Modern Stochastics: Theory and Applications 1 (2014) 129–138
DOI: 10.15559/14-VMSTA14

Heat equation with general stochastic measure colored in time

Vadym Radchenko*

Department of Mathematical Analysis,
Taras Shevchenko National University of Kyiv, Kyiv, Ukraine

vradchenko@univ.kiev.ua (V. Radchenko)

Received: 2 October 2014, Revised: 10 November 2014, Accepted: 13 November 2014,
Published online: 5 December 2014

Abstract A stochastic heat equation on $[0, T] \times \mathbb{R}$ driven by a general stochastic measure $d\mu(t)$ is investigated in this paper. For the integrator μ, we assume the σ-additivity in probability only. The existence, uniqueness, and Hölder regularity of the solution are proved.

Keywords Stochastic measure, stochastic heat equation, mild solution, Hölder regularity, Besov space

2010 MSC 60H15, 60G17, 60G57

1 Introduction

In this paper, we consider a stochastic heat equation that can formally be written as

$$\begin{cases} du(t,x) = a^2 \dfrac{\partial^2 u(t,x)}{\partial x^2}\, dt + f\big(t, x, u(t,x)\big)\, dt + \sigma(t,x)\, d\mu(t), \\ u(0,x) = u_0(x), \end{cases} \tag{1}$$

where $(t, x) \in [0, T] \times \mathbb{R}$, $a \in \mathbb{R}$, $a \neq 0$, and μ is a stochastic measure (SM) defined on the Borel σ-algebra of $[0, T]$. We consider a solution to the formal equation (1) in the mild sense (see Eq. (5)). We prove the existence and uniqueness of the solution and obtain Hölder regularity of its paths under some general conditions for the stochastic part of equation.

*Corresponding author.

A similar problem for μ dependent on the spatial variable x was considered in [5]. The stochastic heat equation on fractals was studied in [7], and a review of results on equations driven by SMs is given in [6].

For equations driven by white noise, the regularity of paths of solutions was considered in [10, Chapter 3]. Equations driven by fractional noise were studied in [9, Chapter 2]. In many papers, the regularity of solutions was considered in appropriate function spaces; see, for example, [2] and references therein.

2 Preliminaries

Let $\mathsf{L}_0 = \mathsf{L}_0(\Omega, \mathcal{F}, \mathsf{P})$ be the set of (equivalence classes of) all real-valued random variables defined on a complete probability space $(\Omega, \mathcal{F}, \mathsf{P})$. The convergence in L_0 is understood as the convergence in probability. Let X be an arbitrary set, and $\mathcal{B}$ be a σ-algebra of subsets of X.

Definition 1. Any σ-additive mapping $\mu : \mathcal{B} \to \mathsf{L}_0$ is called a *stochastic measure* (SM).

In other words, μ is a vector measure with values in L_0. In [3], such μ is called a general SM.

Examples of SMs are the following. Let $\mathsf{X} = [0, T] \subset \mathbb{R}_+$, $\mathcal{B}$ be the σ-algebra of Borel subsets of $[0, T]$, and $N(t)$ be a square-integrable martingale. Then $\mu(\mathsf{A}) = \int_0^T \mathbf{1}_{\mathsf{A}}(t)\, dN(t)$ is an SM. If $W^H(t)$ is a fractional Brownian motion with Hurst index $H > 1/2$ and $f : [0, T] \to \mathbb{R}$ is a bounded measurable function, then $\mu(\mathsf{A}) = \int_0^T f(t)\mathbf{1}_{\mathsf{A}}(t)\, dW^H(t)$ is also an SM, as follows from [4, Theorem 1.1]. An α-stable random measure defined on a σ-algebra is an SM [8, Chapter 3]. Theorem 8.3.1 of [3] states the conditions under which the increments of a real-valued Lévy process generate an SM.

