

# Asymptotics for functionals of powers of a periodogram

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**Abstract** We present large sample properties and conditions for asymptotic normality of linear functionals of powers of the periodogram constructed with the use of tapered data.

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## 1 Introduction

Consider a real-valued measurable zero-mean strictly stationary process  $Y(t)$ ,  $t \in \mathbb{Z}$ , obeying the following assumption.

**Assumption 1.**  $E|Y(t)|^k < \infty$  for all  $k$ , and  $Y(t)$  has (cumulant) spectral densities of orders  $k = 2, 3, \dots$ , that is, there exist the functions  $f_k(\lambda_1, \dots, \lambda_{k-1}) \in L_1(\Lambda^{k-1})$ ,  $\Lambda = (-\pi, \pi]$ ,  $k = 2, 3, \dots$ , such that the cumulant function of order  $k$  is given by

$$c_k(t_1, \dots, t_{k-1}) = \int_{\Lambda^{k-1}} f_k(\lambda_1, \dots, \lambda_{k-1}) e^{i \sum_{j=1}^{k-1} \lambda_j t_j} d\lambda_1 \dots d\lambda_{k-1}.$$

Suppose that we are given the observations  $\{Y(t), t \in K_T\}$ , where  $K_T = \{-T, \dots, T\}$ ,  $T \in \mathbb{Z}$ .

In this paper, we will study large sample properties of the empirical spectral functionals of the form

$$J_{k,T}(\varphi) = \int_{\Lambda} \varphi(\lambda) I_T^k(\lambda) d\lambda \quad (1)$$

for appropriate functions  $\varphi(\lambda)$  with  $\varphi(\lambda) f_2^k(\lambda) \in L_1(\Lambda)$ , where  $I_T^k(\lambda)$  is the  $k$ th power of the periodogram based on the tapered data  $\{h_T(t)Y(t), t \in K_T\}$ , and  $k$  is

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a positive integer. The taper will be of the form  $h_T(t) = h(\frac{t}{T})$  with  $h$  satisfying the following standard assumption.

**Assumption H.** The function  $h(t), t \in R$ , is an even positive function of bounded variation with bounded support:  $h(t) = 0$  for  $|t| > 1$ .

The periodogram corresponding to the tapered data is defined as

$$I_T(\lambda) = \frac{1}{2\pi H_{2,T}(0)} |d_T(\lambda)|^2, \quad \lambda \in \Lambda,$$

where  $d_T(\lambda)$  is the finite Fourier transform based on tapered data:

$$d_T(\lambda) = d_T^h(\lambda) = \sum_{t \in K_T} e^{-i\lambda t} h_T(t) Y(t), \quad \lambda \in \Lambda,$$

$$H_{k,T}(\lambda) = \int_{K_T} h_T^k(t) e^{-i\lambda t} dt,$$

and we suppose that  $H_{2,T}(0) \neq 0$ .

Functionals of the form (1) for  $k = 1$  have been extensively studied in the literature, in particular, due to their applications for parameter estimation in the spectral domain: their behavior as  $T \rightarrow \infty$  is important for establishing asymptotic properties of so-called minimum contrast estimators such as Whittle and Ibragimov estimators (see, e.g., [9, 1], and references therein). The case of the squared periodogram was treated, for example, in [8], with application to a goodness-of-fit statistics, and in [12], with application to minimum contrast estimation.

Asymptotic results for the functionals of the form (1) with general  $k \geq 2$  were studied in [6] and applied to derivation of properties of weighted least squares estimators in the frequency domain and also in [11], with several applications discussed, in particular, a frequency domain goodness-of-fit testing.

In this paper, we derive asymptotic results for functionals (1) with general  $k \geq 2$  in a more general setting, using the tapered data, and under a different set of conditions in comparison with those used in [6] and [11]; in the Gaussian case, we state our results in terms of integrability conditions for the spectral density and weight function. Methods for the proofs are similar to those used in [12] with appropriate modifications required for the more general case under consideration in the present paper.

## 2 Results and discussion

We begin with the following assumptions.

**Assumption 2.** The spectral densities  $f_k(\lambda_1, \dots, \lambda_{k-1}), k = 2, 3, \dots$ , of the stochastic process  $Y(t)$  are bounded and continuous.

**Assumption 3.** The weight function  $\varphi(\lambda)$  is bounded and continuous.

In what follows, we denote the second-order spectral density  $f_2(\lambda)$  simply by  $f(\lambda)$  omitting the subscript 2.

**Theorem 1.** Let Assumptions 1, 2, and H hold, and let the functions  $\varphi, \varphi_1(\lambda), \dots, \varphi_k(\lambda)$  satisfy Assumption 3. Then, as  $T \rightarrow \infty$ ,

$$(1) \quad E J_{k,T}(\varphi) \rightarrow k! \int_{\Lambda} \varphi(\lambda) f^k(\lambda) d\lambda;$$

$$(2) \quad \text{cov}(J_{k,T}(\varphi_1), J_{l,T}(\varphi_2)) \\ \sim \frac{2\pi}{T} e(h) k! l! \left[ \int_{\Lambda} \varphi_1(\lambda) [\overline{\varphi_2}(\lambda) + \overline{\varphi_2}(-\lambda)] f^{k+l}(\lambda) d\lambda \right. \\ \left. + \int_{\Lambda^2} \varphi_1(\lambda_1) \overline{\varphi_2}(\lambda_2) f^{k-1}(\lambda_1) f^{l-1}(\lambda_2) f_4(\lambda_1, -\lambda_1, \lambda_2) d\lambda_1 d\lambda_2 \right],$$

where

$$e(h) = \left\{ \int h^2(t) dt \right\}^{-2} \int h^4(t) dt;$$

$$(3) \quad \text{cum}(J_{m_1,T}(\varphi_1), \dots, J_{m_k,T}(\varphi_k)) = O(T^{1-k}).$$

Suppose now that the process  $Y(t)$  is Gaussian. In this case the above asymptotic results can be stated under the conditions of integrability on the weight function and spectral density.

