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Arithmetic of (independent) sigma-fields on probability spaces

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Abstract This note gathers what is known about, and provides some new results concerning the operations of intersection, of "generated σ-field", and of "complementation" for (independent) complete σ-fields on probability spaces.

Keywords Lattice of complete σ-fields, generated σ-field, intersection of σ-fields, independent complements
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1 Introduction

Let $(\Omega, \mathcal{M}, \mathbb{P})$ be a probability space and let Λ be the collection of all complete sub-σ-fields of $\mathcal{M}$. (We stress here that $\mathbb{P}$ need not itself be complete to begin with. Complete just means containing $0_\Lambda := \mathbb{P}^{-1}(\{0, 1\})$ – the $\mathbb{P}$-trivial events of $\mathcal{M}$.) $\sigma(\times\times)$ (resp. $\overline{\sigma}(\times\times)$) is the smallest (resp. complete) σ-field on Ω containing or making measurable whatever stands in lieu of $\times\times$. Then for $\{\mathcal{X}, \mathcal{Y}, \mathcal{Z}\} \subset \Lambda$ set $\mathcal{X} \wedge \mathcal{Y} := \mathcal{X} \cap \mathcal{Y}$ and $\mathcal{X} \vee \mathcal{Y} := \sigma(\mathcal{X} \cup \mathcal{Y})$; write $\mathcal{X} \perp\!\!\!\perp \mathcal{Y}$ if $\mathcal{X}$ and $\mathcal{Y}$ are independent, in which case set $\mathcal{X} + \mathcal{Y} := \mathcal{X} \vee \mathcal{Y}$; finally, say $\mathcal{X}$ is complemented by $\mathcal{Y}$ in $\mathcal{Z}$, or that $\mathcal{Y}$ is a complement of $\mathcal{X}$ in $\mathcal{Z}$, if $\mathcal{Z} = \mathcal{X} + \mathcal{Y}$.

We are interested in exposing the salient "arithmetical rules" of the operations $\wedge$, $\vee$, and especially of $+$ and the notion of a complement, delineating their scope through (counter)examples. Apart from pure intellectual curiosity, the justification for the interest in such matters — that may seem a bit "dry" at first — can be seen as coming chiefly from the following observations.

(1) Even though the concepts involved are prima facie very simple, the topic is not trivial and intuition can often mislead. The following examples give already a flavor of this; in them, and in the rest of this paper, equiprobable sign means a $(\{-1,1\}, 2^{\{-1,1\}})$-valued random element ξ with $\mathbb{P}(\xi = 1) = \mathbb{P}(\xi = -1) = 1/2$.

Example 1.1 ($\wedge$-$\vee$ distributivity may fail)**.**

(a) If ξ_1 and ξ_2 are independent equiprobable signs, then taking $\mathcal{X} = \overline{\sigma}(\xi_1)$, $\mathcal{Y} = \overline{\sigma}(\xi_1\xi_2)$ and $\mathcal{Z} = \overline{\sigma}(\xi_2)$, the σ-fields $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ are pairwise independent and $(\mathcal{X} \vee \mathcal{Z}) \wedge (\mathcal{Y} \vee \mathcal{Z}) = \overline{\sigma}(\xi_1, \xi_2)$, while $(\mathcal{X} \wedge \mathcal{Y}) \vee \mathcal{Z} = 0_\Lambda \vee \mathcal{Z} = \overline{\sigma}(\xi_2)$; so $(\mathcal{X} \vee \mathcal{Z}) \wedge (\mathcal{Y} \vee \mathcal{Z}) \neq (\mathcal{X} \wedge \mathcal{Y}) \vee \mathcal{Z}$. The same example also shows that one does not in general have $(\mathcal{X} \wedge \mathcal{Z}) \vee (\mathcal{Y} \wedge \mathcal{Z}) = (\mathcal{X} \vee \mathcal{Y}) \wedge \mathcal{Z}$.

(b) [14, Exercise/Warning 4.12] Let $\mathsf{Y} = (\mathsf{Y}_n)_{n\in\mathbb{N}_0}$ be a sequence of independent equiprobable signs. For $n \in \mathbb{N}$ define $\mathsf{X}_n := \mathsf{Y}_0 \cdots \mathsf{Y}_n$; set $\mathcal{Y} := \overline{\sigma}(\mathsf{Y}_1, \mathsf{Y}_2, \ldots)$ and $\mathcal{X}_n := \overline{\sigma}(\mathsf{X}_m : m \in \mathbb{N}_{\geq n})$ for $n \in \mathbb{N}$. Then the $\mathcal{X}_n$, $n \in \mathbb{N}$, are decreasing, but $\wedge_{n\in\mathbb{N}}(\mathcal{X}_n \vee \mathcal{Y}) \neq (\wedge_{n\in\mathbb{N}_0}\mathcal{X}_n) \vee \mathcal{Y}$. Indeed the X_n, $n \in \mathbb{N}$, are independent equiprobable signs, so by Kolmogorov's zero-one law $\wedge_{n\in\mathbb{N}}\mathcal{X}_n = 0_\Lambda$. On the other hand Y_0 is measurable w.r.t. $\overline{\sigma}(\mathsf{Y}) = \wedge_{n\in\mathbb{N}}(\mathcal{X}_n \vee \mathcal{Y})$ and at the same time it is independent of $\mathcal{Y}$. (For another related example see [15].)

Example 1.2 (Complements may not exist)**.** If ξ_1, ξ_2 are independent equiprobable signs, then $\overline{\sigma}(\{\xi_1 = 1\} \cup \{\xi_1 = -1, \xi_2 = 1\})$ has no complement in $\overline{\sigma}(\xi_1, \xi_2)$.

Example 1.3 (Complements may not be unique)**.** Take again a pair of independent equiprobable signs ξ_1 and ξ_2. Then $\overline{\sigma}(\xi_1) + \overline{\sigma}(\xi_2) = \overline{\sigma}(\xi_1, \xi_2)$ but also $\overline{\sigma}(\xi_1) + \overline{\sigma}(\xi_1\xi_2) = \overline{\sigma}(\xi_1, \xi_2)$.

Example 1.4 (Vanishing of information in the limit)**.** [12, Example 1.1; see also the references there]. Let $\Omega = \{-1,1\}^{\mathbb{N}}$, and let ξ_i, $i \in \mathbb{N}$, the canonical projections, be independent equiprobable signs generating $\mathcal{M} = (2^{\{-1,1\}})^{\otimes\mathbb{N}}$. Let $\mathcal{G}_n = \overline{\sigma}(\xi_1\xi_2, \ldots, \xi_n\xi_{n+1})$ and $\mathcal{F}_n = \overline{\sigma}(\xi_{n+1}, \xi_{n+2}, \ldots)$ for $n \in \mathbb{N}$. Then $\mathcal{G}_n + \mathcal{F}_n = \mathcal{F}_0 = \mathcal{M}$ for all $n \in \mathbb{N}$, and by Kolmogorov's zero-one law $\mathcal{F}_\infty := \wedge_{n\in\mathbb{N}}\mathcal{F}_n = 0_\Lambda$. Furthermore, we have $\mathcal{F}_n = \mathcal{F}_{n+1} + \mathcal{H}_{n+1}$ and $\mathcal{G}_{n+1} = \mathcal{G}_n + \mathcal{H}_{n+1}$ for all $n \in \mathbb{N}_0$, if we put $\mathcal{H}_n := \mathcal{G}_n \wedge \mathcal{F}_{n-1} = \overline{\sigma}(\xi_n\xi_{n+1})$ for $n \in \mathbb{N}$. But still $\mathcal{G}_\infty := \vee_{n\in\mathbb{N}}\mathcal{G}_n = \overline{\sigma}(\xi_1\xi_2, \xi_2\xi_3, \ldots) \neq \mathcal{M}$, for instance, because ξ_1 is non-trivial and independent of $\mathcal{G}_\infty$.

Concerning the failure of the equality $\wedge_{n\in\mathbb{N}}(\mathcal{X}_n \vee \mathcal{Y}) = (\wedge_{n\in\mathbb{N}_0}\mathcal{X}_n) \vee \mathcal{Y}$ in Example 1.1(b), Chaumont and Yor [2, p. 30] write: "A number of authors, (including the present authors, separately!!), gave wrong proofs of /this equality/ under various hypotheses. This seems to be one of the worst traps involving σ-fields." According to Williams [14, p. 48]: "The phenomenon illustrated by this example tripped up even Kolmogorov and Wiener. [...] Deciding when we *can* assert [equality] is a tantalizing problem in many probabilistic contexts." Émery and Schachermayer [3, p. 291] call a variant of Example 1.4 "paradigmatic [...], well-known in ergodic theory, [...], independently discovered by several authors".

(2) In spite of the subtleties involved, facts concerning the arithmetic of σ-fields are not very easily accessible in the literature, various partial results being scattered across papers and monographs, as and when the need for them arose.

(3) In broad sense, nondecreasing families of sub-σ-fields — filtrations — model the flow of information in a probabilistic context. They are essential to the modern-day proper understanding of martingales and Markov processes. And since stochastic models are usually specified by some kind of (conditional) independence structure (think i.i.d. sequences, Lévy processes, Markov processes in general), it is therefore important to understand how such information, as embodied by σ-fields, is "aggregated" and/or "intersected" over (conditionally) independent σ-fields. The classical increasing and decreasing martingale convergence theorems [6, Theorem 6.23], for instance, involve the generated and intersected σ-fields in a key way. Kolmogorov's zero-one law and its extensions [6, Corollary 6.25], with their many offsprings, are another example in which the interplay between independence, intersected, and generated σ-fields lies at the very heart of the matter.

(4) More narrowly, the exposition in [12] recognizes stochastic noises (generalizations of Wiener and Poissonian noise) as subsets of the lattice Λ satisfying in particular, and in an essential way, a certain property with respect to independent complements; see also [5, 11].

With the above as motivation, and following the introduction of some further notation and preliminaries in Section 2, we investigate below in Section 3, in depth: (I) the distributivity properties of the pair $\wedge$-$\vee$ for families of σ-fields that, roughly speaking, exhibit at least some independence properties between them; (II) the properties of complements (existence, uniqueness, etc.). In particular, apart from some trivial observations, we confine our attention to those statements concerning the arithmetic of σ-fields, in which a property of (conditional) independence intervenes in a non-trivial way (this is of course automatic for (II)); hence the title. For the most part the paper is of an expository nature; see below for the precise references. In some places a couple of original complements/extensions are provided. Section 4 closes with a brief application; other uses of the presented results are found in the citations that we include, as well as in the literature quoted in those.

2 Further notation and preliminaries

Some general notation and vocabulary. For $M \subset [-\infty, \infty]$, $\mathcal{B}_M$ will denote the Borel σ-field on M for the standard (Euclidean) topology thereon. For σ-fields $\mathcal{F}$ and $\mathcal{G}$, $\mathcal{F}/\mathcal{G}$ is the set of precisely all the $\mathcal{F}/\mathcal{G}$-measurable maps. A measure on a σ-field that contains the singletons of the underlying space will be said to be diffuse, or continuous, if it does not charge any singleton. Throughout "a.s." is short for "$\mathbb{P}$-almost surely" and $\mathbb{E}$ denotes expectation w.r.t. $\mathbb{P}$. A random element valued in $([0, 1], \mathcal{B}_{[0,1]})$ whose law is the (trace of) Lebesgue measure on $[0, 1]$ will be said to have (the) uniform law (on $[0, 1]$).

Let now $\{\mathcal{X}, \mathcal{Y}\} \subset \Lambda$. Then (i) for $\mathsf{M} \in \mathcal{M}/\mathcal{B}_{[-\infty,\infty]}$, $\mathbb{E}[\mathsf{M}|\mathcal{X}]$ is the conditional expectation of M w.r.t. $\mathcal{X}$ (when $\mathbb{E}[\mathsf{M}^+] \wedge \mathbb{E}[\mathsf{M}^-] < \infty$, in which case $\mathbb{E}[\mathsf{M}|\mathcal{X}] \in \mathcal{X}/\mathcal{B}_{[-\infty,\infty]}$)[1] and as usual $\mathbb{P}[F|\mathcal{X}] = \mathbb{E}[\mathbb{1}_F|\mathcal{X}]$ for $F \in \mathcal{M}$; (ii) we will denote by

[1] We will indulge in the usual confusion between measurable functions and their equivalence classes mod $\mathbb{P}$. Because we will only be interested in complete σ-fields this will be of no consequence.

