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Existence and uniqueness of mild solution to fractional stochastic heat equation

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Abstract For a class of non-autonomous parabolic stochastic partial differential equations defined on a bounded open subset $D \subset \mathbb{R}^d$ and driven by an $L^2(D)$-valued fractional Brownian motion with the Hurst index $H > 1/2$, a new result on existence and uniqueness of a mild solution is established. Compared to the existing results, the uniqueness in a fully nonlinear case is shown, not assuming the coefficient in front of the noise to be affine. Additionally, the existence of moments for the solution is established.

Keywords Fractional Brownian motion, stochastic partial differential equation, mild solution, Green's function

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1 Introduction

In this paper we study an initial–Neumann boundary value problem for the following non-autonomous stochastic partial differential equation of parabolic type in a cylinder domain $D \times [0, T]$, driven by an infinite-dimensional fractional noise:

$$du(x,t) = \big(\operatorname{div}\big(k(x,t)\nabla u(x,t)\big) + f\big(u(x,t)\big)\big)dt + h\big(u(x,t)\big)W^H(x,dt),$$
$$(x,t) \in D \times (0,T],$$

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$$u(x,0) = \varphi(x), \quad x \in D,$$
$$\frac{\partial u(x,t)}{\partial n(k)} = 0, \quad (x,t) \in \partial D \times (0,T]. \tag{1}$$

Here $D \subset \mathbb{R}^d$ is a bounded domain with the boundary ∂D of class $C^{2+\beta}$ with some $\beta \in (0,1)$, W^H is an $L^2(D)$-valued fractional Brownian motion with the Hurst index $H \in (\frac{1}{2}, 1)$, $k = \{k_{i,j}\} : \overline{D} \times [0,T] \to \mathbb{R}^{d\times d}$ is a matrix-valued field, $n(k)(x) := k(x,t)n(x)$ denotes the conormal vector-field, and the last relation in (1) refers to the conormal derivative of u relative to k, that is

$$\frac{\partial u(x,t)}{\partial n(k)} = \sum_{i,j=1}^{d} k_{i,j}(x,t) n_i(x) \frac{\partial}{\partial x_j} u(x,t),$$

$n(x) \in \mathbb{R}^d$ is an outer normal vector to ∂D.

Equations similar to (1) were studied extensively in literature, so we will mention only several most relevant articles here. The articles [5] and [7] considered heat equations with additive and multiplicative fractional noise, respectively. The articles [1, 10, 18] are devoted to general non-linear evolution equations with fractional noise; however, the equations are considered in some functional spaces, and the assumptions on the coefficients imposed there do not cover general nonlinear equations of the form (1). The problem (1) was considered in [14] and then in [16], where the notions of *variational* and *mild* solutions were introduced. The article [14] established the existence of a variational solution to this equation, but the uniqueness was shown under the assumption that the function h is affine. In [16], it was shown that a variational solution to (1) is a mild solution too, however, the uniqueness, both in the variational and in the mild sense, was established also under the assumption that h is affine, moreover, a rather restrictive assumption $H > \frac{d+1}{d+2}$ on the Hurst exponent was imposed.

Our goal is to extend the uniqueness results of [16] to the case of arbitrary $H \in (\frac{1}{2}, 1)$ and non-affine h. Specifically, we prove the uniqueness of a mild solution, assuming that h and its derivative h' are Lipschitz continuous. Since the existence of a variational solution is known from [14], and [16] established that each variational solution is a mild solution, we get existence and uniqueness of a variational solution too, thus finally answering a question posed in [14]. We also show that the solution to (1) has finite moments of any order.

It is worth to mention that a similar uniqueness result holds in the case where the function h in front of W^H depends on t sufficiently regularly, say, Hölder continuous with exponent greater than $1/2$. However, since our main reference for existence results are the articles [14, 16], in which h is assumed to be independent of t, we will follow this assumption.

The paper is organized as follows. In Section 2, we formulate the main hypotheses, and give the definition of a mild solution and basic facts on an $L^2(D)$-valued fractional Brownian process and stochastic integration with respect to it. Section 3 contains auxiliary results concerning the parabolic Green's function. In Section 4, we give a priori upper bounds for mild solutions. Finally, the main result on existence and uniqueness of a mild solution is proved in Section 5.

2 Preliminaries

This section is devoted to the precise statement of the problem (1). We introduce necessary notation, give the definition of a mild solution, and formulate the assumptions for its existence and uniqueness.

2.1 Notational conventions

Throughout the article, $|\cdot|$ will denote absolute value of a number, Euclidean norm of a vector or operator norm of a matrix; exact meaning will always be clear from the context. We will use the symbol C for a generic constant, the precise value of which is not important and may vary between different equations and inequalities.

2.2 Norms and spaces

Let $\|\cdot\|_2$ and $\|\cdot\|_\infty$ be the norms in $L^2(D)$ and $L^\infty(D)$, respectively. Denote for $\alpha \in (0,1)$, $u\colon D \times [0,T] \to \mathbb{R}$ and $t \in [0,T]$

$$\|u\|_{\alpha,1,t} := \sup_{x\in D}\sup_{s\in[0,t]}\int_0^s \frac{|u(x,s)-u(x,v)|}{(s-v)^{\alpha+1}}\,dv,$$

$$\|u\|_{\alpha,\infty,t} := \sup_{s\in[0,t]}\|u(\cdot,s)\|_\infty + \|u\|_{\alpha,1,t}\,,$$

$$\|u\|_{\alpha,2,t} := \left(\sup_{s\in[0,t]}\|u(\cdot,s)\|_2^2 + \int_0^t\left(\int_0^s \frac{\|u(\cdot,s)-u(\cdot,v)\|_2}{(s-v)^{\alpha+1}}\,dv\right)^2 ds\right)^{1/2}.$$

Denote by $\mathcal{B}^{\alpha,2}(0,T;L^2(D))$ the Banach space of measurable mappings $u\colon D\times[0,T]\to\mathbb{R}$ such that $\|u\|_{\alpha,2,T}^2 < \infty$.

Let $H^1(D)$ be the Sobolev space of functions $f\colon D\to\mathbb{R}$ equipped with the norm $\|f\|_{1,2} = (\|f\|_2^2 + \|\nabla f\|_2^2)^{1/2}$. Also introduce the space $L^2(0,T;H^1(D))$ of measurable mappings $u\colon [0,T]\to H^1(D)$ such that

$$\int_0^T \|u(t)\|_{1,2}^2\,dt = \int_0^T \big(\|u(t)\|_2^2 + \|\nabla u(t)\|_2^2\big)dt < \infty.$$

For $f\colon[0,T]\to\mathbb{R}$ and $\alpha\in(0,1)$ define a seminorm

$$\|f\|_{\alpha,0,t} = \sup_{0\le u<v<t}\left(\frac{|f(v)-f(u)|}{(v-u)^{1-\alpha}} + \int_u^v \frac{|f(u)-f(z)|}{(z-u)^{2-\alpha}}\,dz\right).$$

2.3 Assumptions on the coefficients and on the initial value

(A1) The coefficients k_{ij} satisfy the following assumptions:

(i) $k_{i,j} = k_{j,i}$ for all $i,j = 1,\dots,d$;

(ii) $k_{i,j} \in C^{\beta,\beta'}(\overline{D}\times[0,T])$ for some $\beta' \in (\frac{1}{2},1]$ and for all $i,j=1,\dots,d$;

(iii) $\frac{\partial}{\partial x_l}k_{i,j} \in C^{\beta,\beta/2}(\overline{D}\times[0,T])$ for all $i,j,l=1,\dots,d$;

(iv) there exists $\underline{k} > 0$ such that

$$\sum_{i,j=1}^{d} k_{i,j}(x,t)q_i q_j \geq \underline{k}\,|q|^2,$$

for all $x \in \overline{D}$, $t \in [0,T]$, $q \in \mathbb{R}^d$;

(v) $(x,t) \mapsto \sum_{i=1}^{d} k_{i,j}(x,t)n_i(x) \in C^{1+\beta,(1+\beta)/2}(\partial D \times [0,T])$ for each j;

(vi) the conormal vector-field $(x,t) \mapsto n(k)(x,t) = k(x,t)n(x)$ is outward pointing, nowhere tangent to ∂D for every t.

