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Martingale-like sequences in Banach lattices

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Abstract Martingale-like sequences in vector lattice and Banach lattice frameworks are defined in the same way as martingales are defined in [Positivity 9 (2005), 437–456]. In these frameworks, a collection of bounded X-martingales is shown to be a Banach space under the supremum norm, and under some conditions it is also a Banach lattice with coordinate-wise order. Moreover, a necessary and sufficient condition is presented for the collection of $\mathcal{E}$-martingales to be a vector lattice with coordinate-wise order. It is also shown that the collection of bounded $\mathcal{E}$-martingales is a normed lattice but not necessarily a Banach space under the supremum norm.

Keywords Banach lattices, martingales, E-martingales, X-martingales
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1 Introduction

The classical definition of martingales is extended to a more general case in the space of Banach lattices by V. Troitsky [6]. In the Banach lattice framework, martingales are defined without a probability space and the famous Doob's convergence theorem was reproduced. Moreover, under certain conditions on the Banach lattice, it was shown that the set of bounded martingales forms a Banach lattice with respect to the point-wise order. In 2011, H. Gessesse and V. Troitsky [2] produced several sufficient

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conditions for the space of bounded martingales on a Banach lattice to be a Banach lattice itself. They also provided examples showing that the space of bounded martingales is not necessarily a vector lattice. Several other works have been done by other authors with regard to martingales in vector lattices, such as [4, 3].

In the theory of random processes, not just the study of martingale convergence is important, but the study of convergence of martingale-like stochastic sequences and processes, and the determination of interrelation between them are also crucial. So it is natural to ask if martingale-like sequences can be defined in a vector lattice or Banach lattice framework. In this article, we define and study martingale-like sequences in Banach lattices along the same lines as martingales are defined and studied in [6].

Classically, a martingale-like sequence is defined as follows (for instance, see a paper by A. Melnikov [5]). Consider a probability space $(\Omega, \mathcal{F}, P)$ and a filtration $(\mathcal{F}_n)_{n=1}^{\infty}$, i.e., an increasing sequence of complete sub-sigma-algebras of $\mathcal{F}$. An integrable stochastic sequence $x = (x_n, \mathcal{F}_n)$ is an L^1**-martingale** if

$$\lim_{n\to\infty} \sup_{m\geqslant n} E\big|E(x_m|\mathcal{F}_n) - x_n\big| = 0.$$

An integrable stochastic sequence $x = (x_n, \mathcal{F}_n)$ is an E**-martingale** if

$$P\big\{\omega : E(x_{n+1}|\mathcal{F}_n) \neq x_n \text{ infinitely often } \big\} = 0.$$

Here we extend the definition of L^1-martingales and E-martingales in a general Banach lattice X following the same lines as the definition of martingales in Banach lattices in [6]. First we mention some terminology and definitions from the theory of Banach lattices for the reader convenience. For more detailed exploration, we refer the reader to [1]. A **vector lattice** is a vector space equipped with a lattice order relation, which is compatible with the linear structure. A **Banach lattice** is a vector lattice with a Banach norm which is monotone, i.e., $0 \leqslant x \leqslant y$ implies $\|x\| \leqslant \|y\|$, and satisfies $\|x\| = \||x|\|$ for any two vectors x and y. A vector lattice is said to be **order complete** if every nonempty subset that is bounded above has a supremum. We say that a Banach lattice has **order continuous norm** if $\|x_\alpha\| \to 0$ for every decreasing net (x_α) with $\inf x_\alpha = 0$. A Banach lattice with order continuous norm is order complete. A sublattice Y of a vector lattice is called an (order) **ideal** if $y \in Y$ and $|x| \leqslant |y|$ imply $x \in Y$. An ideal Y is called a **band** if $x = \sup_\alpha x_\alpha$ implies $x \in Y$ for every positive increasing net (x_α) in Y. Two elements x and y in a vector lattice are said to be **disjoint** whenever $|x| \wedge |y| = 0$ holds. If J is a nonempty subset of a vector lattice, then its **disjoint complement** J^d is the set of all elements of the lattice, disjoint to every element of J. A band Y in a vector lattice X that satisfies $X = Y \otimes Y^d$ is refered to as a **projection band**. Every band in an order complete vector lattice is a projection band. An operator T on a vector lattice X is positive if $Tx \geqslant 0$ for every $x \geqslant 0$. A sequence of positive projections (E_n) on a vector lattice X is called a **filtration** if $E_n E_m = E_{n\wedge m}$. A sequence of positive contractive projections (E_n) on a normed lattice X is called a **contractive filtration** if $E_n E_m = E_{n\wedge m}$. A filtration (E_n) in a normed lattice X is called ***dense*** if $E_n x \to x$ for each x in X. In many articles such as in [6], a ***martingale*** with respect to a filtration (E_n) in a vector lattice X is defined as a sequence (x_n) in X such that $E_n x_m = x_n$ whenever $m \geq n$.