$\mathbb{E}_{|\mathcal{X}}$ the operator, on $L^1(\mathbb{P})$, of the conditional expectation w.r.t. $\mathcal{X}$: so $\mathbb{E}_{|\mathcal{X}}(\mathsf{M}) = \mathbb{E}[\mathsf{M}|\mathcal{X}]$ a.s. for $\mathsf{M} \in L^1(\mathbb{P})$; (iii) $\mathcal{X}$ will be said to be countably generated up to negligible sets, or to be essentially separable, if there is a denumerable $\mathcal{B} \subset \mathcal{X}$ such that $\mathcal{X} = \overline{\sigma}(\mathcal{B})$: manifestly it is so if and only if $L^1(\mathbb{P}|_{\mathcal{X}})$ is separable, in which case every element $\mathcal{Y} \in \Lambda$ with $\mathcal{Y} \subset \mathcal{X}$ is countably generated up to negligible sets, and this is true if and only if there is an $\mathsf{X} \in \mathcal{X}/\mathcal{B}_{\mathbb{R}}$ with $\mathcal{X} = \overline{\sigma}(\mathsf{X})$; (iv) if further $\mathcal{Z} \in \Lambda$, we will write $\mathcal{X} \perp\!\!\!\perp_{\mathcal{Z}} \mathcal{Y}$ to mean that $\mathcal{X}$ and $\mathcal{Y}$ are independent given $\mathcal{Z}$.

Remark 2.1. A warning: separability per se is not hereditary. For instance $\mathcal{B}_{\mathbb{R}}$ is countably generated but the countable-co-countable σ-field on $\mathbb{R}$ is not. In general it is true that completeness will have a major role to play in what follows, and we shall make no apologies for restricting our attention to complete sub-σ-fields from the get go – practically none of the results presented would be true without this assumption (or would be true only "mod $\mathbb{P}$", which amounts to the same thing).

The following basic facts about conditional expectations are often useful; we will use them silently throughout.

Lemma 2.2 (Independent conditioning)**.** *Let* $\{\mathsf{F}, \mathsf{G}\} \subset \mathcal{M}/\mathcal{B}_{[0,\infty]}$ *and let* $\{\mathcal{X}, \mathcal{Y}, \mathcal{Z}\} \subset \Lambda$. *If* $\mathcal{Y} \vee \sigma(\mathsf{G}) \perp\!\!\!\perp \mathcal{X} \vee \sigma(\mathsf{F})$*, then* $\mathbb{E}[\mathsf{FG}|\mathcal{X} \vee \mathcal{Y}] = \mathbb{E}[\mathsf{F}|\mathcal{X}]\mathbb{E}[\mathsf{G}|\mathcal{Y}]$ *a.s.; in particular if* $\mathcal{Y} \perp\!\!\!\perp \mathcal{X} \vee \mathcal{Z}$*, then* $\mathcal{Z} \perp\!\!\!\perp_{\mathcal{X}} \mathcal{Y}$*; finally, if* $\sigma(\mathsf{F}) \perp\!\!\!\perp_{\mathcal{X}} \mathcal{Y}$*, then* $\mathbb{E}[\mathsf{F}|\mathcal{X} \vee \mathcal{Y}] = \mathbb{E}[\mathsf{F}|\mathcal{X}]$ *a.s.*

Proof. For the first claim, by a π/λ-argument it suffices to check that $\mathbb{E}[\mathsf{FG}; X \cap Y] = \mathbb{E}[\mathbb{E}[\mathsf{F}|\mathcal{X}]\mathbb{E}[\mathsf{G}|\mathcal{Y}]; X \cap Y]$ for $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$, which is immediate (both sides are equal to $\mathbb{E}[\mathsf{F}; X]\mathbb{E}[\mathsf{G}; Y]$ on account of $\mathcal{Y} \vee \sigma(\mathsf{G}) \perp\!\!\!\perp \mathcal{X} \vee \sigma(\mathsf{F})$). To obtain the second statement, let $\mathsf{Z} \in \mathcal{Z}/\mathcal{B}_{[0,\infty]}$ and $\mathsf{Y} \in \mathcal{Y}/\mathcal{B}_{[0,\infty]}$; then a.s. $\mathbb{E}[\mathsf{ZY}|\mathcal{X}] = \mathbb{E}[\mathsf{ZY}|\mathcal{X} \vee 0_{\Lambda}] = \mathbb{E}[\mathsf{Z}|\mathcal{X}]\mathbb{E}[\mathsf{Y}] = \mathbb{E}[\mathsf{Z}|\mathcal{X}]\mathbb{E}[\mathsf{Y}|\mathcal{X}]$. For the final claim, by a π/λ-argument it suffices to check that $\mathbb{E}[\mathsf{F}; X \cap Y] = \mathbb{E}[\mathbb{E}[\mathsf{F}|\mathcal{X}]; X \cap Y]$ for all $(X, Y) \in \mathcal{X} \times \mathcal{Y}$. But $\mathbb{E}[\mathbb{E}[\mathsf{F}|\mathcal{X}]; X \cap Y] = \mathbb{E}[\mathbb{E}[\mathsf{F}|\mathcal{X}]\mathbb{P}[Y|\mathcal{X}]; X] = \mathbb{E}[\mathbb{E}[\mathsf{F}\mathbb{1}_Y|\mathcal{X}]; X]$, which is indeed equal to $\mathbb{E}[\mathsf{F}; X \cap Y]$. □

We conclude this section with a statement concerning decreasing convergence for martingales indexed by a directed set (it is also true in its increasing convergence guise [9, Proposition V-1-2] but we shall not find use of that version). In it, and in the remainder of this paper, for a family $(\mathcal{X}_t)_{t\in T}$ in Λ we set $\wedge_{t\in T}\mathcal{X}_t := \cap_{t\in T}\mathcal{X}_t$, provided T is non-empty (similarly, later on, we will use the notation $\vee_{t\in T}\mathcal{X}_t := \overline{\sigma}(\cup_{t\in T}\mathcal{X}_t)$ ($= 0_{\Lambda}$ when T is empty)).

Lemma 2.3 (Decreasing martingale convergence)**.** *Let* $\mathsf{X} \in L^1(\mathbb{P})$ *and let* $(\mathcal{X}_t)_{t\in T}$ *be a non-empty net in* Λ *indexed by a directed set* $(T, \leq)$ *satisfying* $\mathcal{X}_t \subset \mathcal{X}_s$ *whenever* $s \leq t$ *are from* T*. Then the net* $(\mathbb{E}[\mathsf{X}|\mathcal{X}_t])_{t\in T}$ *converges in* $L^1(\mathbb{P})$ *to* $\mathbb{E}[\mathsf{X}| \wedge_{t\in T} \mathcal{X}_t]$.

Remark 2.4. Recall that when $T = \mathbb{N}$ with the usual order, then the convergence is also almost sure.

Proof. According to [9, Lemma V-1-1] and the usual decreasing martingale convergence indexed by $\mathbb{N}$ [9, Corollary V-3-12] the net $(\mathbb{E}[\mathsf{X}|\mathcal{X}_t])_{t\in T}$ is convergent to some X_∞ in $L^1(\mathbb{P})$. Because for each $t \in T$, $L^1(\mathbb{P}|_{\mathcal{X}_t})$ is closed in $L^1(\mathbb{P})$ and since X_∞ is also the limit of the net $(\mathbb{E}[\mathsf{X}|\mathcal{X}_u])_{u\in T_{\geq t}}$, it follows that $\mathsf{X}_\infty \in \mathcal{X}_t/\mathcal{B}_{\mathbb{R}}$; hence $\mathsf{X}_\infty \in$

$(\wedge_{t\in T}\mathcal{X}_t)/\mathcal{B}_{\mathbb{R}}$. Then for any $X \in \wedge_{t\in T}\mathcal{X}_t$, $\mathbb{E}[\mathsf{X}_\infty; X] = \lim_{t\in T}\mathbb{E}[\mathbb{E}[\mathsf{X}|\mathcal{X}_t]; X] = \lim_{t\in T}\mathbb{E}[\mathsf{X}; X] = \mathbb{E}[\mathsf{X}; X]$, which means that a.s. $\mathsf{X}_\infty = \mathbb{E}[\mathsf{X}| \wedge_{t\in T}\mathcal{X}_t]$. □

3 The arithmetic

We begin with some simple observations.

Remark 3.1 (Lattice structure)**.** [12, passim]. The operations $\wedge$, $\vee$ in Λ are clearly associative and commutative, and one has the absorption laws: $(\mathcal{X}\wedge\mathcal{Y})\vee\mathcal{X} = \mathcal{X}$ and $(\mathcal{X}\vee\mathcal{Y})\wedge\mathcal{X} = \mathcal{X}$ for $\{\mathcal{X},\mathcal{Y}\}\subset\Lambda$. Besides, $0_\Lambda\vee\mathcal{X} = \mathcal{X}$ and $\mathcal{X}\wedge\mathcal{M} = \mathcal{X}$ for all $\mathcal{X}\in\Lambda$. Thus $(\Lambda,\wedge,\vee)$ is a bounded algebraic lattice with bottom 0_Λ and top $\mathcal{M}$. However, it is not distributive in general, as we saw in the introduction. While $+$ is not an internal operation on Λ, nevertheless we may assert, for $\{\mathcal{X},\mathcal{Y},\mathcal{Z}\}\subset\Lambda$, that $\mathcal{X}+\mathcal{Y} = \mathcal{Y}+\mathcal{X}$, resp. $(\mathcal{X}+\mathcal{Y})+\mathcal{Z} = \mathcal{X}+(\mathcal{Y}+\mathcal{Z})$, whenever $\mathcal{X}$ and $\mathcal{Y}$ are independent, resp. and independent of $\mathcal{Z}$. Clearly also $\mathcal{X}+0_\Lambda = \mathcal{X}$ for $\mathcal{X}\in\Lambda$.

Proposition 3.2 (Independence and commutativity)**.** [12, Proposition 3.5]. *Let $\{\mathcal{X}, \mathcal{Y}\}\subset\Lambda$. Then the following are equivalent.*

(i) $\mathcal{X}$ and $\mathcal{Y}$ are independent.

(ii) $\mathcal{X}\wedge\mathcal{Y} = 0_\Lambda$ and $\mathcal{X}$ and $\mathcal{Y}$ "commute": $\mathbb{E}_{|\mathcal{X}}\mathbb{E}_{|\mathcal{Y}} = \mathbb{E}_{|\mathcal{Y}}\mathbb{E}_{|\mathcal{X}}$.

(iii) $\mathbb{E}_{|\mathcal{X}}\mathbb{E}_{|\mathcal{Y}} = \mathbb{E}_{|0_\Lambda}$.

Example 3.3. Let ξ_1, ξ_2 be independent equiprobable signs and $\mathcal{X} = \overline{\sigma}(\{\xi_1 = \xi_2 = 1\})$, $\mathcal{Y} = \overline{\sigma}(\xi_1)$. Then $\mathcal{X}$ and $\mathcal{Y}$ are not independent but $\mathcal{X}\wedge\mathcal{Y} = 0_\Lambda$.

Proof. (ii) implies (iii) because $\mathbb{E}_{|\mathcal{X}}\mathbb{E}_{|\mathcal{Y}} = \mathbb{E}_{|\mathcal{Y}}\mathbb{E}_{|\mathcal{X}}$ entails that $\mathbb{E}_{|\mathcal{X}}\mathbb{E}_{|\mathcal{Y}} = \mathbb{E}_{|\mathcal{Y}}\mathbb{E}_{|\mathcal{X}} = \mathbb{E}_{|\mathcal{X}\wedge\mathcal{Y}}$. Also, if $\mathcal{X}$ and $\mathcal{Y}$ are independent, then the basic properties of conditional expectations imply $\mathbb{E}_{|\mathcal{X}}\mathbb{E}_{|\mathcal{Y}} = \mathbb{E}_{|0_\Lambda} = \mathbb{E}_{|\mathcal{Y}}\mathbb{E}_{|\mathcal{X}}$, while clearly $\mathcal{X}\wedge\mathcal{Y} = 0_\Lambda$, i.e. (i) implies (ii). Suppose now (iii). Let $X\in\mathcal{X}$ and $Y\in\mathcal{Y}$. Then $\mathbb{P}(X\cap Y) = \mathbb{E}[\mathbb{P}[Y|\mathcal{X}]; X] = \mathbb{E}[\mathbb{E}[\mathbb{1}_Y|\mathcal{Y}|\mathcal{X}]; X] = \mathbb{E}[\mathbb{P}[Y|0_\Lambda]; X] = \mathbb{P}(X)\mathbb{P}(Y)$, which is (i). □

The next few results deal with the distributivity properties of the pair $\vee$-$\wedge$, when there are strong independence properties.

Proposition 3.4 (Distributivity I)**.** *Let $(\mathcal{X}_{\alpha\beta})_{(\alpha,\beta)\in\mathfrak{A}\times\mathfrak{B}}$ be a family in Λ, $\mathfrak{A}$ non-empty, such that the $\mathcal{Z}_\beta := \vee_{\alpha\in\mathfrak{A}}\mathcal{X}_{\alpha\beta}$, $\beta\in\mathfrak{B}$, are independent. Then*

$$\wedge_{\alpha\in\mathfrak{A}}\vee_{\beta\in\mathfrak{B}}\mathcal{X}_{\alpha\beta} = \vee_{\beta\in\mathfrak{B}}\wedge_{\alpha\in\mathfrak{A}}\mathcal{X}_{\alpha\beta}. \tag{3.1}$$

It is quite agreeable that the preceding statement can be made in such generality. We give some remarks before turning to its proof.

