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Spatial quadratic variations for the solution to a stochastic partial differential equation with elliptic divergence form operator

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Abstract We introduce a stochastic partial differential equation (SPDE) with elliptic operator in divergence form, with measurable and bounded coefficients and driven by space-time white noise. Such SPDEs could be used in mathematical modelling of diffusion phenomena in medium consisting of different kinds of materials and undergoing stochastic perturbations. We characterize the solution and, using the Stein–Malliavin calculus, we prove that the sequence of its recentered and renormalized spatial quadratic variations satisfies an almost sure central limit theorem. Particular focus is given to the interesting case where the coefficients of the operator are piecewise constant.

Keywords Stochastic partial differential equations, divergence form, piecewise constant coefficients, fundamental solution, Stein-Malliavin calculus, almost sure central limit theorem

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1 Introduction

Many diffusion phenomena in various fields of real life are modelled by the following type of partial differential equations (PDEs)

$$\frac{\partial u(t,x)}{\partial t} = \mathcal{L}u(t,x), \tag{1}$$

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where $\mathcal{L}$ is the elliptic divergence form operator defined by

$$\mathcal{L} = \frac{1}{r(x)}\frac{d}{dx}\left(R(x)\frac{d}{dx}\right), \tag{2}$$

R and r are two measurable and bounded functions defined on $\mathbb{R}$ and satisfying

$$\mu_1 \leq R(x) \quad \text{and} \quad \mu_2 \leq r(x) \quad \text{for all } x \in \mathbb{R}$$

where μ_1 and μ_2 are two strictly positive real constants, and $\frac{d}{dx}$ denotes the derivative in the distributional sense. More information on PDEs of the type (1) and their applications can be found, e.g., in [5, 15, 19] and references therein. One interesting example of such PDEs is the one defined by

$$\mathcal{L}_p = \frac{1}{2\rho(x)}\frac{d}{dx}\left(\rho(x)A(x)\frac{d}{dx}\right), \tag{3}$$

$$A(x) = a_1\mathbf{1}_{\{x\leq 0\}} + a_2\mathbf{1}_{\{0<x\}} \quad \text{and} \quad \rho(x) = \rho_1\mathbf{1}_{\{x\leq 0\}} + \rho_2\mathbf{1}_{\{0<x\}}, \tag{4}$$

a_i, ρ_i $(i = 1, 2)$ are strictly positive constants. Operators of the kind (3) are the infinitesimal generators of diffusion processes that have been widely studied in literature (see [7, 12] and references therein). The discontinuity of the coefficients A and ρ reflects the heterogeneity of the media in which the modelled process under study propagates.

In the present paper, we introduce a stochastic partial differential equation (SPDE), that can be considered as a stochastic counterpart of PDE (1). More specifically, we consider

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} &= \mathcal{L}u(t,x) + \dot{W}(t,x); \quad t > 0, x \in \mathbb{R}, \\ u(0,.) &:= 0, \end{cases} \tag{5}$$

where $\mathcal{L}$ is defined by (2) and $\dot{W}$ denotes the formal derivative of a space-time white noise. That is, W is a centered Gaussian field $W = \{W(t, C); t \in [0, T], C \in B_b(\mathbb{R})\}$ with covariance

$$\mathbb{E}\big(W(t,C)W(s,D)\big) = (t \wedge s)\lambda(C \cap D), \tag{6}$$

where λ denotes the Lebesgue measure and $B_b(\mathbb{R})$ is the set of λ-bounded Borel subsets of $\mathbb{R}$. So W behaves as a Wiener process both in time and in space. The solution to Equation (5) is a random field $\{u(t, x), t \geq 0, x \in \mathbb{R}\}$, where t represents the time variable and x is the space variable. In the particular case where the functions r and R are constants $r := 2$ and $R := 1$, the operator $\mathcal{L}$ is reduced to $\frac{1}{2}\frac{\partial^2}{\partial x^2}$. So Equation (5) also represents a natural extension of the stochastic heat equation driven by the space-time white noise, which has been widely studied in the literature (see [22] and the references therein). This can be considered as an important motivation for the investigation of such equation's solution.

This paper has a twofold objective: the first is to lay the first milestone towards the investigation of the stochastic process solution to (5). We prove its existence, and we investigate its spatial quadratic variation. In fact, the study of quadratic variation is motivated by its numerous applications in many fields. For example, in the estimation theory, the analysis of the asymptotic behaviour of the quadratic variations of self-similar processes play an important role in the construction of consistent estimators for the self-similarity parameter (see, e.g., [24] and references therein). In stochastic analysis, quadratic variations are as well one of the main tools used to characterize the semi-martingale property for some mixed Gaussian processes (see, e.g., [10, 14, 29]). Examples of applications of quadratic variation investigation include also the theory of the Itô stochastic calculus with respect to martingales [21] and mathematical finance [3]. We refer to the monograph [22] for a more complete exposition on variations of stochastic processes in general, and of solutions to certain SPDEs in particular. In this paper, under some conditions on the fundamental solution to PDE (1), we fix t and study the limit behaviour in distribution of the sequence $(\sum_{j=0}^{N-1}(u(t,\frac{j+1}{N})-u(t,\frac{j}{N}))^2)_{N\geq 1}$. More precisely, using some elements of the Stein–Malliavin calculus, we show that, after recentralization and renormalization, the above sequence satisfies an Almost Sure Central Limit Theorem (in short: ASCLT). Similar study has been done in the case of stochastic heat equation (see [20, 23]) and also in the case of stochastic wave equation (see [11]). But no similar study has been carried out on SPDEs (5). For more information on ASCLT, see [2] and references therein. The second objective of this paper is to make a further study of the SPDE defined by

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} & = & \mathcal{L}_p u(t,x) + \dot{W}(t,x); \quad t>0, x\in\mathbb{R}, \\ u(0,.) & := & 0, \end{cases} \tag{7}$$

where $\mathcal{L}_p$ is the operator defined by (3). We note that Equation (7) is a particular case of (5), and it could be a good model for diffusion phenomena in a medium consisting of two kind of materials, undergoing stochastic perturbations. Equation (7) has been introduced in [30] where the authors proved the existence of the solution and they presented explicit expressions of its covariance and variance functions. Some regularity properties of the solution sample paths have also been analyzed. In [31], Zili and Zougar expanded the quartic variations in time and the quadratic variations in space of the solution to Equation (7). Both expansions allowed them to deduce an estimation method of the parameters a_1 and a_2 appearing in (4). We make here another step in the study of SPDE (7) by showing that its solution satisfies all conditions under which we can use the ASCLT. In addition to the Stein–Malliavin calculus, our proofs require many integration techniques, calculation, and analysis tools.

The paper is organized as follows. In the next section, we prove the existence of the mild solution to Equation (5) and we give some characterizations of its spatial increments. In Section 3, using some elements of the Stein–Malliavin theory, we establish an almost sure central limit theorem which applies to the solution to SPDE (5) under some conditions on the fundamental solution associated to the operator $\mathcal{L}$. The last section focuses on a further investigation of the solution to SPDE (7). In which case, we show that the ASCLT is statisfied.

2 Existence and some characteristics of the solution

The notion of solution to (5) is defined in the mild sense. We call a mild solution to (5) the stochastic process

$$u(t,x) = \int_0^t \int_{\mathbb{R}} G(t-s,x,y)W(ds,dy), \quad t \in [0,T], x \in \mathbb{R}, \tag{8}$$

where W is the Gaussian noise with covariance given by (53), G is the fundamental solution of the operator $\mathcal{L}$ and the integral in (8) is the Wiener integral with respect to the Gaussian noise W. The existence and many properties of the fundamental solution G of the operator $\mathcal{L}$ have been obtained in many papers (see, e.g., [13] and [19]).

Remark 1. It is well known (see, e.g., [25]) that the mild solution to (5) exists when the Wiener integral (8) is well-defined and this happens when the function $(s,y) \longmapsto G(t-s,x,y)$ belongs to $\mathcal{H}_0 = L^2([0,T]\times\mathbb{R})$, the canonical Hilbert space associated with the Gaussian process W. In fact, $\mathcal{H}_0$ is none other than the closure of the linear span generated by the indicator functions $\mathbf{1}_{[0,t]\times C}$, $t \in [0,T]$, $C \in \mathcal{B}_b(\mathbb{R})$, with respect to the inner product

$$<1_{[0,t]\times C}, 1_{[0,s]\times D}>_{\mathcal{H}_0} = (t \wedge s)\lambda(C \cap D).$$

Moreover, the process $(u(t,x), t \in [0,T], x \in \mathbb{R})$, when it exists, is a centered Gaussian process.

The following proposition deals with the existence of the mild solution to Equation (5).

