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Asymptotics for the sum of three state Markov dependent random variables

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Abstract The insurance model when the amount of claims depends on the state of the insured person (healthy, ill, or dead) and claims are connected in a Markov chain is investigated. The signed compound Poisson approximation is applied to the aggregate claims distribution after $n \in \mathbb{N}$ periods. The accuracy of order $O(n^{-1})$ and $O(n^{-1/2})$ is obtained for the local and uniform norms, respectively. In a particular case, the accuracy of estimates in total variation and non-uniform estimates are shown to be at least of order $O(n^{-1})$. The characteristic function method is used. The results can be applied to estimate the probable loss of an insurer to optimize an insurance premium.

Keywords Signed compound Poisson approximation, insurance model, Markov chain, Kolmogorov norm, local norm, total variation norm, non-uniform estimate

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1 Introduction

This paper is motivated by the insurance model in which the insured is described by a random variable (rv) with three states (healthy, ill, dead), and rvs are connected in a Markov chain. We assume that the insurer pays one unit of money in the case of illness and continuously pays $d \in \mathbb{N}$ units in the case of death. We are interested in aggregate losses for the insurer after $n \in \mathbb{N}$ time periods. More precisely, let $\xi_0, \xi_1, \ldots, \xi_n, \ldots$

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be a non-stationary three-state $\{a_1, a_2, a_3\}$ Markov chain. State a_1 corresponds to being healthy, state a_2 corresponds to being ill, and state a_3 is reached in the case of death. The insurer pays nothing for healthy policy holders, one unit of money for the ill individuals, and constantly pays d units of money ($d \in \mathbb{N}$) in the case of death. We denote the distribution of $S_n = f(\xi_1) + \cdots + f(\xi_n)$ ($n \in \mathbb{N}$) by F_n, that is, $\mathrm{P}(S_n = m) = F_n\{m\}$ for $m \in \mathbb{Z}$. Here $f(a_1) = 0$, $f(a_2) = 1$, $f(a_3) = d$, $d \in \mathbb{N}$. We will analyze a little simplified model by assuming that the probability of a healthy person to die is equal to zero (i.e. we exclude the cases of sudden death). Even though this assumption diminishes model's universality, it is quite reasonable, because usually a person is ill at least for one time period and dies only afterwards.

The matrix of transition probabilities P is defined in the following way

$$P = \begin{pmatrix} 1-\gamma & \gamma & 0 \\ 1-\alpha-\beta & \beta & \alpha \\ 0 & 0 & 1 \end{pmatrix}, \qquad \alpha, \beta, \gamma \in (0, 1).$$

It is assumed that at the beginning the insured person is healthy. Hence, the initial distribution is given by

$$\mathrm{P}(\xi_0 = a_1) = \pi_1 = 1, \qquad \mathrm{P}(\xi_0 = a_2) = \pi_2 = 0, \qquad \mathrm{P}(\xi_0 = a_3) = \pi_3 = 0.$$

Observe, that our Markov chain contains one absorbing state (death).

In this paper, we consider triangular arrays of rvs (the scheme of series), i.e. all transition probabilities α, β, γ can depend on $n \in \mathbb{N}$. Arguably in insurance models the triangular arrays are more natural than the more frequently studied less general scheme of sequences, when it is assumed that the probability to become ill or to die does not change as time passes.

All results are obtained under the condition

$$0 < \beta \leqslant 0.15, \qquad 0 < \gamma \leqslant 0.05, \qquad \alpha \leqslant C_0 < 1, \qquad \alpha + \beta < 1. \tag{1}$$

Here $C_0 \in (0, 1)$ is any maximum possible value of $\alpha(n), n \in \mathbb{N}$ (strictly less than 1), i.e. the maximum probability of an ill individual to die for all time periods $n \in \mathbb{N}$. The condition (1) is not very restrictive, because $\beta \leqslant 0.15$ means that the probability to remain ill during the next time period does not exceed 15%, and $\gamma \leqslant 0.05$ means that the probability of a healthy person to become ill does not exceed 5%, that is, only chronic and epidemic illnesses are excluded.

We denote by C all positive absolute constants, and we denote by θ any complex number satisfying $|\theta| \leqslant 1$. The values of C and θ can vary from line to line or even within the same line. Sometimes, as in (1), we supply constants with indices. Let I_k denote the distribution concentrated at an integer $k \in \mathbb{Z}$, and set $I = I_0$. Let $M_{\mathbb{Z}}$ be a set of finite signed measures concentrated on $\mathbb{Z}$. The Fourier transform and analogue of distribution function for $M \in M_{\mathbb{Z}}$ is denoted by $\widehat{M}(t)$ ($t \in \mathbb{R}$) and $M(x) := \sum_{j=-\infty}^{x} M\{j\}$, respectively. Similarly, $F_n(x) := F_n\{(-\infty, x]\}$. For $y \in \mathbb{R}$ and $j \in \mathbb{N} = \{1, 2, 3, \dots\}$, we set

$$\binom{y}{j} := \frac{1}{j!} y(y-1)\dots(y-j+1), \qquad \binom{y}{0} := 1.$$

If $N, M \in M_{\mathbb{Z}}$, then products and powers of N and M are understood in the convolution sense, that is, for a set $A \subseteq \mathbb{Z}$,

$$NM\{A\} = \sum_{k=-\infty}^{\infty} N\{A-k\}M\{k\}, \quad M^0 = I.$$

The exponential of M is denoted by

$$\mathrm{e}^M = \exp\{M\} := \sum_{k=0}^{\infty} \frac{1}{k!} M^k.$$

We define the local norm, the uniform (Kolmogorov) norm, and the total-variation norm of M respectively by

$$\|M\|_\infty := \sup_{k\in\mathbb{Z}} |M\{k\}|, \quad |M|_{\mathrm{K}} := \sup_{x\in\mathbb{R}} |M\{(-\infty, x]\}|, \quad \|M\| := \sum_{j=-\infty}^{\infty} |M\{j\}|.$$

In the proofs, we apply the following well-known relations:

$$\widehat{MN}(t) = \widehat{M}(t)\widehat{N}(t), \qquad \|MN\| \leqslant \|M\|\|N\|, \qquad |MN|_{\mathrm{K}} \leqslant \|M\||N|_{\mathrm{K}},$$
$$\|MN\|_\infty \leqslant \|M\|\|N\|_\infty, \qquad |\widehat{M}(t)| \leqslant \|M\|, \qquad \widehat{I_a}(t) = \mathrm{e}^{\mathrm{i}ta}, \qquad \widehat{I}(t) = 1.$$

2 Known results

The compound Poisson approximation is frequently used to approximate aggregate losses in risk models (see, for example, [5, 8, 9, 12, 14, 21]); however, in those models it is usually assumed that rvs are independent of time period $n \in \mathbb{N}$. The compound Poisson approximation to sums of Markov dependent rvs was investigated in [6]. Numerous papers were devoted to Markov Binomial distribution, see [1, 3, 4, 7, 10, 18, 19], and the references therein. It seems, however, that the case of Markov chain containing absorbing state was not considered so far. Our research is closely related to the paper [16], in which a non-stationary three-state symmetric Markov chain $\xi_0, \xi_1, \dots \xi_n, \dots$ was investigated with the matrix of transition probabilities

$$\begin{pmatrix} a & 1-2a & a \\ b & 1-2b & b \\ a & 1-2a & a \end{pmatrix}, \quad a, b \in (0, 0.5).$$

Let $\tilde{S}_n = \tilde{f}(\xi_1) + \dots + \tilde{f}(\xi_n)$ $(n \in \mathbb{N})$, $\tilde{f}(a_1) = -1$, $\tilde{f}(a_2) = 0$, $\tilde{f}(a_3) = 1$ and let the initial distribution be $P(\xi_0 = a_1) = \pi_1$, $P(\xi_0 = a_2) = \pi_2$, and $P(\xi_0 = a_3) = \pi_3$. Denote the distribution of $\tilde{S}_n$ by $\tilde{F}_n$. $\tilde{G}$ defines the measure with the Fourier transform:

$$\tilde{g}(t) = \left(\pi_1 + \frac{1-2a\cos t}{1-2a}\pi_2 + \pi_3\right)\frac{1-2(a-b)}{1-2(a-b)-2a(\cos t - 1)}$$
$$\times \exp\left\{\frac{2nb(1-2a)(\cos t - 1)}{(1-2a+2b)(1-2a\cos t)}\right\}. \tag{2}$$

As shown in [16], if $a, b \leqslant 1/30$, then

$$\|\tilde{F}_n - \tilde{G}\| \leqslant C\left(\min\left\{\frac{1}{n}, b\right\} + 0.2^n|a-b|\right). \tag{3}$$

The main part of the approximation $\tilde{G}$ is a compound Poisson distribution with a compounding symmetrized geometric distribution. The accuracy of approximation is at least $O(n^{-1})$. However, due to the symmetry of distribution and possible negative values, it is difficult to find a compatible insurance model.

3 Measures used for approximation

For convenience we present all Fourier transforms of measures used for construction of approximations in a separate table. Note that all measures are denoted by the same capital letters as their Fourier transforms (for example, $\widehat{H}(t)$ is the Fourier transform of H).

The measures can be easily found from their Fourier transforms using the formula

$$M\{k\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{-kit}\widehat{M}(t)\mathrm{d}t \quad \text{for all } k \in \mathbb{Z}.$$

For example,

$$\widehat{H}(t) = \frac{(1-\beta)\mathrm{e}^{\mathrm{i}t}}{1-\beta\mathrm{e}^{\mathrm{i}t}}.$$

Since $\widehat{I_a}(t) = \mathrm{e}^{\mathrm{i}ta}$, for all $k \in \mathbb{Z}$ we have

$$\begin{aligned}
H\{k\} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{-kit}\frac{(1-\beta)\mathrm{e}^{\mathrm{i}t}}{1-\beta\mathrm{e}^{\mathrm{i}t}}\mathrm{d}t = \frac{1-\beta}{2\pi}\int_{-\pi}^{\pi}\mathrm{e}^{-\mathrm{i}kt}\mathrm{e}^{\mathrm{i}t}\sum_{j=0}^{\infty}(\beta\mathrm{e}^{\mathrm{i}t})^j\,\mathrm{d}t \\
&= (1-\beta)\beta^{k-1}\sum_{j=0}^{\infty}\beta^{j-k+1}\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{e}^{-kit}\mathrm{e}^{(j+1)it}\mathrm{d}t \\
&= (1-\beta)\beta^{k-1}\sum_{j=0}^{\infty}\beta^{j-k+1}I_{j+1}\{k\} \\
&= (1-\beta)\sum_{j=0}^{\infty}\beta^{j}I_{j+1}\{k\}.
\end{aligned}$$

The other measures can be calculated analogously using their Fourier transforms presented in Table 1.

