On the infinite divisibility of distributions of some inverse subordinators

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Abstract We consider the infinite divisibility of distributions of some well-known inverse subordinators. Using a tail probability bound, we establish that distributions of many of the inverse subordinators used in the literature are not infinitely divisible. We further show that the distribution of a renewal process time-changed by an inverse stable subordinator is not infinitely divisible, which in particular implies that the distribution of the fractional Poisson process is not infinitely divisible.

Keywords Infinite divisibility, subordinators, inverse subordinators, fractional Poisson process

1 Introduction

Infinitely divisible (ID) distributions were introduced by de Finetti in 1929. Ever since the research literature on these distributions is growing rapidly. A real-valued random variable X with a cumulative distribution function F is said to be ID if for each $n > 1$, there exist independent identically distributed random variables $X_1, X_2, \ldots, X_n$ with a distribution function F_n such that

$$X \stackrel{d}{=} X_1 + X_2 + \cdots X_n.$$

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Well-known examples of ID distributions are normal, Poisson, exponential, t, χ^2 and gamma distributions. Those that are not ID include half normal, discrete normal, inverse normal and inverse t distributions. ID distributions play a central role in the theory of Lévy processes. Note that every continuous-time Lévy process has distributions that are necessarily ID, and conversely every ID distribution generates uniquely a Lévy process (see Steutel and Van Harn, [23]). Further, in several real life situations some models require a random effect to be the sum of several independent random components with the same distribution. In such situations a convenient way is to assume infinite divisibility of the distribution of these random effects. Such situations occur in biology, economics and insurance. It is worth to mention here that to prove or disprove infinite divisibility of a certain distribution is sometimes a very tedious task and it may need an utterly specialized approach. In this article, we only talk about the infinite divisibility of distributions of some selected processes that are studied recently in the literature.

In recent years time-changed stochastic processes are getting increased attention due to their applications in finance, geophysics, fractional partial differential equations and in modeling the anomalous diffusion in statistical physics (see Janczura et al., [7]; Meerschaert et al., [12, 11]; Orsingher and Beghin, [18]). A time-changed stochastic process is obtained by changing the time of the process by another stochastic process. The processes that are used as time-change are generally subordinators, or inverse subordinators. Subordinators are non-decreasing Lévy processes i.e. processes with independent and stationary increments having non-decreasing sample paths. Well-known subordinators are the Poisson process, the compound Poisson processes, the gamma process, the inverse Gaussian process, an α-stable subordinator and a tempered α-stable subordinator. The first-passage time process of a subordinator is called an inverse subordinator. For example, the first-passage times of stable and tempered stable subordinators are called inverse stable and inverse tempered stable subordinators, respectively (see, e.g., Meerschaert and Straka, [15]; Kumar and Vellaisamy, [8]). The most popular inverse subordinator is the inverse α-stable subordinator (ISS). Note that ISS is used as a time-change in the standard Poisson process to define the fractional Poisson process (see, e.g., Meerschaert et al., [10]; Repin and Saichev, [19]; Laskin, [9]; Beghin and Orsingher, [2]). Further, ISS is used as a time-change with the Brownian motion and a stable process to solve fractional diffusion equations with a fractional derivative in time and fractional derivatives both in time and space, respectively (see, e.g., Meerschaert et al., [12]). The time-change with a subordinator $Y(t)$–$X(Y(t))$– is done in the Bochner sense and results in a Lévy process if the process $X(t)$ is a Lévy process.

In this article we study the infinite divisibility of the distribution of some inverse subordinators corresponding to drift-less subordinators. We first obtain a bound on the tail probability of these inverse subordinators. We establish that the distributions of inverse stable, inverse tempered stable and first-exit times of inverse Gaussian subordinators are not ID. Further, we also show that the distribution of a renewal process time-changed by ISS is not ID. In particular we establish that the distribution of the fractional Poisson process is not ID.

