingale which implies

$$\beta\lambda = \int_{\mathbb{R}} z\, d\nu(z)$$

thanks to (2.1). Note that N_t shall not be mixed up with the Poisson random measure $dN(s, z)$. Obviously, we may write $X_t = \gamma t + \sigma W_t + L_t$. We assume that the stochastic processes W_t, N_t and the random variables $Y_1, Y_2, \ldots$ altogether are $\mathbb{Q}$-independent.

Example 2.2. If Y_j is normally distributed with mean μ and variance δ^2 under $\mathbb{Q}$ for all j, then the Lévy measure possesses the Lebesgue density $d\nu(z) = \lambda\varphi_{\mu,\delta^2}(z)\, dz$ where

$$\varphi_{\mu,\delta^2}(z) := \frac{1}{\sqrt{2\pi\delta^2}} e^{-\frac{1}{2}(\frac{z-\mu}{\delta})^2}$$

and $z \in \mathbb{R}$. Here, μ and δ^2 model the mean respectively variance of the jump sizes. In this setup, we receive $\beta = \mu$, $\mathbb{E}_{\mathbb{Q}}[X_t] = t(\gamma + \lambda\mu)$ and $\mathbb{Var}_{\mathbb{Q}}[X_t] = t(\sigma^2 + \lambda\delta^2 + \lambda\mu^2)$. Evidently, choosing a negative μ makes the occurrence of downward jumps more likely than upward jumps and vice versa. We remark that a similar model specification with normally distributed jump sizes has firstly been proposed in [12].

Example 2.3. If Y_j is exponentially distributed with parameter $\alpha > 0$ under $\mathbb{Q}$ for all j, then the Lévy measure possesses the Lebesgue density $d\nu(z) = \lambda p_\alpha(z)\, dz$ where $p_\alpha(z) := \alpha e^{-\alpha z}$ and $z \in [0, \infty[$. We presently find $\beta = 1/\alpha$.

Corollary 2.4. *(a) Suppose that Y_j is normally distributed (cf. [12]) with mean μ and variance δ^2 under $\mathbb{Q}$ for all j. Then for all $t \in [0, T]$ and $x \in \mathbb{R}$ the probability density function of X_t under $\mathbb{Q}$ takes the form*

$$f_{X_t}(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \exp\left\{-iux + t\left(iu\gamma - \frac{1}{2}\sigma^2 u^2 + \lambda e^{iu\mu - \delta^2 u^2/2} - \lambda\right)\right\} du.$$

(b) Suppose that Y_j is exponentially distributed with parameter $\alpha > 0$ under $\mathbb{Q}$ for all j. Then for all $t \in [0, T]$ and $x \in \mathbb{R}$ the probability density function of X_t under $\mathbb{Q}$ takes the form

$$f_{X_t}(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \exp\left\{i(\gamma t - x)u - \frac{1}{2} t\sigma^2 u^2 - \frac{\lambda t u}{u + i\alpha}\right\} du.$$

Proof. Combine (2.6) with Example 2.2 and Example 2.3. □

Proposition 2.5 (distribution function)**.** *Let $X_t = \gamma t + \sigma W_t + \sum_{j=1}^{N_t} Y_j$ and assume that the standard Poisson process N_t jumps m times in the time interval $[0, t]$, that* *is, $N_t = m$ with $m \in \mathbb{N}_0$. As in [12], suppose that Y_j is normally distributed with mean μ and variance δ^2. Then for any Borel set $A \subset \mathbb{R}$ and $t \in [0, T]$ the cumulative probability distribution function of X_t under $\mathbb{Q}$ possesses the representation*

$$\mathbb{Q}(X_t \in A) = \int_A e^{-\lambda t} \sum_{m=0}^{\infty} \frac{(\lambda t)^m}{m! \sqrt{2\pi(\sigma^2 t + m\delta^2)}} \exp\left\{ -\frac{1}{2} \frac{(x - \gamma t - m\mu)^2}{\sigma^2 t + m\delta^2} \right\} dx. \tag{2.7}$$

Proof. Let $A \subset \mathbb{R}$ and $t \in [0, T]$. In accordance to Section 4.3 in [4] and the properties of conditional probabilities, the following ("quickly converging" [4]) series representation for the distribution function of X_t under $\mathbb{Q}$ holds

$$\begin{aligned} \mathbb{Q}(X_t \in A) &= \sum_{m=0}^{\infty} \mathbb{Q}\big(\{X_t \in A\} \cap \{N_t = m\}\big) \\ &= \sum_{m=0}^{\infty} \mathbb{Q}(X_t \in A \mid N_t = m)\mathbb{Q}(N_t = m) \\ &= e^{-\lambda t} \sum_{m=0}^{\infty} \mathbb{Q}(X_t \in A \mid N_t = m) \frac{(\lambda t)^m}{m!} \end{aligned} \tag{2.8}$$

wherein

$$\mathbb{Q}(X_t \in A \mid N_t = m) = \mathbb{Q}\left(\left(\gamma t + \sigma W_t + \sum_{j=1}^{m} Y_j \right) \in A \right).$$

Since $Y_j \sim \mathcal{N}(\mu, \delta^2)$ for all j, we find that the stochastic process $(\gamma t + \sigma W_t + \sum_{j=1}^{m} Y_j)_t$ also is normally distributed under $\mathbb{Q}$ with mean $\gamma t + m\mu$ and variance $\sigma^2 t + m\delta^2$. Thus, by the definition of the cumulative distribution function, we get

$$\mathbb{Q}\left(\left(\gamma t + \sigma W_t + \sum_{j=1}^{m} Y_j \right) \in A \right) = \int_A \varphi_{\gamma t + m\mu, \sigma^2 t + m\delta^2}(x)\, dx$$

where φ denotes the probability density function of the normal distribution (see [12] and Example 2.2 above). Putting the latter equations together, we end up with (2.7). $\square$

Remark 2.6. Verify that the proof of Proposition 2.5 only works, if the random variables Y_j are normally distributed for every j. If Y_j is e.g. exponentially distributed for all $j = 1, \ldots, m$ (as proposed in Example 2.3), then it is unclear how to compute the probability

$$\mathbb{Q}\left(\left(\gamma t + \sigma W_t + \sum_{j=1}^{m} Y_j \right) \in A \right)$$

emerging in the sequel of (2.8).