4 Results

We analyze the scheme of series, when transition probabilities may differ from one time period to another time period, that is, transition probabilities depend on $n \in \mathbb{N}$: $\alpha = \alpha(n)$, $\beta = \beta(n)$, $\gamma = \gamma(n)$. First we formulate a general approximation result for F_n, where possible smallness of α and γ is taken into account.

Table 1. Fourier transforms of used measures.

$\widehat{H}(t)=\dfrac{(1-\beta)\mathrm{e}^{\mathrm{i}t}}{1-\beta\mathrm{e}^{\mathrm{i}t}}$	$\widehat{A}_1(t)=\dfrac{1-\beta}{1+\gamma-\beta}(\widehat{\Psi}(t)-1)$
$\widehat{\Psi}(t)=\dfrac{(1-\alpha-\beta)\mathrm{e}^{\mathrm{i}t}}{1-\beta\mathrm{e}^{\mathrm{i}t}}$	$\widehat{A}_2(t)=-\dfrac{\beta(1-\beta)}{(1+\gamma-\beta)^2}(\widehat{H}(t)-1)(\widehat{\Psi}(t)-1)$
$\widehat{H}(t)-1=\dfrac{\mathrm{e}^{\mathrm{i}t}-1}{1-\beta\mathrm{e}^{\mathrm{i}t}}$	$\widehat{A}_3(t)=\dfrac{\beta^2(1-\beta)(\widehat{H}(t)-1)^2(\widehat{\Psi}(t)-1)}{(1+\gamma-\beta)^3}$
$\widehat{\Psi}(t)-1=\dfrac{(1-\alpha)\mathrm{e}^{\mathrm{i}t}-1}{1-\beta\mathrm{e}^{\mathrm{i}t}}$	$\widehat{A}_4(t)=-\dfrac{(1-\beta)^3(\widehat{\Psi}(t)-1)^2}{(1+\gamma-\beta)^3(1-\beta\mathrm{e}^{\mathrm{i}t})}$
$\widehat{U}(t)=(1-\alpha)\mathrm{e}^{\mathrm{i}t}-1$	$\widehat{A}_5(t)=\dfrac{3\beta(1-\beta)^3(\widehat{\Psi}(t)-1)^2(\widehat{H}(t)-1)}{(1+\gamma-\beta)^4(1-\beta\mathrm{e}^{\mathrm{i}t})}$
$\widehat{\Delta}(t)=1+\widehat{A}_1(t)\gamma$	$\widehat{A}_6(t)=\dfrac{2(1-\beta)^5(\widehat{\Psi}(t)-1)^3}{(1+\gamma-\beta)^5(1-\beta\mathrm{e}^{\mathrm{i}t})^2}$
$\widehat{\Delta}_1(t)=1+\widehat{A}_1(t)\gamma+(\widehat{A}_2(t)+\widehat{A}_4(t))\gamma^2$	
$\widehat{A}(t)=1+\widehat{A}_1(t)\gamma+\widehat{A}_2(t)\gamma^2+\widehat{A}_3(t)\gamma^3+\widehat{A}_4(t)\gamma^2+\widehat{A}_5(t)\gamma^3+\widehat{A}_6(t)\gamma^3$	
$\widehat{V}(t)=\dfrac{(\mathrm{e}^{(d+1)\mathrm{i}t}-1)(\beta-\gamma(1-\alpha))-(\mathrm{e}^{d\mathrm{i}t}-1)\widehat{\Delta}(t)}{(\widehat{A}(t)-\mathrm{e}^{d\mathrm{i}t})(2\widehat{\Delta}(t)-1+\gamma-\beta\mathrm{e}^{\mathrm{i}t})}+\dfrac{(\mathrm{e}^{\mathrm{i}t}-1)[\gamma\widehat{\Delta}(t)-\beta+\gamma(1-\alpha)]}{(\widehat{A}(t)-\mathrm{e}^{d\mathrm{i}t})(2\widehat{\Delta}(t)-1+\gamma-\beta\mathrm{e}^{\mathrm{i}t})}$	
$\widehat{V}_1(t)=\dfrac{(\mathrm{e}^{(d+1)\mathrm{i}t}-1)(\beta-\gamma(1-\alpha))-(\mathrm{e}^{d\mathrm{i}t}-1)\widehat{\Delta}(t)}{(\widehat{\Delta}_1(t)-\mathrm{e}^{d\mathrm{i}t})(2\widehat{\Delta}(t)-1+\gamma-\beta\mathrm{e}^{\mathrm{i}t})}+\dfrac{(\mathrm{e}^{\mathrm{i}t}-1)[\gamma\widehat{\Delta}(t)-\beta+\gamma(1-\alpha)]}{(\widehat{\Delta}_1(t)-\mathrm{e}^{d\mathrm{i}t})(2\widehat{\Delta}(t)-1+\gamma-\beta\mathrm{e}^{\mathrm{i}t})}$	
$\widehat{V}_2(t)=\dfrac{(\mathrm{e}^{(d+1)\mathrm{i}t}-1)(\beta-\gamma(1-\alpha))-(\mathrm{e}^{d\mathrm{i}t}-1)\widehat{\Delta}(t)}{(\widehat{\Delta}_1(t)-\mathrm{e}^{d\mathrm{i}t})(2\widehat{\Delta}_1(t)-1+\gamma-\beta\mathrm{e}^{\mathrm{i}t})}+\dfrac{(\mathrm{e}^{\mathrm{i}t}-1)[\gamma\widehat{\Delta}(t)-\beta+\gamma(1-\alpha)]}{(\widehat{\Delta}_1(t)-\mathrm{e}^{d\mathrm{i}t})(2\widehat{\Delta}_1(t)-1+\gamma-\beta\mathrm{e}^{\mathrm{i}t})}$	
$\widehat{G}(t)=\exp\Big\{\widehat{A}(t)-1-\frac{1}{2}\Big(\widehat{A}_1^2(t)\gamma^2+2\widehat{A}_1(t)(\widehat{A}_2(t)+\widehat{A}_4(t))\gamma^3\Big)+\frac{1}{3}\widehat{A}_1^3(t)\gamma^3\Big\}$	
$\widehat{G}_1(t)=\exp\Big\{\widehat{A}_1(t)\gamma+\Big(\widehat{A}_2(t)+\widehat{A}_4(t)-\dfrac{1}{2}\widehat{A}_1^2(t)\Big)\gamma^2\Big\}$	
$\widehat{E}(t)=\dfrac{\alpha\gamma\mathrm{e}^{(n+1)d\mathrm{i}t}}{(e^{(d-1)\mathrm{i}t}-\beta)(\mathrm{e}^{d\mathrm{i}t}-(1-\gamma))-\gamma(1-\alpha-\beta)}$	

Theorem 1. *Let condition (1) hold. Then, for all* $n=1,2,\dots,$

$$|F_n-(G^nV+E)|_{\mathrm{K}}\leqslant C(d+1)\bigg(\mathrm{e}^{-Cn\gamma\alpha}\sqrt{\frac{\gamma}{n}}+(\beta+4\gamma)^n\bigg), \tag{4}$$

$$\|F_n-(G^nV+E)\|_\infty\leqslant C(d+1)\bigg(\frac{\mathrm{e}^{-Cn\gamma\alpha}}{n}+(\beta+4\gamma)^n\bigg).$$

Remark 1. Observe that, since $\beta+4\gamma\leqslant 0.35$, the second term in (4) tends to zero exponentially.

Unlike (2), there are two components in our approximation: the first one contains n-fold convolution of a signed compound Poisson measure, the second one takes into account the probability of death (the absorbing state). The measures of approximation are chosen in a way ensuring that the accuracy of approximation is at least as good as in the Berry–Esseen theorem.

Corollary 1. *Let condition (1) hold. Then, for all* $n = 1, 2, \dots,$

$$|F_n - (G^n V + E)|_K \leqslant \frac{C(d+1)}{\sqrt{n}}.$$

This accuracy is reached, when $\alpha\gamma = O(n^{-1})$. If $\alpha, \gamma \geqslant C_1 > 0$, the accuracy of approximation is exponentially sharp. That prompts a question: Is it possible to simplify the structure of approximation by imposing more restrictive assumptions? The answer is positive for α uniformly separated from zero for all n.

Theorem 2. *Let condition (1) hold and* $\alpha \geqslant C_2$. *Then, for all* $n = 1, 2, \dots,$

$$|F_n - (G_1^n V_1 + E)|_K \leqslant C(d+1)\big(\gamma e^{-Cn\gamma} + (\beta + 4\gamma)^n\big). \tag{5}$$

Observe that the accuracy of approximation in (5) is at least of order $O(n^{-1})$. This accuracy is reached if $\gamma = O(n^{-1})$.

If both probabilities are uniformly separated from zero, F_n is exponentially close to the measure E.

Theorem 3. *Let condition (1) hold and* $\alpha, \gamma \geqslant C_2$. *Then, for all* $n = 1, 2, \dots,$

$$\|F_n - E\| \leqslant C(d+1)e^{-Cn}. \tag{6}$$

Observe that, if the scheme of sequences is analyzed, all probabilities do not depend on n and hence the conditions of Theorem 3 are satisfied as long as condition (1) holds. Note also that in Theorem 3 the stronger total variation norm is used.