(A2) The initial condition $\varphi \in C^{2+\beta}(\overline{D})$ satisfies the conormal boundary condition relative to k.

(A3) $f, h, h' \colon \mathbb{R} \to \mathbb{R}$ are globally Lipschitz continuous functions.

Remark 1. The global Lipshitz assumption implies that f and h are of linear growth:

$$|f(x)| + |h(x)| \leq C(1 + |x|). \tag{2}$$

It is worth to mention that all results of the article can be proved assuming linear growth and only local Lipschitz continuity of f and h' with some extra technical work. We decided to impose the global Lipschitz continuity assumption for the sake of simplicity and because it does not lead to a considerable loss of generality.

2.4 $L^2(D)$-valued fractional Brownian process and stochastic integration with respect to it

Let us briefly recall the definition of an $L^2(D)$-valued fractional Brownian process and the corresponding stochastic integral, introduced in [10]. Assume that $\{\lambda_j, j \in \mathbb{N}\}$ is a sequence of positive real numbers and $\{e_j, j \in \mathbb{N}\}$ is an orthonormal basis of $L^2(D)$ such that

(A4) $\sup_j \|e_j\|_\infty < \infty$ and $\sum_{j=1}^{\infty} \lambda_j^{1/2} < \infty$.

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a complete probability space. For a fixed $T > 0$ let $\mathbb{F} = \{\mathcal{F}\}_{t\in[0,T]}$ be a filtration satisfying the standard assumptions. Let $B_j^H = \{B_j^H(t), t \geq 0\}$, $j \in \mathbb{N}$, be a sequence of one-dimensional, independent fractional Brownian motions with the Hurst parameter $H \in (1/2, 1)$, defined on $(\Omega, \mathcal{F}, \mathbb{F}, \mathsf{P})$ and starting at the origin. Following [10], define $L^2(D)$-valued fractional Brownian process $W^H = \{W^H(\cdot, t), t \geq 0\}$ by

$$W^H(\cdot, t) = \sum_{j=1}^{\infty} \lambda_j^{1/2} e_j(\cdot) B_j^H(t),$$

where the series converges a. s. in $L^2(D)$.

In this article we consider a pathwise stochastic integration with respect to W^H in the fractional (generalized Lebesgue–Stieltjes) sense. Alternatively, one can look at the so-called Skorokhod (white-noise) integral. However, with the Skorokhod definition, it is difficult to solve even stochastic ordinary differential equations, see e.g. [8].

Fix $\alpha \in (1-H, 1/2)$. Let $\Phi = \{\Phi(t), t \in [0, T]\}$ be an adapted stochastic process taking values in the space of linear bounded operators on $L^2(D)$ such that

$$\sup_{j\in\mathbb{N}} \int_0^T \left(\frac{\left\| \Phi(t)e_j \right\|_2}{t^\alpha} + \int_0^t \frac{\left\| (\Phi(t) - \Phi(s))e_j \right\|_2}{(t-s)^{\alpha+1}} ds \right) dt < \infty.$$

Following [10] (see also [14, 16]), we introduce the integral with respect to an $L^2(D)$-valued fractional Brownian process by

$$\int_a^b \Phi(s)\, dW^H(s) := \sum_{j=1}^{\infty} \lambda_j^{1/2} \int_a^b \Phi(s)e_j\, dB_j^H(s),$$

where the integrals with respect to B_j^H, $j \in \mathbb{N}$, are understood as pathwise generalized Lebesgue–Stieltjes integrals. Such integrals are defined in terms of fractional derivatives, the detailed exposition of this approach can be found, e. g., in the book [12, Section 2.1]. We mention only that under above assumptions, the generalized Lebesgue–Stieltjes integral $\int_a^b \Phi(s)e_j\, dB_j^H(s)$ is well defined and admits the bound

$$\begin{aligned} &\left| \int_a^b \Phi(s)e_j\, dB_j^H(s) \right| \\ &\quad \le C_\alpha \left\| B_j^H \right\|_{\alpha,0,b} \int_a^b \left(\frac{\left|\Phi(s)e_j\right|}{(s-a)^\alpha} + \int_a^s \frac{\left|(\Phi(s) - \Phi(v))e_j\right|}{(s-v)^{\alpha+1}} dv \right) ds \end{aligned} \tag{3}$$

for some constant $C_\alpha > 0$.

2.5 Mild solution

Following [16], we understand a solution to the problem (1) in a mild sense. Its definition uses the notion of the parabolic Green's function $G(x, t, y, s)$, $x, y \in \overline{D}$, $0 \le s < t \le T$, associated with the principal part of (1) (see, e. g., [2–4, 9]). For every $(y, s) \in D \times (0, T]$, $G(x, t, y, s)$ is a classical solution to the linear initial-boundary value problem

$$\begin{aligned} &\partial_t G(x, t; y, s) = \operatorname{div}\big(k(x, t)\nabla_x G(x, t; y, s)\big), \quad (x, t) \in D \times (0, T], \\ &\frac{\partial G(x, t; y, s)}{\partial n(k)} = 0, \quad (x, t) \in \partial D \times (0, T], \end{aligned} \tag{4}$$

with

$$\int_D G(\cdot, s; y, s)\varphi(y)\, dy := \lim_{t\downarrow s} \int_D G(\cdot, t; y, s)\varphi(y)\, dy = \varphi(\cdot).$$

In the next section we consider the properties of G in detail.

Definition 1 ([16])**.** Fix $H \in (1/2, 1)$ and $\alpha \in (1 - H, 1/2)$. An $L^2(D)$-valued random field $\{u(\cdot, t), t \in [0, T]\}$ is a *mild solution* to the problem (1) if the following two conditions are satisfied:

(1) $u \in L^2(0, T; H^1(D)) \cap \mathcal{B}^{\alpha,2}(0, T; L^2(D))$ a. s.

(2) The relation

$$\begin{aligned} u(\cdot, t) = & \int_D G(\cdot, t; y, 0)\varphi(y)\, dy + \int_0^t \int_D G(\cdot, t; y, s) f\big(u(y, s)\big)\, dy\, ds \\ & + \sum_{j=1}^{\infty} \lambda_j^{1/2} \int_0^t \int_D G(\cdot, t; y, s) h\big(u(y, s)\big) e_j(y)\, dy\, dB_j^H(s) \end{aligned} \tag{5}$$

holds a. s. for every $t \in [0, T]$ as an equality in $L^2(D)$.

3 Properties of Green's function

In this section we collect several upper bounds for Green's function G, needed for the proof of the main result.

Denote

$$\Phi_t^C(x) := t^{-d/2} \exp\big\{-C\, |x|^2 / t\big\}, \quad t > 0,\ x \in \mathbb{R}^d.$$

It is known from [2, 3] that under assumptions (A1) and (A2) G is a continuous function, twice continuously differentiable in x, once continuously differentiable in t. Moreover, G satisfies the heat kernel estimates

$$\big|\partial_x^\mu \partial_t^\nu G(x, t; y, s)\big| \le C(t - s)^{-(|\mu|_1 + 2\nu)/2} \Phi_{t-s}^C(x - y) \tag{6}$$

for $\mu = (\mu_1, \dots, \mu_d)$, $\mu_1, \dots, \mu_d, \nu \in \mathbb{N} \cup \{0\}$, and $|\mu|_1 + 2\nu \le 2$ with $|\mu|_1 = \sum_{j=1}^d \mu_j$. In particular, for $|\mu|_1 = \nu = 0$, we have

$$|G(x, t; y, s)| \le C \Phi_{t-s}^C(x - y). \tag{7}$$

The inequality (7) is sometimes called the *Gaussian property* of G ([15, 16]).

Several important properties of the parabolic Green's function G follow from the fact that it is, for every $(x, t) \in D \times [0, T]$, a classical solution to the linear boundary value problem

$$\begin{aligned} & \partial_s G(x, t; y, s) = -\operatorname{div}\big(k(y, s) \nabla_y G(x, t; y, s)\big), \quad (y, s) \in D \times (0, T], \\ & \frac{\partial G(x, t; y, s)}{\partial n(k)} = 0, \quad (y, s) \in \partial D \times (0, T], \end{aligned}$$

dual to (4). In particular, along with (6) we have also

$$\Big|\partial_y^\mu \partial_s^\nu G(x, t; y, s)\Big| \le C(t - s)^{-(|\mu|_1 + 2\nu)/2} \Phi_{t-s}^C(x - y) \tag{8}$$

for $|\mu|_1 + 2\nu \le 2$, and, moreover, the following convolution formula holds:

$$G(x, t; y, s) = \int_D G(x, t; z, \sigma) G(z, \sigma; y, s)\, dz \quad \text{for all } \sigma \in (s, t), \tag{9}$$

see [2, Appendix, p. 232–233] for the details.