**Theorem 2.** Let  $Y(t), t \in Z$ , be a Gaussian stationary process with spectral density  $f(\lambda), \lambda \in \Lambda$ , such that  $f(\lambda) \in L_p(\Lambda)$ , and let the functions  $\varphi, \varphi_1, \dots, \varphi_k \in L_q(\Lambda)$ , where  $1 \leq p, q \leq \infty$ . Suppose also that Assumption H holds.

(1) If  $p$  and  $q$  satisfy the relation

$$\frac{1}{q} + k \frac{1}{p} = 1,$$

then, as  $T \rightarrow \infty$ ,

$$E J_{k,T}(\varphi) \rightarrow k! \int_{\Lambda} \varphi(\lambda) f^k(\lambda) d\lambda.$$

(2) If  $p$  and  $q$  satisfy the relation

$$\frac{1}{q} + \frac{k+l}{2} \cdot \frac{1}{p} = \frac{1}{2},$$

then, as  $T \rightarrow \infty$ ,

$$\text{cov}(J_{k,T}(\varphi_1), J_{l,T}(\varphi_2)) \\ \sim \frac{2\pi}{T} e(h) k! l! \int_{\Lambda} \varphi_1(\lambda) [\overline{\varphi_2}(\lambda) + \overline{\varphi_2}(-\lambda)] f^{k+l}(\lambda) d\lambda.$$

(3) If  $p$  and  $q$  satisfy the relation

$$\frac{1}{q} + k \frac{1}{p} = \frac{1}{2},$$

then the cumulants of orders  $r \geq 3$  of the normalized functionals  $J_{k,T}(\varphi_i)$  tend to zero as  $T \rightarrow \infty$ :

$$\text{cum}(T^{1/2} J_{k,T}(\varphi_1), \dots, T^{1/2} J_{k,T}(\varphi_r)) \rightarrow 0.$$

(4) If  $p$  and  $q$  satisfy the relation

$$\frac{1}{q} + \frac{k_1 + \dots + k_r}{r} \cdot \frac{1}{p} = \frac{1}{2},$$

then the cumulant of  $r$ th order,  $r \geq 3$ , of the normalized functionals  $J_{k_i, T}(\varphi_i)$ ,  $i = 1, \dots, r$ , tends to zero as  $T \rightarrow \infty$ :

$$\text{cum}(T^{1/2}J_{k_1, T}(\varphi_1), \dots, T^{1/2}J_{k_r, T}(\varphi_r)) \rightarrow 0.$$

As corollaries of the above theorems, we obtain the next asymptotic normality results.

Let us fix the weight functions  $\varphi_1, \dots, \varphi_m$  and denote

$$J_T = \{J_{k, T}(\varphi_i)\}_{i=1, \dots, m} \quad \text{and} \quad \tilde{J} = \{\tilde{J}_k(\varphi_i)\}_{i=1, \dots, m},$$

where  $\tilde{J}_k(\varphi) = k! \int_{\Lambda} \varphi(\lambda) f^k(\lambda) d\lambda$ .

Let  $\xi = \{\xi_i\}_{i=1, \dots, m}$  be a Gaussian random vector with zero mean and second-order moments

$$\begin{aligned} w_{ij} &= E\xi_i\bar{\xi}_j \\ &= 2\pi e(h)(kk!)^2 \left( \int_{\Lambda} \varphi_i(\lambda) [\bar{\varphi}_j(\lambda) + \bar{\varphi}_j(-\lambda)] f^{2k}(\lambda) d\lambda \right. \\ &\quad \left. + \int_{\Lambda^2} \varphi_i(\lambda_1) \bar{\varphi}_j(\lambda_2) f^{k-1}(\lambda_1) f^{k-1}(\lambda_2) f_4(\lambda_1, -\lambda_1, \lambda_2) d\lambda_1 d\lambda_2 \right). \end{aligned}$$

**Assumption 4.** The spectral density of the second order  $f(\lambda)$ , the weight function  $\varphi(\lambda)$ , and the taper  $h$  are such that  $T^{1/2}(EJ_{k, T}(\varphi) - \tilde{J}_k(\varphi)) \rightarrow 0$ .