Theorem 4. *Let condition (1) hold and* $\alpha \geqslant C_2$. *Then, for all* $n = 1, 2, \dots,$

$$\|F_n - (G_1^n V_2 + E)\| \leqslant C(d+1)\big(\gamma e^{-Cn\gamma}(1 + \beta/\gamma) + n(\beta + 4\gamma)^n\big). \tag{7}$$

Corollary 2. *Let condition (1) hold and* $\alpha \geqslant C_2$. *Then, for all* $n = 1, 2, \dots,$

$$\|F_n - (G_1^n V_2 + E)\| \leqslant \frac{C(d+1)e^{-Cn\gamma}}{n}\left(1 + \frac{\beta}{\gamma}\right). \tag{8}$$

Remark 2. The local estimates in Theorem 2, 3, and 4 have the same order as in (5), (6), and (7), hence we do no formulate them separately.

In insurance models, tail probabilities are very important, see, for example [11, 17, 20]. Therefore, we formulate some non-uniform estimates for the case when α is uniformly separated from zero.

Theorem 5. *Let condition (1) hold and* $\alpha \geqslant C_2$. *Then, for any integer* $k \geqslant 1$ *and* $n \in \mathbb{N}$,

$$|F_n\{k\} - (G_1^n V_2 + E)\{k\}| \leqslant \frac{C(d+1)e^{-Cn\gamma}(\beta+\gamma)}{n(\beta + (k+1)\gamma)}. \tag{9}$$

$$|F_n(k) - (G_1^n V_2 + E)(k)| \leqslant \frac{Cd^2 e^{-Cn\gamma}}{n(1 + k\gamma^2)}. \tag{10}$$

Remark 3. The non-uniform estimate for distribution functions (10) is quite inaccurate if γ is small. On the other hand, the local non-uniform estimate is at least of order $O(n^{-1}k^{-1})$, when β is of the same order as γ.

When γ is uniformly separated from zero and α is small, estimate (4) could not be simplified.

5 Auxiliary results

We begin from the inversion inequalities.

Lemma 1. *Let $M \in M_{\mathbb{Z}}$. Then*

$$|M|_{\mathrm{K}} \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{|\widehat{M}(t)|}{|\mathrm{e}^{\mathrm{i}t}-1|}\mathrm{d}t, \tag{11}$$

$$\|M\|_{\infty} \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} |\widehat{M}(t)|\,\mathrm{d}t. \tag{12}$$

If, in addition, $\sum_{k\in\mathbb{Z}} |k||M\{k\}| < \infty$, then

$$\|M\| \leqslant (1+b\pi)^{1/2}\biggl(\frac{1}{2\pi}\int_{-\pi}^{\pi} |\widehat{M}(t)|^2 + \frac{1}{b^2}\bigl|\bigl(\mathrm{e}^{-\mathrm{i}ta}\widehat{M}(t)\bigr)'\bigr|^2\mathrm{d}t\biggr)^{1/2}, \tag{13}$$

and, for any $a \in \mathbb{R}$, $b > 0$,

$$|k-a||M\{k\}| \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} \bigl|\bigl(\widehat{M}(t)\mathrm{e}^{-\mathrm{i}ta}\bigr)'\bigr|\mathrm{d}t, \tag{14}$$

$$|k-a||M(k)| \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} \biggl|\biggl(\frac{\widehat{M}(t)}{\mathrm{e}^{-\mathrm{i}t}-1}\mathrm{e}^{-\mathrm{i}ta}\biggr)'\biggr|\mathrm{d}t. \tag{15}$$

Observe that (11) and (15) are trivial if integrals on the right-hand side are infinite. All inequalities are well-known and can be found in [2] Section 6.1 and Section 6.2; see, also [13] and Lemma 3.3 in [15].

The characteristic function method is used for the analysis of the model. Therefore our next step is to obtain $\widehat{F}_n(t)$.

Lemma 2. *Let condition (1) hold. Then the characteristic function $\widehat{F}_n(t)$ can be expressed in the following way:*

$$\widehat{F}_n(t) = \widehat{\Lambda}_1^n(t)\widehat{W}_1(t) + \widehat{\Lambda}_2^n(t)\widehat{W}_2(t) + \widehat{\Lambda}_3^n(t)\widehat{W}_3(t). \tag{16}$$

Here

$$\widehat{\Lambda}_{1,2}(t) = \frac{1-\gamma+\beta\mathrm{e}^{\mathrm{i}t} \pm \sqrt{\widehat{D}(t)}}{2}, \qquad \widehat{\Lambda}_3(t) = \mathrm{e}^{d\mathrm{i}t},$$
$$\widehat{D}(t) = (1-\gamma+\beta\mathrm{e}^{\mathrm{i}t})^2 - 4\mathrm{e}^{\mathrm{i}t}\bigl(\beta-\gamma(1-\alpha)\bigr),$$
$$\widehat{W}_{1,2}(t) = \frac{(\mathrm{e}^{(d+1)\mathrm{i}t}-1)(\beta-\gamma(1-\alpha)) - (\mathrm{e}^{d\mathrm{i}t}-1)\widehat{\Lambda}_{1,2}(t)}{\pm(\widehat{\Lambda}_{1,2}(t)-\mathrm{e}^{d\mathrm{i}t})\sqrt{\widehat{D}(t)}}$$

$$+\frac{(\mathrm{e}^{\mathrm{i}t}-1)[\gamma\widehat{\varLambda}_{1,2}(t)-\beta+\gamma(1-\alpha)]}{\pm(\widehat{\varLambda}_{1,2}(t)-\mathrm{e}^{d\mathrm{i}t})\sqrt{\widehat{D}(t)}},$$

$$\widehat{W}_3(t)=\frac{\alpha\gamma\mathrm{e}^{d\mathrm{i}t}}{(e^{(d-1)\mathrm{i}t}-\beta)(\mathrm{e}^{d\mathrm{i}t}-(1-\gamma))-\gamma(1-\alpha-\beta)}.$$

Proof. The characteristic function $\widehat{F}_n(t)$ can be written as follows, see [16]:

$$\widehat{F}_n(t)=(\pi_1,\pi_2,\pi_3)\big(\widehat{\varLambda}_1^n(t)\vec{y}_1\vec{z}_1^T+\widehat{\varLambda}_2^n(t)\vec{y}_2\vec{z}_2^T+\widehat{\varLambda}_3^n(t)\vec{y}_3\vec{z}_3^T\big)(1,1,1)^T. \tag{17}$$

Expression (16) is known as Perron's formula. Similar expression was used for Markov binomial distribution; see, for example, [3]. $\widehat{\varLambda}_j(t)$ $(j=1,2,3)$ are eigenvalues of the following matrix:

$$\tilde{P}(t)=\begin{pmatrix}1-\gamma & \gamma\mathrm{e}^{\mathrm{i}t} & 0\\ 1-\alpha-\beta & \beta\mathrm{e}^{\mathrm{i}t} & \alpha\mathrm{e}^{d\mathrm{i}t}\\ 0 & 0 & \mathrm{e}^{d\mathrm{i}t}\end{pmatrix}.$$

We find the eigenvalues by solving the following equation:

$$|\tilde{P}(t)-\widehat{\varLambda}(t)I|=0.$$

It is not difficult to prove that

$$\widehat{\varLambda}_{1,2}(t)^2-\widehat{\varLambda}_{1,2}(t)(1-\gamma+\beta\mathrm{e}^{\mathrm{i}t})+\mathrm{e}^{\mathrm{i}t}\big(\beta-\gamma(1-\alpha)\big)=0, \tag{18}$$

and

$$\mathrm{e}^{d\mathrm{i}t}-\widehat{\varLambda}_3(t)=0.$$

Hence,

$$\begin{aligned}\widehat{\varLambda}_{1,2}(t)&=\frac{1-\gamma+\beta\mathrm{e}^{\mathrm{i}t}\pm\widehat{D}^{1/2}(t)}{2},\\ \widehat{D}(t)&=(1-\gamma+\beta\mathrm{e}^{\mathrm{i}t})^2-4\mathrm{e}^{\mathrm{i}t}\big(\beta-\gamma(1-\alpha)\big),\\ \widehat{\varLambda}_3(t)&=\mathrm{e}^{d\mathrm{i}t}.\end{aligned}$$

Eigenvectors $\vec{y}_j$ and $\vec{z}_j$ are obtained by solving the following system of equations:

$$\begin{cases}\tilde{P}(t)\vec{y}_j=\widehat{\varLambda}(t)\vec{y}_j,\\ \vec{z}_j^T\tilde{P}(t)=\widehat{\varLambda}(t)\vec{z}_j^T,\\ \vec{z}_j^T\vec{y}_j=1.\end{cases} \tag{19}$$

From the first equation of system (19) we get that $y_{j,3}=0$, hence the other two equations are equivalent because of equation (18). Therefore,

$$\vec{y}_j^{\,T}=\left(y_{j,1},\frac{1-\alpha-\beta}{\widehat{\varLambda}_j(t)-\beta\mathrm{e}^{\mathrm{i}t}}y_{j,1},0\right),\quad j=1,2. \tag{20}$$

Similarly, from the second equation of system (19) we get

$$\vec{z_j}^T = \left(z_{j,1}, \frac{\widehat{\Lambda}_j(t) - (1-\gamma)}{1-\alpha-\beta} z_{j,1}, \frac{\alpha e^{dit}(\widehat{\Lambda}_j(t) - (1-\gamma))}{(\widehat{\Lambda}_j(t) - e^{dit})(1-\alpha-\beta)} z_{j,1} \right), \quad j = 1, 2. \tag{21}$$

The third equation of system (19) can be written as

$$\begin{aligned} \vec{z_j}^T \vec{y_j} &= 1, \\ y_{j,1}z_{j,1} + y_{j,2}z_{j,2} + y_{j,3}z_{j,3} &= 1, \\ y_{j,1}z_{j,1} + \frac{\widehat{\Lambda}_j(t) - (1-\gamma)}{\widehat{\Lambda}_j(t) - \beta e^{it}} y_{j,1}z_{j,1} + 0 &= 1, \\ 1 + \frac{\gamma e^{it}(1-\alpha-\beta)}{(\widehat{\Lambda}_j(t) - \beta e^{it})^2} &= \frac{1}{y_{j,1}z_{j,1}}. \end{aligned} \tag{22}$$

