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Stochastic models associated to a Nonlocal Porous Medium Equation

Alessandro De Gregorio

Dipartimento di Scienze Statistiche, "Sapienza" University of Rome, P.le Aldo Moro, 5 - 00185, Rome, Italy

alessandro.degregorio@uniroma1.it (A. De Gregorio)

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Abstract The nonlocal porous medium equation considered in this paper is a degenerate nonlinear evolution equation involving a space pseudo-differential operator of fractional order. This space-fractional equation admits an explicit, nonnegative, compactly supported weak solution representing a probability density function. In this paper we analyze the link between isotropic transport processes, or random flights, and the nonlocal porous medium equation. In particular, we focus our attention on the interpretation of the weak solution of the nonlinear diffusion equation by means of random flights.

Keywords Anomalous diffusions, finite speed of propagation, fractional gradient, random flights

1 Introduction

We deal with a Nonlocal Porous Medium Equation (NPME) studied in [3, 4], given by the following degenerate nonlinear and nonlocal evolution equation

$$\partial_t u = \operatorname{div}\big(|u|\nabla^{\alpha-1}\big(|u|^{m-2}u\big)\big), \quad m > 1,\ \alpha \in (0, 2],\ t > 0, \tag{1}$$

subject to the initial condition

$$u(x, 0) = u_0(x), \tag{2}$$

where $u := u(x, t)$, with $x := (x_1, \dots, x_d) \in \mathbb{R}^d$, $d \geq 1$, is a scalar function defined on $\mathbb{R}^d \times \mathbb{R}^+$ and $\partial_t := \partial/\partial t$. The pseudo-differential operator $\nabla^{\alpha-1}$ is the fractional gradient denoting the nonlocal operator defined as $\nabla^{\alpha-1}u := \mathcal{F}^{-1}(i\xi||\xi||^{\alpha-2}\mathcal{F}u)$, where the Fourier transform $\mathcal{F}$ and the inverse transform $\mathcal{F}^{-1}$ of a function $v \in L^1(\mathbb{R}^d)$ are defined by

$$\mathcal{F}v(\xi) = \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} v(\xi)e^{ix\cdot\xi}\mathrm{d}x, \quad \mathcal{F}^{-1}v(\xi) = \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} v(\xi)e^{-ix\cdot\xi}\mathrm{d}x,$$

with $\xi \in \mathbb{R}^d$. This notation highlights that $\nabla^{\alpha-1}$ is a pseudo-differential (vector-valued) operator of order $\alpha - 1$. Equivalently, we can define $\nabla^{\alpha-1}$ as $\nabla(-\Delta)^{\frac{\alpha}{2}-1}$, where $(-\Delta)^{\frac{\alpha}{2}}u = \mathcal{F}^{-1}(||\xi||^{\alpha}\mathcal{F}u)$ is the fractional Laplace operator; i.e. a Fourier multiplier with the symbol $||\xi||^{\alpha}$. We observe that $\nabla^1 = \nabla$ is the classical gradient and that $\mathrm{div}(\nabla^{\alpha-1}) = \nabla^{\frac{\alpha}{2}} \cdot \nabla^{\frac{\alpha}{2}} = -(-\Delta)^{\frac{\alpha}{2}}$. Another equivalent definition of the fractional gradient $\nabla^{\alpha-1}$ involves the Riesz potential; that is $\nabla^{\alpha-1} = \nabla I_{2-\alpha}$ where $I_\beta = (-\Delta)^{-\frac{\beta}{2}}$ is a Fourier multiplier with symbol $||\xi||^{-\beta}$, $\beta \in (0, 2)$ (for more details on this point see [3, 4]).

In [3, 4], explicit and compactly supported nonnegative self-similar solutions of (1) are constructed. These explicit solutions generalize the well-known Barenblatt–Kompanets–Zel'dovich–Pattle solutions of the porous medium equation (4) below. Furthermore, the authors proved the existence of sign-changing weak solution to the Cauchy problem (1)–(2) for $u_0(x) \in L^1(\mathbb{R}^d)$, and the hypercontractivity $L^1 \mapsto L^p$ estimates.

By exploiting Darcy's law, it is possible to interpret the equation (1) as a transport equation $\partial_t u = \mathrm{div}(|u|\mathbf{v})$, where $\mathbf{v} := \nabla\mathbf{p} := \nabla I_{2-\alpha}(|u|^{m-2}u)$ is a vector velocity field with nonlocal and nonlinear pressure $\mathbf{p}$ in the case of nonnegative initial data. We observe that the fractional operator $\nabla I_{2-\alpha}$ represents the long-range diffusion effects. The one-dimensional version of the pseudo-differential equation (1) describes the dynamics of dislocations in crystals (see [2]).

For $\alpha = 2$, (1) becomes the classical nonlinear porous medium equation

$$\partial_t u = \mathrm{div}\big(|u|\nabla\big(|u|^{m-2}u\big)\big) = \mathrm{div}\big((m-1)|u|^{m-1}\nabla u\big). \tag{3}$$

If we restrict our attention to nonnegative solution $u(x, t)$, the equation (3) becomes

$$\partial_t u = \frac{m-1}{m}\Delta\big(u^m\big), \tag{4}$$

which is usually adopted to model the flow of a gas through a porous medium. The reader interested in the theory of porous medium equation can consult, for instance, [33].