Furthermore, according to Eqs. (3.4)–(3.5) from [16], G satisfies the following inequalities for all $x, y \in D$ and $\delta \in (\frac{d}{d+2}, 1)$.

(i) For all $0 < r < v < t < T$ and some $t^* \in (r, v)$,

$$|G(x,t;y,v) - G(x,t;y,r)| \le C(t-v)^{-\delta}(v-r)^{\delta}\, \Phi^C_{t-t^*}(x-y). \tag{10}$$

(ii) For all $0 < v < s < t < T$ and some $v^* \in (s, t)$,

$$|G(x,t;y,v) - G(x,s;y,v)| \le C(t-s)^{\delta}(s-v)^{-\delta}\, \Phi^C_{v^*-v}(x-y). \tag{11}$$

Lemma 1. *Under assumptions (A1)–(A2), for all* $0 < r < v < s < t < T$ *and for all* $x, y \in D$,

$$\begin{aligned}&|G(x,t;y,v) - G(x,s;y,v) - G(x,t;y,r) + G(x,s;y,r)|\\ &\quad \le C\int_r^v\int_s^t (\theta-\tau)^{-2}\Phi^C_{\theta-\tau}(x-y)d\theta\, d\tau.\end{aligned} \tag{12}$$

Proof. Write

$$\begin{aligned}&G(x,t;y,v) - G(x,s;y,v) - G(x,t;y,r) + G(x,s;y,r)\\ &\quad = \int_s^t\int_r^v \frac{\partial^2}{\partial\theta\,\partial\tau}G(x,\theta;y,\tau)\,d\tau\, d\theta.\end{aligned} \tag{13}$$

By (9), the equality

$$\frac{\partial^2}{\partial\theta\,\partial\tau}G(x,\theta;y,\tau) = \int_D \partial_\theta G(x,\theta;z,\sigma)\partial_\tau G(z,\sigma;y,\tau)\,dz$$

holds with $\sigma = \frac{\tau+\theta}{2}$. By applying the bounds (6) and (8) with $|\mu|_1 = 0$ and $\nu = 1$, we see that

$$\begin{aligned}\left|\frac{\partial^2}{\partial\theta\,\partial\tau}G(x,\theta;y,\tau)\right| &\le \int_D |\partial_\theta G(x,\theta;z,\sigma)\partial_\tau G(z,\sigma;y,\tau)|\, dz\, d\sigma\\ &\le C(\theta-\sigma)^{-1}(\sigma-\tau)^{-1}\int_D \Phi^C_{\theta-\sigma}(x-z)\Phi^C_{\sigma-\tau}(z-y)\,dz\\ &\le C(\theta-\tau)^{-2}\Phi^C_{\theta-\tau}(x-y).\end{aligned}$$

Combining this bound with (13), we conclude the proof. □

4 A priori estimates

Fix $H \in (1/2, 1)$ and $\alpha \in (1-H, 1/2)$. Let u be a mild solution to (1), defined by (5). Note that the random variable

$$\xi_{\alpha,H,T} := 1 + \sum_{j=1}^{\infty}\lambda_j^{1/2}\left\|B_j^H\right\|_{\alpha,0,T}$$

is finite a. s., see [11].

The goal of this section is to prove the following result.

Proposition 1. *Under assumptions (A1)–(A4),*

$$\|u\|_{\alpha,\infty,T} \le C \exp\left\{C\xi_{\alpha,H,T}^{1/(1-\alpha)}\right\}.$$

We split the proof of Proposition 1 into two lemmas. In Lemma 2 we establish an upper bound for $\sup_{s\in[0,t]}\sup_{x\in D}|u(x,s)|$. In Lemma 3 we obtain similar estimate for $\|u\|_{\alpha,1,t}$.

In the calculations below we shall often refer to the following simple formulas: for all $a > 0$, $b > 0$, and $0 < v < t$,

$$\int_v^t (t-s)^{a-1}(s-v)^{b-1}\,ds = C(t-v)^{a+b-1}, \tag{14}$$

$$\int_0^v (t-s)^{-a-b}(v-s)^{b-1}\,ds \le C(t-v)^{-a}, \tag{15}$$

where $C = \mathrm{B}(a,b)$, the beta function. The formula (14) follows directly from the definition of the beta function by the substitution $z = \frac{s-v}{t-v}$. The inequality (15) is obtained by the substitution $z = \frac{v-s}{t-v}$ as follows:

$$\begin{aligned}\int_0^v (t-s)^{-a-b}(v-s)^{b-1}\,ds &= (t-v)^{-a}\int_0^{\frac{v}{t-v}} \frac{z^{b-1}}{(1+z)^{a+b}}\,dz\\ &\le (t-v)^{-a}\int_0^\infty \frac{z^{b-1}}{(1+z)^{a+b}}\,dz = \mathrm{B}(a,b)(t-v)^{-a}.\end{aligned}$$

Denote for brevity

$$\|u\|_s = \sup_{s\in[0,t]}\sup_{x\in D}|u(x,s)|.$$

Lemma 2. *Under assumptions (A1)–(A4),*

$$\|u\|_t \le C\xi_{\alpha,H,T}\left(1 + \int_0^t \left(\|u\|_s (t-s)^{-\alpha} + \|u\|_{\alpha,1,s}\right)ds\right), \tag{16}$$

for all $t \in [0,T]$.

Proof. By (5),

$$|u(x,t)| \le I_\varphi + I_f + I_h, \tag{17}$$

where

$$\begin{aligned}I_\varphi &= \int_D |G(x,t;y,0)\varphi(y)|\,dy,\\ I_f &= \int_0^t\int_D \left|G(x,t;y,s)f\big(u(y,s)\big)\right|\,dy\,ds,\\ I_h &= \sum_{j=1}^\infty \lambda_j^{1/2}\left|\int_0^t\int_D G(x,t;y,s)h\big(u(y,s)\big)e_j(y)\,dy\,dB_j^H(s)\right|.\end{aligned}$$

By (A2), φ is bounded. Therefore, the Gaussian property (7) implies that

$$I_\varphi \le C \int_D \Phi_t^C (x-y)\, dy \le C. \tag{18}$$

It follows from the linear growth property (2) that

$$\left| f\big(u(y,s)\big)\right| \le C\big(1 + |u(y,s)|\big) \le C(1 + \|u\|_s). \tag{19}$$

Then, applying (7), we get

$$I_f \le C \int_0^t (1+\|u\|_s) \int_D |G(x,t;y,s)|\, dy\, ds \le C \int_0^t (1+\|u\|_s)\, ds. \tag{20}$$

Using the bound (3) for the integrals with respect to B_j^H, we may write

$$\begin{aligned} I_h \le C \sum_{j=1}^\infty \lambda_j^{1/2} \left\| B_j^H \right\|_{\alpha,0,t} \int_0^t \Bigg(& \frac{\int_D \left|G(x,t;y,s) h(u(y,s)) e_j(y)\right|\, dy}{s^\alpha} \\ & + \int_0^s \frac{\int_D |G(x,t;y,s)h(u(y,s)) - G(x,t;y,v)h(u(y,v))| \left|e_j(y)\right| dy}{(s-v)^{\alpha+1}}\, dv \Bigg) ds. \end{aligned}$$

The assumption (A4) implies that $\sup_j \left|e_j(y)\right| \le C$ for all $y \in D$. Therefore,

$$I_h \le C \xi_{\alpha,H,T} (I_{h1} + I_{h2} + I_{h3}), \tag{21}$$

where

$$\begin{aligned} I_{h1} &= \int_0^t s^{-\alpha} \int_D \left|G(x,t;y,s) h\big(u(y,s)\big)\right|\, dy\, ds, \\ I_{h2} &= \int_0^t \int_0^s (s-v)^{-\alpha-1} \int_D |G(x,t;y,s)| \left|h\big(u(y,s)\big) - h\big(u(y,v)\big)\right| dy\, dv\, ds, \\ I_{h3} &= \int_0^t \int_0^s (s-v)^{-\alpha-1} \int_D |G(x,t;y,s) - G(x,t;y,v)| \left|h\big(u(y,v)\big)\right| dy\, dv\, ds. \end{aligned}$$