According to assumption, $(\pi_1, \pi_2, \pi_3) = (1, 0, 0)$. Substituting (20), (21), and (22) into (17), we obtain

$$\widehat{W}_{1,2}(t) = (1, 0, 0)\vec{y_j}\vec{z_j}^T \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1 + \frac{\widehat{\Lambda}_j(t)-(1-\gamma)}{1-\alpha-\beta}(1 + \frac{\alpha e^{dit}}{\widehat{\Lambda}_j(t)-e^{dit}})}{1 + \frac{\gamma e^{it}(1-\alpha-\beta)}{(\widehat{\Lambda}_j(t)-\beta e^{it})^2}}, \quad j = 1, 2.$$

From equation (18) we get

$$\frac{\widehat{\Lambda}_j(t) - (1-\gamma)}{1-\alpha-\beta} = \frac{\gamma e^{it}}{\widehat{\Lambda}_j(t) - \beta e^{it}}.$$

Hence,

$$\widehat{W}_{1,2}(t) = \frac{1 + \frac{\gamma e^{it}}{\widehat{\Lambda}_{1,2}(t)-\beta e^{it}}(1 + \frac{\alpha e^{dit}}{\widehat{\Lambda}_{1,2}(t)-e^{dit}})}{1 + \frac{(1-\alpha-\beta)\gamma e^{it}}{(\widehat{\Lambda}_{1,2}(t)-\beta e^{it})^2}}. \tag{23}$$

Applying equation (18), we prove that the numerator of $\widehat{W}_{1,2}(t)$ is equal to

$$\begin{aligned} &\frac{(e^{(d+1)it} - 1)(\beta - \gamma(1-\alpha)) - (e^{dit} - 1)\widehat{\Lambda}_{1,2}(t)}{(\widehat{\Lambda}_{1,2}(t) - \beta e^{it})(\widehat{\Lambda}_{1,2}(t) - e^{dit})} \\ &\quad + \frac{(e^{it} - 1)[\gamma \widehat{\Lambda}_{1,2}(t) - (\beta - \gamma(1-\alpha))]}{(\widehat{\Lambda}_{1,2}(t) - \beta e^{it})(\widehat{\Lambda}_{1,2}(t) - e^{dit})}. \end{aligned} \tag{24}$$

It is easy to check that

$$(1 - \gamma - \beta e^{it})^2 + 4\gamma e^{it}(1-\alpha-\beta) = \widehat{D}(t). \tag{25}$$

Similarly

$$\begin{aligned} &\left(\widehat{\Lambda}_{1,2}(t) - \beta e^{it}\right)^2 \\ &\quad = \frac{(1-\gamma-\beta e^{it})^2 \pm 2(1-\gamma-\beta e^{it})\sqrt{\widehat{D}(t)} + \widehat{D}(t)}{4}. \end{aligned} \tag{26}$$

Using (25) and (26), we obtain

$$\begin{aligned}&\big(\widehat{\Lambda}_{1,2}(t)-\beta e^{it}\big)^2+(1-\alpha-\beta)\gamma e^{it}\\&\quad=\frac{\sqrt{\widehat{D}(t)}\big(\sqrt{\widehat{D}(t)}\pm(1-\gamma-\beta e^{it})\big)}{2}.\end{aligned}\tag{27}$$

Notice that

$$2\big(\widehat{\Lambda}_{1,2}(t)-\beta e^{it}\big)=1-\gamma-\beta e^{it}\pm\sqrt{\widehat{D}(t)}.$$

Substituting (24), (26), and (27) into (23), we complete the proof for $\widehat{\Lambda}_{1,2}$ and $\widehat{W}_{1,2}(t)$.

Similarly, system (19) is solved with $\widehat{\Lambda}_3(t)=e^{dit}$. We get

$$\vec{y}_3^{\,T}=\left(y_{3,1},\ \frac{e^{dit}-(1-\gamma)}{\gamma e^{it}}y_{3,1},\ \frac{(e^{dit}-\beta e^{it})y_{3,2}-(1-\alpha-\beta)y_{3,1}}{\alpha e^{dit}}y_{3,1}\right),\tag{28}$$

$$\vec{z}_3^{\,T}=(0,0,z_{3,3}).\tag{29}$$

Hence,

$$\frac{1}{y_{3,1}z_{3,3}}=\frac{(e^{(d-1)it}-\beta)(e^{dit}-(1-\gamma))-\gamma(1-\alpha-\beta)}{\alpha\gamma e^{dit}}.\tag{30}$$

Substituting (28), (29), and (30) into (17), we get

$$\begin{aligned}\widehat{W}_3(t)&=(1,0,0)\vec{y}_3\vec{z}_3^{\,T}\begin{pmatrix}1\\1\\1\end{pmatrix}=y_{3,1}z_{3,3}\\&=\frac{\alpha\gamma e^{dit}}{(e^{(d-1)it}-\beta)(e^{dit}-(1-\gamma))-\gamma(1-\alpha-\beta)}.\end{aligned}\qquad\square$$

It is not difficult to notice that $|\widehat{W}_3(t)|$ is equal to 1 at some points, for example, $\widehat{W}_3(0)=1$, since

$$\widehat{W}_3(0)=\frac{\alpha\gamma}{(1-\beta)(1-(1-\gamma))-\gamma(1-\alpha-\beta)}=\frac{\alpha\gamma}{\alpha\gamma}=1.$$

Therefore, one cannot expect that $\widehat{\Lambda}_3^n(t)\widehat{W}_3$ be small. Therefore we concentrate our research on possible asymptotic behavior of other components of $\widehat{F}_n(t)$. We begin from a short expansion of $\sqrt{\widehat{D}(t)}$.

Observe that $\widehat{D}(t)$ can be written in the following way:

$$\widehat{D}(t)=(1+\gamma-\beta e^{it})^2\left(1+\frac{4\gamma((1-\alpha)e^{it}-1)}{(1+\gamma-\beta e^{it})^2}\right).\tag{31}$$

Lemma 3. *Let condition (1) hold, $|t|\leqslant\pi$. Then*

$$\sqrt{\widehat{D}(t)}=1+\gamma-\beta e^{it}+5.81\theta\gamma.$$

Proof. $\sqrt{\widehat{D}(t)}$ can be expanded and written as

$$\begin{aligned}
\sqrt{\widehat{D}(t)} &= (1+\gamma-\beta \mathrm{e}^{\mathrm{i}t})\sum_{j=0}^{\infty}\binom{1/2}{j}\left(\frac{4\gamma((1-\alpha)\mathrm{e}^{\mathrm{i}t}-1)}{(1+\gamma-\beta\mathrm{e}^{\mathrm{i}t})^2}\right)^j \\
&= (1+\gamma-\beta \mathrm{e}^{\mathrm{i}t}) + \frac{2\gamma((1-\alpha)\mathrm{e}^{\mathrm{i}t}-1)}{1+\gamma-\beta\mathrm{e}^{\mathrm{i}t}} \\
&\quad + \frac{16\gamma^2((1-\alpha)\mathrm{e}^{\mathrm{i}t}-1)^2}{(1+\gamma-\beta\mathrm{e}^{\mathrm{i}t})^3}\sum_{j=2}^{\infty}\binom{1/2}{j}\left(\frac{4\gamma((1-\alpha)\mathrm{e}^{\mathrm{i}t}-1)}{(1+\gamma-\beta\mathrm{e}^{\mathrm{i}t})^2}\right)^{j-2} \\
&= (1+\gamma-\beta \mathrm{e}^{\mathrm{i}t}) + \frac{2\gamma((1-\alpha)\mathrm{e}^{\mathrm{i}t}-1)}{1+\gamma-\beta\mathrm{e}^{\mathrm{i}t}} \\
&\quad + \frac{2\theta\gamma^2|(1-\alpha)\mathrm{e}^{\mathrm{i}t}-1|^2}{|1+\gamma-\beta\mathrm{e}^{\mathrm{i}t}|^3}\sum_{j=0}^{\infty}\left|\frac{4\gamma((1-\alpha)\mathrm{e}^{\mathrm{i}t}-1)}{(1+\gamma-\beta\mathrm{e}^{\mathrm{i}t})^2}\right|^j. \qquad (32)
\end{aligned}$$

Observe that

$$\begin{aligned}
\left|\frac{4\gamma((1-\alpha)\mathrm{e}^{\mathrm{i}t}-1)}{(1+\gamma-\beta\mathrm{e}^{\mathrm{i}t})^2}\right| &\leqslant \frac{8\cdot 0.05}{(0.85+0.05)^2} \leqslant 0.5, \\
\frac{\theta\gamma^2|(1-\alpha)\mathrm{e}^{\mathrm{i}t}-1|^2}{|1+\gamma-\beta\mathrm{e}^{\mathrm{i}t}|^3}\sum_{j=0}^{\infty}\left|\frac{4\gamma((1-\alpha)\mathrm{e}^{\mathrm{i}t}-1)}{(1+\gamma-\beta\mathrm{e}^{\mathrm{i}t})^2}\right|^j &\leqslant 0.55\theta\gamma.
\end{aligned}$$

Therefore

$$\begin{aligned}
\sqrt{\widehat{D}(t)} &= 1+\gamma-\beta\mathrm{e}^{\mathrm{i}t} + \frac{4\theta\gamma}{0.85} + 2\cdot 0.55\theta\gamma \\
&= 1+\gamma-\beta\mathrm{e}^{\mathrm{i}t} + 5.81\theta\gamma. \qquad \square
\end{aligned}$$

Next we prove that $\widehat{\varLambda}_2(t)$ is always small.

Lemma 4. *Let condition (1) hold, $|t| \leqslant \pi$. Then*

$$|\widehat{\varLambda}_2(t)| \leqslant \beta + 4\gamma.$$

Proof. From Lemma 3 we get

$$\begin{aligned}
|\widehat{\varLambda}_2(t)| &= \left|\frac{1-\gamma+\beta\mathrm{e}^{\mathrm{i}t}-\sqrt{\widehat{D}(t)}}{2}\right| \\
&= \frac{1}{2}\left|1-\gamma+\beta\mathrm{e}^{\mathrm{i}t}-(1+\gamma-\beta\mathrm{e}^{\mathrm{i}t}+5.81\theta\gamma)\right| \leqslant \beta+4\gamma. \qquad \square
\end{aligned}$$

Corollary 3. *Let condition (1) hold, $|t| \leqslant \pi$. Then*

$$|\widehat{\varLambda}_2(t)| \leqslant 0.35.$$

The following estimate shows that $\varLambda_1$ behaves similarly to the compound Poisson distribution.