**Theorem 3.** Let Assumptions 1, 2, and H hold, and let the functions  $\varphi_i$ ,  $i = 1, \dots, m$ , satisfy Assumption 3. Then

$$T^{1/2}(J_T - EJ_T) \xrightarrow{D} \xi \quad \text{as } T \rightarrow \infty;$$

moreover, if Assumption 4 holds for the functions  $\varphi_i$ ,  $i = 1, \dots, m$ , then

$$T^{1/2}(J_T - \tilde{J}) \xrightarrow{D} \xi \quad \text{as } T \rightarrow \infty.$$

Let  $\zeta = \{\zeta_i\}_{i=1, \dots, m}$  be a Gaussian random vector with zero mean and second-order moments

$$v_{ij} = E\zeta_i\bar{\zeta}_j = 2\pi e(h)(kk!)^2 \int_{\Lambda} \varphi_i(\lambda) [\bar{\varphi}_j(\lambda) + \bar{\varphi}_j(-\lambda)] f^{2k}(\lambda) d\lambda.$$

**Theorem 4.** Let  $Y(t)$ ,  $t \in \mathbb{Z}$ , be a Gaussian stationary process with spectral density  $f(\lambda) \in L_p(\Lambda)$ , and let the functions  $\varphi_1, \dots, \varphi_m \in L_q(\Lambda)$ , where  $1 \leq p, q \leq \infty$ , be such that  $\frac{1}{q} + k\frac{1}{p} = \frac{1}{2}$ . Suppose also that Assumption H holds. Then

$$T^{1/2}(J_T - EJ_T) \xrightarrow{D} \zeta \quad \text{as } T \rightarrow \infty;$$

moreover, if Assumption 4 holds for the functions  $\varphi_i$ ,  $i = 1, \dots, m$ , then

$$T^{1/2}(J_T - \tilde{J}) \xrightarrow{D} \zeta \quad \text{as } T \rightarrow \infty.$$

We next state more general results, the joint asymptotic normality for functionals of different powers of the periodogram.

Consider

$$J_{k_1, \dots, k_m, T} = \{J_{k_i, T}(\varphi_i)\}_{i=1, \dots, m} \quad \text{and} \quad \tilde{J}_{k_1, \dots, k_m} = \{\tilde{J}_{k_i}(\varphi_i)\}_{i=1, \dots, m},$$

where  $\tilde{J}_k(\varphi) = k! \int_{\Lambda} \varphi(\lambda) f^k(\lambda) d\lambda$ .

Let  $\tilde{\xi} = \{\tilde{\xi}_i\}_{i=1, \dots, m}$  be a Gaussian random vector with zero mean and second-order moments

$$\begin{aligned} \tilde{w}_{ij} &= E\tilde{\xi}_i\tilde{\xi}_j \\ &= 2\pi e(h)k_i k_i! k_j k_j! \left( \int_{\Lambda} \varphi_i(\lambda) [\bar{\varphi}_j(\lambda) + \bar{\varphi}_j(-\lambda)] f^{k_i+k_j}(\lambda) d\lambda \right. \\ &\quad \left. + \int_{\Lambda^2} \varphi_i(\lambda_1) \bar{\varphi}_j(\lambda_2) f^{k_i-1}(\lambda_1) f^{k_j-1}(\lambda_2) f_4(\lambda_1, -\lambda_1, \lambda_2) d\lambda_1 d\lambda_2 \right). \end{aligned}$$

**Theorem 5.** *Let Assumptions 1, 2, and H hold, and let the functions  $\varphi_i, i = 1, \dots, m$ , satisfy Assumption 3. Then*

$$T^{1/2}(J_{k_1, \dots, k_m, T} - E J_{k_1, \dots, k_m, T}) \xrightarrow{D} \xi \quad \text{as } T \rightarrow \infty;$$

moreover, if Assumption 4 holds with  $k = k_i, \varphi = \varphi_i, i = 1, \dots, m$ , then

$$T^{1/2}(J_{k_1, \dots, k_m, T} - \tilde{J}_{k_1, \dots, k_m}) \xrightarrow{D} \xi \quad \text{as } T \rightarrow \infty.$$

Let  $\tilde{\zeta} = \{\tilde{\zeta}_i\}_{i=1, \dots, m}$  be a Gaussian random vector with zero mean and second-order moments

$$\tilde{v}_{ij} = E\tilde{\zeta}_i\tilde{\zeta}_j = 2\pi e(h)k_i k_i! k_j k_j! \int_{\Lambda} \varphi_i(\lambda) [\bar{\varphi}_j(\lambda) + \bar{\varphi}_j(-\lambda)] f^{k_i+k_j}(\lambda) d\lambda.$$

**Theorem 6.** *Let  $Y(t), t \in Z$ , be a Gaussian stationary process with a spectral density  $f(\lambda) \in L_p(\Lambda)$ , and let the functions  $\varphi_1, \dots, \varphi_m \in L_q(\Lambda)$ , where  $1 \leq p, q \leq \infty$ , be such that  $\frac{1}{q} + \min\{k_i\} \frac{1}{p} = \frac{1}{2}$ . Suppose also that Assumption H holds. Then*

$$T^{1/2}(J_{k_1, \dots, k_m, T} - E J_{k_1, \dots, k_m, T}) \xrightarrow{D} \tilde{\zeta} \quad \text{as } T \rightarrow \infty;$$

moreover, if Assumption 4 holds with  $k = k_i, \varphi = \varphi_i, i = 1, \dots, m$ , then

$$T^{1/2}(J_{k_1, \dots, k_m, T} - \tilde{J}_{k_1, \dots, k_m}) \xrightarrow{D} \tilde{\zeta} \quad \text{as } T \rightarrow \infty.$$

**Remark 1.** Integrals of nonlinear functions of the periodogram (including, in particular, powers of positive orders of the periodogram) were studied, for example, in [13] for discrete time processes under the assumption of boundedness of the spectral density. In [8], the integral functionals of the squared periodogram were studied for stationary Gaussian series given by the moving-average representation, and the

asymptotic normality result was stated under the particular assumption of summability of the coefficients of the representation and continuity of the derivative of the spectral density. In [6] and [11], the asymptotic results for functionals of powers of the periodogram of general order have been studied under the conditions of summability of cumulants of the process. In this paper, we state the results for discrete-time non-Gaussian processes under the condition of boundedness of spectral densities of all orders (which are supposed to exist), and we also derive the results for Gaussian case under the conditions of integrability of the spectral density and weight function.