Proposition 1. *The centered Gaussian process* $(u(t,x), t \in [0,T], x \in \mathbb{R})$ *defined by* (8)*, as a solution to Equation* (5)*, exists and satisfies*

$$\sup_{t\in[0,T],x\in\mathbb{R}} \mathbb{E}\big(u(t,x)^2\big) < +\infty.$$

Proof. The existence and some bounds of the fundamental solution to PDE (1) have been established in [1] and [9]. In particular, it has been proved that there exist positive constants C_1 and C_2 such that

$$G(t,x,y) \le \frac{C_1}{\sqrt{2\pi t}} \exp\bigg(-\frac{C_2(x-y)^2}{t}\bigg), \tag{9}$$

for any $t \in [0,T]$ and $(x,y) \in \mathbb{R}^2$. Thus,

$$\begin{aligned}
\int_0^t \int_{\mathbb{R}} G^2(t-s,x,y)dyds &\le \int_0^t \int_{\mathbb{R}} \frac{C_1^2}{2\pi(t-s)} \exp\bigg(-\frac{2C_2(x-y)^2}{t-s}\bigg)dyds \\
&\le C_3 \int_0^t \frac{1}{\sqrt{t-s}} ds \\
&\le C_4\sqrt{T},
\end{aligned}$$

where C_3 and C_4 denote two strictly positive constants. This with Remark 1 and Wiener's isometry allow us to get the existence of the mild solution to (5) and to show that

$$\mathbb{E}\big(u(t,x)^2\big) = \int_0^t \int_{\mathbb{R}} G^2(t-s,x,y)dyds \le C_4\sqrt{T},$$

for every $t \in [0,T]$ and $x \in \mathbb{R}$. □

Now we consider an interval I in $\mathbb{R}$ and denote

$$\Delta_h G(u,x,z) = G(u,x+h,z) - G(u,x,z)$$

and

$$\|\Delta_h G(t-.,x,.)\|^2_{L^2([0,t]\times\mathbb{R})} = \int_0^t \int_{\mathbb{R}} \big(\Delta_h G(t-\sigma,x,y)\big)^2\, d\sigma\, dy$$

for every $u,t \in (0,T]$, $h>0$ and $x,z \in I$. We also consider the conditions:

$H_1(I)$: $\forall t \in (0,T],\ \exists\, C_5 > 0;\ \forall (x,y) \in I^2;\ y > x,$

$$C_5(y-x) \le \|\Delta_{y-x} G(t-.,x,.)\|^2_{L^2((0,t)\times\mathbb{R})}.$$

$H_2(I)$: $\forall t \in (0,T],\ \exists\, C_6 > 0;\ \forall (x,y) \in I^2;\ y > x,$

$$\|\Delta_{y-x} G(t-.,x,.)\|^2_{L^2((0,t)\times\mathbb{R})} \le C_6(y-x).$$

$H_3(I)$: $\forall t \in (0,T], \exists\, C_7 > 0;\ \forall h > 0,\ \forall (x,y) \in I^2,$

$$\int_0^t \int_{\mathbb{R}} \Delta_h G(t-s,x,z)\Delta_h G(t-s,y,z)\, ds\, dz \leqslant C_7\ h^2.$$

The following lemma will play an important role in this paper.

Lemma 1. *Let u be the mild solution to Equation (5).*

1. *If Condition $H_1(I)$ is satisfied then, for every $t > 0$, there exists a positive constant C_8 such that*

$$\forall x,y \in I, \quad C_8\,|y-x| \le \mathbb{E}\big(u(t,y)-u(t,x)\big)^2. \tag{10}$$

2. *If Condition $H_2(I)$ is satisfied then, for every $t > 0$, there exists a positive constant C_9 such that*

$$\forall x,y \in I, \quad \mathbb{E}\big(u(t,y)-u(t,x)\big)^2 \leqslant C_9\,|y-x|. \tag{11}$$

3. *If Condition $H_3(I)$ is satisfied then, for every $t > 0$, $\forall h > 0$,*

$$\forall\, x,y \in I, \quad \mathbb{E}\Big(\big(u(t,x+h)-u(t,x)\big)\big(u(t,y+h)-u(t,y)\big)\Big) \leqslant C_7 h^2. \tag{12}$$

Proof. We first note that if $x = y$, then Inequalities (10) and (11) are trivial. We also note that the proofs in the cases $x > y$ and $y > x$ are similar. So we consider only the case $y > x$. Using Wiener's isometry we get

$$\begin{aligned}
&\mathbb{E}\big(u(t,y)-u(t,x)\big)^2\\
=\;& \mathbb{E}\Big(\int_{(0,t)\times\mathbb{R}} G(t-u,y,z)W(du,dz) - \int_{(0,t)\times\mathbb{R}} G(t-u,x,z)W(du,dz)\Big)^2\\
=\;& \mathbb{E}\Big(\int_{(0,t)\times\mathbb{R}} \big(G(t-u,y,z)-G(t-u,x,z)\big)W(du,dz)\Big)^2\\
=\;& \int_{(0,t)\times\mathbb{R}} \big(G(t-u,y,z)-G(t-u,x,z)\big)^2 du\,dz\\
=\;& \big\|\Delta_{y-x}G(t-.,x,.)\big\|^2_{L^2([0,t]\times\mathbb{R})}.
\end{aligned} \tag{13}$$

Equality (13) and Condition $H_1(I)$ [respectively $H_2(I)$] allow us to get the two first assertions in Lemma 1.

As for the third one, using again Wiener's isometry we get

$$\begin{aligned}
&\mathbb{E}\Big(\big(u(t,x+h)-u(t,x)\big)\big(u(t,y+h)-u(t,y)\big)\Big)\\
=\;& \mathbb{E}\Big(\int_{(0,t)\times\mathbb{R}} \Delta_h G(t-u,x,z)W(du,dz)\\
&\times \int_{(0,t)\times\mathbb{R}} \Delta_h G(t-u,y,z)W(du,dz)\Big)\\
=\;& \int_{(0,t)\times\mathbb{R}} \Delta_h G(t-u,x,z)\,\Delta_h G(t-u,y,z)\,du\,dz.
\end{aligned}$$

Using Condition $H_3(I)$ the proof of the third assertion in Lemma 1 is achieved. □

From Assertion 2 in Lemma 1 and by Kolmogorov's criterion of continuity, we easily get the following corollary.

Corollary 1. *Let u be the mild solution to (5). If Condition $H_2(I)$ is satisfied, then, for every $t \in [0,T]$, the process $(u(t,x))_{x\in I}$ is Hölder continuous of order γ with $0<\gamma<\frac{1}{2}$.*

Remark 2. From Corollary 1, under Condition $H_2(I)$, the process u being a solution to (5) keeps the same Hölder regularity in space as the solution to the standard stochastic heat equation driven by a time-space white noise (see [20] and references therein).

3 Almost sure central limit theorem

Let us start this section with the following definition.

Definition 1. Let $(G_N)_{N\geq 1}$ be a sequence of real-valued random variables defined on a common probability space $(\Omega,\mathcal{F},\mathbb{P})$. We say that the sequence $(G_N)_{N\geq 1}$ satisfies an almost sure central limit theorem (ASCLT), if, almost surely, for every bounded and continuous function $\varphi:\mathbb{R}\to\mathbb{R}$, we have:

$$\frac{1}{logN}\sum_{i=1}^{N}\frac{\varphi(G_i)}{i}\longrightarrow\mathbb{E}\big(\varphi(\mathcal{Z})\big)\quad \text{as } N\longrightarrow\infty,$$

where $\mathcal{Z}$ is an $\mathcal{N}(0,1)$ random variable.

For fixed $t \in (0,T]$, we consider the Gaussian process $(u(t,x))_{x\in[0,1]}$ being the mild solution to Equation (5). We also consider the partition $0 = x_0 < x_1 < \cdots < x_N = 1$ of the interval $[0,1]$ defined by $x_i = \frac{i}{N}$ for every $i = 0, 1, \ldots, N$. We define the centered re-normalized quadratic variation statistic in the following way:

$$V_N = \sum_{i=0}^{N-1} \left[\frac{(u(t,x_{i+1}) - u(t,x_i))^2}{\mathbb{E}(u(t,x_{i+1}) - u(t,x_i))^2} - 1 \right] \quad \text{and} \quad \tilde{V}_N = \frac{1}{\sqrt{2N}} V_N. \tag{14}$$

The aim of this section is to show that the sequence $(\tilde{V}_N)_{N\geq 1}$ satisfies the ASCLT. Let us first recall briefly some basic elements of the Stein–Malliavin theory (see [16]) that will be useful in our proof.

3.1 Elements of the Stein–Malliavin theory

Consider a real separable Hilbert space $(\mathcal{H}, <.,.>_{\mathcal{H}})$ and an isonormal Gaussian process $(B(\varphi), \varphi \in \mathcal{H})$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which is a centered Gaussian family of random variables such that

$$\mathbb{E}\big(B(\varphi), B(\psi)\big) = <\varphi, \psi>_{\mathcal{H}},$$

for every $\varphi, \psi \in \mathcal{H}$. For $q \geq 1$, let $\mathcal{H}^{\otimes q}$ be the qth tensor product of $\mathcal{H}$ and denote $\mathcal{H}^{\odot q}$ the associated qth symmetric tensor product.