The term I_{h1} can be estimated similarly to I_f, using the linear growth of h and the Gaussian property of G:

$$I_{h1} \le C \int_0^t (1+\|u\|_s) s^{-\alpha}\, ds.$$

Since $\|u\|_s$ is non-decreasing and $s^{-\alpha}$ is non-increasing, we can use the rearrangement inequality [6, Theorem 378] to obtain

$$I_{h1} \le C \int_0^t (1+\|u\|_s)(t-s)^{-\alpha}\, ds. \tag{22}$$

By the Lipschitz continuity of h,

$$I_{h2} \le \int_0^t \int_D |G(x,t;y,s)| \int_0^s \frac{|u(y,s) - u(y,v)|}{(s-v)^{\alpha+1}}\, dv\, dy\, ds.$$

The inner integral can be bounded by $\|u\|_{\alpha,1,s}$. Therefore, we get

$$I_{h2} \le \int_0^t \|u\|_{\alpha,1,s} \int_D |G(x,t;y,s)|\,dy\,ds \le C\int_0^t \|u\|_{\alpha,1,s}\,ds. \tag{23}$$

In order to estimate I_{h3}, we use (10) together with the bound

$$\big|h\big(u(y,v)\big)\big| \le C\big(1+|u(y,v)|\big) \le C(1+\|u\|_v). \tag{24}$$

We have

$$\begin{aligned} I_{h3} &\le C\int_0^t (t-s)^{-\delta}\int_0^s (1+\|u\|_v)(s-v)^{\delta-\alpha-1}\int_D \Phi^C_{t-t^*}(x-y)\,dy\,dv\,ds \\ &\le C\int_0^t (t-s)^{-\delta}\int_0^s (1+\|u\|_v)(s-v)^{\delta-\alpha-1}\,dv\,ds \\ &= C\int_0^t (1+\|u\|_v)\int_v^t (t-s)^{-\delta}(s-v)^{\delta-\alpha-1}\,ds\,dv, \end{aligned}$$

where we choose $\delta \in (\frac{d}{d+2},1)$ so that $\delta > \alpha$. Computing the inner integral by (14), we get

$$I_{h3} \le C\int_0^t (1+\|u\|_v)(t-v)^{-\alpha}\,dv. \tag{25}$$

Combining (17), (18), (20)–(23), (25), we obtain

$$|u(x,t)| \le C\xi_{\alpha,H,T}\left(1+\int_0^t \big(\|u\|_s\,(t-s)^{-\alpha}+\|u\|_{\alpha,1,s}\big)ds\right),$$

Since $\|u\|_s$ and $\|u\|_{\alpha,1,s}$ are non-decreasing, the right-hand side here is non-decreasing as well. Indeed, using the substitution $s = zt$, the integral in the right-hand side can be rewritten in the form

$$t^{1-\alpha}\int_0^1 \|u\|_{zt}\,(1-z)^{-\alpha}dz + t\int_0^1 \|u\|_{\alpha,1,zt}\,dz.$$

Therefore, taking suprema, we arrive at (16). □

Lemma 3. *Under assumptions (A1)–(A4),*

$$\|u\|_{\alpha,1,t} \le C\xi_{\alpha,H,T}\left(1+\int_0^t \big(\|u\|_s\,(t-s)^{-2\alpha}+\|u\|_{\alpha,1,s}\,(t-s)^{-\alpha}\big)ds\right), \tag{26}$$

for all $t \in [0,T]$.

Proof. By (5),

$$|u(x,t)-u(x,s)| \le J_\varphi + J_f + J_h,$$

where

$$J_\varphi = \int_D |G(x,t;y,0)-G(x,s;y,0)|\,|\varphi(y)|\,dy,$$

$$J_f = \left|\int_0^t\int_D G(x,t;y,v)f\big(u(y,v)\big)\,dy\,dv - \int_0^s\int_D G(x,s;y,v)f\big(u(y,v)\big)\,dy\,dv\right|,$$

$$J_h = \sum_{j=1}^{\infty} \lambda_j^{1/2} \bigg| \int_0^t \int_D G(x,t;y,v) h\big(u(y,v)\big) e_j(y)\,dy\,dB_j^H(v) \\ - \int_0^s \int_D G(x,s;y,v) h\big(u(y,v)\big) e_j(y)\,dy\,dB_j^H(v) \bigg|.$$

Then

$$\int_0^t \frac{|u(x,t)-u(x,s)|}{(t-s)^{\alpha+1}}\,ds \le K_\varphi + K_f + K_h,$$

where $K_\varphi = \int_0^t \frac{J_\varphi}{(t-s)^{\alpha+1}}\,ds$, $K_f = \int_0^t \frac{J_f}{(t-s)^{\alpha+1}}\,ds$, $K_h = \int_0^t \frac{J_h}{(t-s)^{\alpha+1}}\,ds$.

Let $\delta \in (\frac{d}{d+2} \vee \alpha, 1)$ be fixed throughout the proof. Using the boundedness of φ and (11), we can write

$$J_\varphi \le C s^{-\delta}(t-s)^\delta \int_D \Phi_{v^*}^C(x-y)\,dy \le C s^{-\delta}(t-s)^\delta.$$

Therefore,

$$K_\varphi \le C \int_0^t s^{-\delta}(t-s)^{\delta-\alpha-1}\,ds \le C.$$

Let us consider K_f. We have that

$$J_f \le \int_s^t \int_D \big|G(x,t;y,v) f\big(u(y,v)\big)\big|\,dy\,dv \\ + \int_0^s \int_D |G(x,t;y,v) - G(x,s;y,v)|\,\big|f\big(u(y,v)\big)\big|\,dy\,dv.$$

Consequently,

$$K_f \le K_f' + K_f'',$$

where

$$K_f' = \int_0^t (t-s)^{-\alpha-1} \int_s^t \int_D \big|G(x,t;y,v) f\big(u(y,v)\big)\big|\,dy\,dv\,ds,$$
$$K_f'' = \int_0^t (t-s)^{-\alpha-1} \int_0^s \int_D |G(x,t;y,v) - G(x,s;y,v)|\,\big|f\big(u(y,v)\big)\big|\,dy\,dv\,ds.$$

By (19) and the Gaussian property of G,

$$\begin{aligned} K_f' &\le C \int_0^t (t-s)^{-\alpha-1} \int_s^t (1+\|u\|_v)\,dv\,ds \\ &= C \int_0^t (1+\|u\|_v) \int_0^v (t-s)^{-\alpha-1}\,ds\,dv \\ &\le C \int_0^t (1+\|u\|_v)(t-v)^{-\alpha}\,dv \le C \int_0^t (1+\|u\|_v)(t-v)^{-2\alpha}\,dv. \end{aligned} \tag{27}$$

In order to estimate K_f'', we apply (19) and (11), change the order of integration and then use (14):

$$K_f'' \le C \int_0^t (t-s)^{\delta-\alpha-1} \int_0^s (s-v)^{-\delta}(1+\|u\|_v) \int_D \Phi_{v^*-v}^C(x-y)\,dy\,dv\,ds$$

$$\begin{aligned}
&\le C\int_0^t (t-s)^{\delta-\alpha-1}\int_0^s (s-v)^{-\delta}(1+\|u\|_v)\,dv\,ds\\
&= C\int_0^t (1+\|u\|_v)\int_v^t (t-s)^{\delta-\alpha-1}(s-v)^{-\delta}\,ds\,dv\\
&\le C\int_0^t (1+\|u\|_v)(t-v)^{-\alpha}\,dv \le C\int_0^t (1+\|u\|_v)(t-v)^{-2\alpha}\,dv. \qquad (28)
\end{aligned}$$