**Remark 2.** Conditions on the spectral density under which Assumption 4 will be satisfied can be formulated analogously to the corresponding conditions in [1] for the case where  $h(t) \equiv 1$  and analogously to the conditions in [2] for the general  $h(t)$  of Assumption H.

**Remark 3.** The results on asymptotic properties of the integrals of the powers of the periodogram can be useful for some problems of statistical inference. One possible application is hypothesis testing concerning the form of the spectral density of the process. For example, in [8], a quadratic goodness-of-fit test in the spectral domain was studied for Gaussian processes. Note that the asymptotic normality result for the corresponding test statistic stated in [8] can be also derived from our Theorem 6, that is, under a different set of conditions. More applications of the integrals of the powers of the periodogram for goodness-of-fit testing, peak testing, and assessing model misspecification are presented in [11]. In [6] and [12], the integrals of the squared periodogram were applied for parametric estimation in the spectral domain.

### 3 Proofs

For the proofs of the results of Section 2, we use the technique based the properties of the multidimensional kernels of Fejér type (see, e.g., [5, 1], and references therein) and the Hölder–Young–Brascamp–Lieb inequality (see [3, 4], and references therein; see also [12]). In what follows, we will refer to the latter as the HYBL inequality. The application of these tools leads to very transparent and elegant proofs. We will also use the formula giving expressions for cumulants of products of random variables via products of cumulants of the individual variables (see, e.g., [10, 5]) and the multilinearity property of cumulants. The lines of reasonings are very close to those used in [12].

**Proof of Theorem 1.** Consider

$$\begin{aligned} EI_T^k(\lambda) &= \frac{1}{(2\pi H_{2,T}(0))^k} E[(cum(d_T(\lambda)d_T(-\lambda))^k] \\ &= \frac{1}{(2\pi H_{2,T}(0))^k} E[cum(d_T(\lambda)d_T(-\lambda) \cdots cum(d_T(\lambda)d_T(-\lambda))]. \end{aligned}$$

We apply now the formula for cumulants of products of random variables (see, e.g., [10]); it is convenient to assign the indices to  $\lambda$ s in the following way: we can enumerate all  $\lambda$ s appearing in the above row from 1 to  $2k$ , having in mind that  $\lambda_i$  with

odd indices are simply equal to  $\lambda$ , whereas  $\lambda_i$  with even  $i$  are equal to  $-\lambda$ . Then we can write down the expectation in the following form:

$$\begin{aligned}
 EI_T^k(\lambda) &= \frac{1}{(2\pi H_{2,T}(0))^k} \sum_{\substack{v=(v_1, \dots, v_l) \\ \text{partition of } (1, \dots, 2k)}} \prod_{l=1}^p \text{cum}(d_T(\lambda_i), i \in v_l) \\
 &\quad \times \prod_{i=1}^k \delta(\lambda_{2i-1} - \lambda) \prod_{i=1}^k \delta(\lambda_{2i} + \lambda). \tag{2}
 \end{aligned}$$

The cumulants of the finite Fourier transforms  $d_T(\lambda)$ ,  $\lambda \in \Lambda$ , can be written as follows:

$$\begin{aligned}
 &\text{cum}(d_T(\alpha_1), \dots, d_T(\alpha_k)) \\
 &= \int_{K_T^k} \prod_{i=1}^k h_T(t_i) e^{-i \sum_1^k \alpha_j t_j} c_k(t_1 - t_k, \dots, t_{k-1} - t_k) dt_1 \dots dt_k \\
 &= \int_{\Lambda^{k-1}} f_k(\gamma_1, \dots, \gamma_{k-1}) \\
 &\quad \times \prod_{j=1}^{k-1} H_{1,T}(\gamma_j - \alpha_j) H_{1,T}\left(-\sum_{j=1}^{k-1} \gamma_j - \alpha_k\right) d\gamma_1 \dots d\gamma_{k-1},
 \end{aligned}$$

where

$$H_{1,T}(\lambda) = \int_{K_T} h_T(t) e^{-it\lambda} dt.$$

Correspondingly, we obtain the following formula for the expectation of  $J_{k,T}(\varphi)$ :

$$\begin{aligned}
 EJ_{k,T}(\varphi) &= E \int_{\Lambda} \varphi(\lambda) I_T^k(\lambda) d\lambda \\
 &= \int_{\Lambda} \varphi(\lambda) \frac{1}{(2\pi H_{2,T}(0))^k} \sum_{\substack{v=(v_1, \dots, v_p) \\ \text{partition of } (1, \dots, 2k)}} \int_{\Lambda^{2k-p}} \prod_{i=1}^p f_{|v_i|}(\gamma_j, j \in \tilde{v}_i) \\
 &\quad \times \prod_{j=1}^{2k} H_{1,T}(\gamma_j - \lambda_j) \prod_{l=1}^p \delta\left(\sum_{j \in v_l} \gamma_j\right) \prod_{i=1}^k \delta(\lambda_{2i-1} - \lambda) \prod_{i=1}^k \delta(\lambda_{2i} + \lambda) d\gamma' d\lambda. \tag{3}
 \end{aligned}$$