Denote by I_q the qth multiple stochastic integral with respect to B. This I_q is actually an isometry between the Hilbert space $\mathcal{H}^{\odot q}$ equipped with the scaled norm $\frac{1}{\sqrt{q!}} \|.\|_{\mathcal{H}^{\otimes q}}$ and the Wiener chaos of order q, which is defined as the closed linear span of the random variables $H_q(B(\varphi))$, where $\varphi \in \mathcal{H}$, $\|\varphi\|_{\mathcal{H}} = 1$ and H_q is the Hermite polynomial of degree $q \geqslant 1$ defined by

$$H_q(x) = \frac{(-1)^q}{q!} \exp\left(\frac{x^2}{2}\right) \frac{d^q}{dx^q}\left(\exp\left(-\frac{x^2}{2}\right)\right), \quad x \in \mathbb{R}. \tag{15}$$

The isometry of multiple integrals can be written as follows: for $p, q \geqslant 1$, $f \in \mathcal{H}^{\otimes p}$ and $g \in \mathcal{H}^{\otimes q}$,

$$\mathbb{E}\big(I_p(f) I_q(g)\big) = \begin{cases} q! <\hat{f}, \hat{g}>_{\mathcal{H}^{\otimes q}} & \text{if } p = q \\ 0 & \text{otherwise.} \end{cases} \tag{16}$$

It holds that

$$I_q(f) = I_q(\hat{f}),$$

where $\hat{f}$ denotes the canonical symmetrization of f defined by

$$\hat{f}(x_1, \ldots, x_q) = \frac{1}{q!} \sum_{\sigma \in S_q} f(x_{\sigma(1)}, \ldots, x_{\sigma(q)}), \tag{17}$$

where the sum runs over all permutations σ of $\{1, \ldots, q\}$.

We recall that any square-integrable random variable F, which is measurable with respect to the σ-algebra generated by B, can be expanded into an orthogonal sum of multiple stochastic integrals:

$$F = \mathbb{E}(F) + \sum_{q=1}^{\infty} I_q(f_q), \tag{18}$$

where the series converges in the $L^2(\Omega)$-sense and the kernels f_q, belonging to $\mathcal{H}^{\odot q}$, are uniquely determined by F.

Consider now the class of smooth random variables F that can be written in the form

$$F = g\big(B(\varphi_1), \dots, B(\varphi_n)\big), \tag{19}$$

where $n \geqslant 1$, $g : \mathbb{R}^n \longmapsto \mathbb{R}$ is a $\mathcal{C}^\infty$-function with compact support and $\varphi_1, \dots, \varphi_n \in \mathcal{H}$. The Malliavin derivative of a smooth random variable F of the form (19) is the $\mathcal{H}$-valued random variable given by

$$DF = \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}\big(B(\varphi_1), \dots, B(\varphi_n)\big)\, \varphi_i. \tag{20}$$

The following formula for multiplication of Wiener chaos integrals of any orders p, q will play a basic role in the next section. For any symmetric integrands $f \in \mathcal{H}^{\odot p}$ and $g \in \mathcal{H}^{\odot p}$, we have

$$I_p(f) I_q(g) = \sum_{r=0}^{p\wedge q} r! \binom{p}{r}\binom{q}{r} I_{p+q-2r}(f \otimes_r g), \tag{21}$$

where, in the particular case when $\mathcal{H} = L^2([0,T])$, for $r = 1, \dots, p \wedge q$, the rth contraction $f \otimes_r g$ is the element of $\mathcal{H}^{\otimes(p+q-2r)}$ defined by

$$\begin{aligned} &(f \otimes_r g)(s_1, \dots, s_{p-r}, t_1, \dots, t_{q-r}) \\ &= \int_{[0,T]^r} du_1 \dots du_r\, f(s_1, \dots, s_{p-r}, u_1, \dots, u_r)\, g(t_1, \dots, t_{q-r}, u_1, \dots, u_r). \end{aligned} \tag{22}$$

The following theorem gives a description of the normal approximation of multiple stochastic integrals. We refer to [16–18] and references therein for the proof.

Theorem 1. *Fix $q \geqslant 1$. Assume that $(G_N)_{N\geqslant 1} := (\mathrm{I}_q(g_N))_{N\geqslant 1}$ with $g_N \in \mathcal{H}^{\odot q}$ is a sequence of random variables belonging to the qth Wiener chaos such that*

$$\lim_{N\to\infty} \mathbb{E}\big(G_N^2\big) = \sigma^2.$$

Hence, G_N converges in law to $\mathcal{Z} \sim \mathcal{N}(0,1)$ if and only if

$$\lim_{N\to\infty} \|DG_N\|_{\mathcal{H}}^2 = q\sigma^2.$$

Furthermore, if we denote by d one of the metrics on the space of probability measures on $\mathbb{R}$, including the Kolmogorov, Wasserstein and Total Variation measures, then for N large enough:

$$d\big(G_N, \mathcal{N}(0,1)\big) \leqslant C\Big(\sqrt{\mathbf{Var}\big(\|DG_N\|_{\mathcal{H}}^2\big)} + \sqrt{\mathbb{E}\big(\|DG_N\|_{\mathcal{H}}^2\big) - q\sigma^2}\Big).$$

The following theorem has been introduced in [4]. It gives a sufficient condition for extending Theorem 1 to an ASCLT for multiple stochastic integrals.

Theorem 2. *Fix $q \geqslant 2$, and let $(G_N)_{N\geqslant 1}$ be a sequence of random variables defined by*

$$G_N := \big(\mathrm{I}_q(g_N)\big)_{N\geqslant 1}; \quad g_N \in \mathcal{H}^{\odot q}.$$

Suppose that:

1. *For every $N \geqslant 1$, $\mathbb{E}(G_N^2) = 1$.*
2. *For every $r = 1, \dots, q-1$, $\lim_{N\to\infty} \|g_N \otimes_r g_N\|^2_{\mathcal{H}^{\otimes 2(q-r)}} = 0$.*
3. *For every $r = 1, \dots, q-1$, $\displaystyle\sum_{N\geqslant 2} \frac{1}{N log^2 N} \sum_{l=1}^{N} \frac{1}{l} \|g_l \otimes_r g_l\|^2_{\mathcal{H}^{\otimes 2(q-r)}} < \infty$.*
4. $\displaystyle\sum_{N\geqslant 2} \frac{1}{N log^3 N} \sum_{i,j=1}^{N} \frac{|\mathbb{E}(G_i G_j)|}{ij} < \infty$.

Then, the sequence $(G_N)_{N\geq 1}$ satisfies an ASCLT.

We finish this section with the following useful reduction lemma. For its proof see Lemma 2.2 in [2].

Lemma 2. *Consider a real-valued sequence $(a_n)_{n\geq 1}$ converging to $a_\infty \neq 0$. Consider also a sequence of real valued random variables $(G_n)_{n\geq 1}$. Then the sequence $(G_n)_{n\geq 1}$ satisfies an ASCLT if, and only if, $(a_n G_n)_{n\geq 1}$ does.*

3.2 Limiting behavior of the re-normalized quadratic variation of the spatial solution process

For fixed $t \in (0, T]$, we denote by $\mathcal{H}$ the canonical Hilbert space associated to the Gaussian process $(u(t,x))_{x\in[0,1]}$ being a mild solution to Equation (5). This Hilbert space is defined as the closure of the linear span generated by the indicator functions $\mathbf{1}_{[0,x]},\ x > 0$, with respect to the inner product

$$\mathbb{E}\big(u(t,x)u(t,y)\big) = <\mathbf{1}_{[0,x]}, \mathbf{1}_{[0,y]}>_{\mathcal{H}}. \tag{23}$$

We also denote by I_q, $q \geq 1$, the multiple stochastic integral with respect to the Gaussian process $(u(t,x))_{x\in[0,1]}$. So for every $x < y$ we have

$$u(t,y) - u(t,x) = I_1(\mathbf{1}_{[x,y]}).$$

We start our study of the limit behavior in distribution of the sequence $(\tilde{V}_N)_{N\geq 1}$ by the following main theorem.