One should not conclude from these results that the distributions of inverse subordinators are not ID in general. One counter-example is the Poisson process. Let $N(t)$

be the Poisson process with rate λ. Then the process defined by $T_n = \inf\{t \geq 0 : N(t) \geq n\}$, $n = 1, 2 \cdots$ is called the inverse of the Poisson process. For a fixed n, the random variable T_n is an Erlang random variable of order n, with the probability density function

$$f_{T_n}(t) = \frac{\lambda e^{-\lambda t}(\lambda t)^{n-1}}{(n-1)!}, \quad n = 1, 2, \ldots$$

Note that the Erlang distribution is a special case of the gamma distribution and hence the inverse of the Poisson process $N(t)$ is ID (see, e.g., Steutel and Van Harn, [23]).

Further, the fractional Poisson process, for which applications are suggested in insurance (Biard and Saussereau, [4]), may not be appropriate in situations where one needs to divide the total number of claims in a year (say) in small intervals like months and days with independent identically distributed (i.i.d.) components due to its non-infinite divisibility.

ID distributions are at the heart of the theory of Lévy processes. Every continuous-time Lévy process has distributions that are necessarily ID (see, e.g., Steutel and Van Harn, [23]; Sato, [21]). It is well known in the literature that the inverse stable subordinator $E(t)$, $t \geq 0$, doesn't possess independent and stationary increments and hence is not a Lévy process (see Meerschaert and Scheffler, [13]). Our results conclude that it is not possible even to define a continuous time Lévy process corresponding to the distributions of $E(1)$.

2   Tail probability estimates of inverse subordinators

A subordinator is a one-dimensional Lévy process that is non-decreasing almost sure (a.s.). Such processes can be thought of as a random model of time evolution. If $T(t)$ is a subordinator, then we have

$$\mathbb{E}\left(e^{-uT(t)}\right) = e^{-t\psi(u)}, \tag{2.1}$$

where $\psi(u)$ is called the Laplace exponent and have the following form (see, e.g., Applebaum, [1], p. 53)

$$\psi(u) = bu + \int_0^\infty \left(1 - e^{-uy}\right)\nu(dy). \tag{2.2}$$

The pair (b, ν) is called characteristics of the subordinator T and represents the drift and the Lévy measure respectively. Here we require $\int_0^\infty (1 \wedge |y|)\nu(dy) < \infty$. In this article henceforth we only discuss subordinators with $b = 0$, also called driftless subordinators. For a subordinator $T(t)$, the first-exit time process is defined by

$$E(t) = \inf\{s \geq 0 : T(s) > t\}, \tag{2.3}$$

and we call this process the inverse subordinator. Note that

$$\mathbb{P}\left(E(t) > x\right) = \mathbb{P}\left(T(x) \leq t\right) = \mathbb{P}\left(-uT(x) \geq -ut\right), \quad u > 0$$

$$= \mathbb{P}\left(e^{-uT(x)} \geq e^{-ut}\right) \leq \frac{\mathbb{E}e^{-uT(x)}}{e^{-ut}} \quad \text{(by the Markov inequality)}$$

$$= e^{ut - x\psi(u)} \quad \text{(using (2.1))}. \tag{2.4}$$

Also note that for $b = 0$, $\psi'(u) = \int_0^\infty x e^{-ux} \nu(dx)$. Further, by the dominated convergence theorem $\psi'(u) \downarrow 0$ as $u \uparrow \infty$ and hence ψ' is invertible. Inequality (2.4) is true for all $u > 0$, and hence we can obtain a unique upper bound. It is reached at u such that

$$\frac{d}{du} \left[e^{ut - x\psi(u)} \right] = 0 \implies u = \psi'^{-1}(t/x).$$

Thus, we have the following proposition.

Proposition 2.1. *The tail probabilities for inverse subordinators satisfy*

$$\mathbb{P}\big(E(t) > x\big) \le e^{t\psi'^{-1}(t/x) - x\psi(\psi'^{-1}(t/x))}, \quad \text{for large } x. \tag{2.5}$$

3 Infinite divisibility of distributions of some inverse subordinators

To prove the non-infinite divisibility of inverse subordinators in this article, we use the tail bound (2.5) and a necessary condition for infinite divisibility which is mentioned here (see, e.g., Steutel, [22]): A necessary condition for a cumulative distribution function $F(x)$ to be ID is

$$-\log\big(1 - F(x)\big) \le ax \log x, \tag{3.1}$$

for some $a > 0$ and x sufficiently large.