2 Main definitions

Definition 1. A sequence (x_n) of elements of a normed lattice X is called an X-**martingale** relative to a contractive filtration (E_n) if

$$\lim_{n\to\infty} \sup_{m\geqslant n} \|E_n x_m - x_n\| = 0.$$

Definition 2. A sequence (x_n) of elements of a vector lattice X is called an $\mathcal{E}$-**martingale** relative to a filtration (E_n) if there exists $n \geqslant 1$ such that $E_m x_{m+1} = x_m$ for all $m \geqslant n$.

Note that Definition 2 is equivalent to saying a sequence (x_n) is an $\mathcal{E}$-martingale if there exists $l \geqslant 1$ such that $E_n x_m = x_n$ whenever $m \geqslant n \geq l$. The symbol "$\mathcal{E}$" stands for eventual so when we say (x_n) is an $\mathcal{E}$-martingale, we are saying that after a first few finite elements of the sequence, the sequence becomes a martingale.

Sequences defined by Definition 1 and Definition 2 are collectively called **martingale-like sequences**. Notice that every martingale (x_n) in a vector lattice X with respect to a filtration (E_n) is obviously an $\mathcal{E}$-martingale with respect to the filtration (E_n). Moreover, every $\mathcal{E}$-martingale (x_n) in a Banach lattice X with respect to a contractive filtration (E_n) is an X-martingale with respect to the contrative filtration (E_n). Note that for every x in a vector lattice X and a filtration (E_n) in X, the sequence $(E_n x)$ is an $\mathcal{E}$-martingale with respect to the filtration (E_n). If x is in a normed space X and (E_n) is a contractive filtration, then the sequence $(E_n x)$ is an X-martingale with respect to the contractive filtration (E_n).

By considering any nonzero martingale (x_n) in a Banach lattice X with respect to filtration (E_n) where x_1 is nonzero without loss of generality, we can define a sequence (y_n) such that $y_1 = 2x_1$ and $y_n = x_n$ for all $n \geqslant 2$. Then one can see that (y_n) is an $\mathcal{E}$-martingale with respect to the filtration (E_n). However, (y_n) is not a martingale.

Note that every sequence which converges to zero is an X-martingale with respect to any contractive filtration (E_n) because if $x_n \to 0$ and $m > n$ then $\|E_n x_m - x_n\| \leqslant \|x_m\| + \|x_n\| \to 0$ as $n \to \infty$. So one can easily create an X-martingale (x_n) which is not $\mathcal{E}$-martingale by setting $x_n = \frac{1}{n}x$ where x is a nonzero vector in X.

A martingale-like sequence $A = (x_n)$ with respect to a contractive filtration (E_n) on a normed lattice X is said to be **bounded** if its norm defined by $\|A\| = \sup_n \|x_n\|$ is finite. Given a contractive filtration (E_n) on a normed lattice X, we denote the set of all bounded X-martingales with respect to the contractive filtration (E_n) by $M_X = M_X(X, (E_n))$ and the set of all bounded $\mathcal{E}$-martingales with respect to the contractive filtration (E_n) by $M_E = M_E(X, (E_n))$. With the introduction of the sup norm in these spaces, one can show that M_X and M_E are normed spaces. Keeping the notation M of [6] for all bounded martingales with respect to the contractive filtration (E_n) and from the preceding arguments, these spaces form a nested increasing sequence of linear subspaces $M \subset M_E \subset M_X \subset \ell_\infty(X)$, with the norm being exactly the $\ell_\infty(X)$ norm.