Next, consider K_h. We have

$$\begin{aligned}
J_h &\le \sum_{j=1}^{\infty}\lambda_j^{1/2}\left|\int_s^t\int_D G(x,t;y,v)h\big(u(y,v)\big)e_j(y)\,dy\,dB_j^H(v)\right|\\
&\quad+\sum_{j=1}^{\infty}\lambda_j^{1/2}\left|\int_0^s\int_D \big(G(x,t;y,v)-G(x,s;y,v)\big)h\big(u(y,v)\big)e_j(y)\,dy\,dB_j^H(v)\right|\\
&=: J_h' + J_h''.
\end{aligned}$$

The integrals with respect to fractional Brownian motions can be bounded similarly to I_h, applying (3) and (A4). We obtain

$$\begin{aligned}
J_h' &\le C\xi_{\alpha,H,t}\int_s^t\Bigg(\frac{\int_D |G(x,t;y,v)h(u(y,v))|\,dy}{(v-s)^{\alpha}}\\
&\qquad+\int_s^v \frac{\int_D |G(x,t;y,v)h(u(y,v))-G(x,t;y,r)h(u(y,r))|\,dy}{(v-r)^{\alpha+1}}\,dr\Bigg)dv\\
&\le C\xi_{\alpha,H,T}(J_{h1}+J_{h2}+J_{h3}), \qquad (29)
\end{aligned}$$

where

$$\begin{aligned}
J_{h1} &= \int_s^t (v-s)^{-\alpha}\int_D \left|G(x,t;y,v)h\big(u(y,v)\big)\right|dy\,dv,\\
J_{h2} &= \int_s^t\int_s^v (v-r)^{-\alpha-1}\int_D |G(x,t;y,v)|\left|h\big(u(y,v)\big)-h\big(u(y,r)\big)\right|dy\,dr\,dv,\\
J_{h3} &= \int_s^t\int_s^v (v-r)^{-\alpha-1}\int_D |G(x,t;y,v)-G(x,t;y,r)|\left|h\big(u(y,r)\big)\right|dy\,dr\,dv.
\end{aligned}$$

Similarly,

$$\begin{aligned}
J_h'' &\le C\xi_{\alpha,H,s}\int_0^s\Bigg(v^{-\alpha}\int_D |G(x,t;y,v)-G(x,s;y,v)|\left|h\big(u(y,v)\big)\right|dy\\
&\qquad+\int_0^v (v-r)^{-\alpha-1}\int_D\Big|\big(G(x,t;y,v)-G(x,s;y,v)\big)h\big(u(y,v)\big)\\
&\qquad-\big(G(x,t;y,r)-G(x,s;y,r)\big)h\big(u(y,r)\big)\Big|dy\,dr\Bigg)dv\\
&\le C\xi_{\alpha,H,T}(J_{h4}+J_{h4}+J_{h6}),
\end{aligned}$$

where

$$\begin{aligned}
J_{h4} &= \int_0^s v^{-\alpha} \int_D |G(x,t;y,v) - G(x,s;y,v)| \left|h\big(u(y,v)\big)\right| dy\,dv, \\
J_{h5} &= \int_0^s \int_0^v (v-r)^{-\alpha-1} \int_D |G(x,t;y,v) - G(x,s;y,v)| \\
&\quad \times \left|h\big(u(y,v)\big) - h\big(u(y,r)\big)\right| dy\,dr\,dv, \\
J_{h6} &= \int_0^s \int_0^v (v-r)^{-\alpha-1} \int_D \big|G(x,t;y,v) - G(x,s;y,v) \\
&\quad - G(x,t;y,r) + G(x,s;y,r)\big| \left|h\big(u(y,r)\big)\right| dy\,dr\,dv.
\end{aligned}$$

Now it remains to estimate $K_{hi} = \int_0^t (t-s)^{-\alpha-1} J_{hi}\,ds$, $i = 1, 2, \dots, 6$.

In order to bound K_{h1} we apply successively (24), (7) and (15) (with $a = 2\alpha$, $b = 1-\alpha$):

$$\begin{aligned}
K_{h1} &\le C \int_0^t (t-s)^{-\alpha-1} \int_s^t (v-s)^{-\alpha} (1 + \|u\|_v) \int_D |G(x,t;y,v)|\,dy\,dv\,ds \\
&\le C \int_0^t (t-s)^{-\alpha-1} \int_s^t (v-s)^{-\alpha} (1 + \|u\|_v)\,dv\,ds \\
&= C \int_0^t (1 + \|u\|_v) \int_0^v (t-s)^{-\alpha-1} (v-s)^{-\alpha}\,ds\,dv \\
&\le C \int_0^t (1 + \|u\|_v)(t-v)^{-2\alpha}\,dv.
\end{aligned} \tag{30}$$

By the Lipschitz continuity of h,

$$K_{h2} \le C \int_0^t (t-s)^{-\alpha-1} \int_s^t \int_D |G(x,t;y,v)| \int_s^v \frac{|u(y,v) - u(y,r)|}{(v-r)^{\alpha+1}}\,dr\,dy\,dv\,ds.$$

According to the definition, the inner integral can be bounded by $\|u\|_{\alpha,1,v}$. Then we use the Gaussian property of G to obtain

$$\begin{aligned}
K_{h2} &\le C \int_0^t (t-s)^{-\alpha-1} \int_s^t \|u\|_{\alpha,1,v} \int_D |G(x,t;y,v)|\ dy\,dv\,ds \\
&\le C \int_0^t (t-s)^{-\alpha-1} \int_s^t \|u\|_{\alpha,1,v}\ dv\,ds \\
&= C \int_0^t \|u\|_{\alpha,1,v} \int_0^v (t-s)^{-\alpha-1}\,ds\,dv \\
&\le C \int_0^t \|u\|_{\alpha,1,v} (t-v)^{-\alpha}\,dv.
\end{aligned}$$

In order to estimate K_{h3}, we use (24) and (10), and then (14):

$$\begin{aligned}
K_{h3} &\le C \int_0^t (t-s)^{-\alpha-1} \int_s^t (t-v)^{-\delta} \int_s^v (v-r)^{\delta-\alpha-1} (1 + \|u\|_r) \\
&\quad \times \int_D \Phi^C_{t-t^*}(x-y) dy\,dr\,dv\,ds
\end{aligned}$$

$$
\begin{aligned}
&\le C\int_0^t (t-s)^{-\alpha-1}\int_s^t (t-v)^{-\delta}\int_s^v (v-r)^{\delta-\alpha-1}(1+\|u\|_r)\,dr\,dv\,ds\\
&= C\int_0^t (t-s)^{-\alpha-1}\int_s^t (1+\|u\|_r)\int_r^t (t-v)^{-\delta}(v-r)^{\delta-\alpha-1}\,dv\,dr\,ds\\
&\le C\int_0^t (t-s)^{-\alpha-1}\int_s^t (1+\|u\|_r)(t-r)^{-\alpha}\,dr\,ds\\
&= C\int_0^t (1+\|u\|_r)(t-r)^{-\alpha}\int_0^r (t-s)^{-\alpha-1}\,ds\,dr\\
&\le C\int_0^t (1+\|u\|_r)(t-r)^{-2\alpha}\,dr. \qquad (31)
\end{aligned}
$$

The term K_{h4} can be bounded similarly with the help of (24), (11) and (14):

$$
\begin{aligned}
K_{h4} &\le C\int_0^t (t-s)^{\delta-\alpha-1}\int_0^s (1+\|u\|_v)v^{-\alpha}(s-v)^{-\delta}\int_D \Phi^C_{v^*-v}(x-y)dy\,dv\,ds\\
&\le C\int_0^t (t-s)^{\delta-\alpha-1}\int_0^s (1+\|u\|_v)v^{-\alpha}(s-v)^{-\delta}\,dv\,ds\\
&= C\int_0^t (1+\|u\|_v)v^{-\alpha}\int_v^t (t-s)^{\delta-\alpha-1}(s-v)^{-\delta}\,ds\,dv\\
&\le C\int_0^t (1+\|u\|_v)v^{-\alpha}(t-v)^{-\alpha}\,dv.
\end{aligned}
$$