For a deterministic measurable function $g : \mathsf{X} \to \mathbb{R}$ and SM μ, an integral of the form $\int_{\mathsf{X}} g\, d\mu$ is defined and studied in [3, Chapter 7]; see also [1]. In particular, every bounded measurable g is integrable w.r.t. any μ. An analogue of the Lebesgue dominated convergence theorem holds for this integral [3, Proposition 7.1.1].

We consider the *Besov spaces* $B_{22}^{\alpha}([c, d])$. Recall that the norm in this classical space for $0 < \alpha < 1$ may be introduced by

$$\|g\|_{B_{22}^{\alpha}([c,d])} = \|g\|_{\mathsf{L}_2([c,d])} + \left(\int_0^{d-c} \big(w_2(g, r)\big)^2 r^{-2\alpha-1}\, dr \right)^{1/2}, \tag{2}$$

where

$$w_2(g, r) = \sup_{0 \le h \le r} \left(\int_c^{d-h} \left| g(s+h) - g(s) \right|^2 ds \right)^{1/2}.$$

For all $n \ge 1$, $1 \le k \le 2^n$, put $\Delta_{kn}^{(t)} = ((k-1)2^{-n}t, k2^{-n}t]$.

The following estimate is a key tool for the proof of Hölder regularity of the stochastic integral. In our estimates, C and $C(\omega)$ will denote a constant and a random constant, respectively, which may be different from formula to formula.

Lemma 1 (Lemma 3.2 [5]). *Let SM μ be defined on the Borel σ-algebra of* $[0,t]$*, Z be an arbitrary set, and $q(z,s):\mathsf{Z}\times[0,t]\to\mathbb{R}$ be a function such that for some $1/2<\alpha<1$ and for each $z\in\mathsf{Z}$, $q(z,\cdot)\in B_{22}^{\alpha}([0,t])$. Then the random function*

$$\eta(z)=\int_{[0,t]} q(z,s)\,d\mu(s),\quad z\in\mathsf{Z},$$

has a version $\tilde{\eta}(z)$ such that for some constant C (independent of z, ω) and each $\omega\in\Omega$,

$$\begin{aligned}\left|\tilde{\eta}(z)\right| &\le \left|q(z,0)\mu\big([0,t]\big)\right| \\ &\quad + C\left\|q(z,\cdot)\right\|_{B_{22}^{\alpha}([0,t])}\left\{\sum_{n\ge1}2^{n(1-2\alpha)}\sum_{1\le k\le 2^n}\left|\mu\big(\Delta_{kn}^{(t)}\big)\right|^2\right\}^{1/2}.\end{aligned}\tag{3}$$

From Lemma 3.1 [5] it follows that, for $\varepsilon>0$,

$$\sum_{n\ge1}2^{-n\varepsilon}\sum_{1\le k\le 2^n}\left|\mu\big(\Delta_{kn}^{(t)}\big)\right|^2<+\infty\quad \text{a. s.}\tag{4}$$

3 The problem

Consider equation (1) in the following mild sense:

$$\begin{aligned}u(t,x)=&\int_{\mathbb{R}}p(t,x-y)u_0(y)\,dy+\int_0^t ds\int_{\mathbb{R}}p(t-s,x-y)f\big(s,y,u(s,y)\big)\,dy\\ &+\int_{(0,t]}d\mu(s)\int_{\mathbb{R}}p(t-s,x-y)\sigma(s,y)\,dy.\end{aligned}\tag{5}$$

Here

$$p(t,x)=\frac{1}{2a\sqrt{\pi t}}\,e^{-\frac{x^2}{4a^2t}}\tag{6}$$

is the Gaussian heat kernel, $u(t,x)=u(t,x,\omega):[0,T]\times\mathbb{R}\times\Omega\to\mathbb{R}$ is an unknown measurable random function, and μ is an SM defined on the Borel σ-algebra of $[0,T]$. The integrals of random functions w.r.t. dy and ds are taken for each fixed $\omega\in\Omega$.

Throughout this paper, we will use the following assumptions.