Other types of nonlocal porous medium equations have been proposed in literature. For instance, [5, 6] introduced the porous medium equation with fractional diffusion effects

$$\partial_t u = \mathrm{div}(u\nabla p), \tag{5}$$

with nonlocal pressure $p := (-\Delta)^{-s}u$, $0 < s < 1$, and $u \geq 0$. For $\alpha = 2 - 2s \in (0, 2)$ we obtain the equation (1) with $m = 2$; i.e. $\partial_t u = \mathrm{div}(u\nabla^{\alpha-1}u)$. In [32] the nonlinear diffusion equation (5) is generalized as follows

$$\partial_t u = \mathrm{div}\big(u^{m-1}\nabla p\big), \quad u(x, t) \geq 0, x \in \mathbb{R}^d, t > 0, \tag{6}$$

with $m > 1$ and initial condition $u(x, 0) = u_0(x)$ which is nonnegative bounded with compact support or fast decaying at infinity. The main contribution in [32] concerns the study of the property of finite/infinite speed of propagation of the solutions to (6) with varying m.

The following equation

$$\partial_t u = -(-\Delta)^{\frac{\alpha}{2}}\left(|u|^{m-1}u\right), \quad \alpha \in (0, 2), \tag{7}$$

is studied in [34], where it is also proved that the self-similar solutions of (7) enjoy the L^1-contraction property and then they are unique. Nevertheless, these solutions are not compactly supported. Explicit self-similar solutions to (6) and (7) have been obtained by [20] for particular values of m.

The main goal of this paper is to investigate the relationship between (1) and some random models. In particular, we focus our attention to the probabilistic interpretations of the weak solution to NPME. The idea to study stochastic processes associated to the classical porous medium equation (4) was developed by different authors; see, for instance, [21–23, 12, 13, 24, 29]. In the listed papers the authors introduced different types of Markov chains on lattice and interacting particle systems having a dynamic which macroscopically converges to the solution of (4). By [17], the Barenblatt solution of (4) can be viewed as the mean of the first passage time of a symmetric stable process to exterior of a ball. In [1], the authors provided a probabilistic interpretation of (4) in terms of stochastic differential equations. Recently, [11] highlighted the connection between (4) and the Euler–Poisson–Darboux equations by taking into account time-rescaled random flights.

Up to our knowledge, this paper is the first attempt concerning the probabilistic interpretation of the fractional porous medium equation (1). Similarly to [11], we can exploit stochastic models defined by continuous-time random walks in $\mathbb{R}^d$, $d \geq 1$, arising in the description of the displacements of a particle choosing uniformly its directions; i.e. the so-called isotropic transport processes or random flights. In a suitable time-rescaled frame, the probability law of the above processes is given by the solution (8) below. Therefore, this paper represents a generalizations of some results contained in [11]. We point out that the proposed random processes recover some features of the Barenblatt weak solution (8) to nonlinear evolution equations like finite speed of propagation and the anomalous diffusivity. For this reason the random flights seem to represent a natural way to describe the real phenomena studied by means of (1).

In Section 2, we recall the definition of weak solution to (1) as well as its basic properties. In Section 3 the isotropic transport processes are introduced. Furthermore, Section 3 contains our main results; i.e. Propositions 3.1, 3.2. From these propositions we are able to give a reasonable interpretation of the solutions to (1). In the last section we sum up the main contribution of the paper.

2 A review on the weak solutions to NPME

Let us recall the definition of weak solution to the nonlocal operator equation (1) and its main properties (see [4]).

Definition 2.1. *A function* $u : \mathbb{R}^d \times (0,T) \to \mathbb{R}$ *is a weak solution to the Cauchy problem* (1)–(2) *in* $\mathbb{R}^d \times (0,T)$ *if* $u \in L^1(\mathbb{R}^d \times (0,T))$, $\nabla^{\alpha-1}(|u|^{m-2}u) \in L^1_{loc}(\mathbb{R}^d \times (0,T))$, $|u|\nabla^{\alpha-1}(|u|^{m-2}u) \in L^1_{loc}(\mathbb{R}^d \times (0,T))$ *and*

$$\int_{\mathbb{R}^d}\int_0^T \big(u\partial_t\varphi - |u|\nabla^{\alpha-1}\big(|u|^{m-2}u\big)\cdot\nabla\varphi\big)\mathrm{d}t\mathrm{d}x + \int_{\mathbb{R}^d} u_0(x)\varphi(x,0)\mathrm{d}x = 0,$$

where $\varphi \in C^\infty(\mathbb{R}^d \times (0,T)) \cap C^1(\overline{\mathbb{R}^d \times (0,T)})$ *has a compact support in the space variable* x *and vanishes near* $t = T$.

Let $(x)_+ := \max(x,0)$. The following theorem, proved in [4], represents our starting point.