Corollary 2.7 (Eq. (4.12) in [4]). *Under the assumptions of Proposition 2.5, for all $t \in [0, T]$ and $x \in \mathbb{R}$ the probability density function of X_t under $\mathbb{Q}$ is given by*

$$f_{X_t}(x) = e^{-\lambda t} \sum_{m=0}^{\infty} \frac{(\lambda t)^m}{m!\sqrt{2\pi(\sigma^2 t + m\delta^2)}} \exp\left\{-\frac{1}{2}\frac{(x - \gamma t - m\mu)^2}{\sigma^2 t + m\delta^2}\right\}. \tag{2.9}$$

Proof. The density can directly be read off in (2.7). Also see [12]. □

Corollary 2.8. *If the Borel set $A =]-\infty, a] \subseteq \mathbb{R}$ is an interval, then for any $a \in \mathbb{R}$ and $t \in [0, T]$ the distribution function in (2.7) takes the form*

$$\mathbb{Q}(X_t \leq a) = e^{-\lambda t} \sum_{m=0}^{\infty} \frac{(\lambda t)^m}{m!} \Phi\left(\frac{a - \gamma t - m\mu}{\sqrt{\sigma^2 t + m\delta^2}}\right)$$

where Φ denotes the standard normal cumulative distribution function.

Proof. This representation is an immediate consequence of Proposition 2.5. □

Recall that the stochastic process S_t will serve as our stock price model when it comes to cliquet option pricing in the subsequent section. In this context, for $n \in \mathbb{N}$ we introduce the time partition $\mathcal{P} := \{0 < t_0 < t_1 < \cdots < t_n \leq T\}$ and define the return/revenue process associated with the period $[t_{k-1}, t_k]$ via

$$R_k := \frac{S_{t_k} - S_{t_{k-1}}}{S_{t_{k-1}}} = e^{X_{t_k} - X_{t_{k-1}}} - 1 \tag{2.10}$$

where $k \in \{1, \ldots, n\}$ and X is the Lévy process defined in (2.3). Note that $R_1, \ldots, R_n$ are $\mathbb{Q}$-independent and that $R_k > -1$ $\mathbb{Q}$-almost sure for all k. For the sake of notational simplicity, we always work under the assumption of equidistant time points in the following and define $\tau := t_k - t_{k-1}$. If we want to refrain from this assumption again, then τ simply has to be replaced by the difference $t_k - t_{k-1}$ in all subsequent equations – with (3.17) as an exception.

Proposition 2.9. *Let $\mathcal{P} = \{0 < t_0 < t_1 < \cdots < t_n \leq T\}$ and put $\tau = t_k - t_{k-1}$ for $k \in \{1, \ldots, n\}$ (equidistant time points). Define the return process R_k as in (2.10). Then for any fixed real-valued $\xi > -1$ the distribution function of R_k under $\mathbb{Q}$ admits the series representation*

$$\mathbb{Q}(R_k \leq \xi) = e^{-\lambda\tau} \sum_{m=0}^{\infty} \frac{(\lambda\tau)^m}{m!} \Phi\left(\frac{\ln(1 + \xi) - \gamma\tau - m\mu}{\sqrt{\sigma^2\tau + m\delta^2}}\right) \tag{2.11}$$

where Φ denotes the standard normal cumulative distribution function.

Proof. Since X is a Lévy process under $\mathbb{Q}$, we observe $X_{t_k} - X_{t_{k-1}} \cong X_\tau$ (stationary increments) where $\tau = t_k - t_{k-1}$ and the symbol $\cong$ denotes equality in distribution. Taking (2.10) and (2.9) into account, we obtain

$$\mathbb{Q}(R_k \leq \xi) = \mathbb{Q}\big(X_\tau \leq \ln(1 + \xi)\big)$$

$$= e^{-\lambda\tau} \sum_{m=0}^{\infty} \frac{(\lambda\tau)^m}{m!} \int_{-\infty}^{\ln(1+\xi)} \varphi_{\gamma\tau+m\mu,\sigma^2\tau+m\delta^2}(x)\, dx \tag{2.12}$$

with

$$f_{X_\tau}(x) = e^{-\lambda\tau} \sum_{m=0}^{\infty} \frac{(\lambda\tau)^m}{m!} \varphi_{\gamma\tau+m\mu,\sigma^2\tau+m\delta^2}(x) \tag{2.13}$$

where φ denotes the density function of the normal distribution (recall Example 2.2). We finally perform the integration and end up with (2.11). $\square$

In quantitative risk management, it is often of interest to compute the probability of large drawdowns (shocks) in asset prices like e.g. $\mathbb{Q}(S_u \le \kappa S_t)$, $0 \le t < u \le T$, where κ constitutes some stress scenario percentile like 60%, for instance. Due to (2.2), we find $\mathbb{Q}(S_u \le \kappa S_t) = \mathbb{Q}(X_{u-t} \le \ln \kappa)$ which can easily be computed by Corollary 2.8.

3 Cliquet option pricing

This section is dedicated to the pricing of cliquet options in the Lévy jump-diffusion stock price model presented in Section 2. In accordance to (1.1) in [3], we consider a cliquet option with payoff

$$H_T = K + K \max\left\{g, \sum_{k=1}^{n} \min\{c, R_k\}\right\}$$

where T is the maturity time, K denotes the notional (the initial investment), g is the guaranteed rate at maturity, $c \ge 0$ is the local cap and R_k is the return process defined in (2.10). This option is of monthly sum cap style with credited rate based on the sum of the monthly-capped rates [3]. Verify that the payoff H_T is globally-floored and locally-capped. A popular choice in the insurance industry is to take $g = 0$ (globally-floored by zero) and $n = 12$ (monthly-capped by c). Further, note that the payoff H_T actually is a function of multiple random variables, i.e. $H_T = h(R_1, \dots, R_n) = \overline{h}(S_{t_0}, \dots, S_{t_n})$ wherein h and $\overline{h}$ are appropriately defined functions while the resetting times of the cliquet option are ordered as follows $0 < t_0 < t_1 < \cdots < t_n \le T$. In this regard, a notation like $H_{t_0,\dots,t_n}(T)$ might be more intuitive than simply writing H_T. However, by a case distinction we observe