Remark 3.5. Of course the independence of $\mathcal{Z}_\beta$, $\beta\in\mathfrak{B}$, is far from being necessary in order for (3.1) to prevail. For instance if $\{\mathcal{X},\mathcal{Y},\mathcal{Z}\}\subset\Lambda$, and $\mathcal{Z}\subset\mathcal{X}$ or $\mathcal{Z}\subset\mathcal{Y}$, then $(\mathcal{X}\wedge\mathcal{Z})\vee(\mathcal{Y}\wedge\mathcal{Z}) = \mathcal{Z} = (\mathcal{X}\vee\mathcal{Y})\wedge\mathcal{Z} = (\mathcal{X}\vee\mathcal{Y})\wedge(\mathcal{Z}\vee\mathcal{Z})$, but $\mathcal{X}\vee\mathcal{Z}$ and $\mathcal{Y}\vee\mathcal{Z}$ are not independent unless $\mathcal{Z} = 0_\Lambda$; similarly if $\mathcal{X}\vee\mathcal{Y}\subset\mathcal{Z}$, then $(\mathcal{X}\wedge\mathcal{Y})\vee(\mathcal{Z}\wedge\mathcal{Z}) = (\mathcal{X}\wedge\mathcal{Y})\vee\mathcal{Z} = \mathcal{Z} = (\mathcal{X}\vee\mathcal{Z})\wedge(\mathcal{Y}\vee\mathcal{Z})$, but $\mathcal{X}\vee\mathcal{Y}$ and $\mathcal{Z}$ are not independent unless $\mathcal{X} = \mathcal{Y} = 0_\Lambda$.

Remark 3.6. The generality of a not necessarily denumerable $\mathfrak{B}$ in Proposition 3.4 is of only superficial value. Indeed clearly we have $\vee_{\beta\in\mathfrak{B}} \wedge_{\alpha\in\mathfrak{A}} \mathcal{X}_{\alpha\beta} = \cup_{B \text{ countable} \subset\mathfrak{B}} \vee_{\beta\in B} \wedge_{\alpha\in\mathfrak{A}}\mathcal{X}_{\alpha\beta}$; similarly if $A \in \wedge_{\alpha\in\mathfrak{A}} \vee_{\beta\in\mathfrak{B}} \mathcal{X}_{\alpha\beta}$, then for sure $A \in \vee_{\beta\in B}\mathcal{Z}_\beta$ for some denumerable $B \subset \mathfrak{B}$ so that, by the very statement of this proposition (with $\mathfrak{B}$ a two-point set), $A \in \wedge_{\alpha\in\mathfrak{A}} \vee_{\beta\in B} \mathcal{X}_{\alpha\beta}$, viz. $\wedge_{\alpha\in\mathfrak{A}} \vee_{\beta\in\mathfrak{B}} \mathcal{X}_{\alpha\beta} = \cup_{B \text{ countable} \subset\mathfrak{B}} \wedge_{\alpha\in\mathfrak{A}} \vee_{\beta\in B}\mathcal{X}_{\alpha\beta}$.

Remark 3.7. Proposition 3.4 yields at once Kolmogorov's zero-one law: if $\mathcal{A} = (\mathcal{A}_\gamma)_{\gamma\in\Gamma}$ is an independency (i.e. a family consisting of independent σ-fields) from Λ, independent from a $\mathcal{B} \in \Lambda$ then, setting for cofinite $A \subset \Gamma$, $\vee_A\mathcal{A} := \vee_{\gamma\in A}\mathcal{A}_\gamma$, one obtains $\wedge_{A \text{ cofinite in } \Gamma}(\mathcal{B} \vee (\vee_A\mathcal{A})) = \mathcal{B}$.

Proof. The inclusion $\supset$ in (3.1) is trivial. On the other hand, for $\beta \in \mathfrak{B}$, $\wedge_{\alpha\in\mathfrak{A}} \vee_{\beta'\in\mathfrak{B}} \mathcal{X}_{\alpha\beta'} \subset \wedge_{\alpha\in\mathfrak{A}}(\mathcal{X}_{\alpha\beta}\vee(\vee_{\beta'\in\mathfrak{B}\backslash\{\beta\}}\mathcal{Z}_{\beta'}))$. Hence $\wedge_{\alpha\in\mathfrak{A}}\vee_{\beta'\in\mathfrak{B}}\mathcal{X}_{\alpha\beta'} \subset \wedge_{\beta\in\mathfrak{B}}(\wedge_{\alpha\in\mathfrak{A}}(\mathcal{X}_{\alpha\beta}\vee (\vee_{\beta'\in\mathfrak{B}\backslash\{\beta\}}\mathcal{Z}_{\beta'})))$, and thus it will suffice to prove (3.1) for the following two special cases.

(a) $\mathfrak{B} = \{1, 2\}$, $\mathcal{X}_{\alpha 2} = \mathcal{Z}_2$ for $\alpha \in \mathfrak{A}$.

(b) $\mathfrak{A} = \mathfrak{B}$ and $\mathcal{X}_{\alpha\beta} = \mathcal{Z}_\beta$ for $\alpha \neq \beta$ from $\mathfrak{A}$.

In proving this we will use without special mention the completeness of the members of Λ.

(a). Relabel $\mathcal{X}_{\alpha 1} =: \mathcal{X}_\alpha$, $\alpha \in \mathfrak{A}$, and $\mathcal{Z}_2 =: \mathcal{Y}$. Suppose (3.1) has been established for $\mathfrak{A}$ finite (all the time assuming (a)). Let T consist of the finite non-empty subsets of $\mathfrak{A}$, direct T by inclusion $\subset$, and define $\underline{\mathcal{X}}_A := \wedge_{\alpha\in A}\mathcal{X}_\alpha$ for $A \in T$. Then $\wedge_{\alpha\in\mathfrak{A}}(\mathcal{X}_\alpha \vee \mathcal{Y}) = \wedge_{A\in T}(\underline{\mathcal{X}}_A \vee\mathcal{Y})$ and (of course) $\wedge_{\alpha\in\mathfrak{A}}\mathcal{X}_\alpha = \wedge_{A\in T}\underline{\mathcal{X}}_A$. Let $X \in \vee_{\alpha\in\mathfrak{A}}\mathcal{X}_\alpha =: \mathcal{X}$ and $Y \in \mathcal{Y}$. Using $\mathcal{X} \perp\!\!\!\perp \mathcal{Y}$ and decreasing martingale convergence we see that a.s. $\mathbb{P}[X\cap Y|(\wedge_{A\in T}\underline{\mathcal{X}}_A)\vee\mathcal{Y}] = \mathbb{P}[X|\wedge_{A\in T}\underline{\mathcal{X}}_A]\mathbb{P}[Y|\mathcal{Y}] = (\lim_{A\in T}\mathbb{P}[X|\underline{\mathcal{X}}_A])\mathbb{P}[Y|\mathcal{Y}] = \lim_{A\in T}(\mathbb{P}[X|\underline{\mathcal{X}}_A]\mathbb{P}[Y|\mathcal{Y}]) = \lim_{A\in T}\mathbb{P}[X\cap Y|\underline{\mathcal{X}}_A\vee\mathcal{Y}] = \mathbb{P}[X\cap Y|\wedge_{A\in T}(\underline{\mathcal{X}}_A\vee\mathcal{Y})]$, where the limits are in $L^1(\mathbb{P})$. A π/λ-argument allows to conclude that (3.1) holds true. Suppose now $\mathfrak{A}$ is finite. By induction we may and do consider only the case $\mathfrak{A} = \{1, 2\}$, and so we are to show that $(\mathcal{X}_1 \vee \mathcal{Y}) \wedge (\mathcal{X}_2 \vee \mathcal{Y}) = (\mathcal{X}_1 \wedge \mathcal{X}_2) \vee \mathcal{Y}$. Let again $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. Then using $\mathcal{X} \perp\!\!\!\perp \mathcal{Y}$, convergence of iterated conditional expectations [1, Proposition 3] and bounded convergence, we obtain that a.s. $\mathbb{P}[X \cap Y|(\mathcal{X}_1 \vee \mathcal{Y}) \wedge (\mathcal{X}_2 \vee \mathcal{Y})] = \mathbb{E}[\mathbb{1}_{X\cap Y}|\mathcal{X}_1 \vee \mathcal{Y}|(\mathcal{X}_1 \vee \mathcal{Y}) \wedge (\mathcal{X}_2 \vee \mathcal{Y})] = \mathbb{E}[\mathbb{P}[X|\mathcal{X}_1]\mathbb{1}_Y|(\mathcal{X}_1\vee\mathcal{Y})\wedge(\mathcal{X}_2\vee\mathcal{Y})] = \mathbb{E}[\mathbb{P}[X|\mathcal{X}_1]\mathbb{1}_Y|\mathcal{X}_2\vee\mathcal{Y}|(\mathcal{X}_1\vee\mathcal{Y})\wedge(\mathcal{X}_2\vee\mathcal{Y})] = \mathbb{E}[\mathbb{E}[\mathbb{1}_X|\mathcal{X}_1|\mathcal{X}_2]\mathbb{1}_Y|(\mathcal{X}_1 \vee \mathcal{Y}) \wedge (\mathcal{X}_2 \vee \mathcal{Y})] = \mathbb{E}[\mathbb{E}[\mathbb{1}_X|\mathcal{X}_1|\mathcal{X}_2|\mathcal{X}_1|\mathcal{X}_2]\mathbb{1}_Y|(\mathcal{X}_1 \vee \mathcal{Y}) \wedge (\mathcal{X}_2\vee\mathcal{Y})] = \cdots \to \mathbb{E}[\mathbb{P}[X|\mathcal{X}_1 \wedge \mathcal{X}_2]\mathbb{1}_Y|(\mathcal{X}_1\vee\mathcal{Y})\wedge(\mathcal{X}_2\vee\mathcal{Y})] = \mathbb{P}[X|\mathcal{X}_1\wedge\mathcal{X}_2]\mathbb{1}_Y \in ((\mathcal{X}_1 \wedge \mathcal{X}_2) \vee \mathcal{Y})/\mathcal{B}_{[-\infty,\infty]}$. Again a π/λ-argument allows to conclude.

(b). Relabel $\mathcal{X}_{\alpha\alpha} =: \mathcal{X}_\alpha$ and $\mathcal{Z}_\alpha =: \mathcal{A}_\alpha$, $\alpha \in \mathfrak{A}$. Suppose (3.1) has been shown for $\mathfrak{A}$ finite (all the time assuming (b)). Let T consist of the finite subsets of $\mathfrak{A}$, direct T by inclusion $\subset$, and define $\overline{\mathcal{X}}_A := \vee_{\alpha\in A}\mathcal{X}_\alpha$ for $A \in T$. Then $\wedge_{\alpha\in\mathfrak{A}}(\mathcal{X}_\alpha \vee (\vee_{\alpha'\in\mathfrak{A}\backslash\{\alpha\}}\mathcal{A}_{\alpha'})) = \wedge_{A\in T}(\overline{\mathcal{X}}_A \vee (\vee_{\alpha'\in\mathfrak{A}\backslash A}\mathcal{A}_{\alpha'}))$. Now let $B \in T\backslash\{\emptyset\}$, $A_i \in \mathcal{A}_i$ for $i \in B$. We have by decreasing martingale convergence, a.s. $\mathbb{P}[\cap_{i\in B}A_i| \wedge_{A\in T} (\overline{\mathcal{X}}_A \vee (\vee_{\alpha'\in\mathfrak{A}\backslash A}\mathcal{A}_{\alpha'}))] = \lim_{A\in T}\mathbb{P}[\cap_{i\in B}A_i|\overline{\mathcal{X}}_A \vee (\vee_{\alpha'\in\mathfrak{A}\backslash A}\mathcal{A}_{\alpha'})] = \mathbb{P}[\cap_{i\in B}A_i|\overline{\mathcal{X}}_B] \in (\vee_{\alpha\in\mathfrak{A}}\mathcal{X}_\alpha)/\mathcal{B}_{[-\infty,\infty]}$, where the limit is in $L^1(\mathbb{P})$, and we conclude that (3.1) holds true via a π/λ-argument. So it remains to argue (3.1) for $\mathfrak{A}$ finite, and then by an inductive argument for $\mathfrak{A} = \{1, 2\}$, in which case we are to establish that $(\mathcal{X}_1 \vee \mathcal{A}_2)\wedge$

$(\mathcal{A}_1 \vee \mathcal{X}_2) = \mathcal{X}_1 \wedge \mathcal{X}_2$. To this end let $F \in (\mathcal{X}_1 \vee \mathcal{A}_2) \wedge (\mathcal{A}_1 \vee \mathcal{X}_2)$. Then a.s. $\mathbb{1}_F = \mathbb{P}[F|\mathcal{X}_1 \vee \mathcal{A}_2]$ (because $F \in \mathcal{X}_1 \vee \mathcal{A}_2$), which is $\in (\mathcal{X}_1 \vee \mathcal{X}_2)/\mathcal{B}_{[-\infty,\infty]}$ (because $F \in \mathcal{A}_1 \vee \mathcal{X}_2$, by a π/λ-argument, using $\mathcal{X}_1 \subset \mathcal{A}_1$, $\mathcal{X}_2 \subset \mathcal{A}_2$ and $\mathcal{A}_2 \perp\!\!\!\perp \mathcal{A}_1$: if $A_1 \in \mathcal{A}_1$ and $X_2 \in \mathcal{X}_2$ then a.s. $\mathbb{P}[A_1 \cap X_2|\mathcal{X}_1 \vee \mathcal{A}_2] = \mathbb{1}_{X_2}\mathbb{P}[A_1|\mathcal{X}_1] \in (\mathcal{X}_1 \vee \mathcal{X}_2)/\mathcal{B}_{[-\infty,\infty]}$). □

Corollary 3.8 (Distributivity II)**.**

(i) If $\mathcal{Y} \in \Lambda$ is independent of a nonincreasing sequence $(\mathcal{X}_n)_{n\in\mathbb{N}}$ from Λ, then $\wedge_{n\in\mathbb{N}}(\mathcal{X}_n \vee \mathcal{Y}) = (\wedge_{n\in\mathbb{N}}\mathcal{X}_n) \vee \mathcal{Y}$. [2, Exercise 2.5(1-2)], [10, Exercise 2.15].