Theorem 3. *Let (E_n) be a contractive filtration on a Banach lattice X, then the collection of X-martingales M_X is a closed subspace of $\ell_\infty(X)$, hence a Banach space.*

Proof. Suppose a sequence $(A^m) = (x_n^m)$ of X-martingales converges to A in $\ell_\infty(X)$. We show A is also an X-martingale. Indeed, from $\|A^m - A\| = \sup_n \|x_n^m - x_n\| \to 0$ as $m \to \infty$, we have that for each $n \geqslant 1$, $\|x_n^m - x_n\| \to 0$ as $m \to \infty$. Note that for $l \geq n$,

$$\begin{aligned}\|E_n x_l - x_n\| &= \|E_n x_l - E_n x_l^m + E_n x_l^m - x_n^m + x_n^m - x_n\| \\ &\leqslant \|E_n x_l - E_n x_l^m\| + \|E_n x_l^m - x_n^m\| + \|x_n^m - x_n\|.\end{aligned}$$

From these inequalities and the contractive property of the filtration, we have

$$\lim_{n\to\infty} \sup_{l\geqslant n} \|E_n x_l - x_n\| = 0.$$ □

Corollary 1. *Let (E_n) be a contractive filtration on a Banach lattice X, then $\overline{M_E} \subset M_X$.*

Lemma 1. *Let (E_n) be a contractive filtration on a Banach lattice X and $A = (x_n)$ be in M_X where $x_n \to x$. Then*

$$\lim_{n\to\infty} \sup_{m\geq n} \|E_m x - x_m\| = 0.$$

Proof. Let $A = (x_n)$ be in M_X where $x_n \to x$. Thus, for $m \geqslant n$

$$\|E_n x - x_n\| = \|E_n x - E_n x_m + E_n x_m - x_n\| \leqslant \|x - x_m\| + \|E_n x_m - x_n\|.$$

Taking $\lim_{n\to\infty} \sup_{m\geqslant n}$ on both sides of the inequality completes the proof. □

The following proposition confirms that for any convergent element $A = (x_n)$ of M_X we can find a sequence in M_E that converges to A.

Proposition 4. *Let (E_n) be a contractive filtration on a Banach lattice X and $A = (x_n)$ be a sequence in M_X such that $x_n \to x$. Then there exists a sequence A^m in M_E such that $A^m \to A$ in $\ell_\infty(X)$.*

Proof. Suppose $x_n \to x$ as $n \to \infty$. First note that the sequence $(E_n x)$ is in M. Now define $A^m = (a_n^m)$ such that

$$a_n^m = \begin{cases} x_n, & \text{for } n \leq m, \\ E_n x, & \text{for } n > m. \end{cases}$$

Then $A^m \in M_E$ and $A^m \to A$ in $\ell_\infty(X)$, hence $A \in \overline{M_E}$. Indeed, by Lemma 1,

$$\lim_{m\to\infty} \|A^m - A\| = \lim_{m\to\infty} \sup_j \|E_{m+j} x - x_{m+j}\| = 0.$$ □

In [6] and [2] several sufficient conditions are established where the set of bounded martingales M is a Banach lattice. In [2], counter examples are provided where M is not a Banach lattice. So, one may similarly ask when are M_X and M_E Banach spaces and Banach lattices? We start by showing a counter example that illustrates that M_E is not necessarily a Banach space.

Example 5. Let c_0 be the set of sequences converging to zero. Consider the filtration (E_n) where $E_n \sum_{i=1}^{\infty} \alpha_i e_i = \sum_{i=1}^{n} \alpha_i e_i$. Thus the sequence (y_n) where $y_n = \sum_{i=1}^{n} \frac{1}{i} e_i$ is an E-martingale with respect to this filtration. We define a sequence of E-martingales A^m as $A^m = (x_n^m)$ where

$$x_n^m = \begin{cases} \sum_{i=n}^{\infty} \frac{1}{i} e_i, & \text{for } n \leqslant m, \\ y_n/m, & \text{for } n > m. \end{cases}$$

Define a sequence $A = (x_n)$ where $x_n = \sum_{i=n}^{\infty} \frac{1}{i} e_i$. We can see that A is not an E-martingale. But one can show that A^m converges to A. Indeed,

$$\left\| A^m - A \right\| = \sup_n \left\| x_n^m - x \right\| = \sup_{n \in \{m+1, m+2, \ldots\}} \left\| y_n/m - \sum_{i=n}^{\infty} \frac{1}{i} e_i \right\| \to 0$$

as $m \to \infty$.