Since $(1+\|u\|_v)(t-v)^{-\alpha}$ is non-decreasing and $v^{-\alpha}$ is non-increasing, using the rearrangement inequality, we obtain

$$
K_{h4} \le C\int_0^t (1+\|u\|_v)(t-v)^{-2\alpha}\,dv. \qquad (32)
$$

From the Lipschitz continuity of h we get

$$
\begin{aligned}
K_{h5} \le C\int_0^t (t-s)^{-\alpha-1}\int_0^s\int_D |G(x,t;y,v)-G(x,s;y,v)|\\
\times\int_0^v \frac{|u(y,v)-u(y,r)|}{(v-r)^{\alpha+1}}\,dr\,dy\,dv\,ds.
\end{aligned}
$$

From (11), (14) it follows that

$$
\begin{aligned}
K_{h5} &\le C\int_0^t (t-s)^{\delta-\alpha-1}\int_0^s \|u\|_{\alpha,1,v}\,(s-v)^{-\delta}\int_D \Phi^C_{v^*-v}(x-y)\,dy\,dv\,ds\\
&\le C\int_0^t (t-s)^{\delta-\alpha-1}\int_0^s \|u\|_{\alpha,1,v}\,(s-v)^{-\delta}\,dv\,ds\\
&= C\int_0^t \|u\|_{\alpha,1,v}\int_v^t (t-s)^{\delta-\alpha-1}(s-v)^{-\delta}\,ds\,dv\\
&\le C\int_0^t \|u\|_{\alpha,1,v}\,(t-v)^{-\alpha}\,dv.
\end{aligned}
$$

Finally, we estimate K_{h6}, using (12), (15), (18) and (24):

$$\begin{aligned}
K_{h6} &\le C \int_0^t (t-s)^{-\alpha-1} \int_0^s \int_0^v (1+\|u\|_r)(v-r)^{-\alpha-1} \\
&\quad \times \int_r^v \int_s^t (\theta-\tau)^{-2} \int_D \Phi^C_{\theta-\tau}(x-y) dy\, d\theta\, d\tau\, dy\, dr\, dv\, ds \\
&\le C \int_0^t (1+\|u\|_\theta) \int_0^\theta (t-s)^{-\alpha-1} \\
&\quad \times \int_0^s \int_0^v (\theta-\tau)^{-2} \int_0^\tau (v-r)^{-\alpha-1} dr\, d\tau\, dv\, ds\, d\theta \\
&\le C \int_0^t (1+\|u\|_\theta) \int_0^\theta (t-s)^{-\alpha-1} \int_0^s \int_0^v (\theta-\tau)^{-2}(v-\tau)^{-\alpha} d\tau\, dv\, ds\, d\theta \\
&\le C \int_0^t (1+\|u\|_\theta) \int_0^\theta (t-s)^{-\alpha-1} \int_0^s (\theta-v)^{-\alpha-1} dv\, ds\, d\theta \\
&\le C \int_0^t (1+\|u\|_\theta) \int_0^\theta (t-s)^{-\alpha-1} (\theta-s)^{-\alpha} ds\, d\theta \\
&\le C \int_0^t (1+\|u\|_\theta)(t-\theta)^{-2\alpha} d\theta. \qquad (33)
\end{aligned}$$

Combining the obtained bounds for K_φ, K'_f, K''_f and K_{hi}, $i=1,\dots,6$, we obtain

$$\begin{aligned}
\int_0^t \frac{|u(x,t)-u(x,v)|}{(t-v)^{\alpha+1}} dv &\le C\xi_{\alpha,H,T} \\
&\quad \times \left(1 + \int_0^t \big(\|u\|_v (t-v)^{-2\alpha} + \|u\|_{\alpha,1,v} (t-v)^{-\alpha}\big) dv\right).
\end{aligned}$$

Since the integral in the right-hand side can be rewritten in the form

$$t^{1-2\alpha} \int_0^1 \|u\|_{zt} (1-z)^{-2\alpha} dz + t^{1-\alpha} \int_0^1 \|u\|_{\alpha,1,zt} (1-z)^{-\alpha} dz,$$

it is a non-decreasing function. Therefore, taking suprema, we arrive at (26). □

Proof of Proposition 1. Lemmata 2 and 3 allow us to use a kind of two-dimensional Grönwall argument, proposed in [17, Lemma 4.1]. Namely, for some $\lambda > 0$, which will be chosen later, define

$$f_1(\lambda) = \sup_{t\in[0,T]} e^{-\lambda t} \|u\|_t, \quad f_2(\lambda) = \sup_{t\in[0,T]} e^{-\lambda t} \|u\|_{\alpha,1,t}$$

and denote for shortness $\xi = \xi_{\alpha,H,T}$. From (16) we get

$$\begin{aligned}
f_1(\lambda) &\le C\xi\left(1 + \sup_{t\in[0,T]} e^{-\lambda t} \int_0^t \big(e^{\lambda s} f_1(\lambda)(t-s)^{-\alpha} + e^{\lambda s} f_2(\lambda)\big) ds\right) \\
&\le C\xi\left(1 + f_1(\lambda) \sup_{t\in[0,T]} \int_0^t e^{-\lambda(t-s)}(t-s)^{-\alpha} ds\right.
\end{aligned}$$

$$
\begin{aligned}
&+ f_2(\lambda) \sup_{t\in[0,T]} \int_0^t e^{-\lambda(t-s)} ds \Big) \\
&\le C\xi \left(1 + f_1(\lambda) \int_0^\infty e^{-\lambda u} u^{-\alpha} du + f_2(\lambda) \int_0^\infty e^{-\lambda u} du \right) \\
&\le C\xi \big(1 + \lambda^{\alpha-1} f_1(\lambda) + \lambda^{-1} f_2(\lambda) \big). \qquad (34)
\end{aligned}
$$

Similarly, from (26) we get

$$
f_2(\lambda) \le C\xi \big(1 + f_1(\lambda)\lambda^{2\alpha-1} + f_2(\lambda)\lambda^{\alpha-1} \big). \qquad (35)
$$

Let K be the largest of the constants in (34) and (35); without loss of generality we can assume that $K \ge 1$. Setting $\lambda = (4K\xi)^{1/(1-\alpha)}$ and plugging it into (35), we obtain

$$
\begin{aligned}
f_2(\lambda) &\le K\xi \big(1 + \lambda^{2\alpha-1} f_1(\lambda) + \lambda^{-1} f_2(\lambda) \big) \\
&= \frac{1}{4}\lambda^{1-\alpha} \big(1 + \lambda^{2\alpha-1} f_1(\lambda) + f_2(\lambda) \big) \\
&= \frac{1}{4} \big(\lambda^{1-\alpha} + \lambda^{\alpha} f_1(\lambda) + f_2(\lambda) \big),
\end{aligned}
$$

whence $f_2(\lambda) \le \frac{1}{3}(\lambda^{1-\alpha} + \lambda^{\alpha} f_1(\lambda))$. Plugging this into (34) and noting that $\lambda \ge 1$, we get

$$
\begin{aligned}
f_1(\lambda) &\le \frac{1}{4}\lambda^{1-\alpha} \left(1 + \lambda^{\alpha-1} f_1(\lambda) + \frac{1}{3}\lambda^{-\alpha} + \frac{1}{3}\lambda^{\alpha-1} f_1(\lambda) \right) \\
&\le \lambda^{1-\alpha} + \frac{1}{3} f_1(\lambda),
\end{aligned}
$$

whence $f_1(\lambda) \le \frac{3}{2}\lambda^{1-\alpha} = 6K\xi$. It follows that

$$
\|u\|_T \le f_1(\lambda) e^{\lambda T} \le 6K\xi \exp\left\{ (4K\xi)^{1/(1-\alpha)} T \right\} \le C \exp\left\{ C\xi^{1/(1-\alpha)} \right\}.
$$

Similarly,

$$
\|u\|_{\alpha,1,T} \le f_2(\lambda) e^{\lambda T} \le \lambda^{\alpha} f_1(\lambda) e^{\lambda T} \le C \exp\left\{ C\xi^{1/(1-\alpha)} \right\}.
$$

The statement then follows from adding these estimates. □

By Fernique's theorem, $\mathsf{E} \exp\{a\xi^2\} < \infty$ for some $a > 0$. Since $\frac{1}{1-\alpha} < 2$, we have $C\xi^{1/(1-\alpha)} < a\xi^2 + C$, so Proposition 1 implies existence of moments of the solution.