(ii) For $\{\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}_1, \mathcal{Y}_2\} \subset \Lambda$, if $\mathcal{X}_1 \vee \mathcal{X}_2 \perp\!\!\!\perp \mathcal{Y}_1 \vee \mathcal{Y}_2$, then $(\mathcal{X}_1 \vee \mathcal{Y}_1) \wedge (\mathcal{X}_2 \vee \mathcal{Y}_2) = (\mathcal{X}_1 \wedge \mathcal{X}_2) \vee (\mathcal{Y}_1 \wedge \mathcal{Y}_2)$. [12, Fact 2.18, when $\mathcal{M}$ is countably generated up to negligible sets]. In particular for $\{\mathcal{X}, \mathcal{A}, \mathcal{Y}\} \subset \Lambda$, if $\mathcal{X} \subset \mathcal{A} \perp\!\!\!\perp \mathcal{Y}$, then $(\mathcal{X} \vee \mathcal{Y}) \wedge \mathcal{A} = \mathcal{X}$.

(iii) If $\{\mathcal{X}, \mathcal{Y}, \mathcal{Z}\} \subset \Lambda$, $\mathcal{X} \vee \mathcal{Y} \perp\!\!\!\perp \mathcal{Z}$, then $(\mathcal{X} \vee \mathcal{Z}) \wedge (\mathcal{Y} \vee \mathcal{Z}) = (\mathcal{X} \wedge \mathcal{Y}) \vee \mathcal{Z}$. □

Remark 3.9. [13] discusses the equality in (i) when $\mathcal{X}$ and $\mathcal{Y}$ are not necessarily independent; we have seen in Example 1.1(b) that it fails in general.

Remark 3.10. In (iii) the equality $(\mathcal{X} \wedge \mathcal{Z}) \vee (\mathcal{Y} \wedge \mathcal{Z}) = (\mathcal{X} \vee \mathcal{Y}) \wedge \mathcal{Z}$ is trivial (both sides are equal to 0_Λ). Example 1.1(a) showed that these basic distributivity relations fail in general, even when $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ are pairwise independent.

Remark 3.11. Let $\{\mathcal{A}, \mathcal{B}, \mathcal{C}\} \subset \Lambda$. (I) If $\mathcal{A} \subset \mathcal{B} \vee \mathcal{C}$ and $\mathcal{A} \vee \mathcal{B} \perp\!\!\!\perp \mathcal{C}$, then $\mathcal{A} \subset \mathcal{B}$: $\mathcal{A} = \mathcal{A} \wedge (\mathcal{B} \vee \mathcal{C}) = (\mathcal{A} \vee 0_\Lambda) \wedge (\mathcal{B} \vee \mathcal{C}) = \mathcal{A} \wedge \mathcal{B}$ by (ii), [2, Exercise 2.2(1)]. (II) If $\mathcal{A} \subset \mathcal{B} \vee \mathcal{C}$, $\mathcal{A} \perp\!\!\!\perp \mathcal{C}$, $\mathcal{B} \subset \mathcal{A}$, then $\mathcal{A} = \mathcal{B}$: $\mathcal{A} \subset (\mathcal{B} \vee \mathcal{C}) \wedge (\mathcal{A} \vee 0_\Lambda) = \mathcal{B}$ by (ii) again, [2, Exercise 2.2(3)].

We turn now to complements; we shall resume with the investigation of distributivity later on in Nos. 3.20-3.26.

Proposition 3.12 (Complements I)**.** [4, Proposition 4]. *Let $\{\mathcal{X}, \mathcal{Y}\} \subset \Lambda$. Assume $\mathcal{X}$ is countably generated up to negligible sets and $\mathcal{Y} \subset \mathcal{X}$. Then the following statements are equiveridical.*

(i) Whenever $\mathsf{X} \in \mathcal{X}/\mathcal{B}_\mathbb{R}$ is such that $\mathcal{X} = \overline{\sigma}(\mathsf{X})$, then for every $\mathsf{Y} \in \mathcal{Y}/\mathcal{B}_\mathbb{R}$, $\mathbb{P}(\mathsf{X} = \mathsf{Y}) = 0$.

(ii) There exists $\mathsf{X} \in \mathcal{X}/\mathcal{B}_\mathbb{R}$ such that for every $\mathsf{Y} \in \mathcal{Y}/\mathcal{B}_\mathbb{R}$, $\mathbb{P}(\mathsf{X} = \mathsf{Y}) = 0$.

(iii) There exists $\mathsf{Z} \in \mathcal{X}/\mathcal{B}_\mathbb{R}$ independent of $\mathcal{Y}$ and having a diffuse law.

(iv) There exists $\mathsf{Z} \in \mathcal{X}/\mathcal{B}_{[0,1]}$ independent of $\mathcal{Y}$ with uniform law such that $\mathcal{Y} + \overline{\sigma}(\mathsf{Z}) = \mathcal{X}$.

(v) Every $\mathsf{Z} \in \mathcal{X}/\mathcal{B}_\mathbb{R}$ for which $\mathcal{Y} \vee \overline{\sigma}(\mathsf{Z}) = \mathcal{X}$ has a diffuse law.

Definition 3.13. Let $\{\mathcal{X}, \mathcal{Y}\} \subset \Lambda$, $\mathcal{Y} \subset \mathcal{X}$, $\mathcal{X}$ countably generated up to negligible sets. Following [4] call $\mathcal{X}$ conditionally non-atomic given $\mathcal{Y}$ when the conditions (i)-(v) of Proposition 3.12 prevail.

Example 3.14. Let $\{\mathcal{A}, \mathcal{B}, \mathcal{X}\} \subset \Lambda$, $\mathcal{X} \subset \mathcal{A} + \mathcal{B}$. It can happen that $\mathcal{A}, \mathcal{B}, \mathcal{X}$ are pairwise independent [2, Exercise 2.1(3)], and even when it is so, it may then happen that there is no $\mathcal{X}' \in \Lambda$ with $\mathcal{X}' \subset \mathcal{B}$ and $\mathcal{A} + \mathcal{X} = \mathcal{A} + \mathcal{X}'$, i.e. $\mathcal{X} \subset$

$((\mathcal{A} \vee \mathcal{X}) \wedge \mathcal{B}) \vee \mathcal{A}$ may fail (in particular one can have $\mathcal{X}$ independent of $\mathcal{B}$, but not measurable w.r.t. $\mathcal{A}$ [2, Exercise 2.1(2)]). In the "discrete" setting[2] take, e.g., ξ_i, $i \in \{1, 2, 3, 4\}$, independent equiprobable signs. Let $\mathcal{A} = \overline{\sigma}(\xi_1, \xi_2)$, $\mathcal{B} = \overline{\sigma}(\xi_3, \xi_4)$, $\mathcal{X} = \overline{\sigma}(\xi_1\xi_3 + \xi_2\xi_4)$. Then it is mechanical to check that $(\mathcal{X} \vee \mathcal{A}) \wedge \mathcal{B} = \overline{\sigma}(\xi_3\xi_4)$ (e.g., for inclusion $\supset$ one can notice that $(\xi_1\xi_3 + \xi_2\xi_4)^2 = 2(1 + \xi_1\xi_2\xi_3\xi_4)$; for the reverse inclusion one can consider the behavior of the indicators of the elements of $\sigma(\xi_3, \xi_4)$ on the atoms of $\sigma(\xi_1, \xi_2, \xi_1\xi_3 + \xi_2\xi_4)$). But $\xi_1\xi_3 + \xi_2\xi_4$ is not measurable w.r.t. $\mathcal{A} \vee ((\mathcal{A} \vee \mathcal{X}) \wedge \mathcal{B}) = \overline{\sigma}(\xi_1, \xi_2, \xi_3\xi_4)$, indeed $\xi_1\xi_3 + \xi_2\xi_4$ is not a.s. constant on the atom $\{\xi_1 = 1, \xi_2 = 1, \xi_3\xi_4 = 1\}$ of $\sigma(\xi_1, \xi_2, \xi_3\xi_4)$. To tweak this to the "continuous" case,[3] simply take a sequence $(\xi_i)_{i\in\mathbb{N}}$ of independent equiprobable signs and set $\mathcal{A} = \overline{\sigma}(\xi_{2i} : i \in \mathbb{N})$, $\mathcal{B} = \overline{\sigma}(\xi_{2i+1} : i \in \mathbb{N}_0)$, $\mathcal{X} = \overline{\sigma}(\xi_1\xi_2 + \xi_3\xi_4, \xi_5\xi_6 + \xi_7\xi_8, \ldots)$. By Proposition 3.4 and the preceding, it follows that $(\mathcal{X} \vee \mathcal{A}) \wedge \mathcal{B} = \overline{\sigma}(\xi_1\xi_3, \xi_5\xi_7, \ldots)$, and we see that $\xi_1\xi_2 + \xi_3\xi_4$ is not measurable w.r.t. $((\mathcal{X} \vee \mathcal{A}) \wedge \mathcal{B}) \vee \mathcal{A}$, for, exactly as before, it is not measurable w.r.t. $\overline{\sigma}(\xi_2, \xi_4, \xi_1\xi_3) = [((\mathcal{X} \vee \mathcal{A}) \wedge \mathcal{B}) \vee \mathcal{A}] \wedge \overline{\sigma}(\xi_1, \ldots, \xi_4)$.

Examples 3.15. Let $\{\mathcal{X}, \mathcal{Y}\} \subset \Lambda$, $\mathcal{Y} \subset \mathcal{X}$.

(a) We have already seen in Example 1.2 that in general $\mathcal{Y}$ may fail to have a complement in $\mathcal{X}$, though by Proposition 3.12 this cannot happen when $\mathcal{X}$ is essentially separable and everything is "sufficiently continuous". Example 1.3 shows, in a "discrete" setting, that even when $\mathcal{Y}$ has a complement in $\mathcal{X}$, then it is not necessarily unique. To see the latter also in the "continuous" setting take a doubly infinite sequence $(\xi_i)_{i\in\mathbb{Z}}$ of independent equiprobable signs, and set $\mathcal{X} = \overline{\sigma}(\xi_i : i \in \mathbb{Z})$, $\mathcal{Y} = \overline{\sigma}(\xi_i : i \in \mathbb{N})$. Then $\mathcal{Y} + \overline{\sigma}(\xi_i : i \in \mathbb{Z}_{\leq 0}) = \mathcal{X}$ but also $\mathcal{Y} + \overline{\sigma}(\xi_i\xi_{i+1} : i \in \mathbb{Z}_{\leq 0}) = \mathcal{X}$.