3 When is M_E a vector lattice?

Given a vector (Banach) lattice X and a filtration (respectively contractive) (E_n) on X, we can introduce order structure on the spaces M_E and M_X as follows. For two bounded $\mathcal{E}$-martingales (respectively X-martingales) $A = (x_n)$ and $B = (y_n)$, we write $A \geqslant B$ if $x_n \geqslant y_n$ for each n. With this order M_E and M_X are ordered vector spaces and the monotonicity of the norm follows from the monotonicity of the norm of X, i.e. for two $\mathcal{E}$-martingales (respectively X-martingales) with $0 \leqslant A \leqslant B$, we have $\|A\| \leqslant \|B\|$. For two $\mathcal{E}$-martingales (respectively X-martingales) $A = (x_n)$ and $B = (y_n)$, one may guess that $A \vee B$ (or $A \wedge B$) can be computed by the formulas $A \vee B = (x_n \vee y_n)$ (or $A \wedge B = (x_n \wedge y_n)$). We show in the following theorem that this is in fact the case in order for M_E to be a vector lattice. However, this is not obvious to show in the case of M_X.

Theorem 6. *Let X be a vector lattice. Then the following statements are equivalent.*

(i) M_E *is a vector lattice.*

(ii) For each $A = (x_n)$ *in* M_E*, the sequence* $(|x_n|)$ *is an* $\mathcal{E}$*-martingale and* $|A| = (|x_n|)$.

(iii) M_E *is a sublattice of* $\ell_\infty(X)$.

Proof. First we show $(i) \implies (ii)$. Suppose M_E is a vector lattice and $A = (x_n)$ is in M_E. Since M_E is a vector lattice, $|A|$ exists in M_E, say $|A| = B := (y_n)$. Since $\pm A \leq B$, for each n, $\pm x_n \leq y_n$. So, $|x_n| \leq y_n$ for each n. Since B is in M_E, there exists l such that $E_n y_m = y_n$ whenever $m \geq n \geq l$. Now we claim that $y_n = |x_n|$ for each n. Fix $k > l$. We show $y_n = |x_n|$ for each $n \leq k$.

Indeed, define an $\mathcal{E}$-martingale $C = (z_n)$ where

$$z_n = \begin{cases} |x_n|, & \text{for } n \leq k, \\ y_n, & \text{for } n > k. \end{cases}$$

Since $k > l$ we can easily see that C is an $\mathcal{E}$-martingale. Moreover, $C \geq 0$ and $\pm A \leq C \leq B$. Since $|A| = B$, $C = B$. Thus, for every $n \leq k$, $y_n = |x_n|$. This establishes (ii).

$(ii) \implies (iii) \implies (i)$ is straightforward. □

Using the equivalence in Theorem 6, the following examples illustrate that M_E is not always a vector lattice.

Example 7. Consider the classical martingale (x_n) in $L_1[0,1]$ where $x_n = 2^n \mathbf{1}_{[0,2^{-n}]} - \mathbf{1}$ with the filtration $(\mathcal{F}_n)$ where $\mathcal{F}_n$ is the smallest sigma algebra generated by the set

$$\big\{[0, 2^{-n}], (2^{-n}, 2^{-n+1}], \ldots, (1 - 2^{-n}, 1]\big\}.$$

One can easily show that

$$E_n|x_{n+1}| = E\big[|x_{n+1}||x_n\big] \neq |x_n|$$

for every n and the sequence $(|x_n|)$ fails to be an $\mathcal{E}$-martingale. Hence, Theorem 6 implies that M_E is not a vector lattice.