Lemma 5. *Let condition (1) hold, $|t| \leqslant \pi$. Then*

$$
\begin{aligned}
|\widehat{\Lambda}_1(t)| &\leqslant 1 + 0.4(1-\alpha)\gamma Re\big(\widehat{H}(t)-1\big) - 0.2\alpha\gamma \\
&\leqslant \exp\big\{0.4(1-\alpha)\gamma Re\big(\widehat{H}(t)-1\big) - 0.2\alpha\gamma\big\}.
\end{aligned}
$$

Proof. It is not difficult to check that

$$
\frac{1}{1+\gamma-\beta \mathrm{e}^{\mathrm{i}t}} = \frac{1-\beta}{1+\gamma-\beta}\frac{1}{1-\beta\mathrm{e}^{\mathrm{i}t}} - \frac{\beta\gamma}{1+\gamma-\beta\mathrm{e}^{\mathrm{i}t}}\frac{\mathrm{e}^{\mathrm{i}t}-1}{1-\beta\mathrm{e}^{\mathrm{i}t}}\frac{1}{1+\gamma-\beta}. \tag{33}
$$

From (32) and (33) it follows that

$$
\begin{aligned}
|\widehat{\Lambda}_1(t)| &= \left|\frac{1-\gamma+\beta\mathrm{e}^{\mathrm{i}t}+\sqrt{\widehat{D}(t)}}{2}\right| \\
&\leqslant \left|1+\frac{\gamma(1-\beta)}{1+\gamma-\beta}\big(\widehat{\Psi}(t)-1\big)\right| + \frac{\beta\gamma^2}{(1+\gamma-\beta)^2}|\widehat{\Psi}(t)-1||\mathrm{e}^{\mathrm{i}t}-1| \\
&\quad + 2\gamma^2|\widehat{\Psi}(t)-1|^2\frac{(1+\beta)^2}{(1+\gamma-\beta)^3}.
\end{aligned} \tag{34}
$$

Notice that

$$
\begin{aligned}
|\widehat{\Psi}(t)|^2 &= \big(Re\widehat{\Psi}(t)\big)^2 + \big(Im\widehat{\Psi}(t)\big)^2 \leqslant \left(1-\frac{\alpha}{1-\beta}\right)^2 \leqslant 1, \\
|\widehat{\Psi}(t)-1|^2 &\leqslant 2\big(1-Re\widehat{\Psi}(t)\big) - \frac{\alpha}{1-\beta}\left(2-\frac{\alpha}{1-\beta}\right).
\end{aligned} \tag{35}
$$

For all $0 \leqslant \nu \leqslant 1$, we have

$$
\begin{aligned}
|1+\nu(\widehat{\Psi}(t)-1)| &= \sqrt{(1-\nu)+\nu Re\widehat{\Psi}(t)+\mathrm{i}\nu Im\widehat{\Psi}(t)} \\
&\leqslant 1+\nu(1-\nu)\big(Re\widehat{\Psi}(t)-1\big).
\end{aligned} \tag{36}
$$

Let

$$
\nu = \frac{\gamma(1-\beta)}{1+\gamma-\beta}.
$$

Substituting (35) into (34) and applying inequality (36), we get

$$
\begin{aligned}
|\widehat{\Lambda}_1(t)| &\leqslant 1+\nu(1-\nu)\big(Re\widehat{\Psi}(t)-1\big) + \frac{\beta\gamma^2}{(1+\gamma-\beta)^2}|\widehat{\Psi}(t)-1||\mathrm{e}^{\mathrm{i}t}-1| \\
&\quad + \frac{4\gamma^2(1+\beta)^2}{(1+\gamma-\beta)^3}\big(1-Re\widehat{\Psi}(t)\big) - \frac{2\gamma^2\alpha}{1-\beta}\frac{(1+\beta)^2}{(1+\gamma-\beta)^3}\left(2-\frac{\alpha}{1-\beta}\right).
\end{aligned}
$$

$|\widehat{\Psi}(t)-1|$ can be estimated as

$$
|\widehat{\Psi}(t)-1| \leqslant \frac{2}{1-\beta},
$$

and $|e^{it}-1|$ can be estimated as

$$|e^{it}-1| \leqslant \frac{|(1-\alpha)e^{it}-1|}{|1-\beta e^{it}|}|1-\beta e^{it}|+\alpha \leqslant |\widehat{\Psi}(t)-1|(1+\beta)+\alpha.$$

Then

$$\begin{aligned}|\widehat{\Lambda}_1(t)| \leqslant 1 &+ \big(Re\widehat{\Psi}(t)-1\big)\frac{\gamma}{1+\gamma-\beta}\bigg((1-\beta)\bigg(1-\frac{\gamma(1-\beta)}{1+\gamma-\beta}\bigg)\\ &- \frac{2\gamma\beta(1+\beta)}{1+\gamma-\beta}-\frac{4\gamma(1+\beta)^2}{(1+\gamma-\beta)^2}\bigg)\\ &+ \frac{2\alpha\gamma^2}{(1-\beta)(1+\gamma-\beta)}\bigg(\frac{\beta}{1+\gamma-\beta}-\frac{(1+\beta)^2}{(1+\gamma-\beta)^2}\bigg(2-\frac{\alpha}{1-\beta}\bigg)\bigg).\end{aligned}$$

Notice that

$$Re\widehat{\Psi}(t)-1 = (1-\alpha)Re\big(\widehat{H}(t)-1\big)-\frac{\alpha-\alpha\beta\cos(t)}{|1-\beta e^{it}|^2}.$$

Finally,

$$\begin{aligned}|\widehat{\Lambda}_1(t)| &\leqslant 1+Re\big(\widehat{H}(t)-1\big)\frac{(1-\alpha)\gamma}{1+\gamma-\beta}\bigg((1-\beta)\bigg(1-\frac{\gamma(1-\beta)}{1+\gamma-\beta}\bigg)\\ &\quad -\frac{2\gamma\beta(1+\beta)}{1+\gamma-\beta}-\frac{4\gamma(1+\beta)^2}{(1+\gamma-\beta)^2}\bigg)\\ &\quad -\frac{\alpha\gamma}{1+\gamma-\beta}\bigg[\frac{1-\beta\cos(t)}{|1-\beta e^{it}|^2}\bigg((1-\beta)\bigg(1-\frac{\gamma(1-\beta)}{1+\gamma-\beta}\bigg)\\ &\quad -\frac{2\gamma\beta(1+\beta)}{1+\gamma-\beta}-\frac{4\gamma(1+\beta)^2}{(1+\gamma-\beta)^2}\bigg)\\ &\quad -\frac{2\gamma}{1-\beta}\bigg(\frac{\beta}{1+\gamma-\beta}-\frac{(1+\beta)^2}{(1+\gamma-\beta)^2}\bigg(2-\frac{\alpha}{1-\beta}\bigg)\bigg)\bigg]\\ &\leqslant 1+0.4(1-\alpha)\gamma Re\big(\widehat{H}(t)-1\big)-0.2\alpha\gamma\\ &\leqslant \exp\big\{0.4(1-\alpha)\gamma Re\big(\widehat{H}(t)-1\big)-0.2\alpha\gamma\big\}.\end{aligned} \tag{37}$$

□

Corollary 4. *Let condition (1) hold, $|t|\leqslant\pi$. Then*

$$|\widehat{\Lambda}_1(t)| \leqslant 1+C\gamma\big(Re\widehat{H}(t)-1-\alpha\big) \leqslant \exp\big\{C\gamma\big(Re\widehat{H}(t)-1-\alpha\big)\big\}.$$

Next we demonstrate that $|\widehat{W}_2(t)|$ is always small.

Lemma 6. *Let condition (1) hold, $|t|\leqslant\pi$. Then*

$$|\widehat{W}_2(t)| \leqslant 2(d+1)|e^{it}-1|.$$

Proof. From Lemma 3 we have

$$\left|\sqrt{\widehat{D}(t)}\right| \geqslant 1+\gamma-\beta-5.81\gamma \geqslant 1-4.81\cdot 0.05-0.15 \geqslant 0.6. \tag{38}$$

By applying Corollary 3, we get

$$|\widehat{\Lambda}_2(t)-\mathrm{e}^{dit}| \geqslant 1-|\widehat{\Lambda}_2(t)| \geqslant 1-0.35=0.65. \tag{39}$$

Hence,

$$\begin{aligned}\left|\widehat{W}_2(t)\right| &\leqslant \frac{(d+1)|\mathrm{e}^{it}-1|(2|\beta-\gamma(1-\alpha)|+(1+\gamma)|\widehat{\Lambda}_2(t)|)}{0.65\cdot 0.6}\\ &\leqslant \frac{(d+1)|\mathrm{e}^{it}-1|(2\max\{\beta,\gamma(1-\alpha)\}+(1+\gamma)\cdot 0.35)}{0.39}\\ &\leqslant 2(d+1)|\mathrm{e}^{it}-1|. \end{aligned}\tag{40}$$

□

To approximate $|\widehat{W}_1(t)|$, we need a longer expansion for $\sqrt{\widehat{D}(t)}$.