A1. $u_0(y)=u_0(y,\omega):\mathbb{R}\times\Omega\to\mathbb{R}$ is measurable and ω-wise bounded, $|u_0(y,\omega)|\le C(\omega)$.

A2. $u_0(y)$ is Hölder continuous:

$$\left|u_0(y_1)-u_0(y_2)\right|\le C(\omega)|y_1-y_2|^{\beta(u_0)},\quad \beta(u_0)\ge 1/2.$$

A3. $f(s,y,v):[0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is measurable and bounded: $|f(s,y,v)|\le C$.

A4. $f(s,y,v)$ is uniformly Lipschitz in $y,v\in\mathbb{R}$:

$$\left|f(s,y_1,v_1)-f(s,y_2,v_2)\right|\le C\big(|y_1-y_2|+|v_1-v_2|\big).$$

A5. $\sigma(s, y) : [0, T] \times \mathbb{R} \to \mathbb{R}$ is measurable and bounded: $|\sigma(s, y)| \le C$.

A6. $\sigma(s, y)$ is Hölder continuous:

$$\big|\sigma(s_1, y_1) - \sigma(s_2, y_2)\big| \le C\big(|s_1 - s_2|^{\beta(\sigma)} + |y_1 - y_2|^{\beta(\sigma)}\big), \quad \beta(\sigma) > 1/2.$$

A7. μ is Hölder continuous:

$$\big|\mu\big((s_1, s_2]\big)\big| \le C(\omega)|s_1 - s_2|^{\beta(\mu)}, \quad s_1, s_2 \in [0, T],\ \beta(\mu) > 0.$$

Recall that $\int_{\mathbb{R}} p(t, x)\, dx = 1$.

4 Hölder continuity in *x*

Consider the regularity of paths of the stochastic integral from (5).

Lemma 2. *Let Assumptions A5 and A6 hold. Then, for any fixed* $t \in [0, T]$ *and* $\gamma_1 < \beta(\sigma) - 1/2$*, the stochastic function*

$$\vartheta(x) = \int_{(0,t]} d\mu(s) \int_{\mathbb{R}} p(t - s, x - y)\sigma(s, y)\, dy, \quad x \in \mathbb{R},$$

has a Hölder continuous version with exponent γ_1.

Proof. Denote

$$q(z, s) = \int_{\mathbb{R}} \big(p(t - s, x_1 - y) - p(t - s, x_2 - y)\big)\sigma(s, y)\, dy, \quad z = (x_1, x_2, t),$$

and apply (3) to $\eta(z) = \vartheta(x_1) - \vartheta(x_2)$. We will estimate the Besov space norm in (3). Consider the difference

$$\begin{aligned} &q(z, s + h) - q(z, s) \\ &\quad = \Big(\int_{\mathbb{R}} p(t - s - h, x_1 - y)\sigma(s + h, y)\, dy - \int_{\mathbb{R}} p(t - s, x_1 - y)\sigma(s, y)\, dy\Big) \\ &\qquad - \Big(\int_{\mathbb{R}} p(t - s - h, x_2 - y)\sigma(s + h, y)\, dy - \int_{\mathbb{R}} p(t - s, x_2 - y)\sigma(s, y)\, dy\Big) \\ &\quad := D_1 - D_2. \end{aligned} \tag{7}$$

Using (6) and the change of variables

$$v = \frac{x_1 - y}{2a\sqrt{t - s - h}}, \qquad v = \frac{x_1 - y}{2a\sqrt{t - s}},$$

we get

$$\begin{aligned} |D_1| = C\Big|&\int_{\mathbb{R}} e^{-v^2}\sigma(s + h, x_1 - 2av\sqrt{t - s - h})\, dv \\ &- \int_{\mathbb{R}} e^{-v^2}\sigma(s, x_1 - 2av\sqrt{t - s})\, dv\Big| \end{aligned}$$