Proposition 3.1 (Inverse stable subordinator). *Let $S_\alpha(t)$ be an α-stable subordinator with $\alpha \in (0, 1)$. Then the distribution of ISS defined by $E_\alpha(t) = \inf\{s \ge 0 : S_\alpha(s) > t\}$ is not ID.*

Proof. For an α-stable subordinator the Laplace exponent is given by $\psi(u) = u^\alpha$. Hence, we have $\psi'(u) = \alpha u^{\alpha-1}$, which implies $\psi'^{-1}(u) = (\frac{u}{\alpha})^{\frac{1}{\alpha-1}}$. Further, $\psi(\psi'^{-1}(u)) = (\frac{u}{\alpha})^{\frac{\alpha}{\alpha-1}}$. Thus for large x

$$\mathbb{P}\big(E_\alpha(t) > x\big) \le e^{t(\frac{t}{\alpha x})^{\frac{1}{\alpha-1}} - x(\frac{t}{\alpha x})^{\frac{\alpha}{\alpha-1}}}. \tag{3.2}$$

Further,

$$-\log \mathbb{P}\big(E_\alpha(t) > x\big) \ge x \left(\frac{t}{\alpha x}\right)^{\frac{\alpha}{\alpha-1}} - t \left(\frac{t}{\alpha x}\right)^{\frac{1}{\alpha-1}}$$

$$= (1 - \alpha) \left(\frac{\alpha}{t}\right)^{\frac{\alpha}{1-\alpha}} x^{\frac{1}{1-\alpha}} = d(\alpha, t) x^{\frac{1}{1-\alpha}} \quad \text{(say)}, \tag{3.3}$$

where $d(\alpha, t) = (1 - \alpha)(\frac{\alpha}{t})^{\alpha/(1-\alpha)} > 0$. We have,

$$\lim_{x \to \infty} \frac{-\log \mathbb{P}(E_\alpha(t) > x)}{x \log x} \ge \lim_{x \to \infty} \frac{d(\alpha, t) x^{\frac{1}{1-\alpha}}}{x \log x}$$

$$= \lim_{x \to \infty} \frac{d(\alpha, t) x^{\frac{\alpha}{1-\alpha}}}{\log x} \quad \left(\text{indeterminate } \tfrac{\infty}{\infty} \text{ form} \right)$$

$$= \lim_{x \to \infty} d(\alpha, t) \frac{\alpha}{(1-\alpha)} x^{\frac{\alpha}{1-\alpha}} = \infty.$$

Hence, a finite $a > 0$ that satisfies equation (3.1) does not exist. Therefore the distribution of $E_\alpha(t)$ is not ID. □

Remark 3.1. It is worthwhile to mention the results about $E_\alpha(t)$ from Meerschaert and Scheffler [13]. They showed that the increments of $E_\alpha(t)$ are neither stationary nor independent.

Next we prove the non-infinite divisibility of distributions of inverse tempered stable subordinators (ITSS). Tempered stable subordinators (TSS) are obtained by exponential tempering in distributions of stable subordinators (see, e.g., Rosiński, [20]). TSS have ID distributions, have exponentially decaying tail probabilities and have all moments finite, unlike stable subordinators for which tail probabilities decay polynomially and first moments are infinite. These properties of TSS are derived from their self-similarity. Let $S_{\alpha,\lambda}(t)$ be the TSS with index $\alpha \in (0, 1)$ and tempering parameter $\lambda > 0$. The Laplace transform (LT) of the density of TSS (see Meerschaert et al., [11]) is

$$\mathbb{E}\left(e^{-u S_{\alpha,\lambda}(t)} \right) = e^{-t((u+\lambda)^\alpha - \lambda^\alpha)}. \tag{3.4}$$

TSS are also known as relativistic stable subordinators.