$$H_T = K \max\left\{1 + g, 1 + \sum_{k=1}^{n} \min\{c, R_k\}\right\} = K\left(1 + g + \max\left\{0, \sum_{k=1}^{n} Z_k\right\}\right)$$

where $Z_k := \min\{c, R_k\} - g/n$ denote i.i.d. random variables. Moreover, we introduce a bank account $dB_t = rB_t\, dt$ with constant interest rate $r > 0$ and initial capital $B_0 = 1$, i.e. $B_t = e^{rt}$. Then the price at time $t \le T$ of a cliquet option with payoff H_T at maturity T is the discounted risk-neutral conditional expectation of the payoff, i.e.

$$C_t = e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}}(H_T | \mathcal{F}_t).$$

Combining the latter equations, we obtain

$$C_0 = Ke^{-rT}\left(1 + g + \mathbb{E}_{\mathbb{Q}}\left[\max\left\{0, \sum_{k=1}^{n} Z_k\right\}\right]\right) \tag{3.1}$$

which shows that the considered cliquet option with payoff H_T essentially is a plain-vanilla call option with strike zero written on the basket-style underlying $\sum_{k=1}^{n} Z_k$.

Proposition 3.1 (Cliquet option price). *Let $k \in \{1, \ldots, n\}$ and consider the independent and identically distributed random variables $Z_k = \min\{c, R_k\} - g/n$ where $c \geq 0$ is the local cap, R_k is the return process defined in (2.10) and g is the guaranteed rate at maturity. Denote the maturity time by T, the notional by K and the risk-less interest rate by r. Then the price at time zero of a cliquet option with payoff H_T can be represented as*

$$C_0 = Ke^{-rT}\left(1 + g + \frac{n}{2}\mathbb{E}_{\mathbb{Q}}[Z_1] + \frac{1}{\pi}\int_{0^+}^{\infty} \frac{1 - \mathfrak{Re}(\phi_Z(x))}{x^2}\, dx\right) \tag{3.2}$$

where $\mathfrak{Re}$ denotes the real part and the characteristic function $\phi_Z(x)$ is defined via

$$\phi_Z(x) := \prod_{k=1}^{n} \phi_{Z_k}(x) = \prod_{k=1}^{n} \mathbb{E}_{\mathbb{Q}}\big[e^{ixZ_k}\big] = \big(\phi_{Z_1}(x)\big)^n = \big(\mathbb{E}_{\mathbb{Q}}\big[e^{ixZ_1}\big]\big)^n. \tag{3.3}$$

More explicit expressions for $\phi_Z(x)$ and $\mathbb{E}_{\mathbb{Q}}[Z_1]$ are derived in several propositions below.

Proof. The proof essentially follows the same lines as the proof of Proposition 3.1 in [3] whereas our proof does not make use of the Rademacher random variable introduced in [3]. To begin with, we recall that

$$\max\{0, a\} = \frac{a + |a|}{2},$$
$$|a| = \frac{2}{\pi}\int_{0^+}^{\infty} \frac{1 - \cos(ax)}{x^2}\, dx = \frac{1}{\pi}\int_{0^+}^{\infty} \frac{2 - e^{iax} - e^{-iax}}{x^2}\, dx$$

similar to (3.2) and (3.3) in [3]. As a consequence, we deduce

$$\mathbb{E}_{\mathbb{Q}}\left[\max\left\{0, \sum_{k=1}^{n} Z_k\right\}\right] = \sum_{k=1}^{n} \frac{\mathbb{E}_{\mathbb{Q}}[Z_k]}{2} + \int_{0^+}^{\infty} \frac{2 - \phi_Z(x) - \phi_Z(-x)}{2\pi x^2}\, dx$$

where the characteristic function $\phi_Z(x)$ is such as defined in (3.3). In the derivation of the latter equation, we used the fact that $Z_1, \ldots, Z_n$ are i.i.d. random variables under $\mathbb{Q}$. Since

$$\frac{1}{2}\big(\phi_Z(x) + \phi_Z(-x)\big) = \mathbb{E}_{\mathbb{Q}}\left[\cos\left(x\sum_{k=1}^{n} Z_k\right)\right]$$
$$= \mathfrak{Re}\left(\mathbb{E}_{\mathbb{Q}}\left[\exp\left\{ix\sum_{k=1}^{n} Z_k\right\}\right]\right) = \mathfrak{Re}\big(\phi_Z(x)\big)$$

we get

$$\mathbb{E}_{\mathbb{Q}}\left[\max\left\{0, \sum_{k=1}^{n} Z_k\right\}\right] = \sum_{k=1}^{n} \frac{\mathbb{E}_{\mathbb{Q}}[Z_k]}{2} + \int_{0+}^{\infty} \frac{1 - \mathfrak{Re}(\phi_Z(x))}{\pi x^2}\, dx.$$

Substituting this into (3.1) leads us to (3.2). □

The remaining challenge now consists in finding appropriate computation techniques for the entities $\mathbb{E}_{\mathbb{Q}}[Z_1]$ and $\phi_Z(x)$ emerging in (3.2). In the subsequent sections, we present different methods to derive expressions for the mentioned entities. Similar to the notation introduced in Proposition 2.9, for arbitrary $k \in \{1, \dots, n\}$ we set $\tau = t_k - t_{k-1}$ in the following. We also assume that the jump amplitudes are normally distributed, as pointed out in Example 2.2.

3.1 Cliquet option pricing with distribution functions

Let us first apply a method involving probability distribution functions (cf. [3]). We initially investigate the treatment of $\phi_Z(x)$ as defined in (3.3).