Lemma 7. *Let condition (1) hold, $|t| \leqslant \pi$. Then*

$$\sqrt{\widehat{D}(t)}=2\widehat{A}(t)-1+\gamma-\beta\mathrm{e}^{it}+C\theta\gamma^4\big(\big(1-Re\widehat{H}(t)\big)^2+\alpha^4\big).$$

If also $\alpha \geqslant C_2$, then

$$\left(\sqrt{\widehat{D}(t)}\right)'=\big(2\widehat{\Delta}_1(t)-1+\gamma-\beta\mathrm{e}^{it}\big)'+C\theta\gamma^3.$$

Proof. The expansion of $\widehat{D}(t)$ follows from equations (31) and (33). The second equation of this lemma is proved similarly. □

Corollary 5. *Let condition (1) hold, $|t| \leqslant \pi$. Then*

$$\widehat{\Lambda}_1(t)=\widehat{A}(t)+C\theta\gamma^4\big(\big(1-Re\widehat{H}(t)\big)^2+\alpha^4\big).$$

Corollary 6. *Let condition (1) hold, $\alpha \geqslant C_2$, $|t| \leqslant \pi$. Then*

$$\widehat{\Lambda}_1(t)=1+\widehat{A}_1(t)\gamma+\big(\widehat{A}_2(t)+\widehat{A}_4(t)\big)\gamma^2+C\theta\gamma^3.$$

The following three lemmas are needed for the approximation of W_1.

Lemma 8. *Let condition (1) hold, $|t| \leqslant \pi$. Then*

$$|\widehat{A}(t)| \leqslant 1+C\gamma\big(Re\widehat{H}(t)-1-\alpha\big).$$

If also $\alpha \geqslant C_2$, then there exists C such that

$$|\widehat{\Delta}_1(t)| \leqslant 1-C\gamma.$$

Proof. The proof is very similar to the proof of Lemma 5 and, therefore, is omitted. □

Lemma 9. *Let condition (1) hold, $|t| \leqslant \pi$. Then*

$$|\widehat{W}_1(t) - \widehat{V}(t)| \leqslant C(d+1)\gamma|\mathrm{e}^{\mathrm{i}t} - 1|.$$

Proof. From Corollary 4 and Lemma 8 it follows that

$$|\widehat{\Lambda}_1(t) - \mathrm{e}^{dit}| \geqslant C\gamma\big(1 - Re\widehat{H}(t) + \alpha\big), \tag{41}$$
$$|\widehat{\Lambda}(t) - \mathrm{e}^{dit}| \geqslant C\gamma\big(1 - Re\widehat{H}(t) + \alpha\big). \tag{42}$$

Applying (38), (41), (42), Lemma 7 and Corollary 5, the result follows. □

Lemma 10. *Let condition (1) hold, $\alpha \geqslant C_2$, $|t| \leqslant \pi$. Then*

$$|\widehat{W}_1(t) - \widehat{V}_1(t)| \leqslant C(d+1)\gamma|\mathrm{e}^{\mathrm{i}t} - 1|.$$

Proof. Since $\alpha \geqslant C_2$,

$$|\widehat{\Lambda}_1(t) - \mathrm{e}^{dit}| \geqslant C\gamma\big(1 - Re\widehat{H}(t) + \alpha\big) \geqslant C\gamma(0 + C_2) \geqslant C\gamma.$$

From Corollary 6 it follows that

$$|\widehat{\Lambda}_1(t) - \widehat{\Delta}_1| = C\gamma^3.$$

Also, from Lemma 8 it follows that

$$|\widehat{\Delta}_1 - \mathrm{e}^{dit}| \geqslant 1 - (1 - C\gamma) = C\gamma. \tag{43}$$

Hence, it is easy to check that the inequality of the lemma is correct. □

Lemma 11. *Let condition (1) hold, $|t| \leqslant \pi$. Then*

$$\int_{-\pi}^{\pi} \frac{|\widehat{\Lambda}_1(t)|^n|\widehat{W}_1(t) - \widehat{V}(t)|}{|\mathrm{e}^{\mathrm{i}t} - 1|}\mathrm{d}t \leqslant C(d+1)\sqrt{\frac{\gamma}{n}}\mathrm{e}^{-Cn\gamma\alpha}, \tag{44}$$
$$\int_{-\pi}^{\pi} |\widehat{\Lambda}_1(t)|^n|\widehat{W}_1(t) - \widehat{V}(t)|\mathrm{d}t \leqslant C(d+1)\frac{\mathrm{e}^{-Cn\gamma\alpha}}{n}. \tag{45}$$

Proof. It is obvious that

$$Re\widehat{H}(t) - 1 = \frac{(1+\beta)(\cos(t) - 1)}{|1 - \beta\mathrm{e}^{\mathrm{i}t}|^2} = -2C\sin^2(t/2). \tag{46}$$

We will use the following simple inequality

$$\int_{-\pi}^{\pi} |\sin(t/2)|^k \exp\big\{-2\lambda\sin^2(t/2)\big\}\mathrm{d}t \leqslant C(k)\lambda^{-(k+1)/2}. \tag{47}$$

By applying Lemma 5, Lemma 9, (46), and (47), we get

$$\begin{aligned}
&\int_{-\pi}^{\pi} \frac{|\widehat{\varLambda}_1(t)|^n |\widehat{W}_1(t) - \widehat{V}(t)|}{|\mathrm{e}^{\mathrm{i}t} - 1|} \mathrm{d}t \\
&\quad \leqslant \int_{-\pi}^{\pi} C(d+1)\gamma \exp\bigl\{n\bigl(0.4(1-\alpha)\gamma\bigl(Re\widehat{H}(t) - 1\bigr) - 0.2\gamma\alpha\bigr)\bigr\}\mathrm{d}t \\
&\quad \leqslant \int_{-\pi}^{\pi} C(d+1)\gamma \exp\bigl\{Cn\gamma\bigl(Re\widehat{H}(t) - 1\bigr)\bigr\}\mathrm{e}^{-Cn\gamma\alpha}\mathrm{d}t \\
&\quad \leqslant C(d+1)\sqrt{\frac{\gamma}{n}}\mathrm{e}^{-Cn\gamma\alpha}.
\end{aligned}$$

The second inequality of the lemma is proved similarly. □

Lemma 12. *Let condition (1) hold, $\alpha \geqslant C_2$, $|t| \leqslant \pi$. Then*

$$\int_{-\pi}^{\pi} \frac{|\widehat{\varLambda}_1(t)|^n |\widehat{W}_1(t) - \widehat{V}_1(t)|}{|\mathrm{e}^{\mathrm{i}t} - 1|} \mathrm{d}t \leqslant C(d+1)\gamma\mathrm{e}^{-Cn\gamma}.$$

Proof. From Lemma 5 and Lemma 10 it follows that

$$\begin{aligned}
\int_{-\pi}^{\pi} \frac{|\widehat{\varLambda}_1(t)|^n |\widehat{W}_1(t) - \widehat{V}_1(t)|}{|\mathrm{e}^{\mathrm{i}t} - 1|} \mathrm{d}t &\leqslant \int_{-\pi}^{\pi} C(d+1)\gamma \exp\{-0.2C_2\gamma n\}\mathrm{d}t \\
&\leqslant C(d+1)\gamma\mathrm{e}^{-Cn\gamma}.
\end{aligned}$$ □

Lemma 13. *Let condition (1) hold, $|t| \leqslant \pi$. Then*

$$\int_{-\pi}^{\pi} \frac{|\widehat{V}(t)||\widehat{\varLambda}_1^n(t) - \widehat{G}^n(t)|}{|\mathrm{e}^{\mathrm{i}t} - 1|} \mathrm{d}t \leqslant C(d+1)\gamma\sqrt{\frac{\gamma}{n}}\mathrm{e}^{-Cn\gamma\alpha}, \tag{48}$$

$$\int_{-\pi}^{\pi} |\widehat{V}(t)||\widehat{\varLambda}_1^n(t) - \widehat{G}^n(t)|\mathrm{d}t \leqslant C(d+1)\frac{\gamma}{n}\mathrm{e}^{-Cn\gamma\alpha}. \tag{49}$$

Proof. Notice that

$$|\widehat{V}(t)| \leqslant \frac{C(d+1)|\mathrm{e}^{\mathrm{i}t} - 1|}{\gamma(1 - Re\widehat{H}(t) + \alpha)}, \tag{50}$$

$$|\widehat{\varLambda}_1^n(t) - \widehat{G}^n(t)| \leqslant |\widehat{\varLambda}_1(t) - \widehat{G}(t)| \cdot n \cdot \max\bigl\{|\widehat{\varLambda}_1(t)|^{n-1}, |\widehat{G}(t)|^{n-1}\bigr\}.$$

From Corollary 4 we have $|\widehat{\varLambda}_1| \leqslant \exp\{C\gamma(Re\widehat{H}(t) - 1 - \alpha)\}$. Taking into account that $|\mathrm{e}^{a+b\mathrm{i}}| = \mathrm{e}^a$, $|\widehat{G}(t)|$ can be estimated as

$$|\widehat{G}(t)| \leqslant \exp\bigl\{C\gamma\bigl(Re\widehat{H}(t) - 1 - \alpha\bigr)\bigr\}.$$

Using Corollary 5, we have that

$$\begin{aligned}
|\widehat{\varLambda}_1(t) - \widehat{G}(t)| &= |\exp\{\ln\widehat{\varLambda}_1(t)\} - \exp\{\ln\widehat{G}(t)\}| \\
&\leqslant C|\ln\widehat{\varLambda}_1(t) - \ln\widehat{G}(t)|
\end{aligned}$$

$$
\begin{aligned}
&= C\Big|\big(\widehat{\Lambda}_1(t)-1\big) - \frac{(\widehat{\Lambda}_1(t)-1)^2}{2} \\
&\quad + \frac{(\widehat{\Lambda}_1(t)-1)^3}{3} + \frac{C\theta|\widehat{\Lambda}_1(t)-1|^4}{4} - \ln\widehat{G}(t)\Big| \\
&= C\Big|\big(\widehat{A}(t)-1\big) - \frac{1}{2}\big(\widehat{A}_1^2(t)\gamma^2 + 2\widehat{A}_1(t)\big(\widehat{A}_2(t)+\widehat{A}_4(t)\big)\gamma^3\big) \\
&\quad + \frac{1}{3}\widehat{A}_1^3(t)\gamma^3 + C\theta\gamma^4\big(\big(1-Re\widehat{H}(t)\big)^2+\alpha^4\big) - \ln\widehat{G}(t)\Big| \\
&\leqslant C\gamma^4\big(\big(1-Re\widehat{H}(t)\big)^2+\alpha^4\big). \qquad (51)
\end{aligned}
$$