Proposition 3.2 (ITSS). *The distributions of ITSS defined by* $E_{\alpha,\lambda}(t) = \inf\{s \geq 0 : S_{\alpha,\lambda}(s) > t\}$ *are not ID.*

Proof. The Laplace exponent for ITSS is given by $\psi(u) = (u + \lambda)^\alpha - \lambda^\alpha$. This implies $\psi'^{-1}(u) = (\tfrac{u}{\alpha})^{\frac{1}{\alpha-1}} - \lambda$, $\psi(\psi'^{-1}(u)) = (\tfrac{u}{\alpha})^{\frac{\alpha}{\alpha-1}} - \lambda^\alpha$. Thus

$$\mathbb{P}\left(E_{\alpha,\lambda}(t) > x \right) \leq e^{-\lambda t + t(\frac{t}{\alpha x})^{\frac{1}{\alpha-1}} - x(\frac{t}{\alpha x})^{\frac{\alpha}{\alpha-1}} + \lambda^\alpha x}. \tag{3.5}$$

Hence,

$$-\log \mathbb{P}\left(E_{\alpha,\lambda}(t) > x \right) \geq \lambda t + x \left(\frac{t}{\alpha x} \right)^{\frac{\alpha}{\alpha-1}} - t \left(\frac{t}{\alpha x} \right)^{\frac{1}{\alpha-1}} - \lambda^\alpha x$$

$$= \lambda t - \lambda^\alpha x + (1-\alpha) \left(\frac{\alpha}{t} \right)^{\frac{\alpha}{1-\alpha}} x^{\frac{1}{1-\alpha}} \tag{3.6}$$

$$= \lambda t - \lambda^\alpha x + d(\alpha, t) x^{\frac{1}{1-\alpha}} \quad \text{(say)}, \tag{3.7}$$

It follows that

$$\lim_{x \to \infty} \frac{-\log \mathbb{P}(E_{\alpha,\lambda}(t) > x)}{x \log x} = \infty. \tag{3.8}$$

Using the same argument as in Proposition 3.1, we conclude that distributions of ITSS are not ID. □

Next we discuss the non-infinite divisibility of the distribution of inverse of an inverse Gaussian subordinator. It is worth to mention that an inverse Gaussian subordinator is a particular case of TSS. Let $G(t)$ be an inverse Gaussian subordinator with parameters δ and γ, then its density function is given by

$$f_{G(t)}(y) = \frac{\delta t}{2\pi} e^{\delta \gamma t} y^{-3/2} e^{-\frac{1}{2}(\frac{\delta^2 t^2}{y} + \gamma^2 y)}. \tag{3.9}$$

Further, the Laplace exponent for $G(t)$ is given by $\psi(u) = \delta(\sqrt{2u + \gamma^2} - \gamma)$ (see Applebaum, [1], p. 54). Let $H(t) = \inf\{s \geq 0 : G(s) > t\}$ be the first-passage time process. Using (2.5), it follows

$$\mathbb{P}(H(t) > x) \leq e^{-\frac{\delta^2 x^2}{t} + \delta \gamma x - \frac{\gamma^2 t}{2}}. \tag{3.10}$$

Using the similar argument as earlier, we have the following result.

Proposition 3.3. *The distribution of the first-passage time process $H(t)$ is not ID.*

Remark 3.2. Note that when $\gamma = 0$, the distribution of $H(t)$ is folded Gaussian, which is not ID; the latter is a known result (see, e.g., Steutel and Van Harn, [23], p. 126).

Remark 3.3. A proof of non-infinite divisibility of distribution of $H(t)$ is discussed in Vellaisamy and Kumar [24], where the tail probabilities' bound is obtained by using different techniques.

Next, we discuss the tail probabilities for gamma subordinators. Let $U(t)$ be the gamma subordinator with parameters $a, b > 0$, having the density function

$$f_{U(t)}(x) = \frac{b^{at}}{\Gamma(at)} x^{at-1} e^{-bx}, \quad x > 0. \tag{3.11}$$

The Laplace exponent for the gamma subordinator is given by $\psi(u) = a \log(1 + \frac{u}{b})$ (see Applebaum, [1], p. 55), which implies $\psi'^{-1}(u) = \frac{a-bu}{u}$. Let $V(t)$ be the first-passage time of $U(t)$, then using (2.5)

$$\mathbb{P}(V(t) > x) \leq \left(\frac{bt}{ax}\right)^{ax} e^{ax-bt}, \quad \text{for large } x. \tag{3.12}$$