Theorem 2.1. *Let* $\alpha \in (0,2]$ *and* $m > 1$. *A weak solution in the sense of Definition 2.1 in* $(\eta, T) \times \mathbb{R}^d$, *for every* $0 < \eta < T < \infty$, *is given by the function* $u : \mathbb{R}^d \times (0,T) \to [0,\infty)$ *defined as*

$$u(x,t) = Ct^{-d\beta}\left(1 - k^{\frac{2}{\alpha}}\frac{||x||^2}{t^{2\beta}}\right)_+^{\frac{\alpha}{2(m-1)}}, \tag{8}$$

where $\beta := \beta(\alpha,d,m) := \frac{1}{d(m-1)+\alpha}$,

$$k := k(\alpha,d) := \frac{d\Gamma(d/2)}{(d(m-1)+\alpha)2^\alpha\Gamma(1+\frac{\alpha}{2})\Gamma(\frac{d+\alpha}{2})},$$

$$C := C(\alpha,d,m) := \frac{\Gamma(\frac{d}{2}+\frac{\alpha}{2(m-1)}+1)k^{\frac{d}{\alpha}}}{\pi^{\frac{d}{2}}\Gamma(\frac{\alpha}{2(m-1)}+1)}.$$

Furthermore, $u(x,t)$ *is the pointwise solution of the equation* (1) *for* $||x|| \neq \mathrm{c}t^\beta$ *and is* $\min\{\frac{\alpha}{m-1}, 1\}$*-Hölder continuos at* $||x|| = \mathrm{c}t^\beta$, *where* $\mathrm{c} := \mathrm{c}(\alpha,d) := 1/k^{\frac{1}{\alpha}}$.

It is worth to mention that the family of functions (8) represents a class of non-negative compactly supported solutions of (1). Moreover, (8) is a self-similar solution under a suitable space-time rescaling; i.e.

$$u(x,t) = L^{d\beta}u\big(L^\beta x, Lt\big), \quad L > 0.$$

It is crucial to observe that the constant C appearing in (8) guarantees the mass conservation

$$\int_{\mathbb{R}^d} u(x,t)\mathrm{d}x = \int_{\mathbb{R}^d} u_0(x)\mathrm{d}x = 1,$$

or equivalently

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} u(x,t)\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} u_0(x)\mathrm{d}x = 0,$$

and then $u(x,t)$ (as well as $u_0(x)$) is a probability density function with compact support $\overline{\mathbb{B}}_{\mathrm{c}t^\beta} := \{x \in \mathbb{R}^d : ||x|| \le \mathrm{c}t^\beta\}$. By setting $R^2 = \frac{1}{k^{2/\alpha}}$, the solution (8) coincides with (2.4) in [4].

We point out that NPME has the property of finite speed of propagation. We are able to explain this property as follows. The solution to NPME is a continuous function $u(x,t)$ such that for any $t > 0$ the profile $u(\cdot,t)$ is nonnegative, bounded and compactly supported. Hence, the support expands eventually to penetrate the whole space, but it is bounded at any fixed time. Therefore, for fixed $t > 0$, the support of (8) is given by the closed ball $\overline{\mathbb{B}}_{ct^\beta}$, while the free boundary (that is the set separating the region where the solution is positive) is given by the sphere $\mathbb{S}^{d-1}_{ct^\beta} := \{x \in \mathbb{R}^d : ||x|| = ct^\beta\}$.

The finite speed of propagation of NPME is in contrast with the infinite speed of propagation of the classical heat equation; that is, a nonnegative solution of the heat equation is positive everywhere in $\mathbb{R}^d$.

Remark 2.1. *For $\alpha = 2$, the solution (8) becomes the Barenblatt–Kompanets–Zel'dovich–Pattle solution to the porous medium equation (4) supplemented with the initial condition $u(x,0) = \delta(x)$ (see, for instance, [33]).*

Remark 2.2. *From Theorem 3.1 it follows that $(u(x,t), t \geq 0)$ is a class of rotationally invariant functions; that is, let $O(d)$ be the group of $d \times d$ orthogonal matrices acting in $\mathbb{R}^d$, then we have that $u(M^T x, t) = u(x,t) = u(||x||, t)$, where $M \in O(d)$.*

The next proposition contains the explicit Fourier transform of (8). A similar result has been already proved, for instance, in [4], Lemma 4.1.

Proposition 2.1. *The Fourier transform of the probability density function $u(x,t)$ given by (8) is equal to*

$$
\begin{aligned}
\hat{u}(\xi,t) &:= \mathcal{F}u(\xi,t) \\
&= \frac{1}{(2\pi)^{\frac{d}{2}}} \left(\frac{2k^{1/\alpha}}{t^\beta ||\xi||} \right)^{\frac{d}{2}+\frac{\alpha}{2(m-1)}} \Gamma\left(\frac{d}{2} + \frac{\alpha}{2(m-1)} + 1 \right) J_{\frac{d}{2}+\frac{\alpha}{2(m-1)}} \left(\frac{||\xi|| t^\beta}{k^{1/\alpha}} \right),
\end{aligned}
\tag{9}
$$

where $\xi \in \mathbb{R}^d, d \geq 1$, and $J_\mu(x) = \sum_{k=0}^{\infty} (-1)^k \frac{(x/2)^{2k+\mu}}{k!\Gamma(k+\mu+1)}$, with $\mu \in \mathbb{R}$, is the Bessel function.