By applying (50), (51), and the inequality $x\mathrm{e}^{-x} \leqslant 1$, for all $x > 0$, we can estimate the following integral:

$$
\begin{aligned}
&\int_{-\pi}^{\pi} \frac{|\widehat{V}(t)||\widehat{\Lambda}_1^n(t)-\widehat{G}^n(t)|}{|\mathrm{e}^{\mathrm{i}t}-1|}\mathrm{d}t \\
&\leqslant C(d+1)\int_{-\pi}^{\pi} n\exp\big\{nC\gamma\big(Re\widehat{H}(t)-1-\alpha\big)\big\}\gamma^3\big(\big(1-Re\widehat{H}(t)\big)+1\big)\mathrm{d}t \\
&\leqslant C(d+1)\int_{-\pi}^{\pi} n\gamma^3\frac{\exp\{n\cdot 0.5C\gamma(Re\widehat{H}(t)-1)\}}{n\cdot 0.5C\gamma(-Re\widehat{H}(t)+1)}\mathrm{e}^{-Cn\gamma\alpha}\big(2-Re\widehat{H}(t)\big)\mathrm{d}t \\
&\leqslant C(d+1)\int_{-\pi}^{\pi} \gamma^2\exp\big\{-2Cn\gamma\sin^2(t/2)\big\}\mathrm{e}^{-Cn\gamma\alpha}\mathrm{d}t \\
&\leqslant C(d+1)\gamma\sqrt{\frac{\gamma}{n}}\mathrm{e}^{-Cn\gamma\alpha}. \qquad (52)
\end{aligned}
$$

The second inequality of this lemma is proved similarly. □

Lemma 14. *Let condition (1) hold, $\alpha \geqslant C_2$, $|t| \leqslant \pi$. Then*

$$
\int_{-\pi}^{\pi} \frac{|\widehat{V}_1(t)||\widehat{\Lambda}_1^n(t)-\widehat{G}_1^n(t)|}{|\mathrm{e}^{\mathrm{i}t}-1|}\mathrm{d}t \leqslant C(d+1)\gamma\mathrm{e}^{-Cn\gamma}.
$$

Proof. Since $\alpha \geqslant C_2$,

$$
|\widehat{V}_1(t)| \leqslant \frac{C(d+1)|\mathrm{e}^{\mathrm{i}t}-1|}{\gamma}, \qquad (53)
$$

and

$$
|\widehat{\Lambda}_1^n(t)-\widehat{G}_1^n(t)| \leqslant |\widehat{\Lambda}_1(t)-\widehat{G}_1(t)|\cdot n\cdot\exp\big\{-C\gamma(n-1)\big\}. \qquad (54)
$$

$|\widehat{\Lambda}_1(t)-\widehat{G}_1(t)|$ is estimated by applying Corollary 6:

$$
\begin{aligned}
|\widehat{\Lambda}_1(t)-\widehat{G}_1(t)| &\leqslant C|\ln\widehat{\Lambda}_1(t)-\ln\widehat{G}_1(t)| \\
&= C\Big|\big(\widehat{\Lambda}_1(t)-1\big) - \frac{(\widehat{\Lambda}_1(t)-1)^2}{2} + \frac{C\theta|\widehat{\Lambda}_1(t)-1|^3}{3} - \ln\widehat{G}_1(t)\Big|
\end{aligned}
$$

$$
\begin{aligned}
&= C\Big|\widehat{A}_1(t)\gamma + \big(\widehat{A}_2(t) + \widehat{A}_4(t)\big)\gamma^2 - \frac{1}{2}\widehat{A}_1^2(t)\gamma^2 \\
&\quad + C\theta\gamma^3 - \ln\widehat{G}_1(t)\Big| \\
&\leqslant C\gamma^3. \qquad (55)
\end{aligned}
$$

By applying (53), (55), and the inequality $x\mathrm{e}^{-x} \leqslant 1$, for all $x > 0$, we can estimate the following integral:

$$
\begin{aligned}
\int_{-\pi}^{\pi} \frac{|\widehat{V}_1(t)||\widehat{\Lambda}_1^n(t) - \widehat{G}_1^n(t)|}{|\mathrm{e}^{\mathrm{i}t} - 1|}\mathrm{d}t &\leqslant C(d+1)\int_{-\pi}^{\pi} n\gamma^2 \exp\{-nC\gamma\}\mathrm{d}t \\
&\leqslant C(d+1)\int_{-\pi}^{\pi} n\gamma^2 \frac{\exp\{-n0.5C\gamma\}}{n0.5C\gamma}\mathrm{d}t \\
&\leqslant C(d+1)\gamma\mathrm{e}^{-Cn\gamma}. \qquad \square
\end{aligned}
$$

Lemma 15. *Let condition (1) hold, $\alpha \geqslant C_2$, $|t| \leqslant \pi$. Then*

$$
\begin{aligned}
|\widehat{W}_1(t)| &\leqslant \frac{C(d+1)}{\gamma}, & |\widehat{W}_1'(t)| &\leqslant \frac{C(d+1)(1+\beta/\gamma)}{\gamma}, \\
|\widehat{W}_2(t)| &\leqslant C(d+1), & |\widehat{W}_2'(t)| &\leqslant C(d+1), \\
|\widehat{V}_2(t)| &\leqslant \frac{C(d+1)}{\gamma}, & |\widehat{V}_2'(t)| &\leqslant \frac{C(d+1)(1+\beta/\gamma)}{\gamma}, \\
|\widehat{W}_1(t) - \widehat{V}_2(t)| &\leqslant C(d+1)\gamma, & |\widehat{W}_1'(t) - \widehat{V}_2'(t)| &\leqslant C(d+1)\gamma(1+\beta/\gamma), \\
|\widehat{\Lambda}_1(t)| &\leqslant \mathrm{e}^{-C\gamma}, & |\widehat{G}_1(t)| &\leqslant \mathrm{e}^{-C\gamma}, \\
|\widehat{\Lambda}_1'(t)| &\leqslant C\gamma, & |\widehat{G}_1'(t)| &\leqslant C\gamma, \\
|\widehat{\Lambda}_2(t)| &\leqslant \beta + 4\gamma, & |\widehat{\Lambda}_2'(t)| &\leqslant C(\beta + 4\gamma), \\
|\widehat{\Lambda}_1(t) - \widehat{G}_1(t)| &\leqslant C\gamma^3, & |(\widehat{\Lambda}_1^n(t) - \widehat{G}_1^n(t))'| &\leqslant C\gamma^2\mathrm{e}^{-Cn\gamma}, \\
\frac{|1 - \mathrm{e}^{dit}|}{|\widehat{\Lambda}_1(t) - \mathrm{e}^{dit}|} &\leqslant C, & \frac{|1 - \mathrm{e}^{dit}|}{|\widehat{\Delta}_1(t) - \mathrm{e}^{dit}|} &\leqslant C.
\end{aligned}
$$

Proof. All inequalities are based on the previously obtained estimates of $|\widehat{\Lambda}_1(t)|$, $|\widehat{\Lambda}_2(t)|$, $|\widehat{W}_2(t)|$, $|\widehat{G}_1(t)|$, and the expansion of $\sqrt{\widehat{D}(t)}$. The inequalities containing $\widehat{V}_2(t)$ are proved similarly to those of $\widehat{V}_1(t)$ (see Lemma 10). $\square$

6 Proofs

Proof of Theorem 1. Applying inversion formula (11), Lemma 11, and Lemma 13 we prove

$$
\begin{aligned}
&|F_n - (G^n V + E)|_{\mathrm{K}} \\
&\quad \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{|\widehat{F}_n(t) - \widehat{G}^n(t)\widehat{V}(t) - \widehat{E}(t)|}{|\mathrm{e}^{\mathrm{i}t} - 1|}\mathrm{d}t
\end{aligned}
$$

$$\leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|\widehat{\Lambda}_1^n(t)||\widehat{W}_1(t)-\widehat{V}(t)|}{|\mathrm{e}^{\mathrm{i}t}-1|}\mathrm{d}t + \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|\widehat{V}(t)||\widehat{\Lambda}_1^n(t)-\widehat{G}^n(t)|}{|\mathrm{e}^{\mathrm{i}t}-1|}\mathrm{d}t$$
$$+ \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|\widehat{\Lambda}_2^n(t)\widehat{W}_2(t)|}{|\mathrm{e}^{\mathrm{i}t}-1|}\mathrm{d}t$$
$$\leqslant C(d+1)\sqrt{\frac{\gamma}{n}}\mathrm{e}^{-Cn\gamma\alpha} + C(d+1)(\beta+4\gamma)^n.$$

The local estimate is obtained analogously by applying inversion formula (12). □

Proof of Theorem 2. The proof is similar to the proof of Theorem 1. Lemma 12 and Lemma 14 are applied instead of Lemma 11 and Lemma 13, since $\alpha \geqslant C_2$. □

Proof of Theorem 3. Taking into account Corollary 3 and Lemma 15, we get

$$\begin{aligned}
|\widehat{\Lambda}_{1,2}^n\widehat{W}_{1,2}| &\leqslant C(d+1)\mathrm{e}^{-Cn},\\
|(\widehat{\Lambda}_{1,2}^n\widehat{W}_{1,2})'| &\leqslant |(\widehat{\Lambda}_{1,2}^n)'||\widehat{W}_{1,2}| + |\widehat{\Lambda}_{1,2}^n||\widehat{W}_{1,2}'|\\
&\leqslant nC(d+1)\mathrm{e}^{-C(n-1)} + C(d+1)\mathrm{e}^{-Cn}\\
&\leqslant C(d+1)n\mathrm{e}^{-Cn}.
\end{aligned}$$

From inversion formula (13) applied with $a=0$ and $b=1$ we get

$$\begin{aligned}
\|F_n - E\| &= \|\Lambda_1^n W_1 + \Lambda_2^n W_2\| \leqslant \|\Lambda_1^n W_1\| + \|\Lambda_2^n W_2\|\\
&\leqslant (1+\pi)^{1/2}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|\widehat{\Lambda}_1^n\widehat{W}_1|^2 + |(\widehat{\Lambda}_1^n\widehat{W}_1)'|^2\mathrm{d}t\right)^{1/2}\\
&\quad + (1+\pi)^{1/2}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|\widehat{\Lambda}_2^n\widehat{W}_2|^2 + |(\widehat{\Lambda}_2^n\widehat{W}_2)'|^2\mathrm{d}t\right)^{1/2}\\
&\leqslant C(d+1)\mathrm{e}^{-Cn}.
\end{aligned}$$
□