It follows that $\lim_{x \to \infty} \frac{-\log \mathbb{P}(V(t)>x)}{x \log x} \geq a$ and hence unlike Proposition 3.1 there is no obvious contradiction. So, we can't say anything about the infinite divisibility of first-exit times of gamma subordinators. In this article, we are not able to conclude whether inverse of a gamma subordinator has ID marginals or not. It is worth to mention that inverse Gaussian distributions or, more generally, generalized inverse Gaussian (GIG) distributions are generalized gamma convolutions (Halgreen, [6]) and hence are ID. The inverse gamma subordinator and the first-exit times of a gamma subordinator are different processes. The density of an inverse gamma subordinator is a particular case of GIG densities which are ID.

Next we discuss some transformed processes of the inverse subordinators. Consider the transformed ISS $E(t)^p$, $p > 0$. We have

$$\mathbb{P}(E(t)^p > x) = \mathbb{P}(E(t) > x^{1/p})$$

$$= \mathbb{P}\big(T\big(x^{1/p}\big) < t\big) = \mathbb{P}\big(e^{-uT(x^{1/p})} \geq e^{-ut}\big)$$

$$\leq e^{ut - x^{1/p}\psi(u)}, \quad u > 0$$

$$\leq e^{t\psi'^{-1}(t/x^{1/p}) - x^{1/p}\psi(\psi'^{-1}(t/x^{1/p}))}, \quad \text{for large } x. \tag{3.13}$$

Using (3.13) and the similar argument as in Propositions 3.1 and 3.2, we have the following result.

Proposition 3.4. *The transformed ISS $E_\alpha(t)^p$ and transformed ITSS $E_{\alpha,\lambda}(t)^p$ do not have ID distributions for $p < 1/(1 - \alpha)$. Transformed first-passage times of inverse Gaussian subordinators defined by $H(t)^p$ do not have ID distributions for $p < 2$. Further, transformed first-passage times of gamma subordinators defined by $V(t)^p$ do not have infinitely divisible distributions for $p < 1$.*

Proof. We here provide the proof for an inverse gamma subordinator only. Proofs for other subordinators follow similarly. Note that

$$\mathbb{P}\big(V(t)^p > x\big) \leq \left(\frac{bt}{ax^{1/p}}\right)^{ax^{1/p}} e^{ax^{1/p} - bt}, \quad \text{for large } x.$$

Thus $-\log \mathbb{P}(V(t)^p > x) \geq ax^{1/p}\log(a) + \frac{a}{p}x^{1/p}\log x + bt - ax^{1/p}\log(bt) - ax^{1/p}$, which implies

$$\lim_{x \to \infty} \frac{-\log \mathbb{P}(V(t) > x)}{x \log x} = \infty, \quad \text{for } 0 < p < 1, \tag{3.14}$$

and hence by the necessary condition (3.1), $V(t)^p$ does not have ID distribution for $0 < p < 1$. $\qquad\square$

4   Compositions of ISS

We can easily show that $E_\alpha(t)$ is self-similar with self-similarity index α. Note that

$$\mathbb{P}\big(E_\alpha(ct) \leq x\big) = \mathbb{P}\big(S_\alpha(x) \geq ct\big) = \mathbb{P}\left(\frac{1}{c}S_\alpha(x) \geq t\right)$$

$$= \mathbb{P}\big(S_\alpha(x/c^\alpha) \geq t\big) = \mathbb{P}\left(E_\alpha(t) \leq \frac{x}{c^\alpha}\right)$$

$$= \mathbb{P}\big(c^\alpha E_\alpha(t) \leq x\big),$$

and hence $E_\alpha(ct) \overset{d}{=} c^\alpha E(t)$.