Example 8. Consider the filtration (E_n) defined on c_0 as follows. For each $n = 0, 1, 2, \ldots$

$$E_n = \begin{bmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & 1/2 & 1/2 & & & \\ & & & 1/2 & 1/2 & & & \\ & & & & & 1/2 & 1/2 & \\ & & & & & 1/2 & 1/2 & \\ & & & & & & & \ddots \end{bmatrix}$$

with $2n$ ones in the upper left corner. For each $e_i = (0, \ldots, 0, \underbrace{1}_{i^{\text{th}}}, 0, \ldots)$, $E_n e_i = e_i$ if $i \leqslant 2n$ and $E_n e_{2k-1} = E_n e_{2k} = \frac{1}{2}(e_{2k-1} + e_{2k})$ if $n < k$. Now if we define a sequence $A = (x_n)$ where for each $n = 0, 1, 2, \ldots,$

$$x_n = (\underbrace{-1, 1, \ldots, -1, 1}_{2n\text{-tuple}}, 0, \ldots),$$

one can show this is a martingale as a result an $\mathcal{E}$-martingale. However, $|A| = (|x_n|)$ where

$$|x_n| = (\underbrace{1, \ldots, 1}_{2n\text{-tuple}}, 0, \ldots)$$

is not an $\mathcal{E}$-martingale. So, Theorem 6 implies that M_E is not a vector lattice.

Proposition 9. *If a filtration (E_n) is a sequence of band projections, then M_E is a vector lattice with coordinate-wise lattice operations.*

Proof. If $A = (x_n) \in M_E$, then there exists l such that $E_n x_m = x_n$ whenever $m \geq n \geq l$. Thus, $E_n|x_m| = |E_n x_m| = |x_n|$. So, $|A| = (|x_n|)$ and thus M_E is a vector lattice. □

Theorem 10. *If M_E is a normed lattice and the filtration (E_n) is dense in X, then for each x in X, there exists l such that $|E_n x| = E_n|x|$ whenever $n \geq l$.*

Proof. Let x be in X. Then (E_n) is dense means $E_n x \to x$. Moreover, $(E_n x)$ is a martingale. Since M_E is a vector lattice, by Theorem 6, $(|E_n x|)$ is an E-martingale. Thus there exists l such that for any m and n with $m \geq n \geq l$, $|E_n E_m x| = |E_n x|$ and $E_n|E_m x| = |E_n x|$. So, $|E_n E_m x| = E_n|E_m x|$ and letting $m \to \infty$, we have $|E_n x| = E_n|x|$. □

4 When is M_X a Banach lattice?

Under the pointwise order structure on M_X, for an X-martingale $A = (x_n)$, we can refer to Example 8 to show that the sequence $(|x_n|)$ is not necessarily an X-martingale. However, under certain assumptions, we can show that $(|x_n|)$ is an X-martingale for every X-martingale $A = (x_n)$ making M_X a Banach lattice.

Proposition 11. *If (E_n) is a contractive filtration where E_n is a band projection for every n then M_X is a Banach lattice with coordinate-wise lattice operations.*

Proof. Let $A = (x_n)$ be an X-martingale. For each n and m, E_n is a band projection implies $E_n|x_m| = |E_n x_m|$. Thus, by the fact that $\big||x| - |y|\big| \leqslant |x - y|$, for $m \geqslant n$,

$$\big\|E_n|x_m| - |x_n|\big\| = \big\||E_n x_m| - |x_n|\big\| \leqslant \|E_n x_m - x_n\|.$$

This implies

$$\lim_{n\to\infty} \sup_{m \geqslant n} \big\|E_n|x_m| - |x_n|\big\| = 0$$

which implies $|A| = (|x_n|)$ is also an X-martingale. □

Question. From Theorem 6, M_E is a vector lattice if and only if for each $\mathcal{E}$-martingale (x_n), the sequence $(|x_n|)$ is also an $\mathcal{E}$-martingale. This is the case when the filtration is a sequence of band projections. Can one give a characterization of the filtrations for which M_E is a vector lattice? Or, can one give an example of a filtration which is not a sequence of projections and the corresponding M_E is a vector lattice?

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