Proof of Theorem 4.

$$\|F_n - (G_1^n V_2 + E)\| \leqslant \|(\Lambda_1^n - G_1^n)W_1\| + \|G_1^n(W_1 - V_2)\| + \|\Lambda_2^n W_2\|.$$

From Lemma 15, we get

$$\begin{aligned}
|\widehat{\Lambda}_2^n(t)\widehat{W}_2(t)| &\leqslant C(d+1)(\beta+4\gamma)^n,\\
|(\widehat{\Lambda}_2^n(t)\widehat{W}_2(t))'| &\leqslant |(\widehat{\Lambda}_2^n(t))'\widehat{W}_2(t)| + |\widehat{\Lambda}_2^n(t)\widehat{W}_2'(t)|\\
&\leqslant C(d+1)n(\beta+4\gamma)^n + C(d+1)(\beta+4\gamma)^n\\
&\leqslant C(d+1)n(\beta+4\gamma)^n,\\
|\widehat{G}_1^n(t)(\widehat{W}_1(t)-\widehat{V}_2(t))| &\leqslant C(d+1)\gamma\mathrm{e}^{-Cn\gamma},\\
|(\widehat{G}_1^n(t)(\widehat{W}_1(t)-\widehat{V}_2(t)))'| &\leqslant |(\widehat{G}_1^n(t))'(\widehat{W}_1(t)-\widehat{V}_2(t))| + |\widehat{G}_1^n(t)(\widehat{W}_1(t)-\widehat{V}_2(t))'|\\
&\leqslant C(d+1)n\gamma^2\mathrm{e}^{-C(n-1)\gamma} + C(d+1)\gamma\mathrm{e}^{-Cn\gamma}(1+\beta/\gamma)
\end{aligned}$$

$$
\begin{aligned}
&\leqslant C(d+1)\gamma \mathrm{e}^{-Cn\gamma}(1+\beta/\gamma),\\
|(\widehat{\Lambda}_1^n(t)-\widehat{G}_1^n(t))\widehat{W}_1(t)| &\leqslant n|\widehat{\Lambda}_1(t)-\widehat{G}_1(t)|\mathrm{e}^{-C(n-1)\gamma}\frac{C(d+1)}{\gamma}\\
&\leqslant C(d+1)\gamma \mathrm{e}^{-Cn\gamma},\\
|((\widehat{\Lambda}_1^n(t)-\widehat{G}_1^n(t))\widehat{W}_1(t))'| &\leqslant |(\widehat{\Lambda}_1^n(t)-\widehat{G}_1^n(t))'\widehat{W}_1(t)|+|(\widehat{\Lambda}_1^n(t)-\widehat{G}_1^n(t))\widehat{W}_1'(t)|\\
&\leqslant C(d+1)\gamma \mathrm{e}^{-Cn\gamma}(1+\beta/\gamma).
\end{aligned}
$$

By applying inversion formula (13) with $a=0$ and $b=1$, we prove

$$
\|F_n-(G_1^nV_2+E)\| \leqslant C(d+1)\big(\gamma\mathrm{e}^{-Cn\gamma}(1+\beta/\gamma)+n(\beta+4\gamma)^n\big). \qquad \square
$$

Proof of Theorem 5. We use the inequalities obtained in the proof of Theorem 4 and inversion formula (14) with $a=0$. We have

$$
\begin{aligned}
&k|F_n-(G_1^nV_2+E)\{k\}|\\
&\quad\leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi}|(\widehat{W}_1(t)(\widehat{\Lambda}_1^n(t)-\widehat{G}_1^n(t)))'|\mathrm{d}t\\
&\qquad+\frac{1}{2\pi}\int_{-\pi}^{\pi}|(\widehat{G}_1^n(t)(\widehat{W}_1(t)-\widehat{V}_2(t)))'|\mathrm{d}t+\frac{1}{2\pi}\int_{-\pi}^{\pi}|(\widehat{\Lambda}_2(t)\widehat{W}_2(t))'|\mathrm{d}t\\
&\quad\leqslant C(d+1)\big(\gamma\mathrm{e}^{-0.5Cn\gamma}\mathrm{e}^{-0.5Cn\gamma}(1+\beta/\gamma)+n\mathrm{e}^{n\ln(\beta+4\gamma)}\big).
\end{aligned}
$$

Hence,

$$
k(1+\beta/\gamma)^{-1}|F_n-(G_1^nV_2+E)\{k\}| \leqslant \frac{C(d+1)\mathrm{e}^{-Cn\gamma}}{n}
$$

and

$$
|F_n-(G_1^nV_2+E)\{k\}| \leqslant \frac{C(d+1)\mathrm{e}^{-Cn\gamma}}{n},
$$

since $|M|\leqslant\|M\|_\infty\leqslant\|M\|$.

Summing those inequalities, we get

$$
|F_n-(G_1^nV_2+E)\{k\}| \leqslant \frac{C(d+1)\mathrm{e}^{-Cn\gamma}}{n(1+k(1+\beta/\gamma)^{-1})}=\frac{C(d+1)\mathrm{e}^{-Cn\gamma}(\beta+\gamma)}{n(\beta+(k+1)\gamma)}.
$$

In order to prove the second inequality of the theorem, we apply the inversion formula (15) with $a=0$:

$$
\begin{aligned}
&k|F_n-(G_1^nV_2+E)(k)|\\
&\quad\leqslant\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\left(\frac{\widehat{W}_1(t)}{\mathrm{e}^{-\mathrm{i}t}-1}\big(\widehat{\Lambda}_1^n(t)-\widehat{G}_1^n(t)\big)\right)'\right|\mathrm{d}t\\
&\qquad+\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\left(\widehat{G}_1^n(t)\left(\frac{\widehat{W}_1(t)}{\mathrm{e}^{-\mathrm{i}t}-1}-\frac{\widehat{V}_2(t)}{\mathrm{e}^{-\mathrm{i}t}-1}\right)\right)'\right|\mathrm{d}t\\
&\qquad+\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\left(\widehat{\Lambda}_2(t)\frac{\widehat{W}_2(t)}{\mathrm{e}^{-\mathrm{i}t}-1}\right)'\right|\mathrm{d}t.
\end{aligned}
$$

The summands can be estimated by using the inequalities from the proof of Theorem 4:

$$\bigg|\frac{\widehat{W}_1(t)}{\mathrm{e}^{-\mathrm{i}t}-1}\bigg|\big|(\widehat{\Lambda}_1^n(t)-\widehat{G}_1^n(t))'\big|\leqslant C(d+1)\gamma^2\mathrm{e}^{-Cn\gamma},$$

$$\frac{\mathrm{e}^{(d+1)\mathrm{i}t}-1}{\mathrm{e}^{-\mathrm{i}t}-1}=\frac{(\mathrm{e}^{\mathrm{i}t}-1)(1+\mathrm{e}^{\mathrm{i}t}+\cdots+\mathrm{e}^{d\mathrm{i}t})}{\mathrm{e}^{-\mathrm{i}t}(1-\mathrm{e}^{\mathrm{i}t})}=-\mathrm{e}^{\mathrm{i}t}(1+\mathrm{e}^{\mathrm{i}t}+\cdots+\mathrm{e}^{d\mathrm{i}t}),$$

$$\bigg|\bigg(\frac{\widehat{W}_1(t)}{\mathrm{e}^{-\mathrm{i}t}-1}\bigg)'\bigg|\leqslant\frac{Cd^2}{\gamma^2},\qquad\bigg|\bigg(\frac{\widehat{W}_2(t)}{\mathrm{e}^{-\mathrm{i}t}-1}\bigg)'\bigg|\leqslant Cd^2,$$

$$\bigg|\bigg(\frac{\widehat{W}_1(t)}{\mathrm{e}^{-\mathrm{i}t}-1}\bigg)'\bigg|\big|\widehat{\Lambda}_1^n(t)-\widehat{G}_1^n(t)\big|\leqslant Cn\gamma^3\mathrm{e}^{-Cn\gamma}\frac{d^2}{\gamma^2}\leqslant Cd^2\mathrm{e}^{-Cn\gamma},$$

$$\bigg|\widehat{G}_1^n(t)'\bigg(\frac{\widehat{W}_1(t)-\widehat{V}_2(t)}{\mathrm{e}^{-\mathrm{i}t}-1}\bigg)\bigg|\leqslant C(d+1)\gamma\mathrm{e}^{-Cn\gamma},$$

$$\bigg|\widehat{G}_1^n(t)\bigg(\frac{\widehat{W}_1(t)-\widehat{V}_2(t)}{\mathrm{e}^{-\mathrm{i}t}-1}\bigg)'\bigg|\leqslant\frac{Cd^2\mathrm{e}^{-Cn\gamma}}{\gamma^2},$$

$$\bigg|\widehat{\Lambda}_2^n(t)'\frac{\widehat{W}_2(t)}{\mathrm{e}^{-\mathrm{i}t}-1}\bigg|\leqslant C(d+1)\mathrm{e}^{-Cn},$$

$$\bigg|\widehat{\Lambda}_2^n(t)\bigg(\frac{\widehat{W}_2(t)}{\mathrm{e}^{-\mathrm{i}t}-1}\bigg)'\bigg|\leqslant Cd^2(\beta+4\gamma)^n.$$

Thus, we get

$$k\gamma^2|F_n-(G_1^nV_2+E)(k)|\leqslant\frac{Cd^2\mathrm{e}^{-Cn\gamma}}{n}$$

and

$$|F_n-(G_1^nV_2+E)(k)|\leqslant\frac{C(d+1)\mathrm{e}^{-Cn\gamma}}{n}.$$

By summing the above inequalities we arrive at

$$|F_n-(G_1^nV_2+E)(k)|\leqslant\frac{Cd^2\mathrm{e}^{-Cn\gamma}}{n(1+k\gamma^2)}.\qquad\square$$

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