$$\overset{\text{A6}}{\leq} C\int_{\mathbb{R}} e^{-v^2}\big(|h|^{\beta(\sigma)} + \big|v(\sqrt{t-s-h}-\sqrt{t-s})\big|^{\beta(\sigma)}\big)\,dv$$
$$= C\int_{\mathbb{R}} e^{-v^2}\left(|h|^{\beta(\sigma)} + \frac{|v|^{\beta(\sigma)}|h|^{\beta(\sigma)}}{|\sqrt{t-s-h}+\sqrt{t-s}|^{\beta(\sigma)}}\right)dv$$
$$\leq C|h|^{\beta(\sigma)}\int_{\mathbb{R}} e^{-v^2}\left(1+\frac{|v|^{\beta(\sigma)}}{\sqrt{t-s}^{\beta(\sigma)}}\right)dv = C|h|^{\beta(\sigma)}(t-s)^{-\beta(\sigma)/2}. \tag{8}$$

By a similar way, we can estimate $|D_2|$ and obtain

$$\big|q(z,s+h)-q(z,s)\big| \leq C|h|^{\beta(\sigma)}(t-s)^{-\beta(\sigma)/2}. \tag{9}$$

Further, consider

$$\begin{aligned} q(z,s+h)-q(z,s) = &\left(\int_{\mathbb{R}} p(t-s-h,x_1-y)\sigma(s+h,y)\,dy\right. \\ &\left.-\int_{\mathbb{R}} p(t-s-h,x_2-y)\sigma(s+h,y)\,dy\right) \\ &-\left(\int_{\mathbb{R}} p(t-s,x_1-y)\sigma(s,y)\,dy\right. \\ &\left.-\int_{\mathbb{R}} p(t-s,x_2-y)\sigma(s,y)\,dy\right) := E_1 - E_2. \end{aligned}$$

Using (6) and the substitutions

$$v = \frac{x_1-y}{2a\sqrt{t-s-h}}, \quad v = \frac{x_2-y}{2a\sqrt{t-s-h}},$$

we get

$$\begin{aligned} |E_1| = C&\left|\int_{\mathbb{R}} e^{-v^2}\sigma(s+h,x_1-2av\sqrt{t-s-h})\,dv\right. \\ &\left.-\int_{\mathbb{R}} e^{-v^2}\sigma(s+h,x_2-2av\sqrt{t-s-h})\,dv\right| \\ &\overset{\text{A6}}{\leq} C\int_{\mathbb{R}} e^{-v^2}|x_1-x_2|^{\beta(\sigma)}\,dv = C|x_1-x_2|^{\beta(\sigma)}. \end{aligned}$$

Similarly, we can estimate $|E_2|$ (we consider $|E_1|$ for $h=0$) and obtain

$$\big|q(z,s+h)-q(z,s)\big| \leq C|x_1-x_2|^{\beta(\sigma)}. \tag{10}$$

The product of (9) raised to the power λ and (10) raised to the power $1-\lambda$, $0<\lambda<1$, now satisfies

$$\begin{aligned} \big|q(z,s+h)-q(z,s)\big| &\leq C|h|^{\lambda\beta(\sigma)}(t-s)^{-\beta(\sigma)\lambda/2}|x_1-x_2|^{(1-\lambda)\beta(\sigma)}, \\ w_2\big(q(z,\cdot),r\big) &\leq Cr^{\lambda\beta(\sigma)}|x_1-x_2|^{(1-\lambda)\beta(\sigma)}. \end{aligned}$$

If $\lambda\beta(\sigma) > 1/2$, then the integral from (2) is finite for some $\alpha > 1/2$. In this case, the integral does not exceed $C|x_1 - x_2|^{(1-\lambda)\beta(\sigma)}$.