(b) Even when the equivalent conditions of Proposition 3.12 are met, and a $\mathcal{Z} \in \Lambda$ satisfies $\mathcal{Y} \vee \mathcal{Z} = \mathcal{X}$, there may be no $\mathcal{Z}' \in \Lambda$ with $\mathcal{Z}' \subset \mathcal{Z}$ and $\mathcal{Y} + \mathcal{Z}' = \mathcal{X}$. The following example of this situation is essentially verbatim from [4, p. 11, Remark (b)]. Let $\Omega = ([0, \frac{1}{2}] \times [0, 1]) \cup ([\frac{1}{2}, 1] \times [0, \frac{1}{2}]) \cup ([1, \frac{3}{2}] \times [\frac{1}{2}, 1])$, $\mathcal{M} = \mathcal{B}_\Omega$, and $\mathbb{P}$ be the (restriction of the) Lebesgue measure. Let Y be the projection onto the first coordinate and Z be the projection onto the second coordinate, $\mathcal{Y} = \overline{\sigma}(\mathsf{Y})$, $\mathcal{Z} = \overline{\sigma}(\mathsf{Z})$, $\mathcal{X} = \overline{\sigma}(\mathsf{Y}, \mathsf{Z}) = \mathcal{M}$. Then $|\mathsf{Z} - \frac{1}{2}|$ is independent of $\mathcal{Y}$, verifying (iii), though $\mathcal{Y}$ and $\mathcal{Z}$ are not independent. Suppose that $\mathcal{Z}' \in \Lambda$ satisfies $\mathcal{Z}' \subset \mathcal{Z}$ and $\mathcal{Y} \vee \mathcal{Z}' = \mathcal{X}$. The σ-field $\mathcal{X}$ and hence $\mathcal{Z}'$ is countably generated up to negligible sets so there is $\mathsf{Z}' \in \mathcal{Z}'/\mathcal{B}_\mathbb{R}$ such that $\mathcal{Z}' = \overline{\sigma}(\mathsf{Z}')$. By the Doob–Dynkin lemma there are $f \in \mathcal{B}_{[0,1]}/\mathcal{B}_\mathbb{R}$ and $g \in \mathcal{B}_{[0,\frac{3}{2}]\times\mathbb{R}}/\mathcal{B}_{[0,1]}$ such that a.s. $\mathsf{Z}' = f(\mathsf{Z})$ and $\mathsf{Z} = g(\mathsf{Y}, \mathsf{Z}')$. Then $\mathsf{Z} = g(\mathsf{Y}, f(\mathsf{Z}))$ a.s.; consequently by Tonelli's theorem for Lebesgue-almost every $y \in [0, \frac{1}{2}]$, $z = g(y, f(z))$ for Lebesgue-almost all $z \in [0, 1]$. Fix such y. Then because Z is absolutely continuous, one obtains $\mathsf{Z} = g(y, f(\mathsf{Z})) = g(y, \mathsf{Z}')$ a.s.; this forces $\mathcal{Z}' = \mathcal{Z}$, preventing $\mathcal{Z}' \perp\!\!\!\perp \mathcal{Y}$.

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[2] In precise terms, by "discrete", we mean here, and in what follows, that every σ-field under consideration is generated up to negligible sets by a discrete random variable.

[3] To be precise, by "continuous", we mean to say here, and in what follows, that every σ-field under consideration is generated up to negligible sets by a diffuse random variable.

(c) If the equivalent conditions of Proposition 3.12 are met and if $\mathsf{Z} \in \mathcal{X}/\mathcal{B}_{\mathbb{R}}$ has diffuse law and is independent of $\mathcal{Y}$, there may exist no $\mathcal{Z}' \in \Lambda$ such that $\mathcal{Y} + \mathcal{Z}' = \mathcal{X}$ and $\overline{\sigma}(\mathsf{Z}) \subset \mathcal{Z}'$ (however this cannot happen if ceteris paribus Z is discrete rather than continuous – see Corollary 3.16(ii)(b)). We repeat here for the reader's convenience [4, p. 11, Remark (a)] exemplifying this scenario. Let $\mathsf{X}, \mathsf{Y}, \mathsf{Z}$ be independent random variables with uniform law on $[0, 1]$ and let $\mathcal{Y} = \overline{\sigma}(\mathsf{Y})$, $\mathcal{X} = \overline{\sigma}(\mathsf{Y}, \mathsf{Z}, \mathsf{X}\mathbb{1}_{\{\mathsf{Y}<\frac{1}{2}\}})$. Clearly $\mathcal{X}$ is countably generated up to negligible sets; Z has a diffuse law and is independent of $\mathcal{Y}$; in particular (iii) is verified. Let $\mathcal{Z}' \in \Lambda$ be such that $\mathcal{Y} \perp\!\!\!\perp \mathcal{Z}' \supset \overline{\sigma}(\mathsf{Z})$, $\mathcal{Z}' \subset \mathcal{X}$. There is a $\mathsf{Z}' \in \mathcal{Z}'/\mathcal{B}_{\mathbb{R}}$ such that $\mathcal{Z}' = \overline{\sigma}(\mathsf{Z}')$. By the Doob–Dynkin lemma there are $f \in \mathcal{B}_{\mathbb{R}}/\mathcal{B}_{[0,1]}$ and $g \in \mathcal{B}_{[0,1]^3}/\mathcal{B}_{\mathbb{R}}$ such that a.s. $\mathsf{Z} = f(\mathsf{Z}')$ and $\mathsf{Z}' = g(\mathsf{Y}, \mathsf{Z}, \mathsf{X}\mathbb{1}_{\{\mathsf{Y}<\frac{1}{2}\}})$. Then on $\{\mathsf{Y} \geq \frac{1}{2}\}$, $\mathsf{Z}' = g(\mathsf{Y}, \mathsf{Z}, 0) = g(\mathsf{Y}, f(\mathsf{Z}'), 0)$ a.s.; hence by Tonelli's theorem for Lebesgue-almost every $y \in [\frac{1}{2}, 1]$, $z' = g(y, f(z'), 0)$ for $\mathsf{Z}'_\star\mathbb{P}$-almost every $z' \in \mathbb{R}$. Fix such y. It follows that $\mathsf{Z}' = g(y, f(\mathsf{Z}'), 0) = g(y, \mathsf{Z}, 0)$ a.s.; this forces $\mathcal{Z}' = \overline{\sigma}(\mathsf{Z})$, which precludes $\mathcal{Y} \vee \mathcal{Z}' = \mathcal{X}$.

Proof of Proposition 3.12. We follow closely the proof of [4, Proposition 4].

(i) $\Rightarrow$ (ii) because $\mathcal{X}$ is countably generated up to negligible sets.

(iv) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (ii) by Tonelli's theorem.

(v) $\Rightarrow$ (i). Let $\mathsf{X} \in \mathcal{X}/\mathcal{B}_{\mathbb{R}}$ be such that $\mathcal{X} = \overline{\sigma}(\mathsf{X})$, take $\mathsf{Y} \in \mathcal{Y}/\mathcal{B}_{\mathbb{R}}$. Fix $x_0 \in \mathbb{R}$ for which $\mathbb{P}(\mathsf{X} = x_0) = 0$. Then $\mathcal{Y} \vee \overline{\sigma}(\mathsf{X}\mathbb{1}_{\{\mathsf{X}\neq\mathsf{Y}\}} + x_0\mathbb{1}_{\{\mathsf{X}=\mathsf{Y}\}}) = \mathcal{X}$, hence by (v) $\mathsf{X}\mathbb{1}_{\{\mathsf{X}\neq\mathsf{Y}\}} + x_0\mathbb{1}_{\{\mathsf{X}=\mathsf{Y}\}}$ has a diffuse law, and therefore $\mathbb{P}(\mathsf{X} = \mathsf{Y}) = 0$.

(ii) $\Rightarrow$ (v). Let $\mathsf{X} \in \mathcal{X}/\mathcal{B}_{\mathbb{R}}$ be such that for every $\mathsf{Y} \in \mathcal{Y}/\mathcal{B}_{\mathbb{R}}$, $\mathbb{P}(\mathsf{X} = \mathsf{Y}) = 0$ and let $\mathsf{Z} \in \mathcal{X}/\mathcal{B}_{\mathbb{R}}$ be such that $\mathcal{Y} \vee \overline{\sigma}(\mathsf{Z}) = \mathcal{X}$. Because $\mathcal{Y}$ is countably generated up to negligible sets, there is $\mathsf{Y} \in \mathcal{Y}/\mathcal{B}_{\mathbb{R}}$ such that $\mathcal{Y} = \overline{\sigma}(\mathsf{Y})$. Then $\overline{\sigma}(\mathsf{Y}, \mathsf{Z}) = \mathcal{X}$ and by the Doob–Dynkin lemma there is $f \in \mathcal{B}_{\mathbb{R}^2}/\mathcal{B}_{\mathbb{R}}$ such that a.s. $\mathsf{X} = f(\mathsf{Y}, \mathsf{Z})$. We conclude that for each $z_0 \in \mathbb{R}$, $\mathbb{P}(\mathsf{Z} = z_0) \subset \mathbb{P}(\mathsf{X} = f(\mathsf{Y}, z_0)) = 0$.

(ii) $\Rightarrow$ (iv). Let again $\mathsf{X} \in \mathcal{X}/\mathcal{B}_{\mathbb{R}}$ be such that for every $\mathsf{Y} \in \mathcal{Y}/\mathcal{B}_{\mathbb{R}}$, $\mathbb{P}(\mathsf{X} = \mathsf{Y}) = 0$. Take also $\mathsf{Y} \in \mathcal{Y}/\mathcal{B}_{\mathbb{R}}$ such that $\mathcal{Y} = \overline{\sigma}(\mathsf{Y})$ and $\mathsf{X}' \in \mathcal{X}/\mathcal{B}_{\mathbb{R}}$ such that $\overline{\sigma}(\mathsf{X}') = \mathcal{X}$. Let μ be the law of Y and let $(\nu_y)_{y\in\mathbb{R}}$ be a version of the conditional law of X' given Y: $(\mathbb{R} \ni y \mapsto \nu_y(A)) \in \mathcal{B}_{\mathbb{R}}/\mathcal{B}_{[0,1]}$ for each $A \in \mathcal{B}_{\mathbb{R}}$; ν_y is a law on $\mathcal{B}_{\mathbb{R}}$ for each $y \in \mathbb{R}$; and $\mathbb{E}[f(\mathsf{X}', \mathsf{Y})] = \int f(x', y)\nu_y(dx')\mu(dy)$ for $f \in \mathcal{B}_{\mathbb{R}^2}/\mathcal{B}_{[0,\infty]}$. Remark that in particular ($\star$) a.s. X' cannot fall into a maximal non-degenerate interval that is negligible for ν_Y. Besides, by the Doob–Dynkin lemma, there is $g \in \mathcal{B}_{\mathbb{R}}/\mathcal{B}_{\mathbb{R}}$ such that $\mathsf{X} = g(\mathsf{X}')$ a.s. Then $\mathbb{P}(\mathsf{Y}' = \mathsf{X}') \subset \mathbb{P}(\mathsf{X} = g(\mathsf{Y}')) = 0$ for any $\mathsf{Y}' \in \mathcal{Y}/\mathcal{B}_{\mathbb{R}}$. From this it follows that ($\star\star$) ν_y is diffuse for μ-almost every $y \in \mathbb{R}$. Set now $\mathsf{Z} := \nu_{\mathsf{Y}}((-\infty, \mathsf{X}']) \in \mathcal{X}/\mathcal{B}_{[0,1]}$; then for $\phi \in \mathcal{B}_{\mathbb{R}}/\mathcal{B}_{[0,\infty]}$ and $z \in [0, 1]$,

$$\begin{aligned}\mathbb{E}\big[\phi(\mathsf{Y}); \mathsf{Z} \leq z\big] &= \int\int \phi(y)\mathbb{1}_{[0,z]}(\nu_y\big((-\infty, x']\big))\nu_y\big(dx'\big)\mu(dy)\\ &= z\int \phi d\mu = \mathbb{P}(\mathsf{Z} \leq z)\mathbb{E}\big[\phi(\mathsf{Y})\big],\end{aligned}$$

because of ($\star\star$). On account of ($\star$), it also follows from the equality $\mathsf{Z} = \nu_Y((-\infty, \mathsf{X}'])$ that $\mathsf{X}' \in \overline{\sigma}(\mathsf{Z}, \mathsf{Y})$. Thus Z meets all the requisite properties. □

Several "stability" properties of conditionally non-atomic σ-fields can be noted:

Corollary 3.16 (Conditionally non-atomic σ-fields)**.** [4, Corollaries 3 and 4] *Let $\{\mathcal{X}, \mathcal{Y}, \mathcal{Z}\} \subset \Lambda$, $\mathcal{Y} \subset \mathcal{X}$. Assume $\mathcal{X} \vee \mathcal{Z}$ is countably generated up to negligible sets.*

(i) If $\mathcal{X} \vee \mathcal{Z}$ is conditionally non-atomic given $\mathcal{Y} \vee \mathcal{Z}$, then $\mathcal{X}$ is conditionally non-atomic given $\mathcal{Y}$.

(ii) Suppose $\mathcal{X}$ is conditionally non-atomic given $\mathcal{Y}$.

(a) If $\mathcal{X}$ and $\mathcal{Z}$ are independent, then $\mathcal{X} \vee \mathcal{Z}$ is conditionally non-atomic given $\mathcal{Y} \vee \mathcal{Z}$.

(b) If $\mathcal{P} \subset \mathcal{X}$ is a denumerable partition of Ω, then $\mathcal{X}$ is conditionally non-atomic given $\mathcal{Y} \vee \overline{\sigma}(\mathcal{P})$; if further $\overline{\sigma}(\mathcal{P}) \perp\!\!\!\perp \mathcal{Y}$, then there exists $\mathsf{Z} \in \mathcal{X}/\mathcal{B}_{[0,1]}$ with uniform law such that $\mathcal{Y} + \overline{\sigma}(\mathsf{Z}) = \mathcal{X}$ and $\overline{\sigma}(\mathcal{P}) \subset \overline{\sigma}(\mathsf{Z})$.