Proof. We prove the theorem for $d \geq 2$. The case $d = 1$ follows by simple calculations. Let σ be the measure on $\mathbb{S}^{d-1}_1$. We recall that (see (2.12), p. 690, [10]),

$$
\int_{\mathbb{S}^{d-1}_1} e^{i\rho\xi\cdot\theta} \mathrm{d}\sigma(\theta) = (2\pi)^{d/2} \frac{J_{\frac{d}{2}-1}(\rho||\xi||)}{(\rho||\xi||)^{\frac{d}{2}-1}}
\tag{10}
$$

One has that

$$
\begin{aligned}
\hat{u}(\xi,t) &= \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{i\xi\cdot x} u(x,t) \mathrm{d}x \\
&= (\text{by Remark 2.2}) \\
&= \frac{1}{(2\pi)^{\frac{d}{2}}} \int_0^{\frac{t^\beta}{k^{1/\alpha}}} \rho^{d-1} C t^{-d\beta} \left(1 - \frac{k^{2/\alpha}\rho^2}{t^{2\beta}} \right)^{\frac{\alpha}{2(m-1)}} \mathrm{d}\rho \int_{\mathbb{S}^{d-1}_1} e^{i\rho\xi\cdot\theta} \mathrm{d}\sigma(\theta)
\end{aligned}
$$

$$
\begin{aligned}
&= (\text{by } (10)) \\
&= \int_0^{\frac{t^\beta}{k^{1/\alpha}}} \rho^{d-1} C t^{-d\beta} \left(1 - \frac{k^{2/\alpha}\rho^2}{t^{2\beta}}\right)^{\frac{\alpha}{2(m-1)}} \frac{J_{\frac{d}{2}-1}(\rho||\xi||)}{(\rho||\xi||)^{\frac{d}{2}-1}} \mathrm{d}\rho \\
&= \frac{C t^{\beta(1-\frac{d}{2})}}{(k^{1/\alpha})^{\frac{d}{2}+1} ||\xi||^{\frac{d}{2}-1}} \int_0^1 \left(1 - w^2\right)^{\frac{\alpha}{2(m-1)}} w^{d/2} J_{d/2-1}\left(\frac{||\xi|| t^\beta}{k^{1/\alpha}} w\right) \mathrm{d}w.
\end{aligned}
$$

In view of formula 6.567(1) on p. 688 of [19],

$$
\int_0^1 x^{\nu+1} \left(1 - x^2\right)^{\mu} J_\nu(bx) \mathrm{d}x = 2^\mu \Gamma(\mu + 1) b^{-(\mu+1)} J_{\nu+\mu+1}(b) \tag{11}
$$

where $b > 0$, $\mathrm{Re}\nu > -1$, $\mathrm{Re}\mu > -1$, we obtain (9). $\square$

3 Isotropic transport processes related to NPME

In this section, we analyze the link between the weak solution of the nonlocal equation (1) and the transport processes. We follow the approach developed in [11]. Let us start with introducing isotropic transport processes and recalling their main features.

An isotropic transport process, also called random flight, is a continuous-time random walk in $\mathbb{R}^d$ described by a particle starting at the origin with a randomly chosen direction and with finite speed $c > 0$. The direction of the particle changes whenever a collision with some scattered obstacles in the environment happens and then a new direction of motion is taken. For $d \geq 2$, all the directions are independent and identically distributed. The directions are chosen uniformly on the sphere $\mathbb{S}_1^{d-1} = \{x \in \mathbb{R}^d : ||x|| = 1\}$. For $d = 1$, we have two possible directions alternatively taken by the moving particle. The random flights have been studied, for instance, in [30, 31, 9, 26, 27, 10, 7, 18, 28]. Recently, in [14, 15] the relationship between the isotropic transport processes and some fractional Klein–Gordon equations has been analyzed. Furthermore, stochastic models like random flights are associated to the Euler–Poisson–Darboux partial differential equations as argued in [16].

Rigorously speaking, we introduce the isotropic transport processes as follows. Let $(T_k, k \in \mathbb{N}_0)$ be a sequence of random arrival epochs with $T_0 := 0$. Furthermore, let $(V_k, k \in \mathbb{N}_0)$ be a sequence of random variables defined for $d = 1$, by $V_k := V(0)(-1)^k$, where V(0) is a uniform r.v. on $\{-1, +1\}$, while for $d \geq 2$ they are independent $(\mathbb{S}_1^{d-1}, \mathcal{B}(\mathbb{S}_1^{d-1}))$-valued random variables where $\mathcal{B}(\mathbb{S}_1^{d-1})$ denotes the Borel class on $\mathbb{S}_1^{d-1}$. We assume that during the interval $[0, t]$ the particle takes a new direction, $V_0, V_1, \ldots, V_n$, $n + 1$ times at random moments $T_0, T_1, \ldots, T_n$, respectively. Therefore, we can define an isotropic random flight on $(\Omega, (\mathcal{F}_t^n, t \geq 0))$ as follows

$$
X^n := \left(X^n(t) = \left(X_1^n(t), \ldots, X_d^n(t)\right), t \geq 0\right), \quad X^n(0) = 0, \quad n \in \mathbb{N},
$$

where $X^n(t)$ stands for the position, at time $t \geq 0$, reached by the moving particle according to the mechanism described above and $(V^n(t), t \geq 0)$ is the jump process

$$
V^n(t) := V_k, \quad T_k \leq t < T_{k+1},
$$

with $0 \le k \le n$; i.e.

$$X^n(t) := c \int_0^t V^n(s)\mathrm{d}s = c \sum_{k=0}^{n-1} V_k(T_{k+1} - T_k) + c(t - T_n)V_n, \quad n \in \mathbb{N}. \tag{12}$$