Corollary 1. *Under assumptions (A1)–(A4), the solution u to (1) satisfies*

$$
\mathsf{E} \|u\|_{\alpha,\infty,T}^p < \infty
$$

for all $p > 0$. In particular,

$$
\mathsf{E} \sup_{t\in[0,T], x\in D} |u(t,x)|^p < \infty
$$

for all $p > 0$.

5 Existence and uniqueness of mild solution

The following theorem is the main result of the article.

Theorem 1. *Let $H \in (1/2, 1)$, $\alpha \in (1-H, 1/2)$. Assume that Hypotheses (A1)–(A4) hold. Then the problem (1) has a unique mild solution.*

Under the standing assumptions, the existence of a mild solution was established in [16, Th. 2.3(a)]. Hence, it remains to prove the uniqueness.

Let u and $\tilde{u}$ be two mild solutions to the problem (1). In order to prove that u and $\tilde{u}$ coincide we shall establish that the norm

$$\|u-\tilde{u}\|_{\alpha,\infty,T} = \|u-\tilde{u}\|_T + \|u-\tilde{u}\|_{\alpha,1,T} \tag{36}$$

is equal to zero. The proof of this fact is carried out similarly to that of Proposition 1, using the bounds

$$\begin{aligned} &\left|f\big(u(y,s)\big) - f\big(\tilde{u}(y,s)\big)\right| + \left|h\big(u(y,s)\big) - h\big(\tilde{u}(y,s)\big)\right| \\ &\quad \le C\,|u(y,s)-\tilde{u}(y,s)| \le C\,\|u-\tilde{u}\|_s \end{aligned} \tag{37}$$

instead of (19) and (24). Therefore we omit some details. As above, we first obtain the upper bounds for each of two terms in the right-hand side of the norm (36).

Let $\xi = \xi_{\alpha,H,T}$. Denote also

$$\eta = 1 + \|u\|_{\alpha,1,T} + \|\tilde{u}\|_{\alpha,1,T}\,.$$

Corollary 1 implies that η is finite a. s.

Lemma 4. *Under assumptions (A1)–(A4),*

$$\|u-\tilde{u}\|_t \le C\xi\eta \int_0^t \|u-\tilde{u}\|_{\alpha,\infty,s}\,(t-s)^{-\alpha} ds$$

for all $t \in [0,T]$.

Proof. By (5),

$$|u(x,t)-\tilde{u}(x,t)| \le P_f + P_h, \tag{38}$$

where

$$P_f = \int_0^t \int_D |G(x,t;y,s)|\left|f\big(u(y,s)\big) - f\big(\tilde{u}(y,s)\big)\right| dy\,ds,$$

$$P_h = \sum_{j=1}^{\infty} \lambda_j^{1/2} \left|\int_0^t \int_D G(x,t;y,s)\big(h\big(u(y,s)\big) - h\big(\tilde{u}(y,s)\big)\big) e_j(y)\,dy\,dB_j^H(s)\right|.$$

Using the bound (37) and the Gaussian property of G we immediately get

$$P_f \le C \int_0^t \|u-\tilde{u}\|_s\,ds. \tag{39}$$

Further, similarly to (21),

$$P_h \le C\xi(P_{h1} + P_{h2} + P_{h3}),$$

where

$$
\begin{aligned}
P_{h1} &= \int_0^t s^{-\alpha} \int_D |G(x,t;y,s)| \left|h\big(u(y,s)\big) - h\big(\tilde{u}(y,s)\big)\right| dy\,ds,\\
P_{h2} &= \int_0^t \int_0^s (s-v)^{-\alpha-1} \int_D |G(x,t;y,s)|\\
&\quad \times \left|h\big(u(y,s)\big) - h\big(\tilde{u}(y,s)\big) - h\big(u(y,v)\big) + h\big(\tilde{u}(y,v)\big)\right| dy\,dv\,ds, \qquad (40)\\
P_{h3} &= \int_0^t \int_0^s (s-v)^{-\alpha-1} \int_D |G(x,t;y,s) - G(x,t;y,v)|\\
&\quad \times \left|h\big(u(y,v)\big) - h\big(\tilde{u}(y,v)\big)\right| dy\,dv\,ds.
\end{aligned}
$$

The bounds

$$P_{h1} \le C \int_0^t (t-s)^{-\alpha} \|u-\tilde{u}\|_s\,ds \tag{41}$$

and

$$P_{h3} \le C \int_0^t (t-s)^{-\alpha} \|u-\tilde{u}\|_s\,ds \tag{42}$$

are obtained analogously to the bounds (22) and (25).

According to [13, Lemma 7.1], the assumption (A3) implies that for any x_1, x_2, x_3, x_4,

$$
\begin{aligned}
|h(x_1) - h(x_2) - h(x_3) + h(x_4)| &\le C\,|x_1 - x_2 - x_3 + x_4|\\
&\quad + C\,|x_1 - x_3|\,(|x_1 - x_2| + |x_3 - x_4|).
\end{aligned}
$$

Therefore, we can write

$$
\begin{aligned}
&\int_0^s \frac{|h(u(y,s)) - h(\tilde{u}(y,s)) - h(u(y,v)) + h(\tilde{u}(y,v))|}{(s-v)^{\alpha+1}}\,dv\\
&\quad \le C \int_0^s \frac{|u(y,s) - \tilde{u}(y,s) - u(y,v) + \tilde{u}(y,v)|}{(s-v)^{\alpha+1}}\,dv\\
&\qquad + C\,|u(y,s) - \tilde{u}(y,s)| \int_0^s \frac{|u(y,s) - u(y,v)| + |\tilde{u}(y,s) - \tilde{u}(y,v)|}{(s-v)^{\alpha+1}}\,dv\\
&\quad \le C\,\|u-\tilde{u}\|_{\alpha,1,s} + C\,\|u-\tilde{u}\|_s\,(\|u\|_{\alpha,1,s} + \|\tilde{u}\|_{\alpha,1,s})\\
&\quad \le C\eta\,\|u-\tilde{u}\|_{\alpha,\infty,s}\,.
\end{aligned}
\tag{43}
$$

Inserting the bound (43) into (40), we get

$$P_{h2} \le C\eta \int_0^t \|u-\tilde{u}\|_{\alpha,\infty,s} \int_D |G(x,t;y,s)|\,dy\,ds \le C\eta \int_0^t \|u-\tilde{u}\|_{\alpha,\infty,s}\,ds. \tag{44}$$

Combining (38), (39), (41), (42), and (44) we arrive at

$$|u(x,t) - \tilde{u}(x,t)| \le C\xi\eta \int_0^t \|u-\tilde{u}\|_{\alpha,\infty,s}\,(t-s)^{-\alpha} ds.$$

We conclude the proof similarly to that of Lemma 2, t using the monotonicity of the right-hand side. □

Lemma 5. *Under assumptions (A1)–(A4),*

$$\|u-\tilde{u}\|_{\alpha,1,t} \le C\xi\eta \int_0^t \|u-\tilde{u}\|_{\alpha,\infty,s}\,(t-s)^{-2\alpha} ds \tag{45}$$

for all $t \in [0, T]$.