From the estimate of E_1 for $h = 0$ we obtain

$$\left|q(z, 0)\right| \le C|x_1 - x_2|^{\beta(\sigma)}, \qquad \left\|q(z, \cdot)\right\|_{\mathsf{L}_2([0,t])} \le C|x_1 - x_2|^{\beta(\sigma)}.$$

Therefore, we have

$$\left|\vartheta(x_1) - \vartheta(x_2)\right| \le C(\omega)|x_1 - x_2|^{\gamma_1}, \quad \gamma_1 = (1 - \lambda)\beta(\sigma).$$

Under the restriction $\lambda\beta(\sigma) > 1/2$, we can get any $\gamma_1 < \beta(\sigma) - 1/2$. □

5 Hölder continuity in *t*

Lemma 3. *Assume that Assumptions A5, A6, and A7 hold. Then, if $\gamma_2 \le \beta(\mu)$ and $\gamma_2 < \beta(\sigma) - 1/2$, then for any fixed $x \in \mathbb{R}$, the stochastic process*

$$\bar{\vartheta}(t) = \int_{(0,t]} d\mu(s) \int_{\mathbb{R}} p(t - s, x - y)\sigma(s, y)\, dy, \quad t \in [0, T],$$

has a Hölder continuous version with exponent γ_2.

Proof. For $t_1 < t_2$, we have

$$\begin{aligned}\bar{\vartheta}(t_2) - \bar{\vartheta}(t_1) &= \int_{(t_1,t_2]} d\mu(s) \int_{\mathbb{R}} p(t_2 - s, x - y)\sigma(s, y)\, dy \\ &\quad + \int_{(0,t_1]} d\mu(s) \int_{\mathbb{R}} \big(p(t_2 - s, x - y) - p(t_1 - s, x - y)\big)\sigma(s, y)\, dy \\ &:= F_1 + F_2.\end{aligned}$$

Step 1. Estimation of F_1. Consider segments $[0, T]$, $\Delta_{kn}^{(T)} = ((k-1)2^{-n}T, k2^{-n}T]$, and the function

$$\bar{q}(z, s) = \int_{\mathbb{R}} p(t_2 - s, x - y)\sigma(s, y)\, dy, \quad s \in [t_1, t_2],\ z = (x, t_2).$$

From the estimates of D_1 in (7) and (8) it follows that

$$\left|\bar{q}(z, s + h) - \bar{q}(z, s)\right| \le C|h|^{\beta(\sigma)}(t_2 - s)^{-\beta(\sigma)/2}, \quad s \in [t_1 - h, t_2 - h). \tag{11}$$

Let k_{n1} and k_{n2} be such that $t_1 \in \Delta_{k_{n1}n}^{(T)}$, $t_2 \in \Delta_{k_{n2}n}^{(T)}$. For the functions

$$\bar{q}_n(z, s) = \sum_{k=1}^{2^n} \bar{q}\big(z, (k-1)2^{-n}T \vee t_1 \wedge t_2\big)\mathbf{1}_{\Delta_{kn}^{(T)}}(s),$$

the analogue of the Lebesgue theorem [3, Proposition 7.1.1] implies that

$$\left|\int_{(t_1,t_2]} \bar{q}(z, s)\, d\mu(s)\right| = \left|\mathrm{p}\lim_{n\to\infty} \int_{(t_1,t_2]} \bar{q}_n(z, s)\, d\mu(s)\right|$$

$$= \left| \int_{(t_1,t_2]} \bar{q}_0(z,s)\, d\mu(s) + \sum_{n=1}^{\infty} \left(\int_{(t_1,t_2]} \bar{q}_n(z,s)\, d\mu(s) - \int_{(t_1,t_2]} \bar{q}_{n-1}(z,s)\, d\mu(s) \right) \right|$$
$$\le \left| \bar{q}(z,t_1)\mu\big((t_1,t_2]\big) \right| + \sum_{n=n_0}^{\infty} \left| \big(\bar{q}\big(z, k_{n1}2^{-n}T\big) - \bar{q}(z,t_1)\big)\mu\big(\Delta^{(T)}_{(k_{n1}+1)n}\big) \right|$$
$$+ \sum_{n=n_0}^{\infty} \sum_{k: k_{n1} \le 2k-2 < k_{n2}-2} \left| \big(\bar{q}\big(z,(2k-1)2^{-n}T\big) - \bar{q}\big(z,(2k-2)2^{-n}T\big)\big)\mu\big(\Delta^{(T)}_{(2k)n}\big) \right|$$
$$+ \sum_{n=n_0}^{\infty} \left| \big(\bar{q}\big(z,(k_{n2}-1)2^{-n}T\big) - \bar{q}\big(z,(k_{n2}-2)2^{-n}T\big)\big)\mu\big(\big((k_{n2}-1)2^{-n}T, t_2\big]\big) \right|. \tag{12}$$