X^n is adapted to the filtration $(\mathcal{F}_t^n, t \ge 0)$ where

$$\mathcal{F}_t^n := \sigma\big((T_k \le t) \cap (V_0, V_1, \dots, V_k) \in B, \forall B \in \mathfrak{B}^{\otimes k+1}, 0 \le k \le n\big),$$

where $\mathfrak{B} := \{-1, +1\}$, if $d = 1$, or $\mathfrak{B} := \mathcal{B}(\mathbb{S}_1^{d-1})$, if $d > 1$. Therefore $X^n(t)$ represents a random motion with finite velocity c and $X^n(t) \in \mathbb{B}_{ct}$ a.s. for a fixed $t > 0$. The components of $X^n(t)$ can be written explicitly as in formula (1.6) of [10]. Important assumptions in our paper are: the random vector of the renewal times $(\tau_1, \dots, \tau_n)$, where $\tau_{k+1} := T_{k+1} - T_k$, has the joint density equal to

$$f_1(\tau_1, \dots, \tau_n) = \frac{n!}{t^n} 1_{S_n}(\tau_1, \dots, \tau_n), \quad \text{for } d = 1, \tag{13}$$

and

$$f_2(\tau_1, \dots, \tau_n) = \frac{\Gamma((n+1)(d-1))}{(\Gamma(d-1))^{n+1}} \frac{1}{t^{(n+1)(d-1)-1}} \left(\prod_{j=1}^{n+1} \tau_j^{d-2}\right) 1_{S_n}(\tau_1, \dots, \tau_n), \tag{14}$$

for $d \ge 2$, or

$$f_3(\tau_1, \dots, \tau_n) = \frac{\Gamma((n+1)(\frac{d}{2}-1))}{(\Gamma(\frac{d}{2}-1))^{n+1}} \frac{1}{t^{(n+1)(\frac{d}{2}-1)-1}} \left(\prod_{j=1}^{n+1} \tau_j^{\frac{d}{2}-2}\right) 1_{S_n}(\tau_1, \dots, \tau_n), \tag{15}$$

for $d \ge 3$, where

$$S_n := \Bigg\{(\tau_1, \dots, \tau_n) \in \mathbb{R}^d : 0 < \tau_j < t - \sum_{k=0}^{j-1} \tau_k, \\ 1 \le j \le n, \tau_0 = 0, \tau_{n+1} = t - \sum_{j=1}^{n} \tau_j\Bigg\}.$$

The distributions (14) and (15) are rescaled Dirichlet distributions, with parameters $(d-1, \dots, d-1)$, $d \ge 2$, and $(\frac{d}{2}-1, \dots, \frac{d}{2}-1)$, $d \ge 3$, respectively. Generalized versions of the Dirichlet density functions (13) and (14) have been used in [8] to generalize the family of random walks defined above.

In the one-dimensional case the process (12) is the well-known telegraph process and admits the density given by (see [9])

$$\frac{P(X_1^n(t) \in \mathrm{d}x_1)}{\mathrm{d}x_1} = \begin{cases} \frac{\Gamma(n+1)}{(\Gamma(\frac{n+1}{2}))^2 2^n ct}(1 - \frac{x_1^2}{c^2t^2})_+^{\frac{n-1}{2}}, & n \text{ odd}, \\ \frac{\Gamma(n+1)}{\Gamma(\frac{n}{2}+1)\Gamma(\frac{n}{2})2^n ct}(1 - \frac{x_1^2}{c^2t^2})_+^{\frac{n}{2}-1}, & n \text{ even}. \end{cases} \tag{16}$$

We observe that for n odd, we have that

$$P\big(X_1^n(t) \in \mathrm{d}x_1\big) = P\big(X_1^{n+1}(t) \in \mathrm{d}x_1\big).$$

Under the assumptions (14) and (15), [10] provides (Theorem 2 in [10]) the explicit density functions of the random flights $X^n(t)$; that is,

$$\frac{P(X^n(t) \in \mathrm{d}x)}{\mathrm{d}x} = \begin{cases} \frac{\Gamma(\frac{n+1}{2}(d-1)+\frac{1}{2})}{\Gamma(\frac{n}{2}(d-1))\pi^{\frac{d}{2}}(ct)^d}(1-\frac{\|x\|^2}{c^2t^2})_+^{\frac{n}{2}(d-1)-1}, & \text{if (14) holds,} \\ \frac{\Gamma((n+1)(\frac{d}{2}-1)+1)}{\Gamma(n(\frac{d}{2}-1))\pi^{\frac{d}{2}}(ct)^d}(1-\frac{\|x\|^2}{c^2t^2})_+^{n(\frac{d}{2}-1)-1}, & \text{if (15) holds.} \end{cases} \tag{17}$$

Remark 3.1. *It is easy to check that the sequence of random flights $X^n, n \in \mathbb{N}$, admits the following scaling property*

$$P\big(aX^n(t/a) \in \mathrm{d}x\big) = P\big(X^n(t) \in \mathrm{d}x\big), \quad a > 0.$$

Hereafter, we discuss the main results of the paper; i.e. Propositions 3.1, 3.2 below. Therefore, we provide a reasonable probabilistic interpretation of the weak solution (8) in terms of a time-rescaled random flights. From the features of X^n it emerges that the random flights share with (8) the crucial property of finite speed of propagation in the space. For this reason the transport process (12) seems to represent a fine choice to model phenomena described by nonlinear diffusion equation with nonlocal pressure (1). Our first result is the following theorem and it represents a generalization of Theorem 1 in [11].