Theorem 3. *Let u be the mild solution to Equation (5), G be the fundamental solution associated to the operator $\mathcal{L}$ and $\tilde{V}_N$ be given by (14). If G satisfies Conditions $H_1([0,1])$ and $H_3([0,1])$, then*

$$\lim_{N\to\infty} \mathbb{E}\big(\tilde{V}_N^2\big) = 1.$$

Proof. By using Formula (21), we can write

$$\begin{aligned} V_N &= \sum_{j=0}^{N-1}\left[\frac{(u(t,x_{j+1})-u(t,x_j))^2}{\mathbb{E}(u(t,x_{j+1})-u(t,x_j))^2}-1\right] \\ &= \sum_{j=0}^{N-1}\left[\frac{\mathrm{I}_1{}^2(\mathbf{1}_{[x_j,x_{j+1}]})}{\mathbb{E}(u(t,x_{j+1})-u(t,x_j))^2}-1\right] \\ &= \sum_{j=0}^{N-1}\frac{\mathrm{I}_2(\mathbf{1}^{\otimes 2}_{[x_j,x_{j+1}]})}{\mathbb{E}(u(t,x_{j+1})-u(t,x_j))^2}. \end{aligned}$$

By virtue of the isometry formula (16), we get

$$\begin{aligned} \mathbb{E}\big(V_N^2\big) &= \mathbb{E}\Bigg(\sum_{j=0}^{N-1}\frac{\mathrm{I}_2(\mathbf{1}^{\otimes 2}_{[x_j,x_{j+1}]})}{\mathbb{E}(u(t,x_{j+1})-u(t,x_j))^2}\Bigg)^2 \\ &= \sum_{j,k=0}^{N-1}\frac{\mathbb{E}(\mathrm{I}_2(\mathbf{1}^{\otimes 2}_{[x_j,x_{j+1}]})\,\mathrm{I}_2(\mathbf{1}^{\otimes 2}_{[x_k,x_{k+1}]}))}{\mathbb{E}(u(t,x_{j+1})-u(t,x_j))^2\,\mathbb{E}(u(t,x_{k+1})-u(t,x_k))^2} \\ &= 2\sum_{j,k=0}^{N-1}\frac{<\mathbf{1}_{[x_j,x_{j+1}]},\mathbf{1}_{[x_k,x_{k+1}]}>^2_{\mathcal{H}}}{\mathbb{E}(u(t,x_{j+1})-u(t,x_j))^2\,\mathbb{E}(u(t,x_{k+1})-u(t,x_k))^2}. \end{aligned}$$

Thus,

$$\mathbb{E}\big(V_N{}^2\big) = T_{1,N} + T_{2,N},$$

where

$$T_{1,N} = 2\sum_{j=0}^{N-1}\frac{<\mathbf{1}_{[x_j,x_{j+1}]},\mathbf{1}_{[x_j,x_{j+1}]}>^2_{\mathcal{H}}}{[\mathbb{E}(u(t,x_{j+1})-u(t,x_j))^2]^2} \tag{24}$$

and

$$T_{2,N} = 2\sum_{j,k=0;\, j\neq k}^{N-1}\frac{<\mathbf{1}_{[x_j,x_{j+1}]},\mathbf{1}_{[x_k,x_{k+1}]}>^2_{\mathcal{H}}}{\mathbb{E}(u(t,x_{j+1})-u(t,x_j))^2\,\mathbb{E}(u(t,x_{k+1})-u(t,x_k))^2}. \tag{25}$$

On the one hand we clearly have $T_{1,N} = 2N$. On the other hand, since Conditions $H_1([0,1])$ and $H_3([0,1])$ are satisfied, by virtue of Lemma 1 we get

$$T_{2,N} \leqslant 2\,N^2\sum_{j,k=0;\, j\neq k}^{N-1}<\mathbf{1}_{[x_j,x_{j+1}]},\mathbf{1}_{[x_k,x_{k+1}]}>^2_{\mathcal{H}} \leqslant C\,N^2\sum_{j,k=0;\, j\neq k}^{N-1}\left(\frac{1}{N^2}\right)^2 \leq C, \tag{26}$$

where C denotes a universal positive constant. Thus, we deduce that the dominant term for $\mathbb{E}(\tilde{V}_N^2)$ is obviously $T_{1,N}$. Consequently, we obtain, for a fixed $t \in (0,T]$,

$$\mathbb{E}\big(\tilde{V}_N^2\big) = \frac{1}{2N}\,\mathbb{E}\big(V_N^2\big) \longrightarrow 1 \quad \text{as } N \longrightarrow \infty. \qquad \square$$

In the following theorem we establish the convergence in law of the sequence $(\tilde{V}_N)_N$.

Theorem 4. *Consider the sequence of random variables $\tilde{V}_N$ defined in (14). If G satisfies Conditions $H_1([0,1])$ and $H_3([0,1])$, then*

$$\tilde{V}_N \xrightarrow{Law} \mathcal{N}(0,1).$$

Moreover, if we denote by d one of the metrics on the space of probability measures on $\mathbb{R}$, including the Kolmogorov, Wasserstein and Total Variation measures, then for N large enough

$$d\big(\tilde{V}_N, \mathcal{N}(0,1)\big) \leqslant \frac{C}{\sqrt{N}}.$$

Proof. By virtue of Formula (20) we get

$$D\tilde{V}_N = \frac{1}{\sqrt{2N}} \sum_{j=0}^{N-1} \frac{\mathrm{I}_1(\mathbf{1}_{[x_j,x_{j+1}]})\,\mathbf{1}_{[x_j,x_{j+1}]}}{\mathbb{E}(u(t,x_{j+1})-u(t,x_j))^2}.$$

Hence, for every fixed $t \in [0,T]$, using Formula (21), we get

$$\begin{aligned}\|D\tilde{V}_N\|^2_{\mathcal{H}} &= \frac{2}{N}\sum_{j,k=0}^{N-1} \frac{\mathrm{I}_2(\mathbf{1}_{[x_j,x_{j+1}]}\otimes \mathbf{1}_{[x_k,x_{k+1}]}) <\mathbf{1}_{[x_j,x_{j+1}]}, \mathbf{1}_{[x_k,x_{k+1}]}>_{\mathcal{H}}}{\mathbb{E}(u(t,x_{j+1})-u(t,x_j))^2\,\mathbb{E}(u(t,x_{k+1})-u(t,x_k))^2}\\ &\quad + \mathbb{E}\big(\|D\tilde{V}_N\|^2_{\mathcal{H}}\big),\end{aligned}$$

and consequently,

$$\begin{aligned}&\mathbf{Var}\big(\|D\tilde{V}_N\|^2_{\mathcal{H}}\big)\\ &= \mathbb{E}\big[\|D\tilde{V}_N\|^2_{\mathcal{H}} - \mathbb{E}\big(\|D\tilde{V}_N\|^2_{\mathcal{H}}\big)\big]^2\\ &= \mathbb{E}\Bigg[\frac{2}{N}\sum_{j,k=0}^{N-1} \frac{\mathrm{I}_2(\mathbf{1}_{[x_j,x_{j+1}]}\otimes \mathbf{1}_{[x_k,x_{k+1}]}) <\mathbf{1}_{[x_j,x_{j+1}]}, \mathbf{1}_{[x_k,x_{k+1}]}>_{\mathcal{H}}}{\mathbb{E}(u(t,x_{j+1})-u(t,x_j))^2\,\mathbb{E}(u(t,x_{k+1})-u(t,x_k))^2}\Bigg]^2\\ &= \frac{8}{N^2}\sum_{j,k,m,l=0}^{N-1} \frac{\mathbb{E}(\mathrm{I}_2(\mathbf{1}_{[x_j,x_{j+1}]}\otimes \mathbf{1}_{[x_k,x_{k+1}]})\,\mathrm{I}_2(\mathbf{1}_{[x_m,x_{m+1}]}\otimes \mathbf{1}_{[x_l,x_{l+1}]}))}{\mathbb{E}(u(t,x_{j+1})-u(t,x_j))^2\,\mathbb{E}(u(t,x_{k+1})-u(t,x_k))^2}\\ &\quad\times \frac{<\mathbf{1}_{[x_j,x_{j+1}]}, \mathbf{1}_{[x_k,x_{k+1}]}>_{\mathcal{H}} <\mathbf{1}_{[x_m,x_{m+1}]}, \mathbf{1}_{[x_l,x_{l+1}]}>_{\mathcal{H}}}{\mathbb{E}(u(t,x_{m+1})-u(t,x_m))^2\,\mathbb{E}(u(t,x_{l+1})-u(t,x_l))^2}\\ &= \frac{8}{N^2}\sum_{j,k,m,l=0}^{N-1} \frac{<\mathbf{1}_{[x_j,x_{j+1}]}\tilde{\otimes}\mathbf{1}_{[x_k,x_{k+1}]}, \mathbf{1}_{[x_m,x_{m+1}]}\tilde{\otimes}\mathbf{1}_{[x_l,x_{l+1}]}>_{\mathcal{H}^{\otimes 2}}}{\|\mathbf{1}_{[x_j,x_{j+1}]}\|^2_{\mathcal{H}}\,\|\mathbf{1}_{[x_k,x_{k+1}]}\|^2_{\mathcal{H}}}\\ &\quad\times \frac{<\mathbf{1}_{[x_j,x_{j+1}]}, \mathbf{1}_{[x_k,x_{k+1}]}>_{\mathcal{H}} <\mathbf{1}_{[x_m,x_{m+1}]}, \mathbf{1}_{[x_l,x_{l+1}]}>_{\mathcal{H}}}{\|\mathbf{1}_{[x_m,x_{m+1}]}\|^2_{\mathcal{H}}\,\|\mathbf{1}_{[x_l,x_{l+1}]}\|^2_{\mathcal{H}}},\end{aligned}$$

where $f\tilde{\otimes}g$ denotes the symmetrization of the tensor product $f\otimes g$ that satisfies

$$f\tilde{\otimes}g = \frac{1}{2}(f\otimes g + g\otimes f)$$

and

$$<f\tilde{\otimes}g, f'\tilde{\otimes}g'>_{\mathcal{H}} = \frac{1}{2}\big(<f, f'>_{\mathcal{H}}<g, g'>_{\mathcal{H}} + <f, g'>_{\mathcal{H}}<g, f'>_{\mathcal{H}}\big).$$