Proof. We follow closely the proofs of [4, Corollaries 3 and 4].

(i). Let $\mathsf{Z} \in \mathcal{X}/\mathcal{B}_{\mathbb{R}}$ be such that $\mathcal{X} = \mathcal{Y} \vee \overline{\sigma}(\mathsf{Z})$; then $\mathcal{X} \vee \mathcal{Z} = (\mathcal{Y} \vee \mathcal{Z}) \vee \overline{\sigma}(\mathsf{Z})$. Thus if $\mathcal{X} \vee \mathcal{Z}$ is conditionally non-atomic given $\mathcal{Y} \vee \mathcal{Z}$, then by Proposition 3.12(v) Z is diffuse, which makes $\mathcal{X}$ conditionally non-atomic given $\mathcal{Y}$ by the very same argument.

(ii)(a). Let $\mathcal{X}$ and $\mathcal{Z}$ be independent. By Proposition 3.12(iii), there exists $\mathsf{Z} \in \mathcal{X}/\mathcal{B}_{\mathbb{R}}$ independent of $\mathcal{Y}$ and having a diffuse law; such Z is then also independent of $\mathcal{Y} \vee \mathcal{Z}$, so that by the very same condition $\mathcal{X} \vee \mathcal{Z}$ is conditionally non-atomic given $\mathcal{Y} \vee \mathcal{Z}$.

(ii)(b). There is a random variable $\mathsf{P} \in \mathcal{X}/2^{\mathbb{N}}$ for which $\overline{\sigma}(\mathcal{P}) = \overline{\sigma}(\mathsf{P})$. If $\mathsf{Z} \in \mathcal{X}/\mathcal{B}_{\mathbb{R}}$ is such that $\mathcal{X} = (\mathcal{Y} \vee \overline{\sigma}(\mathsf{P})) \vee \overline{\sigma}(\mathsf{Z}) = \mathcal{Y} \vee \overline{\sigma}(\mathsf{P}, \mathsf{Z})$, then (P, Z) has a diffuse law by Proposition 3.12(v), hence (because P has a denumerable range) Z has a diffuse law, which entails the desired conclusion by the very same argument. Now suppose P is independent of $\mathcal{Y}$. Via Proposition 3.12(iv) let $\mathsf{Z}' \in \mathcal{X}/\mathcal{B}_{[0,1]}$ have uniform law and be a complement for $\mathcal{Y}+\overline{\sigma}(\mathsf{P})$ in $\mathcal{X}$. Of course $\overline{\sigma}(\mathsf{Z}', \mathsf{P})$ is essentially separable so there is $\mathsf{Z} \in \sigma(\mathsf{Z}', \mathsf{P})/\mathcal{B}_{\mathbb{R}}$ with $\overline{\sigma}(\mathsf{Z}) = \overline{\sigma}(\mathsf{Z}', \mathsf{P})$. Z is diffuse, because Z' is, hence may be chosen to be uniform on $[0, 1]$. □

The next proposition investigates to what extent complements are "hereditary".

Proposition 3.17 (Complements II)**.** *Let $\{\mathcal{X}, \mathcal{Y}, \mathcal{Z}\} \subset \Lambda$, $\mathcal{Z} \subset \mathcal{X} + \mathcal{Y}$. Then the following statements are equivalent.*

(i) $\mathcal{Z} = (\mathcal{X} \wedge \mathcal{Z}) \vee (\mathcal{Y} \wedge \mathcal{Z})$, i.e. $\mathcal{X} \wedge \mathcal{Z}$ is a complement of $\mathcal{Y} \wedge \mathcal{Z}$ in $\mathcal{Z}$.

(ii) $\mathcal{X}$ and $\mathcal{Y}$ are conditionally independent given $\mathcal{Z}$, and $\mathbb{P}[Y|\mathcal{Z}] \in \mathcal{Y}/\mathcal{B}_{[-\infty,\infty]}$ for $Y \in \mathcal{Y}$, $\mathbb{P}[X|\mathcal{Z}] \in \mathcal{X}/\mathcal{B}_{[-\infty,\infty]}$ for $X \in \mathcal{X}$.

Remark 3.18. Dropping, ceteris paribus, the condition that $\mathcal{X} \perp\!\!\!\perp \mathcal{Y}$, then (i) no longer implies (ii) (because one can have $\mathcal{Z} \subset \mathcal{X}$ or $\mathcal{Z} \subset \mathcal{Y}$, without $\mathcal{X}$ and $\mathcal{Y}$ being conditionally independent given $\mathcal{Z}$); however, (ii) still implies (i) (this will be clear from the proof, and at any rate Proposition 3.21 will provide a more general statement, that will subsume this implication as a special case).

Examples 3.19.

(a) The situation described by (i), equivalently (ii) is not trivial. For instance if $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ are independent members of Λ, then one can take $\mathcal{X} = \mathcal{A} + \mathcal{B}$, $\mathcal{Y} = \mathcal{C} + \mathcal{D}$, $\mathcal{Z} = \mathcal{B} + \mathcal{C}$. Of course in this case $\mathcal{Z} = (\mathcal{X} \wedge \mathcal{Z}) \vee (\mathcal{Y} \wedge \mathcal{Z})$ can be seen (slightly indirectly) from Proposition 3.4 as much as (directly) from the validity of (ii).

(b) But there are cases when Proposition 3.4 does not apply (or applies only (very) indirectly), while Proposition 3.17 does. A trivial example of this is when $\mathcal{Z} \subset \mathcal{X}$ or $\mathcal{Z} \subset \mathcal{Y}$.

(c) For a less trivial example of the situation described in (b) let ξ_i, $i \in \{1, 2, 3, 4\}$, be independent equiprobable signs. Let $\mathcal{X} = \overline{\sigma}(\xi_1, \{\xi_1 = \xi_2 = 1\})$, $\mathcal{Y} = \overline{\sigma}(\xi_3, \{\xi_3 = \xi_4 = 1\})$ and $\mathcal{Z} = \overline{\sigma}(\xi_1, \xi_3)$. In this case, unlike in (a), it is not the case that $\mathcal{Z} \wedge \mathcal{X} = \overline{\sigma}(\xi_1)$ would have a complement in $\mathcal{X}$ and $\mathcal{Z} \wedge \mathcal{Y} = \overline{\sigma}(\xi_3)$ would have a complement in $\mathcal{Y}$. For this reason Proposition 3.4 cannot be (indirectly) applied to deduce $(\mathcal{X} \wedge \mathcal{Z}) \vee (\mathcal{Y} \wedge \mathcal{Z}) = \mathcal{Z}$. Yet this equality does prevail and can indeed be seen directly and a priori from the validity of (ii).

Proof. Suppose (i) hods true. Let $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. Then because $\mathcal{X} \perp\!\!\!\perp \mathcal{Y}$, a.s. $\mathbb{P}[X \cap Y|\mathcal{Z}] = \mathbb{P}[X \cap Y|(\mathcal{X} \wedge \mathcal{Z}) \vee (\mathcal{Y} \wedge \mathcal{Z})] = \mathbb{P}[X|\mathcal{X} \wedge \mathcal{Z}]\mathbb{P}[Y|\mathcal{Y} \wedge \mathcal{Z}]$. Taking $Y = \Omega$ and $X = \Omega$ shows that $\mathbb{P}[X|\mathcal{Z}] = \mathbb{P}[X|\mathcal{X} \wedge \mathcal{Z}]$ a.s. and $\mathbb{P}[Y|\mathcal{Y} \wedge \mathcal{Z}] = \mathbb{P}[Y|\mathcal{Z}]$ a.s., which concludes the argument. Conversely, suppose that (ii) holds true. Let $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. Then a.s. $\mathbb{P}[X \cap Y|\mathcal{Z}] = \mathbb{P}[X|\mathcal{Z}]\mathbb{P}[Y|\mathcal{Z}]$ and $\mathbb{P}[X|\mathcal{Z}] = \mathbb{P}[X|\mathcal{X} \wedge \mathcal{Z}]$, $\mathbb{P}[Y|\mathcal{Z}] = \mathbb{P}[Y|\mathcal{Y} \wedge \mathcal{Z}]$. Hence $\mathbb{P}[X \cap Y|\mathcal{Z}] \in ((\mathcal{X} \wedge \mathcal{Z}) \vee (\mathcal{Y} \wedge \mathcal{Z}))/\mathcal{B}_{[-\infty,\infty]}$. A π/λ-argument allows to conclude that $\mathbb{P}[Z|\mathcal{Z}] \in ((\mathcal{X} \wedge \mathcal{Z}) \vee (\mathcal{Y} \wedge \mathcal{Z}))/\mathcal{B}_{[-\infty,\infty]}$ for all $Z \in \mathcal{X} \vee \mathcal{Y}$ and therefore, because $\mathcal{Z} \subset \mathcal{X} \vee \mathcal{Y}$, for all $Z \in \mathcal{Z}$. Thus $\mathcal{Z} \subset (\mathcal{X} \wedge \mathcal{Z}) \vee (\mathcal{Y} \wedge \mathcal{Z})$, while the reverse inclusion is trivial. □

More generally (in the sufficiency part):

Proposition 3.20 (Distributivity III). *Let $(\mathcal{X}_\alpha)_{\alpha \in \mathfrak{A}}$ be a family in Λ consisting of independent σ-fields. Then*

$$(\vee_{\alpha \in \mathfrak{A}} \mathcal{X}_\alpha) \wedge \mathcal{Z} = \vee_{\alpha \in \mathfrak{A}} (\mathcal{X}_\alpha \wedge \mathcal{Z})$$

provided (i) the $\mathcal{X}_\alpha$, $\alpha \in \mathfrak{A}$, are conditionally independent given $\mathcal{Z}$ and (ii) $\mathbb{P}[X_\alpha|\mathcal{Z}] \in \mathcal{X}_\alpha/\mathcal{B}_{[-\infty,\infty]}$ for all $X_\alpha \in \mathcal{X}_\alpha$, $\alpha \in \mathfrak{A}$.

Proof. Set $\mathcal{X} := \vee_{\alpha \in \mathfrak{A}} \mathcal{X}_\alpha$. Condition (ii) entails that a.s. $\mathbb{P}[X_\alpha|\mathcal{X} \wedge \mathcal{Z}] = \mathbb{P}[X_\alpha|\mathcal{X}_\alpha \wedge \mathcal{Z}] = \mathbb{P}[X_\alpha|\mathcal{Z}]$ for all $\alpha \in \mathfrak{A}$; combining this with (i) shows via a π/λ-argument that a.s. $\mathbb{P}[X|\mathcal{X} \wedge \mathcal{Z}] = \mathbb{P}[X|\mathcal{Z}]$ for all $X \in \mathcal{X}$: if B is a finite non-empty subset of $\mathfrak{A}$, then a.s. $\mathbb{P}[\cap_{\beta \in B} X_\beta|\mathcal{Z}] = \prod_{\beta \in B} \mathbb{P}[X_\beta|\mathcal{Z}] = \prod_{\beta \in B} \mathbb{P}[X_\beta|\mathcal{X} \wedge \mathcal{Z}] \in (\mathcal{X} \wedge \mathcal{Z})/\mathcal{B}_{[-\infty,\infty]}$. Replacing $\mathcal{Z}$ by $\mathcal{Z} \wedge \mathcal{X}$ if necessary, we may and do assume $\mathcal{Z} \subset \mathcal{X}$. Then $\vee_{\alpha \in \mathfrak{A}} (\mathcal{X}_\alpha \wedge \mathcal{Z}) \subset \mathcal{Z} = \mathcal{X} \wedge \mathcal{Z}$ is trivial. For the reverse inclusion, let B be a finite non-empty subset of $\mathfrak{A}$, and let $X_\beta \in \mathcal{X}_\beta$ for $\beta \in B$. Then a.s. $\mathbb{P}[\cap_{\beta \in B} X_\beta|\mathcal{Z}] = \prod_{\beta \in B} \mathbb{P}[X_\beta|\mathcal{Z}] = \prod_{\beta \in B} \mathbb{P}[X_\beta|\mathcal{X}_\beta \wedge \mathcal{Z}] \in (\vee_{\alpha \in \mathfrak{A}} (\mathcal{X}_\alpha \wedge \mathcal{Z}))/\mathcal{B}_{[-\infty,\infty]}$. By a π/λ-argument we conclude that $\mathbb{P}[Z|\mathcal{Z}] \in (\vee_{\alpha \in \mathfrak{A}} (\mathcal{X}_\alpha \wedge \mathcal{Z}))/\mathcal{B}_{[-\infty,\infty]}$ for all $Z \in \vee_{\alpha \in \mathfrak{A}} \mathcal{X}_\alpha$, and therefore for all $Z \in \mathcal{Z}$. It means that also $\mathcal{X} \wedge \mathcal{Z} = \mathcal{Z} \subset \vee_{\alpha \in \mathfrak{A}} (\mathcal{X}_\alpha \wedge \mathcal{Z})$. □

Parallel to Proposition 3.20 we have:

Proposition 3.21 (Distributivity IV). *Let $(\mathcal{X}_\alpha)_{\alpha\in\mathfrak{A}}$ be a family in Λ, with $\mathfrak{A}$ containing at least two elements, consisting of σ-fields that are conditionally independent given $\mathcal{Z}\in\Lambda$. Then*

$$\mathcal{Z}=\wedge_{\alpha\in\mathfrak{A}}(\mathcal{X}_\alpha\vee\mathcal{Z});$$

in particular $\wedge_{\alpha\in\mathfrak{A}}\mathcal{X}_\alpha\subset\mathcal{Z}$.