Proposition 3.1. *Let $Y^n := (Y^n(t), t \geq 0), n \in \mathbb{N}$, be the sequence of random flights $Y^n(t) := X^n(t^\beta)$, with speed $\mathrm{c} := \mathrm{c}(\alpha, d) := 1/k^{\frac{1}{\alpha}}$. Y^n is adapted to the filtration $(\mathcal{G}_t^n, t \geq 0)$, where $\mathcal{G}_t^n := \mathcal{F}_{t^\beta}^n$, progressively measurable and*

$$P\big(Y^n(t) \in \mathrm{d}x\big) = u(x,t)\mathrm{d}x, \quad t > 0,$$

where $u(x,t)$ is the weak solution (8) to the equation (1). The relationships between the number n of changes of velocity of Y^n and the parameters $m > 1$ and $\alpha \in (0,2]$ of NPME, are given by:

(i) for $d = 1$,

$$m = \begin{cases} \frac{\alpha}{n-1} + 1 = \frac{\alpha}{2k} + 1, & n = 2k+1, \\ \frac{\alpha}{n-2} + 1 = \frac{\alpha}{2k} + 1, & n = 2k+2, \end{cases} \quad k \geq 1;$$

(ii) for $d \geq 2$, (14) holds and $m = \frac{\alpha}{n(d-1)-2} + 1$ with $d > \frac{2}{n} + 1$;

(iii) for $d \geq 3$, (15) holds and $m = \frac{\alpha}{n(d-2)-2} + 1$ with $d > \frac{2}{n} + 2$.

Proof. Let $n \in \mathbb{N}$. We observe that path map

$$t \mapsto Y^n(t,\omega) = \mathrm{c}\int_0^{t^\beta} V^n(s,\omega)\mathrm{d}s, \quad \omega \in \Omega,$$

is continuous and then Y^n is a continuous process. Therefore, Y^n is progressively measurable if it is adapted to $(\mathcal{G}_t^n, t \geq 0)$ (see, e.g., Proposition 1.13, [25]). Let $t^\beta > 0$, $(s, \omega) \mapsto V^n(s, \omega), \omega \in \Omega, s \leq t^\beta$ is a $\mathcal{B}([0, t^\beta]) \otimes \mathcal{G}_t^n$-measurable function. Hence, by Fubini's theorem one has that the map $\omega \mapsto \mathrm{c} \int_0^{t^\beta} V^n(s, \omega) \mathrm{d}s$ is $\mathcal{G}_t^n$-measurable and then the process Y^n is adapted to the filtration $(\mathcal{G}_t^n, t \geq 0)$.

By rescaling the time coordinate as follows

$$t' := t^\beta,$$

the solution (8) to NPME becomes

$$u(x, t') = \frac{\Gamma(\frac{d}{2} + \frac{\alpha}{2(m-1)} + 1)}{\Gamma(\frac{\alpha}{2(m-1)} + 1)\pi^{\frac{d}{2}}} \frac{1}{(\mathrm{c}t')^d} \left(1 - \frac{||x||^2}{(\mathrm{c}t')^2}\right)_+^{\frac{\alpha}{2(m-1)}}, \tag{18}$$

where $\mathrm{c} := \mathrm{c}(\alpha, d) := 1/k^{\frac{1}{\alpha}}$.

Let us deal with a telegraph process defined by (12) with time scale t' and speed c. By exploiting the duplication formula for the Gamma function we can write the solution (18) for $d = 1$ as follows

$$u(x_1, t') = \frac{\Gamma(2 + \frac{\alpha}{m-1}) 2^{1-2(\frac{\alpha}{2(m-1)}+1)}}{(\Gamma(\frac{\alpha}{2(m-1)} + 1))^2} \frac{1}{\mathrm{c}t'} \left(1 - \frac{x_1^2}{(\mathrm{c}t')^2}\right)_+^{\frac{\alpha}{2(m-1)}}. \tag{19}$$

For

$$\frac{\alpha}{2(m-1)} = \frac{n-1}{2}, \quad \text{that is} \quad m = \frac{\alpha}{n-1} + 1,$$

the solution (19) coincides with the first part of (16), while for

$$\frac{\alpha}{2(m-1)} = \frac{n}{2} - 1, \quad \text{that is} \quad m = \frac{\alpha}{n-2} + 1,$$

the solution (19) coincides with the second part of (16). For $n > 2$, in both cases $m \in (1, \infty)$. Therefore, we can conclude that

$$P\big(X_1^n(t') \in \mathrm{d}x_1\big) = P\big(X_1^{n+1}(t') \in \mathrm{d}x_1\big) = u(x_1, t')\mathrm{d}x_1.$$