Proposition 3.2. *Suppose that $Z_k = \min\{c, R_k\} - g/n$ where $k \in \{1, \dots, n\}$. Then the characteristic function of Z_k under $\mathbb{Q}$ can be represented as*

$$\begin{aligned}\phi_{Z_k}(x) = e^{-ix(1+g/n)}\Bigg(&e^{ix(1+c)} \\ &- ixe^{-\lambda\tau}\sum_{m=0}^{\infty}\frac{(\lambda\tau)^m}{m!}\int_{0+}^{1+c} e^{ixw}\Phi\left(\frac{\ln(w) - \gamma\tau - m\mu}{\sqrt{\sigma^2\tau + m\delta^2}}\right)dw\Bigg)\end{aligned} \tag{3.4}$$

where Φ denotes the standard normal cumulative distribution function.

Proof. By a case distinction, we find that the distribution function of Z_k is given by

$$\mathbb{Q}(Z_k > \xi) = \mathbb{Q}(R_k - g/n > \xi) \tag{3.5}$$

if $R_k \le c$ and $\xi \le c - g/n$, whereas $\mathbb{Q}(Z_k > \xi) = 0$ otherwise (cf. (3.15) in [3]). Since $R_k > -1$ $\mathbb{Q}$-a.s. for all k, we deduce $Z_k > -1 - g/n$ $\mathbb{Q}$-a.s. for all k. Thus, $Z_k + 1 + g/n > 0$ $\mathbb{Q}$-a.s. for all k. With respect to (3.5), we obtain

$$\mathbb{Q}(Z_k + 1 + g/n > w) = \mathbb{Q}(Z_k > w - 1 - g/n) = \mathbb{Q}(R_k > w - 1) \tag{3.5a}$$

if $R_k \le c$ and $w \le 1 + c$, whereas $\mathbb{Q}(Z_k + 1 + g/n > w) = 0$ otherwise. Further on, verify that for the characteristic function the following relation holds

$$\phi_{Z_k+1+g/n}(x)e^{-ix(1+g/n)} = \phi_{Z_k}(x). \tag{3.6}$$

Moreover, we recall that for any random variable $\Lambda \ge 0$ with finite first moment, its characteristic function can be represented as

$$\phi_\Lambda(x) = 1 + ix\int_0^{\infty} e^{ixu}\mathbb{Q}(\Lambda > u)\, du.$$

(This equality follows from integration by parts; cf. Eq. (3.14) in [3].) Combining the latter equation with (3.6) and (3.5a), we deduce

$$\phi_{Z_k}(x) = e^{-ix(1+g/n)}\left(1 + ix\int_0^{1+c} e^{ixw}\mathbb{Q}(R_k > w-1)\,dw\right) \tag{3.7}$$

which can be rewritten as

$$\phi_{Z_k}(x) = e^{-ix(1+g/n)}\left(e^{ix(1+c)} - ix\int_0^{1+c} e^{ixw}\mathbb{Q}(R_k \le w-1)\,dw\right).$$

Merging (2.11) into the latter equation while noting that in (2.11) it holds $\xi > -1$, we finally end up with (3.4). □

If we insert (3.4) into (3.3), we eventually get a representation for the characteristic function $\phi_Z(x)$. Let us proceed with the computation of $\mathbb{E}_{\mathbb{Q}}[Z_k]$.

Proposition 3.3. *Suppose that $Z_k = \min\{c, R_k\} - g/n$ where $k \in \{1, \ldots, n\}$. Then the first moment of Z_k under $\mathbb{Q}$ is given by*

$$\mathbb{E}_{\mathbb{Q}}[Z_k] = c - \frac{g}{n} - e^{-\lambda\tau}\sum_{m=0}^{\infty}\frac{(\lambda\tau)^m}{m!}\int_{0+}^{1+c}\Phi\left(\frac{\ln(w) - \gamma\tau - m\mu}{\sqrt{\sigma^2\tau + m\delta^2}}\right)dw \tag{3.8}$$

where Φ denotes the standard normal distribution function.

Proof. In accordance to Proposition 2.4 in [4], we have

$$\mathbb{E}_{\mathbb{Q}}[Z_k] = \frac{1}{i}\frac{\partial}{\partial x}\big(\phi_{Z_k}(x)\big)\Big|_{x=0}. \tag{3.9}$$

A substitution of (3.7) into (3.9) yields

$$\mathbb{E}_{\mathbb{Q}}[Z_k] = c - \frac{g}{n} - \int_{0+}^{1+c}\mathbb{Q}(R_k \le w-1)\,dw.$$

We ultimately put (2.11) into the latter equation and receive (3.8). □

3.2 Cliquet option pricing with Fourier transform techniques

There is an alternative method to derive expressions for $\mathbb{E}_{\mathbb{Q}}[Z_k]$, $\phi_Z(x)$ and C_0 involving Fourier transforms and the Lévy–Khinchin formula. In the following, we present this method.

Proposition 3.4. *Suppose that $Z_k = \min\{c, R_k\} - g/n$ where $k \in \{1, \ldots, n\}$ and let $a > 0$ be a finite real-valued dampening parameter. Then the first moment of Z_k under $\mathbb{Q}$ can be represented as*

$$\mathbb{E}_{\mathbb{Q}}[Z_k] = c - \frac{g}{n} - \frac{1}{2\pi}\int_{\mathbb{R}}\frac{(c+1)^{1+a+iy}}{(a+iy)(1+a+iy)}\phi_{X_\tau}(ia-y)\,dy \tag{3.10}$$

where the characteristic function ϕ_{X_τ} is given by

$$\phi_{X_\tau}(ia-y) = e^{-\lambda\tau} \sum_{m=0}^{\infty} \frac{(\lambda\tau)^m}{m!} \times \exp\left\{(a+iy)\left(\frac{1}{2}\left(\sigma^2\tau + m\delta^2\right)(a+iy) - \gamma\tau - m\mu\right)\right\}. \tag{3.11}$$

Proof. First of all, verify that

$$\min\{c, R_k\} = -\max\{-c, -R_k\} = c - \max\{0, c - R_k\} = c - [c - R_k]^+ \tag{3.12}$$

which implies

$$\mathbb{E}_{\mathbb{Q}}[Z_k] = c - g/n - \mathbb{E}_{\mathbb{Q}}\left[(c - R_k)^+\right].$$