Remark 3.22. The converse is not true, because, for instance, one can have $\mathcal{X}$ and $\mathcal{Y}$ dependent with $\mathcal{X}\wedge\mathcal{Y}=0_\Lambda$ (then $\mathcal{Z}=(\mathcal{X}\vee\mathcal{Z})\wedge(\mathcal{Y}\vee\mathcal{Z})$ for $\mathcal{Z}=0_\Lambda$, but $\mathcal{X}$ and $\mathcal{Y}$ are not independent given $\mathcal{Z}$) – see Example 3.3. The condition on the conditional independence of course cannot be dropped, not even if the $\mathcal{X}_\alpha$, $\alpha\in\mathfrak{A}$, and $\mathcal{Z}$ are pairwise independent – see Example 1.1(a).

Remark 3.23. By Proposition 3.4 the equality

$$(\wedge_{\alpha\in\mathfrak{A}}\mathcal{X}_\alpha)\vee\mathcal{Z}=\wedge_{\alpha\in\mathfrak{A}}(\mathcal{X}_\alpha\vee\mathcal{Z})$$

also prevails when the $\mathcal{X}_\alpha$, $\alpha\in\mathfrak{A}$, are independent of $\mathcal{Z}$, however the scope of this result is clearly different from that of Proposition 3.21.

Proof. It is clear that $\mathcal{Z}\subset\wedge_{\alpha\in\mathfrak{A}}(\mathcal{X}_\alpha\vee\mathcal{Z})$. For the reverse inclusion we may assume $\mathfrak{A}=\{1,2\}$. Let $F\in(\mathcal{X}_1\vee\mathcal{Z})\wedge(\mathcal{X}_2\vee\mathcal{Z})$. Then a.s. $\mathbb{1}_F=\mathbb{P}[F|\mathcal{X}_1\vee\mathcal{Z}]$ (because $F\in\mathcal{X}_1\vee\mathcal{Z}$). Let us now show that if $F\in\mathcal{X}_2\vee\mathcal{Z}$, then $\mathbb{P}[F|\mathcal{X}_1\vee\mathcal{Z}]\in\mathcal{Z}/\mathcal{B}_{[-\infty,\infty]}$; this will conclude the argument. Take $X_2\in\mathcal{X}_2$ and $Z\in\mathcal{Z}$. Then a.s. $\mathbb{P}[X_2\cap Z|\mathcal{X}_1\vee\mathcal{Z}]=\mathbb{1}_Z\mathbb{P}[X_2|\mathcal{X}_1\vee\mathcal{Z}]$. Thus by a π/λ-argument it will suffice to establish that $\mathbb{P}[X_2|\mathcal{X}_1\vee\mathcal{Z}]\in\mathcal{Z}/\mathcal{B}_{[-\infty,\infty]}$. For this, just argue that a.s. $\mathbb{P}[X_2|\mathcal{X}_1\vee\mathcal{Z}]=\mathbb{P}[X_2|\mathcal{Z}]$: let $X_1\in\mathcal{X}_1$ and $Z\in\mathcal{Z}$; then $\mathbb{P}(X_2\cap X_1\cap Z)=\mathbb{E}[\mathbb{P}[X_2|\mathcal{Z}];\,X_1\cap Z]$ because $\mathcal{X}_1$ is conditionally independent of $\mathcal{X}_2$ given $\mathcal{Z}$; another π/λ-argument allows to conclude. □

A further substantial statement involving conditional independence and distributivity is the following. It generalizes Proposition 3.4 in the case when $\mathfrak{B}$ is a two-point set.

Proposition 3.24 (Distributivity V). *Let $(\mathcal{X}_{\alpha i})_{(\alpha,i)\in\mathfrak{A}\times\{1,2\}}$ be a family in Λ, $\mathfrak{A}$ non-empty. Set $\mathcal{X}_i:=\vee_{\alpha\in\mathfrak{A}}\mathcal{X}_{\alpha i}$ for $i\in\{1,2\}$. Assume that for each finite non-empty $A\subset\mathfrak{A}$, $\mathcal{X}_1$ is conditionally independent of $\mathcal{X}_2$ given $\wedge_{\alpha\in A}\mathcal{X}_{\alpha 1}$ and also given $\wedge_{\alpha\in A}\mathcal{X}_{\alpha 2}$. Then*

$$\wedge_{\alpha\in\mathfrak{A}}(\mathcal{X}_{\alpha 1}\vee\mathcal{X}_{\alpha 2})=(\wedge_{\alpha\in\mathfrak{A}}\mathcal{X}_{\alpha 1})\vee(\wedge_{\alpha\in\mathfrak{A}}\mathcal{X}_{\alpha 2}).\tag{3.2}$$

Proof. By decreasing martingale convergence, $\mathcal{X}_1$ is conditionally independent of $\mathcal{X}_2$ given $\wedge_{\alpha\in\mathfrak{A}}\mathcal{X}_{\alpha 1}$ and also given $\wedge_{\alpha\in\mathfrak{A}}\mathcal{X}_{\alpha 2}$. Therefore, by the same reduction as at the start of the proof of Proposition 3.4, it suffices to establish the claim in the following two cases.

(A) $\mathcal{X}_{\alpha 2}=\mathcal{X}_2$ for all $\alpha\in\mathfrak{A}$.

(B) $\mathcal{A}=\{1,2\}$, $\mathcal{X}_{11}\subset\mathcal{X}_{21}$, $\mathcal{X}_{22}\subset\mathcal{X}_{12}$.

(A). Suppose (3.2) has been established for $\mathfrak{A}$ finite (all the time assuming (A), of course). Let $A\subset\mathfrak{A}$ be finite and non-empty and $(X_1,X_2)\in\mathcal{X}_1\times\mathcal{X}_2$. Then, because

$\mathcal{X}_1 \perp\!\!\!\perp_{\wedge_{\alpha\in A}\mathcal{X}_{\alpha 1}} \mathcal{X}_2$, a.s. $\mathbb{P}[X_1 \cap X_2|(\wedge_{\alpha\in A}\mathcal{X}_{\alpha 1}) \vee \mathcal{X}_2] = \mathbb{P}[X_1| \wedge_{\alpha\in A} \mathcal{X}_{\alpha 1}]\mathbb{1}_{X_2}$. By decreasing martingale convergence and the assumption made, it follows that $\mathbb{P}[X_1 \cap X_2| \wedge_{\alpha\in\mathfrak{A}} (\mathcal{X}_{\alpha 1} \vee \mathcal{X}_2)] \in ((\wedge_{\alpha\in\mathfrak{A}}\mathcal{X}_{\alpha 1}) \vee \mathcal{X}_2)/\mathcal{B}_{[-\infty,\infty]}$, and we conclude as usual. Then it remains to establish the claim for finite $\mathcal{A}$, and by induction for $\mathcal{A} = \{1, 2\}$. The remainder of the proof is now the same as in the proof of item (a) of Proposition 3.4, except that, as appropriate, one appeals to conditional independence in lieu of independence.

(B). This is proved just as in the final part of the proof of item (b) of Proposition 3.4 (only the final part is relevant because here a priori $\mathcal{A} = \mathcal{B} = \{1, 2\}$), except that again one appeals to conditional independence in lieu of independence, as appropriate. □

Corollary 3.25 (Distributivity VI)**.** *[7], [2, Exercise 2.5(1)]. If $\mathcal{Y} \in \Lambda$ and a non-increasing sequence $(\mathcal{X}_n)_{n\in\mathbb{N}}$ from Λ are such that $\mathcal{Y} \perp\!\!\!\perp_{\mathcal{X}_n} \mathcal{X}_1$ for all $n \in \mathbb{N}$, then $\wedge_{n\in\mathbb{N}}(\mathcal{X}_n \vee \mathcal{Y}) = (\wedge_{n\in\mathbb{N}}\mathcal{X}_n) \vee \mathcal{Y}$.* □

Remark 3.26. The generalization to a general $\mathfrak{B}$ in lieu of $\{1, 2\}$ in Proposition 3.24 seems too cumbersome to be of any value, and we omit making it explicit.

Finally we return yet again to complements. In the following it is investigated what happens if one is given $\mathcal{A} \perp\!\!\!\perp \mathcal{B}$ from Λ, and one enlarges $\mathcal{A}$ by an independent complement $\mathcal{X}$ to form $\mathcal{A}' = \mathcal{A} + \mathcal{X}$, while reducing $\mathcal{B}$ to $\mathcal{B}'$ through an independent complement $\mathcal{Y}$, $\mathcal{B}' + \mathcal{Y} = \mathcal{B}$, in such a manner that $\mathcal{A}' \perp\!\!\!\perp \mathcal{B}'$, and that between them $\mathcal{A}'$ and $\mathcal{B}'$ generate the same σ-field as $\mathcal{A}$ and $\mathcal{B}$ do. (We will see in Section 4 why this is an interesting situation to consider.)

Proposition 3.27 (Two-sided complements)**.** *Let $\{\mathcal{A}, \mathcal{B}, \mathcal{A}', \mathcal{B}'\} \subset \Lambda$ be such that $\mathcal{A} + \mathcal{B} = \mathcal{A}' + \mathcal{B}'$.*

(i) There is at most one $\mathcal{X} \in \Lambda$ such that $\mathcal{A} + \mathcal{X} = \mathcal{A}'$ and $\mathcal{B}' + \mathcal{X} = \mathcal{B}$, namely $\mathcal{A}' \wedge \mathcal{B}$.

(ii) Let $\{\mathcal{X}, \mathcal{Y}\} \subset \Lambda$ be such that $\mathcal{A} + \mathcal{X} = \mathcal{A}'$ and $\mathcal{B}' + \mathcal{Y} = \mathcal{B}$. The following statements are equivalent:

(a) There is $\mathcal{Z} \in \Lambda$ with $\mathcal{A} + \mathcal{Z} = \mathcal{A}'$ and $\mathcal{B}' + \mathcal{Z} = \mathcal{B}$.

(b) $\mathcal{A} + (\mathcal{A}' \wedge \mathcal{B}) + \mathcal{B}' = \mathcal{A} + \mathcal{B}$ $(= \mathcal{A}' + \mathcal{B}')$.

(c) $\mathcal{X} \subset \mathcal{A} \vee (\mathcal{A}' \wedge \mathcal{B})$ and $\mathcal{Y} \subset \mathcal{B}' \vee (\mathcal{A}' \wedge \mathcal{B})$.

(d) There is $\mathcal{X}' \in \Lambda$ with $\mathcal{X}' \subset \mathcal{B}$ and $\mathcal{A} + \mathcal{X}' = \mathcal{A}'$ and there is $\mathcal{Y}' \in \Lambda$ with $\mathcal{Y}' \subset \mathcal{A}'$ and $\mathcal{B}' + \mathcal{Y}' = \mathcal{B}$.

(e) $\mathbb{P}[B|\mathcal{A}'] \in \mathcal{B}/\mathcal{B}_{[-\infty,\infty]}$ for $B \in \mathcal{B}$ and $\mathbb{P}[A'|\mathcal{B}] \in \mathcal{A}'/\mathcal{B}_{[-\infty,\infty]}$ for $A' \in \mathcal{A}'$.

Example 3.28. Let ξ_i, $i \in \{1, 2, 3\}$, be independent equiprobable signs. Let $\mathcal{A} := \overline{\sigma}(\xi_1)$, $\mathcal{B}' := \overline{\sigma}(\xi_2)$, $\mathcal{X} := \overline{\sigma}(\xi_3)$, $\mathcal{Y} := \overline{\sigma}(\{\xi_1 = \xi_3 = 1 \text{ or } \xi_3\xi_2 = \xi_1 = -1\})$, $\mathcal{A}' := \mathcal{A} + \mathcal{X}$, $\mathcal{B} := \mathcal{B}' + \mathcal{Y}$. It is then straightforward to check, for instance by considering the induced partitions, that $\mathcal{A} + \mathcal{B} = \overline{\sigma}(\xi_1, \xi_2, \xi_3) = \mathcal{A}' + \mathcal{B}'$, while $\mathcal{A}' \wedge \mathcal{B} \subset 0_\Lambda$, so that in particular $\mathcal{A} + (\mathcal{A}' \wedge \mathcal{B}) + \mathcal{B}' \neq \mathcal{A} + \mathcal{B}$. This "discrete" example can be tweaked to a "continuous" one, just like it was done in Example 3.14.