Here and in similar formulas below, we use the following notation: for a set of natural numbers  $v$ , we denote by  $|v|$  the number of elements in  $v$  and by  $\tilde{v}$  the subset of  $v$  that contains all elements of  $v$  except the last one. Integration in the inner integral in the above formula is understood with respect to  $(2k - p)$ -dimensional vector  $\gamma'$  obtained from the vector  $\gamma = (\gamma_1, \dots, \gamma_{2k})$  due to  $p$  restrictions on the variables  $\gamma_j$ ,  $j = 1, \dots, 2k$ , described by the Kronecker delta functions  $\delta$ .

Now we note that the products  $\prod_{j=1}^k H_{1,T}(\lambda_j)$  in the case where  $\sum_{j=1}^k \lambda_j = 0$  give rise to a class of  $\delta$ -type kernels (or Fejér-type kernels). Namely, if Assumption H

holds and  $H_{k,T}(0) \neq 0$ , then

$$\Phi_{k,T}^h(\lambda_1, \dots, \lambda_{k-1}) := \frac{1}{(2\pi)^{k-1} H_{k,T}(0)} \prod_{j=1}^{k-1} H_{1,T}(\lambda_j) H_{1,T}\left(-\sum_{j=1}^{k-1} \lambda_j\right) \quad (4)$$

is a kernel over  $\Lambda^{k-1}$ , which is an approximate identity for convolution (see, e.g., [7]), and

$$\begin{aligned} \lim_{T \rightarrow \infty} \int_{\Lambda^{k-1}} G(u_1 - v_1, \dots, u_{k-1} - v_{k-1}) \Phi_{k,T}^h(u_1, \dots, u_{k-1}) du_1 \dots du_{k-1} \\ = G(v_1, \dots, v_{k-1}), \end{aligned} \quad (5)$$

provided that the function  $G(\cdot, \dots, \cdot)$  is bounded and continuous at the point  $(v_1, \dots, v_{k-1})$ .

The asymptotic behavior of the right-hand side of (3) can be evaluated basing on the property (5).

Let us first consider the partitions  $\nu$  composed by pairs. For those partitions, when the products of only the cumulants of the form  $cum(d_T(\lambda), d_T(-\lambda))$  appear in (2), we obtain under the integral sign in (3) the terms of the form

$$\begin{aligned} \frac{1}{(2\pi H_{2,T}(0))^k} \left\{ \int f(\gamma) H_{1,T}(\gamma - \lambda) H_{1,T}(-\gamma + \lambda) d\gamma \right\}^k \\ = \left\{ \int f(\gamma) \Phi_{2,T}^h(\gamma - \lambda) d\gamma \right\}^k. \end{aligned}$$

We note that there are  $k!$  such terms, therefore, in the expression for  $E J_{k,T}(\varphi)$ , we have the term

$$k! \int_{\Lambda} \varphi(\lambda) \left\{ \int_{\Lambda} f(\gamma) \Phi_{2,T}^h(\gamma - \lambda) d\gamma \right\}^k d\lambda, \quad (6)$$

and this is the only case where we have  $k$  kernels, and all  $k$  factors  $\frac{1}{2\pi H_{2,T}(0)}$  are used to compose these kernels  $\Phi_{2,T}^h(\cdot)$ .

In all other partitions, we will be able to compose from 1 to  $k - 1$  kernels taking combination of  $H_{1,T}(\cdot)$  with suitable arguments: for each 2nd-order kernel, we will use one of the factors  $\frac{1}{2\pi H_{2,T}(0)}$  from  $\frac{1}{(2\pi H_{2,T}(0))^k}$ ; otherwise, when composing the  $l$ th order kernel with  $l \neq 2$ , we will need the normalizing factor  $\frac{1}{(2\pi)^{l-1} H_{l,T}(0)}$ , and therefore we will modify the factor  $\frac{1}{(2\pi H_{2,T}(0))^k}$  by taking, instead,  $\frac{(2\pi)^{l-1} H_{l,T}(0)}{(2\pi H_{2,T}(0))^k}$ .

So, for those partitions, when we compose kernels of orders, say,  $l_1, \dots, l_r$ , with  $\sum_{i=1}^r l_i = 2k$ , the corresponding integral in (3) will be represented in the form of a generalized convolution of some product of spectral densities of different orders with the product of kernels of orders  $l_1, \dots, l_r$ , and the factor

$$\frac{\prod_{i=1}^r (2\pi)^{l_i-1} H_{l_i,T}(0)}{(2\pi H_{2,T}(0))^k} \quad (7)$$



will be supplied to the integral. For example, in the simplest case where  $p = 1$ , the corresponding term in (3) can be represented as follows:

$$\begin{aligned} & \frac{(2\pi)^{2k-1} H_{2k,T}(0)}{(2\pi H_{2,T}(0))^k} \int_{\Lambda} \varphi(\lambda) \int_{\Lambda^{2k-1}} f_{2k}(\gamma_1, \dots, \gamma_{k-1}) \\ & \times \Phi_{2k,T}^h(\gamma_1 - \lambda_1, \dots, \gamma_{2k-1} - \lambda_{2k-1}) \\ & \times \prod_{i=1}^k \delta(\lambda_{2i-1} - \lambda) \prod_{i=1}^k \delta(\lambda_{2i} + \lambda) \prod_{i=1}^{2k-1} d\gamma_i d\lambda. \end{aligned}$$