Hence, the evaluation of $\mathbb{E}_{\mathbb{Q}}[Z_k]$ is equivalent to the evaluation of a put option with underlying R_k and strike $c \geq 0$. Taking (2.10) into account, we receive

$$\mathbb{E}_{\mathbb{Q}}[Z_k] = c - g/n - \mathbb{E}_{\mathbb{Q}}\left[\left(c + 1 - e^{X_\tau}\right)^+\right]$$

where $\tau = t_k - t_{k-1}$ and X is the real-valued Lévy process introduced in (2.3). Furthermore, we define the function

$$\zeta(u) := e^{au}\left(c + 1 - e^u\right)^+$$

with a finite real-valued dampening parameter $a > 0$. Since $\zeta \in \mathcal{L}^1(\mathbb{R})$, its Fourier transform exists and reads as

$$\hat{\zeta}(y) = \frac{(c+1)^{1+a+iy}}{(a+iy)(1+a+iy)}.$$

Using the inverse Fourier transform along with Fubini's theorem, we get

$$\mathbb{E}_{\mathbb{Q}}\left[\left(c + 1 - e^{X_\tau}\right)^+\right] = \mathbb{E}_{\mathbb{Q}}\left[e^{-aX_\tau}\zeta(X_\tau)\right] = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{\zeta}(y)\mathbb{E}_{\mathbb{Q}}\left[e^{-(a+iy)X_\tau}\right] dy$$

which implies (3.10). What remains is the computation of the characteristic function ϕ_{X_τ}. It holds

$$\phi_{X_\tau}(ia-y) = \mathbb{E}_{\mathbb{Q}}\left[e^{-(a+iy)X_\tau}\right] = \int_{\mathbb{R}} e^{-(a+iy)x} f_{X_\tau}(x)\, dx,$$

such that (2.13) yields

$$\phi_{X_\tau}(ia-y) = e^{-\lambda\tau} \sum_{m=0}^{\infty} \frac{(\lambda\tau)^m}{m!} \int_{\mathbb{R}} e^{-(a+iy)x} \varphi_{\gamma\tau+m\mu,\sigma^2\tau+m\delta^2}(x)\, dx.$$

We finally perform the integration while noting that

$$\int_{\mathbb{R}} e^{bx}\varphi_{\mu,\sigma^2}(x)\, dx = \exp\left\{\mu b + \frac{1}{2}\sigma^2 b^2\right\} \tag{3.13}$$

(with arbitrary $b \in \mathbb{C}$, $\mu \in \mathbb{R}$, $\sigma \in \mathbb{R}^+$) and end up with (3.11). □

It is possible to derive an alternative representation for the characteristic function ϕ_{X_τ} by using (2.4), (2.5), Example 2.2 and the equality $(ia-y)^2 = -(a+iy)^2$. If we do so, we obtain

$$\phi_{X_\tau}(ia-y) = \exp\left\{\tau\left(-(a+iy)\gamma + \frac{1}{2}\sigma^2(a+iy)^2 + \lambda e^{(a+iy)[(a+iy)\delta^2/2-\mu]} - \lambda\right)\right\}$$

instead of (3.11). In contrast to (3.11), in the latter equation the series expansion has vanished.

Our argumentation in the proof of Proposition 3.4 motivates the following considerations.

Proposition 3.5. *Suppose that* $Z_k = \min\{c, R_k\} - g/n$ *with* $k \in \{1,\dots,n\}$ *and* $c \geq 0$. *Then the characteristic function of* Z_k *under* $\mathbb{Q}$ *reads as*

$$\phi_{Z_k}(x) = e^{-ixg/n}\left(e^{ixc} + \int_{-\infty}^{\ln(1+c)} \left[e^{ix(e^u-1)} - e^{ixc}\right] f_{X_\tau}(u)\, du\right) \tag{3.14}$$

where the density f_{X_τ} *of* X_τ *under* $\mathbb{Q}$ *is such as given in (2.9).*

Proof. By the definition of the characteristic function (recall (2.4)), we get

$$\phi_{Z_k}(x) = e^{-ixg/n}\mathbb{E}_{\mathbb{Q}}\left[e^{ix\min\{c,R_k\}}\right].$$

Taking (3.12) and (2.10) into account, the latter equation can be expressed as

$$\phi_{Z_k}(x) = e^{-ixg/n}\mathbb{E}_{\mathbb{Q}}\left[e^{ix(c-[1+c-e^{X_\tau}]^+)}\right] = e^{-ixg/n}\int_{\mathbb{R}} e^{ix(c-[1+c-e^u]^+)} f_{X_\tau}(u)\, du$$

where the density f_{X_τ} is such as given in (2.9). Next, verify that

$$\left[1+c-e^u\right]^+ = \left(1+c-e^u\right)\mathbb{1}_{u\leq \ln(1+c)}$$

(where $\mathbb{1}$ denotes the indicator function) which implies

$$\phi_{Z_k}(x) = e^{-ixg/n}\left(\int_{-\infty}^{\ln(1+c)} e^{ix(e^u-1)} f_{X_\tau}(u)\, du + e^{ixc}\int_{\ln(1+c)}^{\infty} f_{X_\tau}(u)\, du\right).$$

Since the last integral can be rewritten as

$$\int_{\ln(1+c)}^{\infty} f_{X_\tau}(u)\, du = 1 - \int_{-\infty}^{\ln(1+c)} f_{X_\tau}(u)\, du$$

we eventually obtain (3.14). □

There is an alternative method involving (3.9) to derive an expression for $\mathbb{E}_{\mathbb{Q}}[Z_k]$ which is presented in the following.