For a strictly increasing subordinator $T(t)$ with the Laplace exponent $\psi(u)$, the density function $q(x, t)$ of the inverse subordinator has the LT with respect to time variable (see Meerschaert and Scheffler, [14])

$$\mathcal{L}_t\big(q(x, t)\big) = \int_0^\infty e^{-st} q(x, t)dt = \frac{1}{s}\psi(s)e^{-x\psi(s)}.$$

Let $g(x, t)$ be the density function of the ISS $E_\alpha(t)$. Then $\mathcal{L}_t(g(x, t)) = s^{\alpha-1}e^{-xs^\alpha}$. Let $E^*(t) = E_{\alpha_1}(E_{\alpha_2}(t))$ represent the composition of two independent inverse stable subordinators. Further, let $h(x, t)$ be the density function of $E^*(t)$ and let $h_1(x, t)$ and $h_2(x, t)$ be the density functions of $E_{\alpha_1}(t)$ and $E_{\alpha_2}(t)$ respectively. Then

$$h(x, t) = \int_0^\infty h_1(x, r)h_2(r, t)dr.$$

Thus

$$\begin{aligned}
\mathcal{L}_t\big(h(x, t)\big) &= \int_0^\infty h_1(x, r)\mathcal{L}_t\big(h_2(r, t)\big)dr \\
&= \int_0^\infty h_1(x, r)s^{\alpha_2-1}e^{-rs^{\alpha_2}}dr \\
&= s^{\alpha_2-1}\int_0^\infty h_1(x, r)e^{-rs^{\alpha_2}}dr \\
&= s^{\alpha_2-1}\big(s^{\alpha_2}\big)^{\alpha_1-1}e^{-xs^{\alpha_1\alpha_2}} \\
&= s^{\alpha_1\alpha_2-1}e^{-xs^{\alpha_1\alpha_2}}.
\end{aligned}$$

Hence $E^*(t) = E_{\alpha_1} \circ E_{\alpha_2}(t) = E_{\alpha_1}(E_{\alpha_2}(t))$ is the same in distribution as an ISS of index $\alpha_1\alpha_2$. In general, let $E_{\alpha_1}(t), E_{\alpha_2}(t), \ldots, E_{\alpha_n}(t)$ be independent ISS with indices $\alpha_1, \alpha_2, \ldots, \alpha_n$ respectively. Then the process defined by the composition $E^*(t) = E_{\alpha_1} \circ E_{\alpha_2} \circ \cdots \circ E_{\alpha_n}(t)$ is the same in distribution as an ISS with index $\alpha_1\alpha_2 \cdots \alpha_n$. Further, the distribution of the process $E^*(t)$ is not infinitely divisible. Next, we prove the non-infinite divisibility of the distribution of a time-changed ISS where the time-change is a general subordinator.

Remark 4.1. Nane [17] has considered the composition of independent inverse stable subordinators of index $\alpha = 1/2$. He observed that for fixed $t \geq 0$, the k-iterated Brownian motion

$$|I_k(t)| = \big|B_1\big(\big|B_2\big(\big|\cdots\big(|B_k(t)|\big)\cdots\big|\big)\big|\big)\big|$$

and $E^{1/2^k}(t) = E_{1/2} \circ E_{1/2}(t) \cdots \circ E_{1/2}(t)$ have the same one-dimensional distributions.

Proposition 4.1. *Let $T(t)$ be a general subordinator with finite mean i.e. a positive Lévy process with non-decreasing sample paths having $\mathbb{E}(T(1)) < \infty$. Then the time-changed process $E_\alpha(T(t))$ does not have ID distributions.*

Proof. By self-similarity of $E_\alpha(t)$ we have

$$\frac{E_\alpha(T(t))}{t^\alpha} \stackrel{d}{=} \left(\frac{T(t)}{t}\right)^\alpha E_\alpha(1)$$

$$\stackrel{a.s.}{\longrightarrow} \big(\mathbb{E}(T(1))\big)^\alpha E_\alpha(1),$$

as $t \to \infty$. Here we have used the fact that for a subordinator $T(t)/t \to \mathbb{E}(T(1))$ a.s. as $t \to \infty$ (see, e.g., Bertoin, [3], p. 92). Since the a.s. convergence implies the convergence in distribution (see, e.g., Chung, [5]), it follows $\frac{E_\alpha(T(t))}{t^\alpha} \stackrel{d}{\to} (\mathbb{E}(T(1)))^\alpha E_\alpha(1)$.