Here n_0 is such that

$$2^{-n_0}T < t_2 - t_1 \le 2^{-n_0+1}T. \tag{13}$$

We have

$$k_{n2} - k_{n1} \le (t_2 - t_1)2^{n+2}/T, \quad n \ge n_0. \tag{14}$$

Applying Assumptions A5, A7, (11), and the Cauchy inequality from (12) for $0 < \varepsilon < 2\beta(\sigma) - 1$, we obtain

$$\left| \int_{(t_1,t_2]} \bar{q}(z,s)\, d\mu(s) \right|$$
$$\le C(\omega)(t_2-t_1)^{\beta(\mu)} + C(\omega) \sum_{n=n_0}^{\infty} 2^{-n\beta(\mu)}$$
$$+ C \sum_{n=n_0}^{\infty} \sum_{k: k_{n1} \le 2k-2 < k_{n2}-2} 2^{-n\beta(\sigma)} \big(t_2 - (2k-2)2^{-n}T\big)^{-\beta(\sigma)/2} \left| \mu\big(\Delta^{(T)}_{(2k)n}\big) \right|$$
$$+ C(\omega) \sum_{n=n_0}^{\infty} 2^{-n\beta(\mu)}$$
$$\le C(\omega)(t_2-t_1)^{\beta(\mu)} + C(\omega)2^{-n_0\beta(\mu)}$$
$$+ C\left(\sum_{n=n_0}^{\infty} 2^{\varepsilon n} 2^{-2n\beta(\sigma)} \sum_{k: k_{n1} \le 2k-2 < k_{n2}-2} \big(t_2 - (2k-2)2^{-n}T\big)^{-\beta(\sigma)} \right)^{1/2}$$
$$\times \left(\sum_{n=0}^{\infty} 2^{-\varepsilon n} \sum_{k=1}^{2^n} \mu^2\big(\Delta^{(T)}_{kn}\big) \right)^{1/2}$$
$$\overset{(4),(13)}{\le} C(\omega)(t_2-t_1)^{\beta(\mu)}$$
$$+ C(\omega)\left(\sum_{n=n_0}^{\infty} 2^{\varepsilon n} 2^{-2n\beta(\sigma)} \sum_{1 \le i < (k_{n2}-k_{n1})/2} \big(i 2^{-n}T\big)^{-\beta(\sigma)} \right)^{1/2}$$
$$\le C(\omega)(t_2-t_1)^{\beta(\mu)} + C(\omega)\left(\sum_{n=n_0}^{\infty} 2^{n(\varepsilon - \beta(\sigma))} (k_{n2}-k_{n1})^{1-\beta(\sigma)} \right)^{1/2}$$

$$\overset{(14)}{\leq} C(\omega)(t_2-t_1)^{\beta(\mu)} + C(\omega)(t_2-t_1)^{(1-\beta(\sigma))/2} 2^{-n_0(2\beta(\sigma)-\varepsilon-1)/2}$$

$$\overset{(13)}{\leq} C(\omega)(t_2-t_1)^{\beta(\mu)} + C(\omega)(t_2-t_1)^{(\beta(\sigma)-\varepsilon)/2} \leq C(\omega)(t_2-t_1)^{\gamma_2}.$$