Corollary 3.6. *In the setup of Proposition 3.5, we receive the representation*

$$\mathbb{E}_{\mathbb{Q}}[Z_k] = c - \frac{g}{n} + e^{-\lambda\tau}\sum_{m=0}^{\infty}\frac{(\lambda\tau)^m}{m!}\left[\exp\left\{\left(\gamma + \frac{\sigma^2}{2}\right)\tau + \left(\mu + \frac{\delta^2}{2}\right)m\right\}\Phi\left(\kappa_1^m\right)\right.$$

$$- (1+c)\Phi\left(\kappa_2^m\right)\Big] \tag{3.15}$$

wherein Φ denotes the standard normal distribution function and

$$\kappa_1^m := \frac{\ln(1+c) - \gamma\tau - m\mu - \sigma^2\tau - m\delta^2}{\sqrt{\sigma^2\tau + m\delta^2}}, \qquad \kappa_2^m := \frac{\ln(1+c) - \gamma\tau - m\mu}{\sqrt{\sigma^2\tau + m\delta^2}}.$$

Proof. A substitution of (3.14) into (3.9) yields

$$\mathbb{E}_{\mathbb{Q}}[Z_k] = c - \frac{g}{n} + \int_{-\infty}^{\ln(1+c)} \left[e^u - 1 - c\right] f_{X_\tau}(u)\, du. \tag{3.16}$$

Taking (2.13) into account, we obtain the equalities

$$\int_{-\infty}^{\ln(1+c)} e^u f_{X_\tau}(u)\, du = e^{(\gamma-\lambda+\frac{\sigma^2}{2})\tau} \sum_{m=0}^{\infty} e^{(\mu+\frac{\delta^2}{2})m} \frac{(\lambda\tau)^m}{m!} \Phi\left(\kappa_1^m\right),$$

$$\int_{-\infty}^{\ln(1+c)} f_{X_\tau}(u)\, du = e^{-\lambda\tau} \sum_{m=0}^{\infty} \frac{(\lambda\tau)^m}{m!} \Phi\left(\kappa_2^m\right)$$

where the arguments κ_1^m and κ_2^m are such as defined in the sequel of (3.15). Putting the latter equations into (3.16), we ultimately get (3.15). □

Note that the expressions in (3.4), (3.8), (3.10), (3.14) and (3.15) altogether are independent of k. This is not a surprising observation, since $Z_1, \dots, Z_n$ are i.i.d. random variables and we have chosen equidistant resetting times with $\tau = t_k - t_{k-1}$.

Inspired by the Fourier transform techniques applied in the proof of Proposition 3.4, we now focus on the derivation of an alternative representation for the cliquet option price C_0 given in Eq. (3.1).

Theorem 3.7 (Fourier transform cliquet option price)**.** *Let $k \in \{1, \dots, n\}$ and consider the independent and identically distributed random variables $Z_k = \min\{c, R_k\} - g/n$ where $c \geq 0$ is the local cap, g is the guaranteed rate at maturity and R_k is the return process defined in (2.10). For arbitrary $n \in \mathbb{N}$ we set $\varrho := nc - g$ and denote the maturity time by T, the notional by K and the risk-less interest rate by r. Then the price at time zero of a cliquet option paying*

$$H_T = K\left(1 + g + \max\left\{0, \sum_{k=1}^{n} Z_k\right\}\right)$$

at maturity can be represented as

$$C_0 = Ke^{-rT}\Bigg[1 + g + \int_{0^+}^{\infty} \frac{1 + iy\varrho - e^{iy\varrho}}{2\pi y^2} \left(1 + \int_{-\infty}^{\ln(1+c)} \left[e^{iy(e^u-1-c)} - 1\right] f_{X_\tau}(u)\, du\right)^n dy\Bigg] \tag{3.17}$$

where $f_{X_\tau}(u)$ constitutes the probability density function claimed in (2.9).

Proof. Suppose that the cliquet option price C_0 is such as given in (3.1). We only need to evaluate the expectation

$$J := \mathbb{E}_{\mathbb{Q}}\left[\left(\sum_{k=1}^{n} Z_k\right)^+\right]$$

appearing in (3.1). Taking the definition of Z_k and (3.12) into account, we obtain

$$J = \mathbb{E}_{\mathbb{Q}}\left[\left(\varrho - \sum_{k=1}^{n}[c - R_k]^+\right)^+\right]$$

where $\varrho = nc - g$ is a constant. Note that in the latter equation we observe a basket-style composition of put options. Let us further introduce the function $\vartheta(x) := (\varrho - x)^+ \in \mathcal{L}^1(\mathbb{R}^+)$ as well as the non-negative random variable

$$D := \sum_{k=1}^{n}[c - R_k]^+$$

such that we may write

$$J = \mathbb{E}_{\mathbb{Q}}\big[\vartheta(D)\big].$$

An application of the inverse Fourier transform yields

$$J = \frac{1}{2\pi}\int_{\mathbb{R}^+} \hat{\vartheta}(y)\mathbb{E}_{\mathbb{Q}}\big[e^{-iyD}\big]\, dy$$

where

$$\hat{\vartheta}(y) = \frac{1 + iy\varrho - e^{iy\varrho}}{y^2}$$

constitutes the Fourier transform of ϑ. In the next step, we compute the characteristic function of D. Taking the definition of D and the $\mathbb{Q}$-independence of the random variables $R_1, \ldots, R_n$ into account, we deduce

$$\mathbb{E}_{\mathbb{Q}}\big[e^{-iyD}\big] = \prod_{k=1}^{n} \mathbb{E}_{\mathbb{Q}}\big[e^{-iy[c-R_k]^+}\big].$$

With respect to (2.10), we obtain

$$\mathbb{E}_{\mathbb{Q}}\big[e^{-iyD}\big] = \prod_{k=1}^{n} \mathbb{E}_{\mathbb{Q}}\big[e^{-iy[1+c-e^{X_\tau}]^+}\big] = \prod_{k=1}^{n} \int_{\mathbb{R}} e^{-iy[1+c-e^u]^+} f_{X_\tau}(u)\, du$$

where $\tau = t_k - t_{k-1}$ and $f_{X_\tau}(u)$ is such as given in (2.9). By a case distinction, we get

$$\mathbb{E}_{\mathbb{Q}}\big[e^{-iyD}\big] = \prod_{k=1}^{n}\left(\int_{-\infty}^{\ln(1+c)} e^{-iy(1+c-e^u)} f_{X_\tau}(u)\, du + \int_{\ln(1+c)}^{\infty} f_{X_\tau}(u)\, du\right)$$

$$= \prod_{k=1}^{n} \left(1 + \int_{-\infty}^{\ln(1+c)} \left[e^{iy(e^u-1-c)} - 1 \right] f_{X_\tau}(u)\, du \right).$$

Verify that the emerging integrand $[e^{iy(e^u-1-c)} - 1] f_{X_\tau}(u)$ is finite for all $u \in [-\infty, \ln(1+c)]$. Also note that the appearing factors altogether are independent of k. Combining the latter equations with (3.1), we finally get the asserted cliquet option price formula (3.17). □

We recall that Fourier transform techniques have also been applied in the context of cliquet option pricing in e.g. [9] and [11].