Now, let us consider a random flight defined in $\mathbb{R}^d$, $d \geq 2$, by (12) with time scale t' and speed c defined above. Under the assumption (14), for

$$\frac{\alpha}{2(m-1)} = \frac{n}{2}(d-1) - 1, \quad \text{that is} \quad m = \frac{\alpha}{n(d-1) - 2} + 1,$$

the function (18) coincides with the first part of (17). Since $m \in (1, \infty)$, we infer that

$$d > \frac{2}{n} + 1. \tag{20}$$

For $d = 2$ the inequality (20) holds for $n \geq 3$; for $d = 3$, it holds for $n \geq 2$; for $d > 3$, (20) holds for all $n \geq 1$. Therefore, under the condition (20)

$$P\big(X^n(t') \in \mathrm{d}x\big) = u(x, t')\mathrm{d}x.$$

Analogously, under the assumption (15), for

$$\frac{\alpha}{2(m-1)} = n\left(\frac{d}{2}-1\right)-1, \quad \text{that is} \quad m = \frac{\alpha}{n(d-2)-2}+1,$$

the function (18) coincides with the second part of (17). Since $m \in (1, \infty)$, we infer that

$$d > \frac{2}{n}+2. \tag{21}$$

For $d = 3$ the inequality (20) holds for $n \geq 3$; for $d = 4$, it holds for $n \geq 2$; for $d > 4$, (20) holds for all $n \geq 1$. Therefore, under the condition (21)

$$P\big(X^n\big(t'\big) \in \mathrm{d}x\big) = u\big(x, t'\big)\mathrm{d}x. \qquad \square$$

To enhance the features of the random models Y^n, $n \geq 1$, it is useful to introduce the Euclidean distance process $\mathsf{R}^n := (\mathsf{R}^n(t), t \geq 0)$; that is $\mathsf{R}^n(t) := ||Y^n(t)||$. For a fixed $t \geq 0$, $\mathsf{R}^n(t) \in [0, \mathrm{c}t^\beta]$ a.s. The next result will be useful for arguing on the anomalous diffusivity of Y^n.

Proposition 3.2. *Under the conditions (i), (ii) and (iii) of Proposition 3.1, the following results hold:*

1) the probability density function of R^n becomes:

$$\frac{P(\mathsf{R}^n(t) \in \mathrm{d}r)}{\mathrm{d}r} = \frac{2\Gamma(\frac{d}{2}+\frac{\alpha}{2(m-1)}+1)}{\Gamma(\frac{\alpha}{2(m-1)}+1)\Gamma(\frac{d}{2})}\frac{r^{d-1}}{(\mathrm{c}t^\beta)^d}\left(1-\frac{r^2}{\mathrm{c}^2t^{2\beta}}\right)_+^{\frac{\alpha}{2(m-1)}}, \quad t>0; \tag{22}$$

2) let $p \geq 1$ and $d \geq 2$; then

$$E\big(\mathsf{R}^n(t)\big)^p = \frac{\Gamma(\frac{d}{2}+\frac{\alpha}{2(m-1)}+1)\Gamma(\frac{d+p}{2})}{\Gamma(\frac{\alpha}{2(m-1)}+1+\frac{d+p}{2})\Gamma(\frac{d}{2})}\big(\mathrm{c}t^\beta\big)^p, \tag{23}$$

while for $d = 1$

$$E\big(\mathsf{R}^n(t)\big) = \begin{cases} 0, & p \text{ odd}, \\ \frac{\Gamma(\frac{1}{2}+\frac{\alpha}{2(m-1)}+1)\Gamma(\frac{1+p}{2})}{\Gamma(\frac{\alpha}{2(m-1)}+1+\frac{p+1}{2})\sqrt{\pi}}(\mathrm{c}t^\beta)^p, & p \text{ even}; \end{cases} \tag{24}$$

3) the rescaled process $(\frac{X^n(t^\beta)}{\mathrm{c}t^\beta}, t \geq 0)$ has the distribution law independent from the time t and with compact support $\overline{\mathbb{B}}_1$; i.e.

$$w(x,t) := \frac{P(\frac{X^n(t^\beta)}{\mathrm{c}t^\beta} \in \mathrm{d}x)}{\mathrm{d}x} = \frac{\Gamma(\frac{d}{2}+\frac{\alpha}{2(m-1)}+1)}{\pi^{d/2}\Gamma(\frac{\alpha}{2(m-1)}+1)}\big(1-||x||^2\big)_+^{\frac{\alpha}{2(m-1)}}.$$

Furthermore

$$\begin{aligned}\hat{w}(\xi,t) &:= \mathcal{F}w(\xi,t) \\ &= \frac{1}{(2\pi)^{\frac{d}{2}}}\left(\frac{2}{||\xi||}\right)^{\frac{d}{2}+\frac{\alpha}{2(m-1)}}\Gamma\left(\frac{d}{2}+\frac{\alpha}{2(m-1)}+1\right)J_{\frac{d}{2}+\frac{\alpha}{2(m-1)}}(||\xi||);\end{aligned}$$

4) the rescaled distance process $(\frac{\mathsf{R}^n(t)}{\mathrm{c}t^\beta}, t \geq 0)$ *admits probability density function given by a Beta r.v. with parameters* $\frac{d}{2}$ *and* $\frac{\alpha}{2(m-1)} + 1$.