Now we take into account the following asymptotics for  $H_{k,T}(0)$ :  $H_{k,T}(0) \sim T H_k(0)$ , where  $H_k(0) = \int h^k(\lambda) d\lambda$ , and conclude that, in the case of  $r$  kernels,  $1 \leq r \leq k - 1$ , the factor (7) is asymptotically of order  $\frac{1}{T^{k-r}}$ ; the corresponding integrals containing these kernels will converge to finite limits under the conditions of the theorem according to (5). Therefore, the expectation is obtained as the limit of (6). This gives statement (1) of the theorem.

Consider

$$\begin{aligned} & cov(J_{k,T}(\varphi_1), J_{l,T}(\varphi_2)) \\ & = \frac{1}{(2\pi H_{2,T}(0))^{k+l}} \\ & \times \int_{\Lambda^2} \varphi_1(\alpha) \overline{\varphi_2}(\beta) cum((d_T(\alpha) d_T(-\alpha))^k, (d_T(\beta) d_T(-\beta))^l) d\alpha d\beta. \end{aligned} \tag{8}$$

The cumulant under the integral sign in (8) according to the formula for calculation of cumulants of products of random variables can be written in the form

$$\sum_{\nu=(\nu_1, \dots, \nu_p)} \prod_{i=1}^p cum(d_T(\mu_j), \mu_j \in \nu_i), \tag{9}$$

where the summation is taken over all indecomposable partitions  $\nu = (\nu_1, \dots, \nu_p)$ ,  $|\nu_j| > 1$ , of the table  $T_2$  with two rows,  $\{\alpha, -\alpha, \dots, \alpha, -\alpha\}$  (of length  $2k$ ) and  $\{\beta, -\beta, \dots, \beta, -\beta\}$  (of length  $2l$ ). For asymptotic analysis of expression (8), we can use the reasonings analogous to those for the case of functionals of squared periodogram in [12], but now, dealing with the tapered case, we need to keep track of normalizing factors for appearing kernels. Again, similarly to the previous consideration of the expectation, we analyze all possible partitions and kernels that can be composed for every particular partition. Let us first consider the terms in (9) that correspond to partitions by pairs, that is,

$$\prod_{i=1}^{k+l} cum(d_T(\mu_i), d_T(\lambda_i)), \tag{10}$$

where  $\mu_i, \lambda_i \in \{\alpha, -\alpha, \beta, -\beta\}$ , and  $\nu = \{(\mu_i, \lambda_i), i = 1, \dots, k + l\}$  forms an indecomposable partition of the table  $T_2$ .

In the case where we have the  $k - 1$  cumulants  $cum(d_T(\alpha), d_T(-\alpha))$  and  $l - 1$  cumulants  $cum(d_T(\beta), d_T(-\beta))$  in the product (10), according to formula (4), we can compose  $k+l-1$  kernels ( $k+l-2$  kernels of order 2 and one of order 4), and the factor before the integral in (8) becomes of the form  $\frac{2\pi H_{4,T}(0)}{(H_{2,T}(0))^2}$ , which is asymptotically of order  $\frac{1}{T}$ . Only the terms of this kind in (9) give the main contribution (of order  $\frac{1}{T}$ ) into the covariance (8); all other terms in (10) produce a smaller-order contribution to the covariance (8). More precisely, in order to describe the asymptotics of the covariance, we have to consider, among the terms in (9), the following ones:

$$\begin{aligned} & (cum(d_T(\alpha), d_T(-\alpha)))^{k-1} (cum(d_T(\beta), d_T(-\beta)))^{l-1} \\ & \times [cum(d_T(\alpha), d_T(\beta))cum(d_T(-\alpha), d_T(-\beta)) \\ & + cum(d_T(\alpha), d_T(-\beta))cum(d_T(-\alpha), d_T(\beta))]. \end{aligned} \tag{11}$$