Remark 3.29. One would call $\mathcal{X}$ satisfying the relations stipulated by (i) a two-sided complement of $(\mathcal{A}, \mathcal{B})$ in $(\mathcal{A}', \mathcal{B}')$. Unlike the usual "one-sided" complement, it is always unique, if it exists. However, by Example 3.28, the "existence of one-sided complements on both sides", i.e. what is the starting assumption of (ii), does not ensure the existence of a two-sided complement (which is (ii)(a)).

Proof. (i). Suppose the two relations are also satisfied by a $\mathcal{Y} \in \Lambda$ in lieu of $\mathcal{X}$. Then $\mathcal{Y} \subset \mathcal{B} = \mathcal{B}' + \mathcal{X}$ and $\mathcal{Y} \subset \mathcal{A}' = \mathcal{A} + \mathcal{X}$; hence $\mathcal{Y} \subset (\mathcal{B}' + \mathcal{X}) \wedge (\mathcal{A} + \mathcal{X})$. But $\mathcal{B}'$ is independent of $\mathcal{A}'$, and $\mathcal{A}' = \mathcal{A} + \mathcal{X}$; hence $\mathcal{B}'$, $\mathcal{A}$ and $\mathcal{X}$ are independent, so Corollary 3.8(iii) entails that $(\mathcal{B}' + \mathcal{X}) \wedge (\mathcal{A} + \mathcal{X}) = \mathcal{X}$. Thus $\mathcal{Y} \subset \mathcal{X}$ and by symmetry $\mathcal{X} \subset \mathcal{Y}$, also; hence $\mathcal{X} = \mathcal{Y}$. If $\mathcal{X}$ satisfies the relations, then they are also a fortiori satisfied by $\mathcal{A}' \wedge \mathcal{B}$; by uniqueness $\mathcal{X} = \mathcal{A}' \wedge \mathcal{B}$.

(ii). Suppose (a) holds. Then by (i) $\mathcal{Z} = \mathcal{A}' \wedge \mathcal{B}$ and (b)-(c)-(d) follow at once. To see (e), let $B' \in \mathcal{B}'$ and $Z \in \mathcal{Z}$. Then a.s. $\mathbb{P}[B' \cap Z|\mathcal{A}'] = \mathbb{P}[B' \cap Z|\mathcal{A} \vee \mathcal{Z}] = \mathbb{1}_Z\mathbb{P}[B'|\mathcal{A} \vee \mathcal{Z}] = \mathbb{1}_Z\mathbb{P}(B') \in \mathcal{Z}/\mathcal{B}_{[-\infty,\infty]} \subset \mathcal{B}/\mathcal{B}_{[-\infty,\infty]}$. The general case obtains by a π/λ-argument and then the second part by symmetry. Conversely, if any of (b)-(c)-(d) obtains, then it is straightforward to check that one can take $\mathcal{Z} = \mathcal{A}' \wedge \mathcal{B}$ in (a) (of course by (i) there is no other choice for $\mathcal{Z}$). Finally we verify that (e) implies $\mathcal{X} \subset \mathcal{A} \vee (\mathcal{A}' \wedge \mathcal{B})$ (by (c) and symmetry it will be enough). The assumption entails that $\mathbb{P}[B|\mathcal{A}'] = \mathbb{P}[B|\mathcal{A}' \wedge \mathcal{B}]$ a.s. for $B \in \mathcal{B}$. Let $X \in \mathcal{X}$; it will be sufficient to show that a.s. $\mathbb{P}[X|\mathcal{A} \vee (\mathcal{A}' \wedge \mathcal{B})] = \mathbb{1}_X$, and then by a π/λ-argument, that $\mathbb{E}[\mathbb{P}[X|\mathcal{A} \vee (\mathcal{A}' \wedge \mathcal{B})]; A \cap B] = \mathbb{P}(X \cap A \cap B)$ for $A \in \mathcal{A}$, $B \in \mathcal{B}$. Now because $(\mathcal{A}' \wedge \mathcal{B}) \vee \sigma(B) \subset \mathcal{B} \perp\!\!\!\perp \mathcal{A}$, we find indeed that $\mathbb{E}[\mathbb{P}[X|\mathcal{A} \vee (\mathcal{A}' \wedge \mathcal{B})]; A \cap B] = \mathbb{E}[\mathbb{P}[X \cap A|\mathcal{A} \vee (\mathcal{A}' \wedge \mathcal{B})]; B] = \mathbb{E}[\mathbb{P}[B|\mathcal{A} \vee (\mathcal{A}' \wedge \mathcal{B})]; X \cap A] = \mathbb{E}[\mathbb{P}[B|\mathcal{A}' \wedge \mathcal{B}]; X \cap A] = \mathbb{E}[\mathbb{P}[B|\mathcal{A}']; X \cap A] = \mathbb{P}(X \cap A \cap B)$. □

4 An application to the problem of innovation

Let $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{N}}$ be a nonincreasing sequence in Λ and let $\mathcal{G} = (\mathcal{G}_n)_{n\in\mathbb{N}}$ be a nondecreasing sequence in Λ such that $\mathcal{F}_n \vee \mathcal{G}_n = \mathcal{F}_1 \vee \mathcal{G}_1$ for all $n \in \mathbb{N}$. Set $\mathcal{F}_\infty := \wedge_{n\in\mathbb{N}}\mathcal{F}_n$ and $\mathcal{G}_\infty := \vee_{n\in\mathbb{N}}\mathcal{G}_n$, as well as (for convenience) $\mathcal{G}_0 := 0_\Lambda$, $\mathcal{F}_0 := \mathcal{F}_1 \vee \mathcal{G}_1$. We are interested in specifying (equivalent) conditions under which $\mathcal{F}_\infty \vee \mathcal{G}_\infty = \mathcal{F}_0$. We have of course a priori the inclusion $\mathcal{F}_\infty \vee \mathcal{G}_\infty \subset \mathcal{F}_0$.

Remark 4.1. Since $\mathcal{F}_n \vee \mathcal{G}_\infty = \mathcal{F}_0$ for all $n \in \mathbb{N}$, the statement $\mathcal{F}_\infty \vee \mathcal{G}_\infty = \mathcal{F}_0$ is equivalent to $(\wedge_{n\in\mathbb{N}}\mathcal{F}_n) \vee \mathcal{G}_\infty = \wedge_{n\in\mathbb{N}}(\mathcal{F}_n \vee \mathcal{G}_\infty)$, and the conditions of the theorem of [13] apply. For instance, assume (i) $\mathcal{F}_0$ is countably generated up to negligible sets; and (ii) $\mathcal{F}_\infty = 0_\Lambda$. Take a regular version $(\mathbb{P}^\omega_{\mathcal{G}_\infty})_{\omega\in\Omega}$ of the conditional probability on $\mathcal{F}_0$ given $\mathcal{G}_\infty$ [it means that $\mathcal{G}_\infty/\mathcal{B}_{[0,1]} \ni \mathbb{P}^{\cdot}_{\mathcal{G}_\infty}(A) = \mathbb{P}[A|\mathcal{G}_\infty]$ a.s. for all $A \in \mathcal{F}_0$, and $\mathbb{P}^\omega_{\mathcal{G}_\infty}$ is a probability measure on $\mathcal{F}_0$ for each $\omega \in \Omega$]. Then we can write Theorem.e in [13] as $\mathcal{F}_\infty \vee \mathcal{G}_\infty = \mathcal{F}_0$ iff $\mathbb{P}^\omega_{\mathcal{G}_\infty}$ is trivial on $\mathcal{F}_\infty$ a.s. in $\omega \in \Omega$.

We will restrict our attention to the case when there are strong independence properties. A typical example of the type of situation that we have in mind and when the equality $\mathcal{F}_\infty \vee \mathcal{G}_\infty = \mathcal{F}_0$ (nevertheless) fails was the content of Example 1.4 in the introduction.

Example 1.4 continued. With regard to Remark 4.1, note that (in the context of Example 1.4) $\mathcal{G}_\infty = \overline{\sigma}(\{A \in \mathcal{M} : A = -A\})$. Indeed one checks easily that $\sigma(\xi_1\xi_2,$

$\xi_2\xi_3,\dots) \subset \{A \in \mathcal{M} : A = -A\}$. Conversely, if for a $C \in (2^{\{-1,1\}})^{\otimes\mathbb{N}}$, $A = (\xi_1, \xi_1\xi_2, \xi_2\xi_3,\dots)^{-1}(C) = -A$, then $A = (\xi_1, \xi_1\xi_2, \xi_2\xi_3,\dots)^{-1}(C) = (-\xi_1, \xi_1\xi_2, \xi_2\xi_3,\dots)^{-1}(C) = (\xi_1\xi_2, \xi_2\xi_3,\dots)^{-1}(\mathrm{pr}_{2,3,\dots}(C))$; as a consequence, Blackwell's theorem [8, Theorem III.17] shows that $A \in \sigma(\xi_1\xi_2, \xi_2\xi_3,\dots)$, so that also $\sigma(\xi_1\xi_2, \xi_2\xi_3,\dots) \supset \{A \in \mathcal{M} : A = -A\}$. Thus in Remark 4.1 we may take $\mathbb{E}_{\mathbb{P}^{\cdot}_{\mathcal{G}_\infty}}[f] = (f + f \circ (-\mathrm{id}_\Omega))/2$ for $f \in ((2^{\{-1,1\}})^{\otimes\mathbb{N}})/\mathcal{B}_{[0,\infty]}$. For this choice $\mathbb{P}^{\omega}_{\mathcal{G}_\infty}$ is non-trivial on $\mathcal{F}_\infty$ for arbitrary $\omega \in \Omega$ (take, e.g., f equal to the indicator of the event $A_\omega := \{\xi_n = \omega(n)$ for all sufficiently large $n \in \mathbb{N}\}$).

Here is now a general result that motivates the investigation of two-sided complements in Proposition 3.27.

Proposition 4.2. *Let $\mathcal{H} = (\mathcal{H}_n)_{n\in\mathbb{N}}$ be a sequence in Λ such that $\mathcal{F}_n = \mathcal{F}_{n+1} + \mathcal{H}_{n+1}$ and $\mathcal{G}_{n+1} = \mathcal{G}_n + \mathcal{H}_{n+1}$ for all $n \in \mathbb{N}_0$. (One would say that the sequence $\mathcal{H}$ "innovates" $(\mathcal{F}, \mathcal{G})$.) Then $\mathcal{H}_n = \mathcal{G}_n \wedge \mathcal{F}_{n-1}$ for all $n \in \mathbb{N}$, and the following statements are equivalent.*

(i) $\mathcal{F}_\infty \vee \mathcal{G}_\infty = \mathcal{F}_0$.

(ii) $\mathcal{F}_n = \mathcal{F}_\infty \vee [\vee_{k\in\mathbb{N}_{>n}} \mathcal{H}_k]$ *for all* $n \in \mathbb{N}_0$.

(iii) $\mathcal{F}_n = \mathcal{F}_\infty \vee [\vee_{k\in\mathbb{N}_{>n}} \mathcal{H}_k]$ *for some* $n \in \mathbb{N}_0$.

Proof. We have $\mathcal{F}_n + \mathcal{G}_n = \mathcal{F}_{n+1} + \mathcal{G}_{n+1}$ for all $n \in \mathbb{N}_0$. Now the expressions for the $\mathcal{H}_n$, $n \in \mathbb{N}$, follow from Proposition 3.27(i). Note also that $\mathcal{G}_n = \mathcal{H}_1 \vee \cdots \vee \mathcal{H}_n$ for all $n \in \mathbb{N}_0$.

The implication (ii) $\Rightarrow$ (iii) is trivial.

(i) $\Rightarrow$ (ii). The inclusion $\supset$ is clear. Conversely, if $F \in \mathcal{F}_n$, then a.s. $\mathbb{1}_F = \mathbb{P}[F|\mathcal{F}_0] = \mathbb{P}[F|\mathcal{F}_\infty \vee \mathcal{G}_\infty] = \mathbb{P}[F|\mathcal{F}_\infty \vee \mathcal{G}_n \vee [\vee_{k\in\mathbb{N}_{>n}} \mathcal{H}_k]] = \mathbb{P}[F|\mathcal{F}_\infty \vee [\vee_{k\in\mathbb{N}_{>n}} \mathcal{H}_k]]$, since $\mathcal{G}_n \perp\!\!\!\perp \mathcal{F}_n \supset \sigma(F) \vee \mathcal{F}_\infty \vee [\vee_{k\in\mathbb{N}_{>n}} \mathcal{H}_k]$.

(iii) $\Rightarrow$ (i). $\mathcal{F}_\infty \vee \mathcal{G}_\infty = \mathcal{F}_\infty \vee \mathcal{G}_n \vee [\vee_{k\in\mathbb{N}_{>n}} \mathcal{H}_k] = \mathcal{F}_n \vee \mathcal{G}_n = \mathcal{F}_0$. □

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