Step 2. Estimation of F_2. Now denote

$$\tilde{q}(z,s) = \int_{\mathbb{R}} \big(p(t_2-s, x-y) - p(t_1-s, x-y)\big)\sigma(s,y)\,dy,$$
$$s \in [0,t_1],\ z = (x,t_1,t_2).$$

Using the change of variables

$$v = \frac{x-y}{2a\sqrt{t_2-s}}, \qquad v = \frac{x-y}{2a\sqrt{t_1-s}},$$

we get

$$\begin{aligned} \big|\tilde{q}(z,s)\big| &= C\bigg|\int_{\mathbb{R}} e^{-v^2}\sigma(s, x-2av\sqrt{t_2-s})\,dv \\ &\quad - \int_{\mathbb{R}} e^{-v^2}\sigma(s, x-2av\sqrt{t_1-s})\,dv\bigg| \\ &\overset{\text{A6}}{\leq} C\int_{\mathbb{R}} e^{-v^2}\big|v(\sqrt{t_2-s}-\sqrt{t_1-s})\big|^{\beta(\sigma)}\,dv \\ &\overset{(8)}{\leq} C(t_2-t_1)^{\beta(\sigma)}(t_2-s)^{-\beta(\sigma)/2}. \end{aligned} \tag{15}$$

Also, analogously to (7) and (8), we have

$$\big|\tilde{q}(z,s+h)-\tilde{q}(z,s)\big| \leq C|h|^{\beta(\sigma)}(t_1-s)^{-\beta(\sigma)/2}. \tag{16}$$

From (15) and (16) for $0<\lambda<1$ and $0 \leq s \leq t_1-h$, we obtain

$$\begin{aligned} &\big|\tilde{q}(z,s+h)-\tilde{q}(z,s)\big| \\ &\quad \leq C|h|^{\lambda\beta(\sigma)}(t_1-s)^{-\beta(\sigma)\lambda/2}(t_2-t_1)^{(1-\lambda)\beta(\sigma)}(t_2-s-h)^{-(1-\lambda)\beta(\sigma)/2}, \\ &w_2\big(\tilde{q}(z,\cdot),r\big) \leq Cr^{\lambda\beta(\sigma)}(t_2-t_1)^{(1-\lambda)\beta(\sigma)}. \end{aligned}$$

If $\lambda\beta(\sigma) > 1/2 \Leftrightarrow (1-\lambda)\beta(\sigma) < \beta(\sigma)-1/2$, then the integral from (2) is finite for some $\alpha > 1/2$. In this case, the integral does not exceed $C(t_2-t_1)^{(1-\lambda)\beta(\sigma)}$.

From (15) we get

$$\big|\tilde{q}(z,0)\big| \leq C(t_2-t_1)^{\beta(\sigma)/2}, \qquad \big\|\tilde{q}(z,\cdot)\big\|_{\mathsf{L}_2([0,t_1])} \leq C(t_2-t_1)^{\beta(\sigma)/2}.$$

Therefore, from (3) we have $|F_2| \leq C(\omega)(t_2-t_1)^{\gamma_2}$, which finishes the proof. □

6 Solution to the equation

Theorem 1. *Suppose that Assumptions A1–A6 hold.*

1. *Equation (5) has a solution $u(t,x)$. If $v(t,x)$ is another solution to (5), then for all t and x, $u(t,x)=v(t,x)$ a.s.*

2. *For any fixed $t \in [0, T]$ and $\gamma_1 < \beta(\sigma) - 1/2$, the stochastic function $u(t, x)$, $x \in \mathbb{R}$, has a Hölder continuous version with exponent γ_1.*