Therefore,

$$\begin{aligned}
&\mathbf{Var}\big(\|D\tilde{V}_N\|^2_{\mathcal{H}}\big)\\
= &\ \frac{8}{N^2}\sum_{j,k,m,l=0}^{N-1} \frac{<\mathbf{1}_{[x_j,x_{j+1}]}, \mathbf{1}_{[x_k,x_{k+1}]}>_{\mathcal{H}} <\mathbf{1}_{[x_m,x_{m+1}]}, \mathbf{1}_{[x_l,x_{l+1}]}>_{\mathcal{H}}}{\|\mathbf{1}_{[x_j,x_{j+1}]}\|^2_{\mathcal{H}}\ \|\mathbf{1}_{[x_k,x_{k+1}]}\|^2_{\mathcal{H}}}\\
&\times \frac{<\mathbf{1}_{[x_j,x_{j+1}]}, \mathbf{1}_{[x_k,x_{k+1}]}>_{\mathcal{H}} <\mathbf{1}_{[x_m,x_{m+1}]}, \mathbf{1}_{[x_l,x_{l+1}]}>_{\mathcal{H}}}{\|\mathbf{1}_{[x_m,x_{m+1}]}\|^2_{\mathcal{H}}\ \|\mathbf{1}_{[x_l,x_{l+1}]}\|^2_{\mathcal{H}}}\\
= &\ D_{4,N} + D_{3,N} + D_{2,N} + D_{1,N}
\end{aligned}$$

where $D_{i,N}$, for every $i \in \{1, 2, 3, 4\}$, contains all the terms with i equal indices. So, $D_{4,N}$ contains all the summands above with $j = k = m = l$; that is

$$D_{4,N} = \frac{8}{N^2}\sum_{j=0}^{N-1} 1 = \frac{8}{N}.$$

As for $D_{3,N}$, it contains all the terms corresponding to $j = k = l \neq m$; so, since G satisfies Conditions $H_1([0, 1])$ and $H_3([0, 1])$, using Lemma 1 we get

$$\begin{aligned}
D_{3,N} &\leq \frac{8}{N^2}\sum_{l,m=0}^{N-1} \frac{\|\mathbf{1}_{[x_l,x_{l+1}]}\|^4_{\mathcal{H}}<\mathbf{1}_{[x_l,x_{l+1}]}, \mathbf{1}_{[x_m,x_{m+1}]}>^2_{\mathcal{H}}}{\|\mathbf{1}_{[x_l,x_{l+1}]}\|^6_{\mathcal{H}}\ \|\mathbf{1}_{[x_m,x_{m+1}]}\|^2_{\mathcal{H}}}\\
&\leq \frac{C}{N^2}\sum_{l,m=0}^{N-1} \frac{(\frac{1}{N^2})^2}{(\frac{1}{N})^2} = \frac{C}{N^2}.
\end{aligned}$$

By the same way, and using again Lemma 1, we show that

$$D_{2,N} \leqslant \frac{C}{N^2} \quad \text{and} \quad D_{1,N} \leqslant \frac{C}{N^2}.$$

All this allow us to get

$$\mathbf{Var}\big(\|D\tilde{V}_N\|^2_{\mathcal{H}}\big) \leqslant \frac{C}{N}.$$

Moreover, we have

$$\mathbb{E}\big(\|D\tilde{V}_N\|^2_{\mathcal{H}}\big) = 2\mathbb{E}(\tilde{V}_N)^2 = \frac{\mathbb{E}(V_N^2)}{N} = \frac{1}{N}(T_{1,N} + T_{2,N}) = 2 + \frac{T_{2,N}}{N}$$

where $T_{1,N}$ and $T_{2,N}$ are defined by (24) and (25). This and Inequality (26) allow us to deduce that

$$\mathbb{E}\big(\|D\tilde{V}_N\|^2_{\mathcal{H}}\big) - 2 \leqslant \frac{C}{N}.$$

By virtue of Theorem 1, the proof of Theorem 4 is completed. □

3.3 Almost sure central limit theorem

The following theorem is a kind of extension of Theorem 4.

Theorem 5. *If G satisfies Conditions $H_1([0,1])$ and $H_3([0,1])$, then the sequence $(\tilde{V}_N)_{N\geq 1}$ satisfies an ASCLT.*

Proof. Denoting $\sigma_N = \sqrt{\mathbb{E}(\tilde{V}_N^2)}$, for every $N \geq 1$, according to Theorem 3, we have $\lim_{N\to\infty} \sigma_N = 1$. So without loss of generality, we assume that $\inf_{N\geq 1} \sigma_N = \sigma_0 > 0$ and we consider $G_N = \frac{\tilde{V}_N}{\sigma_N}$, for every $N \geq 1$.

According to Lemma 2, to obtain Theorem 5 it suffices to show that the sequence $(G_N)_{N\geq 1}$ satisfies an ASCLT. To this end, since for every $N \geq 1$ we have $G_N = \mathrm{I}_2(g_N)$ with

$$g_N := \frac{1}{\sigma_N\sqrt{2N}} \sum_{j=0}^{N-1} \frac{\mathbf{1}_{[x_j,x_{j+1}]}^{\otimes 2}}{\mathbb{E}(u(t,x_{j+1})-u(t,x_j))^2},$$

and since we obviously have $\mathbb{E}(G_N^2) = 1$, for every $N \geq 1$, it suffices to check the three last assumptions in Theorem 2.

By the 1st contraction defined by (22), we obtain

$$\begin{aligned} g_l \otimes_1 g_l &= \frac{1}{2\sigma_l^2 l} \sum_{j,k=0}^{l-1} \frac{\mathbf{1}_{[x_j,x_{j+1}]}^{\otimes 2} \otimes_1 \mathbf{1}_{[x_k,x_{k+1}]}^{\otimes 2}}{\mathbb{E}(u(t,x_{j+1})-u(t,x_j))^2\,\mathbb{E}(u(t,x_{k+1})-u(t,x_k))^2} \\ &= \frac{1}{2\sigma_l^2 l} \sum_{j,k=0}^{l-1} \frac{<\mathbf{1}_{[x_j,x_{j+1}]}, \mathbf{1}_{[x_k,x_{k+1}]}>_{\mathcal{H}} \mathbf{1}_{[x_j,x_{j+1}]} \otimes \mathbf{1}_{[x_k,x_{k+1}]}}{\mathbb{E}(u(t,x_{j+1})-u(t,x_j))^2\,\mathbb{E}(u(t,x_{k+1})-u(t,x_k))^2}. \end{aligned}$$

Therefore,

$$\begin{aligned} &\|g_l \otimes_1 g_l\|_{\mathcal{H}^{\otimes 2}}^2 \\ &= \frac{1}{4\sigma_l^4 l^2} \sum_{j,k,m,p=0}^{l-1} \frac{<\mathbf{1}_{[x_j,x_{j+1}]}, \mathbf{1}_{[x_k,x_{k+1}]}>_{\mathcal{H}} <\mathbf{1}_{[x_m,x_{m+1}]}, \mathbf{1}_{[x_p,x_{p+1}]}>_{\mathcal{H}}}{\mathbb{E}(u(t,x_{j+1})-u(t,x_j))^2\,\mathbb{E}(u(t,x_{k+1})-u(t,x_k))^2} \\ &\quad \times \frac{<\mathbf{1}_{[x_j,x_{j+1}]}\tilde{\otimes}\mathbf{1}_{[x_k,x_{k+1}]}, \mathbf{1}_{[x_m,x_{m+1}]}\tilde{\otimes}\mathbf{1}_{[x_p,x_{p+1}]}>_{\mathcal{H}}}{\mathbb{E}(u(t,x_{m+1})-u(t,x_m))^2\,\mathbb{E}(u(t,x_{p+1})-u(t,x_p))^2} \\ &= \frac{1}{4\sigma_l^2 l^2} \sum_{j,k,m,p=0}^{l-1} \frac{<\mathbf{1}_{[x_j,x_{j+1}]}, \mathbf{1}_{[x_k,x_{k+1}]}>_{\mathcal{H}} <\mathbf{1}_{[x_m,x_{m+1}]}, \mathbf{1}_{[x_p,x_{p+1}]}>_{\mathcal{H}}}{\mathbb{E}(u(t,x_{j+1})-u(t,x_j))^2\,\mathbb{E}(u(t,x_{k+1})-u(t,x_k))^2} \\ &\quad \times \frac{<\mathbf{1}_{[x_j,x_{j+1}]}, \mathbf{1}_{[x_m,x_{m+1}]}>_{\mathcal{H}} <\mathbf{1}_{[x_k,x_{k+1}]}, \mathbf{1}_{[x_p,x_{p+1}]}>_{\mathcal{H}}}{\mathbb{E}(u(t,x_{m+1})-u(t,x_m))^2\,\mathbb{E}(u(t,x_{p+1})-u(t,x_p))^2}. \end{aligned}$$

Since $\frac{1}{\sigma_l^4} \leq \frac{1}{\sigma_0^4}$ for every $l \geq 1$, and since G satisfies Conditions $H_1([0,1])$ and $H_3([0,1])$, proceeding in the same way as in the proof of Theorem 4, we get

$$\|g_l \otimes_1 g_l\|_{\mathcal{H}^{\otimes 2}}^2 \leqslant \frac{C}{l}, \tag{27}$$

and consequently the second assumption in Theorem 2 is satisfied.