Their contribution to the covariance is of the form

$$\begin{aligned} & \frac{1}{(2\pi H_{2,T}(0))^{k+l}} \int \int \varphi_1(\alpha) \overline{\varphi_2}(\beta) \\ & \times \left[ \int f(\gamma_1) H_{1,T}(\gamma_1 - \alpha) H_{1,T}(-\gamma_1 + \alpha) d\gamma_1 \right]^{k-1} \\ & \times \left[ \int f(\gamma_2) H_{1,T}(\gamma_2 - \beta) H_{1,T}(-\gamma_2 + \beta) d\gamma_2 \right]^{l-1} \\ & \times \left[ \int f(\gamma_3) H_{1,T}(\gamma_3 - \alpha) H_{1,T}(-\gamma_3 - \beta) d\gamma_3 \right. \\ & \times \int f(\gamma_4) H_{1,T}(\gamma_4 + \alpha) H_{1,T}(-\gamma_4 + \beta) d\gamma_4 \\ & \left. + \int f(\gamma_3) H_{1,T}(\gamma_3 - \alpha) H_{1,T}(-\gamma_3 + \beta) d\gamma_3 \right. \\ & \left. \times \int f(\gamma_4) H_{1,T}(\gamma_4 + \alpha) H_{1,T}(-\gamma_4 - \beta) d\gamma_4 \right] d\alpha d\beta \\ & = \frac{2\pi H_{4,T}(0)}{(H_{2,T}(0))^2} \int \int \varphi_1(\alpha) \overline{\varphi_2}(\beta) \left[ \int f(\gamma_1) \Phi_{2,T}^h(\gamma_1 - \alpha) d\gamma_1 \right]^{k-1} \\ & \times \left[ \int f(\gamma_2) \Phi_{2,T}^h(\gamma_2 - \beta) d\gamma_2 \right]^{l-1} \\ & \times \int \int f(\gamma_3) f(\gamma_4) [\Phi_{4,T}^h(\gamma_3 - \alpha, -\gamma_3 - \beta, \gamma_4 + \alpha) \\ & + \Phi_{4,T}^h(\gamma_3 - \alpha, -\gamma_3 + \beta, \gamma_4 + \alpha)] d\gamma_3 d\gamma_4 d\alpha d\beta. \end{aligned}$$

Taking into account formula (5), we can evaluate the latter as

$$\sim \frac{2\pi}{T} \frac{\int h^4(t) dt}{(\int h^2(t) dt)^2} \int_{\Lambda} \varphi_1(\lambda) [\overline{\varphi_2}(\lambda) + \overline{\varphi_2}(-\lambda)] f^{k+l}(\lambda) d\lambda \quad \text{as } T \rightarrow \infty.$$

We note also that there are  $klk!l!$  terms of the form (11) in (9).

Let us now consider the terms in (9) that correspond to other partitions. We can see that one more possibility is left to compose  $k + l - 1$  kernels, namely, the case of the terms of the form

$$\begin{aligned} & [cum(d_T(\alpha), d_T(-\alpha))]^{k-1} [cum(d_T(\beta), d_T(-\beta))]^{l-1} \\ & \times cum(d_T(\alpha), d_T(-\alpha), d_T(\beta), d_T(-\beta)) \end{aligned}$$

in the sum (9) (there are  $klk!!$  such terms), and the corresponding contribution to the covariance (8) is of the form

$$\begin{aligned} & \frac{1}{(2\pi H_{2,T}(0))^{k+l}} \int \int \varphi_1(\alpha) \overline{\varphi_2}(\beta) \\ & \times \left[ \int f(\gamma_1) H_{1,T}(\gamma_1 - \alpha) H_{1,T}(-\gamma_1 + \alpha) d\gamma_1 \right]^{k-1} \\ & \times \left[ \int f(\gamma_2) H_{1,T}(\gamma_2 - \beta) H_{1,T}(-\gamma_2 + \beta) d\gamma_2 \right]^{l-1} \\ & \times \int \int \int f_4(\mu_1, \mu_2, \mu_3) H_{1,T}(\mu_1 - \alpha) H_{1,T}(\mu_2 + \alpha) \\ & \times H_{1,T}(\mu_3 - \beta) H_1^T \left( -\sum_{i=1}^3 \mu_i + \beta \right) d\mu_1 d\mu_2 d\mu_3 d\alpha d\beta \\ & = \frac{2\pi H_{4,T}(0)}{(H_{2,T}(0))^2} \int \int \varphi_1(\alpha) \overline{\varphi_2}(\beta) \\ & \times \int f(\gamma_1) \Phi_{2,T}^h(\gamma_1 - \alpha) d\gamma_1 \int f(\gamma_2) \Phi_{2,T}^h(\gamma_2 - \beta) d\gamma_2 \\ & \times \int \int \int f_4(\mu_1, \mu_2, \mu_3) \Phi_{4,T}^h(\mu_1 - \alpha, \mu_2 + \alpha, \mu_3 - \beta) d\mu_1 d\mu_2 d\mu_3 d\alpha d\beta \\ & \sim \frac{2\pi}{T} \frac{\int h^4(t) dt}{(\int h^2(t) dt)^2} \int \int \varphi_1(\alpha) \overline{\varphi_2}(\beta) f(\alpha) f(\beta) f_4(\alpha, -\alpha, \beta) d\alpha d\beta \quad \text{as } T \rightarrow \infty. \end{aligned}$$

In all other cases, we can compose less than  $k + l - 1$  kernels; the corresponding integrals will converge to finite limits supplied by the factor of orders not exceeding  $\frac{1}{T^2}$ .

Summarizing the above reasonings, we come to the asymptotics for the covariance as given in statement (2) of the theorem.

We now evaluate the asymptotic behavior of the cumulant of order  $k \geq 3$ :

$$\begin{aligned} & cum(J_{m_1,T}(\varphi_1), \dots, J_{m_k,T}(\varphi_k)) \\ & = \frac{1}{(2\pi H_{2,T}(0))^M} \int_{\Lambda^k} \varphi_1(\alpha_1) \cdots \varphi_k(\alpha_k) \\ & \times cum((d_T(\alpha_1) d_T(-\alpha_1))^{m_1}, \dots, (d_T(\alpha_k) d_T(-\alpha_k))^{m_k}) d\alpha_1 \dots d\alpha_k, \end{aligned}$$

where  $M = \sum_{i=1}^k m_i$ .