4 Hedging and Greeks

In this section, we are concerned with sensitivity analysis and the computation of Greeks in our cliquet option pricing context. Let us start with an investigation of the Greek *Rho* which is defined as the derivative of the cliquet option price C_0 with respect to the interest rate r. Due to (3.2), respectively (3.17), we find

$$\rho := \frac{\partial C_0}{\partial r} = -T C_0$$

where T denotes the maturity time of the option. Further on, both the *Delta* and *Gamma* of the cliquet option vanish, i.e.

$$\Delta := \frac{\partial C_0}{\partial S_0} = 0, \qquad \Gamma := \frac{\partial^2 C_0}{\partial S_0^2} = 0$$

since we assumed $t_0 \neq 0$ in (2.10) such that neither R_1 nor Z_1 contains S_0. In accordance to Section 3.2 in [3], we claim that in any cliquet option pricing context the most important Greek to study is the *Vega* which is defined as

$$V := \frac{\partial C_0}{\partial \sigma}$$

where $\sigma > 0$ denotes the volatility parameter of the Lévy process X defined in (2.3). In the Fourier transform framework studied in Section 3.2, we get the following result.

Proposition 4.1 (Vega; Fourier transform case)**.** *Presume that the density function f_{X_τ} is such as given in (2.13) and denote the density function of the normal distribution by φ. In the setup of Theorem 3.7, we then find the following expression for the Vega of the cliquet option*

$$V = \frac{nKe^{-rT}}{2\pi} \int_{0^+}^{\infty} \frac{1 + iy(nc-g) - e^{iy(nc-g)}}{y^2} F_y(\sigma)^{n-1} F_y'(\sigma)\, dy \tag{4.1}$$

where

$$F_y(\sigma) := 1 + \int_{-\infty}^{\ln(1+c)} \left[e^{iy(e^u-1-c)} - 1 \right] f_{X_\tau}(u)\, du,$$

$$F'_y(\sigma) = \sigma\tau e^{-\lambda\tau} \sum_{m=0}^{\infty} \frac{(\lambda\tau)^m}{m!(\sigma^2\tau + m\delta^2)^2} \int_{-\infty}^{\ln(1+c)} \varphi_{\gamma\tau+m\mu,\sigma^2\tau+m\delta^2}(u) G_y(u)\, du,$$

$$G_y(u) := \big[e^{iy(e^u-1-c)} - 1\big]\big[(u-\gamma\tau-m\mu)^2 - \sigma^2\tau - m\delta^2\big].$$

Proof. First of all, note that the only ingredient in (3.17) which contains the parameter σ is the density function f_{X_τ}. Thus, from (3.17) we deduce

$$V = \frac{nKe^{-rT}}{2\pi} \int_{0+}^{\infty} \frac{1 + iy\varrho - e^{iy\varrho}}{y^2} F_y(\sigma)^{n-1} F'_y(\sigma)\, dy$$

where $F_y(\sigma)$ is as defined in the proposition and the derivative $F'_y(\sigma) := \partial F_y(\sigma)/\partial\sigma$ reads as

$$F'_y(\sigma) = \int_{-\infty}^{\ln(1+c)} \big[e^{iy(e^u-1-c)} - 1\big] \frac{\partial f_{X_\tau}(u)}{\partial\sigma}\, du.$$

Taking (2.13) into account, we find

$$\frac{\partial f_{X_\tau}(u)}{\partial\sigma} = \sigma\tau e^{-\lambda\tau} \sum_{m=0}^{\infty} \frac{(\lambda\tau)^m}{m!} \frac{(u-\gamma\tau-m\mu)^2 - \sigma^2\tau - m\delta^2}{(\sigma^2\tau + m\delta^2)^2} \varphi_{\gamma\tau+m\mu,\sigma^2\tau+m\delta^2}(u).$$

Putting the latter equations together, we obtain (4.1) which completes the proof. □

In the distribution function context studied in Section 3.1, we find the following expression for the Vega.

Proposition 4.2 (Vega; distribution function case)**.** *Let us denote by $\varphi_{0,1} = \Phi'$ the probability density function of the standard normal distribution. Then, in the setup of Proposition 3.1, we get the following representation for the Vega*

$$V = n\tau\sigma K e^{-rT-\lambda\tau} \sum_{m=0}^{\infty} \frac{(\lambda\tau)^m}{m!} \times \int_{0+}^{1+c} \Psi(w) \left[\frac{1}{2} - \frac{1}{\pi} \int_{0+}^{\infty} \frac{\mathfrak{Re}(ie^{ix[w-1-g/n]}\phi_{Z_1}(x)^{n-1})}{x}\, dx\right] dw \tag{4.2}$$

where the characteristic function $\phi_{Z_1}(x)$ is such as given in (3.4) and

$$\Psi(w) := \varphi_{0,1}\left(\frac{\ln(w) - \gamma\tau - m\mu}{\sqrt{\sigma^2\tau + m\delta^2}}\right) \frac{\ln(w) - \gamma\tau - m\mu}{\sqrt{(\sigma^2\tau + m\delta^2)^3}}.$$

Proof. Taking (3.2) into account, we get

$$V = Ke^{-rT}\left(\frac{n}{2}\frac{\partial \mathbb{E}_{\mathbb{Q}}[Z_1]}{\partial\sigma} - \frac{1}{\pi}\int_{0+}^{\infty} \mathfrak{Re}\left(\frac{\partial\phi_Z(x)}{\partial\sigma}\right) x^{-2}\, dx\right).$$