From Inequality (27) we also deduce that

$$\begin{aligned}\sum_{N\geqslant 2}\frac{1}{Nlog^2N}\sum_{l=1}^{N}\frac{1}{l}\|g_l\otimes_1 g_l\|^2_{\mathcal{H}^{\otimes 2}} &\leqslant C\sum_{N\geqslant 2}\frac{1}{Nlog^2N}\sum_{l=1}^{\infty}\frac{1}{l^2}\\ &\leqslant C\sum_{N\geqslant 2}\frac{1}{Nlog^2N}<\infty,\end{aligned}$$

that means that the third assumption in Theorem 2 is also satisfied.

Let us now check the last assumption in Theorem 2. Since we have

$$\mathbb{E}(G_iG_j)=2<g_i,g_j>_{\mathcal{H}^{\otimes 2}}$$

and since Conditions $H_1([0,1])$ and $H_3([0,1])$ are satisfied, using Lemma 1 we get:

$$\text{If } i=j,\quad <g_i,g_i>_{\mathcal{H}^{\otimes 2}}\leqslant\frac{C}{i}\quad\text{and}\quad\text{if } i>j,\quad <g_i,g_j>_{\mathcal{H}^{\otimes 2}}\leqslant C\sqrt{\frac{j}{i}}.$$

Therefore,

$$\begin{aligned}&\sum_{N\geqslant 2}\frac{1}{Nlog^3N}\sum_{i,j=1}^{N}\frac{|\mathbb{E}(G_iG_j)|}{ij}\\ =&\sum_{N\geqslant 2}\frac{1}{Nlog^3N}\left[\sum_{i\neq j=1}^{N}\frac{|\mathbb{E}(G_iG_j)|}{ij}+\sum_{i=1}^{N}\frac{|\mathbb{E}(G_i^2)|}{i^2}\right]\\ \leqslant&\ 2\sum_{N\geqslant 2}\frac{1}{Nlog^3N}\left[2\sum_{i>j=1}^{N}\frac{|<g_i,g_j>_{\mathcal{H}^{\otimes 2}}|}{ij}+\sum_{i=1}^{N}\frac{|<g_i,g_i>_{\mathcal{H}^{\otimes 2}}|}{i^2}\right]\\ \leqslant&\ C\sum_{N\geqslant 2}\frac{2}{Nlog^3N}\left[\sum_{i>j=1}^{N}\frac{2}{i\sqrt{ij}}+\sum_{i=1}^{N}\frac{1}{i^3}\right]\\ <&\ \infty.\end{aligned}$$

□

4 Stochastic heat equation with piecewise constant coefficients

The study done in the previous section allows us to make a new step in the investigation of the solution to the SPDE (7). Equation (7) is obviously a particular case of (5). Indeed, the operator $\mathcal{L}_p$ defined by (3) can be written in the form (2) with

$$r(x)=2\rho(x)\text{ and }R(x):=\rho(x)A(x).$$

In the following proposition we present the expression of the fundamental solution associated to the operator $\mathcal{L}_p$. For a proof see, e.g., [8, 26, 27] and [28].

Proposition 2. *There exists a unique fundamental solution $G(t-s,x,y)$ associated to the operator $\mathcal{L}_p$. It can be explicitly expressed as*

$$G(u,x,z)=m(u)\left[\frac{1}{\sqrt{a_1}}A^-(u,x,z)1_{\{z\leqslant 0\}}+\frac{1}{\sqrt{a_2}}A^+(u,x,z)1_{\{z>0\}}\right]\tag{28}$$

with

$$m(u) = \frac{1}{\sqrt{2\pi u}} 1_{\{u>0\}}, \tag{29}$$

$$\begin{cases} A^-(u,x,z) = E^-(u,x,z) - \beta E^+(u,x,z), \\ A^+(u,x,z) = E^-(u,x,z) + \beta E^+(u,x,z), \end{cases} \tag{30}$$

$$\begin{cases} E^-(u,x,z) = \exp\left(-\frac{(f(z)-f(x))^2}{2u}\right), \\ E^+(u,x,z) = \exp\left(-\frac{(|f(z)|+|f(x)|)^2}{2u}\right), \end{cases} \tag{31}$$

$$f(z) = \frac{z}{\sqrt{a_1}}\mathbf{1}_{\{z\leqslant 0\}} + \frac{z}{\sqrt{a_2}}\mathbf{1}_{\{z>0\}} \quad \textit{and} \quad \beta = \frac{\rho_2\sqrt{a_2} - \rho_1\sqrt{a_1}}{\rho_2\sqrt{a_2} + \rho_1\sqrt{a_1}}. \tag{32}$$

In this section, by making an in-depth study of the terms f, A^- and A^+ defined in Expressions (30) and (32), we will prove the following theorem.

Theorem 6. *Let u be the mild solution to Equation* (7) *and* $\tilde{V}_N$ *be the sequence given by* (14)*. Suppose that the coefficients A and* ρ *defined in* (4) *satisfy*

$$\max\left(1, \frac{\sqrt{a_1}}{\sqrt{a_2}}\right) \leq \frac{\rho_2}{\rho_1}. \tag{33}$$

Then the following is valid:

1.

$$\tilde{V}_N \xrightarrow{Law} \mathcal{N}(0,1).$$

Moreover, if we denote by d one of the metrics on the space of probability measures on $\mathbb{R}$*, including the Kolmogorov, Wasserstein and Total Variation measures, then for N large enough*

$$d\big(\tilde{V}_N, \mathcal{N}(0,1)\big) \leqslant \frac{C}{\sqrt{N}}.$$

2. The sequence $(\tilde{V}_N)_{N\geq 1}$ *satisfies an ASCLT.*

Remark 3. If $a_1 = a_2 = 1$ and $\rho_1 = \rho_2 = 2$, then Condition (33) is well satisfied. Thus, the result of Theorem 6 applies to the standard stochastic heat equation with the time-space white noise and it corresponds exactly to that obtained in [23].

To prove Theorem 6, we shall first establish the following lemmas.

4.1 Preliminary lemmas

Lemma 3. *Consider f, the function defined in* (32)*. For every* $x, y \in \mathbb{R}$*,*

$$\min\left(\frac{1}{\sqrt{a_2}}, \frac{1}{\sqrt{a_1}}\right)|y-x| \leq \big|f(y)-f(x)\big| \leq \max\left(\frac{1}{\sqrt{a_2}}, \frac{1}{\sqrt{a_1}}\right)|y-x|.$$

Proof. Expression (32) allows to get

$$f(y)-f(x)=\begin{cases} \dfrac{y-x}{\sqrt{a_2}} & \text{if } y>0\ x>0,\\ \dfrac{y-x}{\sqrt{a_1}} & \text{if } y\le 0\ x\le 0,\\ \dfrac{y}{\sqrt{a_1}}-\dfrac{x}{\sqrt{a_2}} & \text{if } y\le 0\ x>0,\\ \dfrac{y}{\sqrt{a_2}}-\dfrac{x}{\sqrt{a_1}} & \text{if } y>0\ x\le 0. \end{cases} \tag{34}$$

If $xy\ge 0$, both inequalities in Lemma 3 are directly obtained from (34). If $y>0$ and $x<0$,

$$\begin{aligned} &\max\left(\frac{1}{\sqrt{a_2}},\frac{1}{\sqrt{a_1}}\right)|y-x|-\left|f(y)-f(x)\right|\\ =\ &\max\left(\frac{1}{\sqrt{a_2}},\frac{1}{\sqrt{a_1}}\right)(y-x)-\frac{y}{\sqrt{a_2}}+\frac{x}{\sqrt{a_1}}\\ =\ &y\left[\max\left(\frac{1}{\sqrt{a_2}},\frac{1}{\sqrt{a_1}}\right)-\frac{1}{\sqrt{a_2}}\right]-x\left[\max\left(\frac{1}{\sqrt{a_2}},\frac{1}{\sqrt{a_1}}\right)-\frac{1}{\sqrt{a_1}}\right]\\ >\ &0 \end{aligned}$$

and

$$\begin{aligned} &\min\left(\frac{1}{\sqrt{a_2}},\frac{1}{\sqrt{a_1}}\right)|y-x|-\left|f(y)-f(x)\right|\\ =\ &\min\left(\frac{1}{\sqrt{a_2}},\frac{1}{\sqrt{a_1}}\right)(y-x)-\frac{y}{\sqrt{a_2}}+\frac{x}{\sqrt{a_1}}\\ =\ &y\left[\min\left(\frac{1}{\sqrt{a_2}},\frac{1}{\sqrt{a_1}}\right)-\frac{1}{\sqrt{a_2}}\right]-x\left[\min\left(\frac{1}{\sqrt{a_2}},\frac{1}{\sqrt{a_1}}\right)-\frac{1}{\sqrt{a_1}}\right]\\ <\ &0. \end{aligned}$$