The cumulant under the integral sign can be represented as the sum

$$\sum_{v=(v_1, \dots, v_p)} \prod_{i=1}^p cum(d_T(\mu_j), \mu_j \in v_i), \tag{12}$$

where the summation now is taken over all indecomposable partitions  $\nu = (\nu_1, \dots, \nu_p)$ ,  $|\nu_i| > 1$ , of the table  $T_k$  with  $k$  rows  $\{\alpha_i, -\alpha_i, \dots, \alpha_i, -\alpha_i\}$ ,  $i = 1, \dots, k$ , the length of the  $i$ th row being  $2m_i$ . Starting again with consideration of partitions by pairs, we can see that with these partitions we can compose at most  $\sum_{i=1}^k (m_i - 1) + 1 = M - k + 1$  kernels ( $M - k$  kernels  $\Phi_{2,T}^h$  and one kernel  $\Phi_{4,T}^h$ ), and the corresponding integrals will converge to finite limits supplied with the factor  $\frac{(2\pi)^3 H_{4,T}(0)}{(2\pi H_{2,T}(0))^{M-(M-k)}} = \frac{(2\pi)^3 H_{4,T}(0)}{(2\pi H_{2,T}(0))^k}$ , which is asymptotically of order  $\frac{1}{T^{k-1}}$ . With all other partitions, we will be able to compose no more that  $M - k + 1$  kernels; therefore, their contribution to cumulant (12) will be of order less than  $\frac{1}{T^{k-1}}$ . This gives statement (3) of the theorem.  $\square$

**Proof of Theorem 2.** We use the same calculations as those in the proof of Theorem 1, but to analyze the limit behavior of the integrals representing the cumulants, we will appeal to the HYBL inequality (see [3, 4, 12]). The reasonings follow the same lines as in [12], so here we just point out the key steps.

For the Gaussian case, we will have only partitions by pairs in (3):

$$\begin{aligned}
 E J_{k,T}(\varphi) &= E \int_{\Lambda} \varphi(\lambda) I_T^k(\lambda) d\lambda \\
 &= \int_{\Lambda} \varphi(\lambda) \frac{1}{(2\pi H_{2,T}(0))^k} \sum_{\substack{\nu=(\nu_1, \dots, \nu_k), |\nu_k|=2, \\ \text{partition of } (1, \dots, 2k)}} \int_{\Lambda^k} \prod_{i=1}^k f(\gamma_j, j \in \tilde{\nu}_i) \\
 &\quad \times \prod_{j=1}^{2k} H_{1,T}(\gamma_j - \lambda_j) \prod_{l=1}^k \delta\left(\sum_{j \in \nu_l} \gamma_j\right) \prod_{i=1}^k \delta(\lambda_{2i-1} - \lambda) \prod_{i=1}^k \delta(\lambda_{2i} + \lambda) d\gamma' d\lambda.
 \end{aligned} \tag{13}$$

We consider separately the term (6):

$$\begin{aligned}
 &k! \int_{\Lambda} \varphi(\lambda) \left\{ \int_{\Lambda} f(\gamma) \Phi_{2,T}^h(\gamma - \lambda) d\gamma \right\}^k d\lambda \\
 &= k! \int_{\Lambda^k} \left[ \int_{\Lambda} \varphi(\lambda) \prod_{j=1}^k f(\gamma_j - \lambda) d\lambda \right] \prod_{j=1}^k \Phi_{2,T}^h(\gamma_j) \prod_{j=1}^k d\gamma_j.
 \end{aligned}$$

Note that the convergence to the finite limit  $k! \int_{\Lambda} \varphi(\lambda) f^k(\lambda) d\lambda$  will be assured if we assume the conditions for statement (1) of the theorem.

Now consider the remaining terms: we have the integrals over  $\Lambda^{k+1}$  with integrands composed by products of the functions  $\varphi$  with  $k$  functions  $f$  and  $2k$  functions  $H_{1,T}$  with some linear relations between the arguments of these functions; these integrals are supplied with the factor  $\frac{1}{(2\pi H_{2,T}(0))^k}$ .

Applying the HYBL inequality, we can bound each such integral by the expression

$$\frac{1}{(2\pi H_{2,T}(0))^k} \text{const} \|\varphi\|_q \|f\|_p^k \|H_{1,T}\|_r^{2k}, \tag{14}$$

provided that  $\varphi(\lambda) \in L_q(\Lambda)$ ,  $f(\lambda) \in L_p(\Lambda)$ , and  $H_{1,T}(\lambda) \in L_r(\Lambda)$  with

$$\frac{1}{q} + k\frac{1}{p} + 2k\frac{1}{r} = k + 1.$$

If we choose  $r = 2$  and take into account that, under Assumption H, we have  $\|H_{1,T}\|_r \leq CT^{1-1/r}$  and  $H_{1,T}(0) \sim T$ , then from (14) we arrive at the bound  $\text{const}\|\varphi\|_q\|f\|_p^k$  as  $T \rightarrow \infty$ , with the restrictions on  $p$  and  $q$  as in statement (1) of the theorem. From this point we can repeat the same arguments as in [12] to show that, in fact, this bound can be strengthened to  $o(1)$  as  $T \rightarrow \infty$ .

The similar reasonings are applied to derive statements (2)–(4) of Theorem 2.  $\square$

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