Using (3.8) and the ordinary chain rule, we obtain

$$\frac{\partial \mathbb{E}_{\mathbb{Q}}[Z_1]}{\partial\sigma} = \tau\sigma e^{-\lambda\tau} \sum_{m=0}^{\infty} \frac{(\lambda\tau)^m}{m!}$$

$$\times \int_{0^+}^{1+c} \varphi_{0,1}\left(\frac{\ln(w) - \gamma\tau - m\mu}{\sqrt{\sigma^2\tau + m\delta^2}}\right) \frac{\ln(w) - \gamma\tau - m\mu}{\sqrt{(\sigma^2\tau + m\delta^2)^3}}\, dw$$

where $\varphi_{0,1} = \Phi'$ denotes the probability density function of the standard normal distribution. On the other hand, by (3.3) and (3.4) we deduce

$$\begin{aligned}\frac{\partial}{\partial\sigma}\phi_Z(x) &= n\phi_{Z_1}(x)^{n-1}\frac{\partial}{\partial\sigma}\phi_{Z_1}(x)\\ &= \tau\sigma nix\phi_{Z_1}(x)^{n-1}e^{-ix(1+g/n)-\lambda\tau}\sum_{m=0}^{\infty}\frac{(\lambda\tau)^m}{m!}\\ &\quad\times \int_{0^+}^{1+c} e^{ixw}\varphi_{0,1}\left(\frac{\ln(w) - \gamma\tau - m\mu}{\sqrt{\sigma^2\tau + m\delta^2}}\right) \frac{\ln(w) - \gamma\tau - m\mu}{\sqrt{(\sigma^2\tau + m\delta^2)^3}}\, dw.\end{aligned}$$

Putting the latter equations together, we end up with the asserted representation (4.2). □

5 Conclusions

In this paper, we investigated the pricing of a monthly sum cap style cliquet option with underlying stock price modeled by a jump-diffusion Lévy process with compound Poisson jumps. In Section 2, we derived representations for the probability density and distribution function of the involved Lévy process. Moreover, we obtained semi-analytic expressions for the cliquet option price by using the probability distribution function of the driving Lévy process in Section 3.1 and by an application of Fourier transform techniques in Section 3.2. With view on existing literature, the main contribution of the paper consists of the Fourier transform cliquet option price formula provided in Theorem 3.7. In Section 4, we concentrated on sensitivity analysis and computed the Greeks Rho, Delta, Gamma and Vega.

A future research topic might consist in a transformation of the presented techniques and results to a time-inhomogeneous Lévy process setup which, in particular, contains a time (and state) dependent volatility coefficient $\sigma(t)$, respectively $\sigma(t, X_t)$, in order to obtain more realistic (implied) volatility surfaces. In this context, we refer to Section 4 in [16] as well as Sections 1.2.1 and 11.1.2 in [4].

To read more on cliquet option pricing in a pure-jump Meixner–Lévy process setup, the reader is referred to the accompanying article [8].

References

[1] Ballotta, L.: A Lévy process-based framework for the fair valuation of participating life insurance contracts. Insur. Math. Econ. **37**(2), 173–196 (2005). MR2172097. https://doi.org/10.1016/j.insmatheco.2004.10.001

[2] Bernard, C., Boyle, P., Gornall, W.: Locally-capped contracts and the retail investor. J. Deriv. **18**(4), 72–88 (2011). https://doi.org/10.3905/jod.2011.18.4.072

[3] Bernard, C., Li, W.: Pricing and hedging of cliquet options and locally-capped contracts. SIAM J. Financ. Math. **4**, 353–371 (2013). MR3038023. https://doi.org/10.1137/100818157

[4] Cont, R., Tankov, P.: Financial Modeling with Jump Processes, 1st edn. Chapman & Hall/CRC, (2004). MR2042661

[5] Cui, Z., Kirkby, J., Nguyen, D.: Equity-linked annuity pricing with cliquet-style guarantees in regime-switching and stochastic volatility models with jumps. Insur. Math. Econ. **74**, 46–62 (2017). MR3648875. https://doi.org/10.1016/j.insmatheco.2017.02.010

[6] Di Nunno, G., Øksendal, B., Proske, F.: Malliavin Calculus for Lévy Processes with Applications to Finance, 1st edn. Springer, (2009). MR2460554. https://doi.org/10.1007/978-3-540-78572-9

[7] Haifeng, Y., Jianqi, Y., Limin, L.: Pricing cliquet options in jump-diffusion models. Stoch. Models **21**, 875–884 (2005). MR2179304. https://doi.org/10.1080/15326340500294587

[8] Hess, M.: Cliquet option pricing with Meixner processes. Mod. Stoch. Theory Appl. **5**(1), 81–97 (2018). MR3784039. https://doi.org/10.15559/18-VMSTA96. Preprint: https://ssrn.com/abstract=3016326.

[9] Hieber, P.: Cliquet-style return guarantees in a regime switching Lévy model. Insur. Math. Econ. **72**, 138–147 (2017). MR3600688. https://doi.org/10.1016/j.insmatheco.2016.11.009

[10] Iseger, den Oldenkamp E, P.: Cliquet options: Pricing and Greeks in deterministic and stochastic volatility models (2005). SSRN working paper. https://ssrn.com/abstract=1013510

[11] Kassberger, S., Kiesel, R., Liebmann, T.: Fair valuation of insurance contracts under Lévy process specifications. Insur. Math. Econ. **42**(1), 419–433 (2008). MR2392098. https://doi.org/10.1016/j.insmatheco.2007.04.007

[12] Merton, R.: Option pricing when underlying stock returns are discontinuous. J. Financ. Econ. **3**(1–2), 125–144 (1976). https://doi.org/10.1016/0304-405X(76)90022-2

[13] Protter, P.: Stochastic Integration and Differential Equations, 2nd edn. Springer, (2005). MR2273672. https://doi.org/10.1007/978-3-662-10061-5

[14] Sato, K.: Lévy Processes and Infinitely Divisible Distributions. Cambridge Studies in Advanced Mathematics, vol. 68 (1999). MR1739520

[15] Schoutens, W.: Lévy Processes in Finance: Pricing Financial Derivatives. John Wiley & Sons, Ltd., (2003)

[16] Wilmott, P.: Cliquet options and volatility models, Wilmott magazine (December 2002).