Assume that $E_\alpha(T(t))$ has an ID distribution, then $E_\alpha(T(t))/t^\alpha$ will also have an ID distribution for each t (see, e.g., Steutel and Van Harn, [23], Prop. 2.1, p. 94). Next we recall Prop. 2.2 from Steutel and Van Harn ([23], p. 94): If a sequence of $\mathbb{R}_+$-valued random variables $X_n, n \geq 0$ with ID distributions converges in distribution to X, then X has ID distribution. Hence, the limit in distribution i.e. $(\mathbb{E}(T(1)))^\alpha E_\alpha(1)$ will also have an ID distribution or, equivalently, $E_\alpha(1)$ will also have an ID distribution. This is a contradiction. $\qquad\square$

Corollary 4.1. *Let $S_{\alpha,\lambda}(t)$, $G(t)$ and $U(t)$ be tempered stable, inverse Gaussian and gamma subordinators. Then the distributions of time-changed processes $E_\alpha(S_{\alpha,\lambda}(t))$, $E_\alpha(G(t))$ and $E_\alpha(U(t))$ are not infinitely divisible. Further, $E^*(S_{\alpha,\lambda}(t))$, $E^*(G(t))$ and $E^*(U(t))$ also do not have ID distributions, where E^* is the composition of n independent inverse stable subordinators.*

5 Time-changed renewal processes

Let W_i, $i = 1, 2, \ldots$ be a sequence of i.i.d. a.s. positive random variables. Then the random walk $T_0 = 0$, $T_n = W_1 + \cdots + W_n, n \geq 1$, is said to be a renewal sequence and the counting process $N(t) = \max\{i : T_i \leq t\}$ is called the corresponding renewal process (see, e.g., Mikosch, [16], p. 59). We have the following result for the time-changed renewal process.

Proposition 5.1. *Let $N(t)$ be a renewal process with finite expectations of the inter-arrival times W_i, $\mathbb{E}W_1 = \lambda^{-1}$. The renewal process time-changed by an ISS defined by $N(E_\alpha(t))$ does not have ID distribution.*

Proof. Note that $E_\alpha(1) > 0$ a.s. Hence, as $t \to \infty$, it follows $t^\alpha E_\alpha(1) \to \infty$ a.s. By an application of the renewal theorem, we have

$$\lim_{t\to\infty} \frac{N(t^\alpha E_\alpha(1))}{t^\alpha} = \lim_{t\to\infty} \frac{N(t^\alpha E_\alpha(1))}{t^\alpha E_\alpha(1)} E_\alpha(1)$$
$$\xrightarrow{a.s.} \lambda E_\alpha(1).$$

Further due to self-similarity of $E_\alpha(t)$, it follows that $N(E_\alpha(t)) \overset{d}{=} N(t^\alpha E_\alpha(1))$ and hence $N(E_\alpha(t))/t^\alpha \overset{d}{\to} \lambda E_\alpha(1)$. Suppose that $N(E_\alpha(t))$ has infinitely divisible distribution then $N(E_\alpha(t))/t^\alpha$ will also have an ID distribution (see, e.g., Steutel and Van Harn, [23], Prop. 2.1, p. 94). Since the limit of a sequence of random variables with ID distributions has an infinitely divisible distribution (see, e.g., Sato [21]; Steutel and Van Harn, [23]), we have that $E_\alpha(1)$ has an ID distribution, and hence contradiction. $\qquad\square$

Meerschaert et al. [12] establish that the fractional Poisson process introduced by Laskin [9] can be obtained from the standard Poisson process by the time-change with an ISS. Since the Poisson process is a renewal process, we have the following result.

Corollary 5.1. *Let $M(t)$ be the standard Poisson process. Then the fractional Poisson process which is defined by $M^*(t) = M(E_\alpha(t))$ does not have an ID distribution. Further, since $M^*(t)$ is also a renewal process with Mittag-Leffler waiting times, the time-changed process defined by $M^{**}(t) = M^*(E_\beta(t))$ does not have an ID distribution.*

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