The proof of both inequalities in the case where $y<0$ and $x>0$ is similar. □

Lemma 4. *There exists a universal positive constant C, such that*

$$\int_0^t \frac{1}{2\pi u}\int_{\mathbb{R}}\left|E^+(u,y,z)-E^+(u,x,z)\right|^2 dz\,du \leqslant C\,|y-x|$$

and

$$\int_0^t \frac{1}{2\pi u}\int_{\mathbb{R}}\left|E^-(u,y,z)-E^-(u,x,z)\right|^2 dz\,du \leqslant C\,|y-x|$$

for every $t>0$ and $x,y\in\mathbb{R}$.

Proof. We present only the proof of the first inequality; the proof of the second one is similar. By using Expression (31), we get

$$\int_0^t \frac{1}{2\pi u}\int_{\mathbb{R}}\left|E^+(u,y,z)-E^+(u,x,z)\right|^2 dz\,du$$

$$
\begin{aligned}
= & \int_0^t \int_{\mathbb{R}} \Big[\frac{1}{\sqrt{2\pi u}} \exp\Big(-\frac{(|f(z)| + |f(y)|)^2}{2u} \Big) \\
& - \frac{1}{\sqrt{2\pi u}} \exp\Big(-\frac{(|f(z)| + |f(x)|)^2}{2u} \Big) \Big]^2 \, dz\, du \\
= & \int_0^t \int_0^\infty \Big[\frac{1}{\sqrt{2\pi u}} \exp\Big(\frac{(z/\sqrt{a_2} + |f(y)|)^2}{2u} \Big) \\
& - \frac{1}{\sqrt{2\pi u}} \exp\Big(-\frac{(z/\sqrt{a_2} + |f(x)|)^2}{2u} \Big) \Big]^2 \, dz\, du \\
& + \int_0^t \int_{-\infty}^0 \Big[\frac{1}{\sqrt{2\pi u}} \exp\Big(-\frac{(-z/\sqrt{a_1} + |f(y)|)^2}{2u} \Big) \\
& - \frac{1}{\sqrt{2\pi u}} \exp\Big(-\frac{(-z/\sqrt{a_1} + |f(x)|)^2}{2u} \Big) \Big]^2 \, dz\, du.
\end{aligned}
$$

The changes of variables $Z = z/\sqrt{a_2} + |f(x)|$ in the first integral and $Z = -z/\sqrt{a_1} + |f(x)|$, in the second one give

$$
\begin{aligned}
& \int_0^t \frac{1}{2\pi u} \int_{\mathbb{R}} \big| E^+(u, y, z) - E^+(u, x, z) \big|^2 \, dz\, du \\
= & \sqrt{a_2} \int_0^t \int_{|f(x)|}^{+\infty} \Big[\frac{1}{\sqrt{2\pi u}} \exp\Big(-\frac{(Z + (|f(y)| - |f(x)|))^2}{2u} \Big) \\
& - \frac{1}{\sqrt{2\pi u}} \exp\Big(-\frac{Z^2}{2u} \Big) \Big]^2 \, dZ\, du \\
& + \sqrt{a_1} \int_0^t \int_{|f(x)|}^{+\infty} \Big[\frac{1}{\sqrt{2\pi u}} \exp\Big(-\frac{(Z + (|f(y)| - |f(x)|))^2}{2u} \Big) \\
& - \frac{1}{\sqrt{2\pi u}} \exp\Big(-\frac{Z^2}{2u} \Big) \Big]^2 \, dZ\, du. \qquad (35)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& \int_0^t \frac{1}{2\pi u} \int_{\mathbb{R}} \big| E^+(u, y, z) - E^+(u, x, z) \big|^2 \, dz\, du \\
\leqslant & \; 2 \max(\sqrt{a_1}, \sqrt{a_2}) \int_0^t \int_{\mathbb{R}} \Big[\frac{1}{\sqrt{2\pi u}} \exp\Big(-\frac{(Z + \tilde{H})^2}{2u} \Big) \\
& - \frac{1}{\sqrt{2\pi u}} \exp\Big(-\frac{Z^2}{2u} \Big) \Big]^2 \, dZ\, du \\
= & \; 2 \max(\sqrt{a_1}, \sqrt{a_2}) \int_0^t \int_{\mathbb{R}} \big[p_u(Z + \tilde{H}) - p_u(Z) \big]^2 \, dZ\, du, \qquad (36)
\end{aligned}
$$

where $\tilde{H} = |f(y)| - |f(x)|$ and p_u denotes the heat kernel defined by

$$
p_t(x) = \frac{1}{\sqrt{2\pi t}} \exp\Big(\frac{-x^2}{2t} \Big), \quad \text{for every } t > 0 \text{ and } x \in \mathbb{R}. \qquad (37)
$$

It is known that the Fourier transform of p_u is

$$\mathcal{F}(p_t)(\xi) = e^{-t\xi^2/2} \quad \forall \xi \in \mathbb{R},\ t > 0.$$

By virtue of the Plancherel theorem we can write

$$\begin{aligned} & \int_0^t ds \int_{\mathbb{R}} \left[p_s(v+h) - p_s(v)\right]^2 dv \\ = & \frac{1}{2\pi} \int_0^t ds \int_{-\infty}^{\infty} \left|e^{-s\xi^2/2 + i\xi h} - e^{-s\xi^2/2}\right|^2 d\xi \\ = & \frac{1}{\pi} \int_0^t ds \int_{-\infty}^{\infty} e^{-s\xi^2} \big(1 - \cos(h\xi)\big)\, d\xi \end{aligned}$$

for every $h \in \mathbb{R}$. Applying Fubini's Theorem and using the fact that the functions cosine and $\xi \longmapsto \frac{1-\cos(h\xi)}{\xi^2}$ are even we get:

$$\begin{aligned} \int_0^t ds \int_{\mathbb{R}} \left[p_s(v+h) - p_s(v)\right]^2 dv &= \frac{1}{\pi} \int_{-\infty}^{\infty} \left[\int_0^t e^{-s\xi^2}\, ds\right] \big(1 - \cos(h\xi)\big)\, d\xi \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \big(1 - e^{-t\xi^2}\big) \frac{1 - \cos(h\xi)}{\xi^2}\, d\xi \\ &= \frac{2}{\pi} \int_0^{\infty} \big(1 - e^{-t\xi^2}\big) \frac{1 - \cos(|h|\xi)}{\xi^2}\, d\xi. \quad (38) \end{aligned}$$

Suppose that $h \neq 0$. By a simple change of variables in (38), using the fact that

$$\forall \theta \geq 0 \quad 1 - \exp(-\theta) \leqslant 1,$$

we obtain

$$\begin{aligned} \int_0^t ds \int_{\mathbb{R}} \left[p_s(v+h) - p_s(v)\right]^2 dv &= \frac{2\,|h|}{\pi} \int_0^{\infty} \big(1 - e^{-t\frac{\xi^2}{h^2}}\big) \frac{1 - \cos(\xi)}{\xi^2}\, d\xi \\ &\leqslant \frac{2\,|h|}{\pi} \int_0^{\infty} \frac{1 - \cos(\xi)}{\xi^2}\, d\xi. \quad (39) \end{aligned}$$

The function $g : \xi \longmapsto \frac{1-\cos(\xi)}{\xi^2}$ is continuous on the interval $(0, +\infty)$ and consequently it is locally integrable. In addition, on the one hand, $\lim_{\xi \to 0} g(\xi) = \frac{1}{2}$, which implies that g is integrable in a neighbourhood of 0. On the other hand, $|g(\xi)| \leq \frac{2}{\xi^2}$ for every $\xi > 1$ and $\int_1^{+\infty} \frac{1}{\xi^2} d\xi < \infty$, which entails the integrability of g on a neighbourhood of $+\infty$. From all this, one can deduce that the integral $\int_0^{\infty} \frac{1-\cos(\xi)}{\xi^2}\, d\xi$ is convergent and, consequently, by using (39),

$$\int_0^t ds \int_{\mathbb{R}} \left[p_s(v+h) - p_s(v)\right]^2 dv \leqslant C|h|, \quad (40)$$

for every real $h \neq 0$. Morover, Inequality (40) is obviously true for $h = 0$. Thus, (40) is statisfied for every $h \in \mathbb{R}$.