Proof. 1) By exploiting Remark 2.2 and $P(\mathsf{R}^n(t) \leq r) = P(Y^n(t) \in \mathbb{B}_r)$, it is not hard to prove that

$$P\big(\mathsf{R}^n(t) \in \mathrm{d}r\big) = \mathrm{area}\big(\mathbb{S}_1^{d-1}\big) r^{d-1} u(r,t)\mathrm{d}r,$$

where $\mathrm{area}(\mathbb{S}_1^{d-1}) = 2\pi^{d/2}/\Gamma(d/2)$, and then (22) immediately follows.

2) From point 1), we have

$$\begin{aligned} E\big(\mathsf{R}^n(t)\big)^p &= \int_0^\infty \mathrm{area}\big(\mathbb{S}_1^{d-1}\big) r^{p+d-1} u(r,t)\mathrm{d}r \\ &= \frac{2\Gamma(\frac{d}{2} + \frac{\alpha}{2(m-1)} + 1)}{\Gamma(\frac{\alpha}{2(m-1)} + 1)\Gamma(\frac{d}{2})} \int_0^{\mathrm{c}t^\beta} \frac{r^{p+d-1}}{(\mathrm{c}t^\beta)^d} \left(1 - \frac{r^2}{\mathrm{c}^2 t^{2\beta}}\right)^{\frac{\alpha}{2(m-1)}} \mathrm{d}r \\ &= \frac{\Gamma(\frac{d}{2} + \frac{\alpha}{2(m-1)} + 1)}{\Gamma(\frac{\alpha}{2(m-1)} + 1)\Gamma(\frac{d}{2})} \big(\mathrm{c}t^\beta\big)^p \int_0^1 w^{\frac{d+p}{2}-1}(1-w)^{\frac{\alpha}{2(m-1)}} \mathrm{d}w \\ &= \frac{\Gamma(\frac{d}{2} + \frac{\alpha}{2(m-1)} + 1)\Gamma(\frac{d+p}{2})}{\Gamma(\frac{\alpha}{2(m-1)} + 1 + \frac{d+p}{2})\Gamma(\frac{d}{2})} \big(\mathrm{c}t^\beta\big)^p. \end{aligned}$$

For $d = 1$ the result (24) follows by similar calculations.

3) For fixed $t > 0$, the result (3.2) is derived from (18), by applying the Jacobian theorem to the bijection $g : \mathbb{R}^d \to \mathbb{R}^d$ with $g(x) = \frac{1}{\mathrm{c}t^\beta}x$. By the same calculations leading to (9), we can prove that the Fourier transform $\hat{w}(\xi,t)$ holds true.

4) It is an immediate consequence of the point 1). □

The Barenblatt–Kompanets–Zel'dovich–Pattle solution to the classical PME does not spread in the space linearly over the time and then we can argue that the phenomena described by the equation (4) represent anomalous diffusion (see, for instance, [33]). Similar considerations hold for (8). By means of Theorem 3.2, we infer that the stochastic models Y^n, $n \geq 1$, behave similarly to an anomalous diffusion. From (23) and (24), we observe that

$$\mathrm{Var}\big(\mathsf{R}^n(t)\big) = O\big(t^{2\beta}\big), \quad t > 0.$$

For $d = 1$, one has $2\beta = \frac{2}{m-1+\alpha} = \frac{4k}{\alpha(2k+1)}$, $k \geq 1$ (condition (i) in Proposition 3.1). Therefore, for a fixed $k \in \mathbb{N}$, we can find the values $\alpha \in (0, 2]$, such that the process Y^n spreads over the real line like a sub-diffusion or a super-diffusion. Therefore Y^n has the following properties:

- scatters in the space as a sub-diffusion; i.e. $2\beta < 1$, if and only if $\frac{4k}{(2k+1)} < \alpha$;
- is a super-diffusion process; i.e. $2\beta > 1$, if and only if $\frac{4k}{(2k+1)} > \alpha$;
- represents a classical diffusion if and only if $\frac{4k}{(2k+1)} = \alpha$ (i.e. $2\beta = 1$).

Analogous remarks hold in higher dimensions. For a fixed $n \in \mathbb{N}$ and $d \geq 2$ (resp. $d \geq 3$), under the condition (ii) (resp. (iii)) in Proposition 3.1, the random process Y^n has the following properties:

- behaves similarly to a sub-diffusion if and only if $\frac{2n(d-1)-4}{(n+1)(d-1)-1} < \alpha$ (resp. $\frac{2n(d-2)-4}{(n+1)(d-2)-2} < \alpha$);
- spreads over the space like a super-diffusion if and only if $\frac{2n(d-1)-4}{(n+1)(d-1)-1} > \alpha$ (resp. $\frac{2n(d-2)-4}{(n+1)(d-2)-1} > \alpha$);
- represents a diffusion if and only if $\frac{2n(d-1)-4}{(n+1)(d-1)-1} = \alpha$ (resp. $\frac{2n(d-2)-4}{(n+1)(d-2)-1} = \alpha$).

4 Conclusions

We are able to provide a probabilistic interpretation of the weak solution (8) to NPME. In particular, we deal with random flight models (12) with a suitable rescaling of the time coordinate. These random processes enjoy the main features of (8), at least for particular values of m:

- finite speed of propagation property with compact support given by a closed ball;
- spread over the space like $t^{2\beta}$; i.e. anomalous diffusivity depending on the values of the fractional parameter α.

In conclusion, the isotropic transport processes seem to describe well the real phenomena studied by means of the degenerate nonlinear diffusion equation with fractional pressure (1).

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