Proof. As in the proof of Lemma 3, we can write

$$\begin{aligned}
&\int_0^t \frac{|u(x,t)-\tilde{u}(x,t)-u(x,s)+\tilde{u}(x,s)|}{(t-s)^{\alpha+1}}\,ds \\
&\le \int_0^t (t-s)^{-\alpha-1}\bigg(\bigg|\int_0^t\int_D G(x,t;y,v)\big(f\big(u(y,v)\big)-f\big(\tilde{u}(y,v)\big)\big)\,dy\,dv \\
&\quad -\int_0^s\int_D G(x,s;y,v)\big(f\big(u(y,v)\big)-f\big(\tilde{u}(y,v)\big)\big)\,dy\,dv\bigg| \\
&\quad +\sum_{j=1}^{\infty}\lambda_j^{1/2}\bigg|\int_0^t\int_D G(x,t;y,v)\big(h\big(u(y,v)\big)-h\big(\tilde{u}(y,v)\big)\big)e_j(y)\,dy\,dB_j^H(v) \\
&\quad -\int_0^s\int_D G(x,s;y,v)\big(h\big(u(y,v)\big)-h\big(\tilde{u}(y,v)\big)\big)e_j(y)\,dy\,dB_j^H(v)\bigg|\bigg)ds \\
&\le Q'_f+Q''_f+Q'_h+Q''_h,
\end{aligned}$$

where

$$\begin{aligned}
Q'_f &= \int_0^t (t-s)^{-\alpha-1}\int_s^t\int_D |G(x,t;y,v)|\,\big|f\big(u(y,v)\big)-f\big(\tilde{u}(y,v)\big)\big|\,dy\,dv\,ds, \\
Q''_f &= \int_0^t (t-s)^{-\alpha-1}\int_0^s\int_D |G(x,t;y,v)-G(x,s;y,v)| \\
&\qquad \times \big|f\big(u(y,v)\big)-f\big(\tilde{u}(y,v)\big)\big|\,dy\,dv\,ds, \\
Q'_h &= \sum_{j=1}^{\infty}\lambda_j^{1/2}\int_0^t (t-s)^{-\alpha-1} \\
&\qquad \times \bigg|\int_s^t\int_D G(x,t;y,v)\big(h\big(u(y,v)\big)-h\big(\tilde{u}(y,v)\big)\big)e_j(y)\,dy\,dB_j^H(v)\bigg|\,ds, \\
Q''_h &= \sum_{j=1}^{\infty}\lambda_j^{1/2}\int_0^t (t-s)^{-\alpha-1}\bigg|\int_s^t\int_D \big(G(x,t;y,v)-G(x,s;y,v)\big) \\
&\qquad \times \big(h\big(u(y,v)\big)-h\big(\tilde{u}(y,v)\big)\big)e_j(y)\,dy\,dB_j^H(v)\bigg|\,ds.
\end{aligned}$$

Similarly to (27), (28) and (29), we get

$$\begin{aligned}
Q'_f &\le C\int_0^t \|u-\tilde{u}\|_v\,(t-v)^{-2\alpha}\,dv, \\
Q''_f &\le C\int_0^t \|u-\tilde{u}\|_v\,(t-v)^{-2\alpha}\,dv,
\end{aligned}$$

and

$$Q'_h \le C\xi(Q_{h1} + Q_{h2} + Q_{h3}),$$

where

$$\begin{aligned}
Q_{h1} &= \int_0^t (t-s)^{-\alpha-1} \int_s^t (v-s)^{-\alpha} \\
&\quad \times \int_D |G(x,t;y,v)| \left|h\big(u(y,v)\big) - h\big(\tilde{u}(y,v)\big)\right| dy\,dv\,ds, \\
Q_{h2} &= \int_0^t (t-s)^{-\alpha-1} \int_s^t \int_s^v (v-r)^{-\alpha-1} \int_D |G(x,t;y,v)| \\
&\quad \times \left|h\big(u(y,v)\big) - h\big(\tilde{u}(y,v)\big) - h\big(u(y,r)\big) + h\big(\tilde{u}(y,r)\big)\right| dy\,dr\,dv\,ds, \\
Q_{h3} &= \int_0^t (t-s)^{-\alpha-1} \int_s^t \int_s^v (v-r)^{-\alpha-1} \\
&\quad \times \int_D |G(x,t;y,v) - G(x,t;y,r)| \left|h\big(u(y,r)\big) - h\big(\tilde{u}(y,r)\big)\right| dy\,dr\,dv\,ds.
\end{aligned}$$

Further, we estimate

$$\begin{aligned}
Q_{h1} &\le C \int_0^t \|u-\tilde{u}\|_v \,(t-v)^{-2\alpha}\,dv, \\
Q_{h3} &\le C \int_0^t \|u-\tilde{u}\|_v \,(t-v)^{-2\alpha}\,dv,
\end{aligned}$$

similarly to (30) and (31). Then applying (43), the Gaussian property of G, and integrating with respect to s, we obtain

$$\begin{aligned}
Q_{h2} &\le C\eta \int_0^t (t-s)^{-\alpha-1} \int_s^t \|u-\tilde{u}\|_{\alpha,\infty,v} \int_D |G(x,t;y,v)|\,dy\,dv\,ds, \\
&\le C\eta \int_0^t (t-v)^{-\alpha} \|u-\tilde{u}\|_{\alpha,\infty,v}\,dv \le C\eta \int_0^t (t-v)^{-2\alpha} \|u-\tilde{u}\|_{\alpha,\infty,v}\,dv.
\end{aligned}$$

Finally,

$$Q''_h \le C\xi(Q_{h4} + Q_{h5} + Q_{h6}),$$

where

$$\begin{aligned}
Q_{h4} &= \int_0^t (t-s)^{-\alpha-1} \int_0^s v^{-\alpha} \int_D |G(x,t;y,v) - G(x,s;y,v)| \\
&\quad \times \left|h\big(u(y,v)\big) - h\big(\tilde{u}(y,v)\big)\right| dy\,dv\,ds, \\
Q_{h5} &= \int_0^t (t-s)^{-\alpha-1} \int_0^s \int_0^v (v-r)^{-\alpha-1} \int_D |G(x,t;y,v) - G(x,s;y,v)| \\
&\quad \times \left|h\big(u(y,v)\big) - h\big(\tilde{u}(y,v)\big) - h\big(u(y,r)\big) + h\big(\tilde{u}(y,r)\big)\right| dy\,dr\,dv\,ds, \\
Q_{h6} &= \int_0^t (t-s)^{-\alpha-1} \int_0^s \int_0^v (v-r)^{-\alpha-1} \int_D \big|G(x,t;y,v) - G(x,s;y,v) \\
&\quad - G(x,t;y,r) + G(x,s;y,r)\big| \left|h\big(u(y,r)\big) - h\big(\tilde{u}(y,r)\big)\right| dy\,dr\,dv\,ds.
\end{aligned}$$

Arguing as in the proof of Lemma 3, we can prove the bounds

$$Q_{h4} \le C \int_0^t \|u - \tilde{u}\|_v \,(t - v)^{-2\alpha}\, dv,$$
$$Q_{h6} \le C \int_0^t \|u - \tilde{u}\|_v \,(t - v)^{-2\alpha}\, dv,$$

which are similar to (32) and (33). Finally, using (43), (11), and (14), we get

$$\begin{aligned} Q_{h5} &\le C\eta \int_0^t (t-s)^{-\alpha-1} \int_0^s \|u - \tilde{u}\|_{\alpha,\infty,v} \\ &\quad \times \int_D |G(x,t;y,v) - G(x,s;y,v)|\, dy\, dv\, ds \\ &\le C\eta \int_0^t (t-s)^{\delta-\alpha-1} \int_0^s \|u - \tilde{u}\|_{\alpha,\infty,v} (s-v)^{-\delta}\, dv\, ds \\ &\le C\eta \int_0^t \|u - \tilde{u}\|_{\alpha,\infty,v} (t-v)^{-\alpha}\, dv \\ &\le C\eta \int_0^t \|u - \tilde{u}\|_{\alpha,\infty,v} (t-v)^{-2\alpha}\, dv. \end{aligned}$$

Combining the bounds for Q'_f, Q''_f and Q_{hi}, $i = 1, \ldots, 6$, we arrive at

$$\int_0^t \frac{|u(x,t) - \tilde{u}(x,t) - u(x,s) + \tilde{u}(x,s)|}{(t-s)^{\alpha+1}} ds \le C\xi\eta \int_0^t \|u - \tilde{u}\|_{\alpha,\infty,s} (t-s)^{-2\alpha} ds.$$

Taking suprema we get (45). □

Proof of Theorem 1. Adding the bounds from Lemmata 4–5, we obtain that for all $t \in [0,T]$

$$\begin{aligned} \|u - \tilde{u}\|_{\alpha,\infty,t} &\le C\xi\eta \int_0^t \|u - \tilde{u}\|_{\alpha,\infty,s} (t-s)^{-2\alpha} ds \\ &\le C\xi\eta\, t^{2\alpha} \int_0^t \|u - \tilde{u}\|_{\alpha,\infty,s} (t-s)^{-2\alpha} s^{-2\alpha} ds. \end{aligned}$$

Then $\|u - \tilde{u}\|_{\alpha,\infty,T} = 0$ by the generalized Grönwall lemma [13, Lemma 7.6]. □

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