3. *In addition, let Assumption A7 hold. Then for any fixed $\delta > 0$ and γ_1, γ_2 such that $\gamma_1 < \beta(\sigma) - 1/2$, $\gamma_2 \le \beta(\mu)$, and $\gamma_2 < \beta(\sigma) - 1/2$, the stochastic function $u(t, x)$ has a version $\tilde{u}(t, x)$ such that*

$$\begin{aligned} &\bigl|\tilde{u}(t_1, x_1) - \tilde{u}(t_2, x_2)\bigr| \le C(\omega)\bigl(|t_1 - t_2|^{\gamma_2} + |x_1 - x_2|^{\gamma_1}\bigr), \\ &t_i \in [\delta, T],\ x_i \in \mathbb{R}. \end{aligned} \tag{17}$$

Proof. Consider the standard iteration process. Take $u^{(0)}(t, x) = 0$ and set

$$\begin{aligned} u^{(n+1)}(t, x) = &\int_{\mathbb{R}} p(t, x - y) u_0(y)\, dy \\ &+ \int_0^t ds \int_{\mathbb{R}} p(t - s, x - y) f\bigl(s, y, u^{(n)}(s, y)\bigr)\, dy \\ &+ \int_{(0,t]} d\mu(s) \int_{\mathbb{R}} p(t - s, x - y) \sigma(s, y)\, dy. \end{aligned}$$

Further, we can repeat the proof of Theorem [5]. Instead of reference to Lemmas 5.1 and 6.1 of [5], we can refer to Lemmas 2 and 3 of this paper. □

Remark 1. For u, we obtained less regularity than for elements of equation (5). However, a solution to a heat equation usually has the same regularity or even more regular than the coefficients. One may expect that using other methods gives (17) with exponents $\gamma_2 \le \beta(\mu) \wedge \gamma_2 < \beta(\sigma)$ and $\gamma_1 < \beta(\sigma)$.

References

[1] Curbera, G., Delgado, O.: Optimal domains for L^0-valued operators via stochastic measures. Positivity **11**, 399–416 (2007). MR2336205. doi:10.1007/s11117-007-2071-0

[2] Kruse, R., Larsson, S.: Optimal regularity for semilinear stochastic partial differential equations with multiplicative noise. Electron. J. Probab. **17**, 65–119 (2012). MR2968672. doi:10.1214/EJP.v17-2240

[3] Kwapień, S., Woyczyński, W.A.: Random Series and Stochastic Integrals: Single and Multiple. Birkhäuser, Boston (1992). MR1167198. doi:10.1007/978-1-4612-0425-1

[4] Mémin, J., Mishura, Y., Valkeila, E.: Inequalities for the moments of Wiener integrals with respect to a fractional Brownian motion. Stat. Probab. Lett. **51**, 197–206 (2001). MR1822771. doi:10.1016/S0167-7152(00)00157-7

[5] Radchenko, V.: Mild solution of the heat equation with a general stochastic measure. Studia Math. **194**, 231–251 (2009). MR2539554. doi:10.4064/sm194-3-2

[6] Radchenko, V.: Stochastic partial differential equations driven by general stochastic measures. In: Korolyuk, V., Limnios, N., Mishura, Y., Sakhno, L., Shevchenko, G. (eds.) Modern Stochastics and Applications, pp. 143–156. Springer, Cham Heidelberg (2014). MR3236073. doi:10.1007/978-3-319-03512-3_9

[7] Radchenko, V., Zähle, M.: Heat equation with a general stochastic measure on nested fractals. Stat. Probab. Lett. **82**, 699–704 (2012). MR2887489. doi:10.1016/j.spl.2011.12.013

[8] Samorodnitsky, G., Taqqu, M.S.: Stable Non-Gaussian Random Processes. Chapman and Hall, London (1994). MR1280932

[9] Tudor, C.A.: Solutions to the linear stochastic heat and wave equation. In: Analysis of Variations for Self-similar Processes. Probability and Its Applications, pp. 27–75. Springer, Cham Heidelberg (2013). doi:10.1007/978-3-319-00936-0_2

[10] Walsh, J.B.: An Introduction to Stochastic Partial Differential Equations. Lect. Notes Math., vol. 1180, pp. 265–439 (1986). MR0876085. doi:10.1007/BFb0074920