This and Inequality (35) imply that

$$\begin{aligned}\int_0^t \frac{1}{2\pi u}\int_{\mathbb{R}}\left|E^+(u,y,z)-E^+(u,x,z)\right|^2 dz\, du &\leq C\big|\left|f(y)\right|-\left|f(x)\right|\big| \\ &\leq C\left|f(y)-f(x)\right| \\ &\leq C|y-x|,\end{aligned}$$

where in the last inequality we used Lemma 3. □

Lemma 5. *For every $A > 0$ and $t \in [0, T]$, there exists a positive constant c such that*

$$\int_0^t \frac{1}{2\pi u}\int_{\mathbb{R}}\left|E^-(t-s,y,z)-E^-(t-s,x,z)\right|^2 dz\, du \geqslant c\,|y-x|$$

and

$$\int_0^t \frac{1}{2\pi u}\int_{\mathbb{R}}\left|E^+(t-s,y,z)-E^+(t-s,x,z)\right|^2 dz\, du \geqslant c\,|y-x|$$

for every $x, y \in [0, A]$.

Proof. We present the proof just for the first inequality. The second is obtained in the same way. We have

$$\begin{aligned}&\int_0^t \frac{1}{2\pi u}\int_{\mathbb{R}}\left|E^-(t-s,y,z)-E^-(t-s,x,z)\right|^2 dz\, du \\ = &\int_0^t\int_{\mathbb{R}}\Big[\frac{1}{\sqrt{2\pi u}}\exp\Big(-\frac{(f(z)-f(y))^2}{2u}\Big) \\ &-\frac{1}{\sqrt{2\pi u}}\exp\Big(-\frac{(f(z)-f(x))^2}{2u}\Big)\Big]^2 dz\, du \\ = &\int_0^t\int_0^\infty\Big[\frac{1}{\sqrt{2\pi u}}\exp\Big(-\frac{(z/\sqrt{a_2}-f(y))^2}{2u}\Big) \\ &-\frac{1}{\sqrt{2\pi u}}\exp\Big(-\frac{(z/\sqrt{a_2}-f(x))^2}{2u}\Big)\Big]^2 dz\, du \\ &+\int_0^t\int_{-\infty}^0\Big[\frac{1}{\sqrt{2\pi u}}\exp\Big(-\frac{(z/\sqrt{a_1}-f(y))^2}{2u}\Big) \\ &-\frac{1}{\sqrt{2\pi u}}\exp\Big(-\frac{(z/\sqrt{a_1}-f(x))^2}{2u}\Big)\Big]^2 dz\, du.\end{aligned}$$

Applying the changes of variables $Z = z/\sqrt{a_2} - f(x)$ in the first integral and $Z = z/\sqrt{a_1} - f(x)$ in the second one we obtain

$$\begin{aligned}&\int_0^t \frac{1}{2\pi u}\int_{\mathbb{R}}\left|E^-(t-s,y,z)-E^-(t-s,x,z)\right|^2 dz\, du \\ = &\sqrt{a_2}\int_0^t\int_{-f(x)}^\infty\Big[\frac{1}{\sqrt{2u}}\exp\Big(-\frac{(Z-(f(y)-f(x)))^2}{2u}\Big)\end{aligned}$$

$$
\begin{aligned}
& - \frac{1}{\sqrt{2\pi u}} \exp\Big(-\frac{Z^2}{2u}\Big)\Big]^2 dZ\, du \\
+ \quad & \sqrt{a_1} \int_0^t \int_{\infty}^{-f(x)} \Big[\frac{1}{\sqrt{2u}} \exp\Big(-\frac{(Z-(f(y)-f(x)))^2}{2u}\Big) \\
& - \frac{1}{\sqrt{2\pi u}} \exp\Big(-\frac{Z^2}{2u}\Big)\Big]^2, dZ\, du \\
\geqslant \quad & \min(\sqrt{a_1}, \sqrt{a_2}) \int_0^t \int_{\mathbb{R}} \Big[\frac{1}{\sqrt{2u}} \exp\Big(-\frac{(Z-(f(y)-f(x)))^2}{2u}\Big) \\
& - \frac{1}{\sqrt{2\pi u}} \exp\Big(-\frac{Z^2}{2u}\Big)\Big]^2 dZ\, du \\
= \quad & \min(\sqrt{a_1}, \sqrt{a_2}) \int_0^t \int_{\mathbb{R}} \big[p_u(Z-\tilde{K}) - p_u(Z)\,\big]^2 dZ\, du,
\end{aligned}
$$

where $\tilde{K} = f(y) - f(x)$ and p_u is the heat kernel defined by (37). Without loss of generality we can suppose that $x < y$. So,

$$
0 < \tilde{K} = \frac{y-x}{\sqrt{a_2}} < \frac{A}{\sqrt{a_2}}.
$$

Applying the same technique as that used in (38), we get

$$
\begin{aligned}
& \int_0^t \int_{\mathbb{R}} \big[p_u(Z-h) - p_u(Z)\,\big]^2 dZ\, du \\
= \quad & \frac{1}{2\pi} \int_0^t ds \int_{-\infty}^{\infty} \big|e^{-s\xi^2/2 - i\xi h} - e^{-s\xi^2/2}\big|^2\, d\xi \\
= \quad & \frac{1}{\pi} \int_0^t ds \int_{-\infty}^{\infty} e^{-s\xi^2}\big(1-\cos(h\xi)\big)\, d\xi \\
= \quad & \frac{2}{\pi} \int_0^{\infty} \big(1-e^{-t\xi^2}\big) \frac{1-\cos(|h|\xi)}{\xi^2}\, d\xi
\end{aligned}
$$

for every $h \in \mathbb{R}$. Thus,

$$
\begin{aligned}
& \int_0^t ds \int_{\mathbb{R}} \big[p_s(Z-\tilde{K}) - p_s(Z)\big]^2 dy \\
= \quad & \frac{2}{\pi} \int_0^{\infty} \big(1-e^{-tz^2}\big) \frac{1-\cos(\tilde{K}z)}{z^2}\, dz \\
= \quad & \frac{2}{\pi} \int_0^{\frac{1}{\tilde{K}}} \big(1-e^{-tz^2}\big) \frac{1-\cos(\tilde{K}z)}{z^2}\, dz + \frac{2}{\pi} \int_{\frac{1}{\tilde{K}}}^{\infty} \big(1-e^{-tz^2}\big) \frac{1-\cos(\tilde{K}z)}{z^2}\, dz \\
\geq \quad & \frac{2}{\pi} \int_{\frac{1}{\tilde{K}}}^{\infty} \big(1-e^{-tz^2}\big) \frac{1-\cos(\tilde{K}z)}{z^2}\, dz, \qquad (41)
\end{aligned}
$$

where in the last inequality we used the fact that

$$
\frac{2}{\pi} \int_0^{\frac{1}{\tilde{K}}} \big(1-e^{-tz^2}\big) \frac{1-\cos(\tilde{K}z)}{z^2}\, dz \geq 0.
$$

Since $1 - e^{-tz^2} \geq 1 - e^{-t\tilde{K}^{-2}}$ for every $z \geq \frac{1}{\tilde{K}}$, from (41) we get

$$\begin{aligned}\int_0^t ds \int_{\mathbb{R}} \left[p_s(Z - \tilde{K}) - p_s(Z)\right]^2 dy &\geq \frac{2}{\pi}\left(1 - e^{-t\tilde{K}^{-2}}\right) \int_{\frac{1}{\tilde{K}}}^{\infty} \frac{1 - \cos(\tilde{K}z)}{z^2}\, dz \\ &\geq \tilde{K}\frac{2}{\pi}\left(1 - e^{-t\tilde{K}^{-2}}\right) \int_1^{\infty} \frac{1 - \cos(\xi)}{\xi^2}\, d\xi,\end{aligned}$$

where the last inequality is obtained after applying the change of variables $\xi = \tilde{K}z$.

Now, since

$$0 < \tilde{K} = \frac{y - x}{\sqrt{a_2}} < \frac{A}{\sqrt{a_2}},$$

we have

$$1 - e^{-t\tilde{K}^{-2}} \geq 1 - e^{-ta_2 A^{-2}}$$

and consequently

$$\int_0^t \int_{\mathbb{R}} \left[p_u(Z - \tilde{K}) - p_u(Z)\,\right]^2 dZ\, du \geqslant c|y - x|$$

with

$$c = \frac{2}{\sqrt{a_2}\pi}\left(1 - e^{-ta_2 A^{-2}}\right) \int_1^{\infty} \frac{1 - \cos(z)}{z^2}\, dz. \qquad \square$$

4.2 Proof of Theorem 6

Since Equation (7) is a particular case of (5), by Theorems 4 and 5, to get Theorem 6, it suffices to show that the fundamental solution associated to the operator $\mathcal{L}_p$ satisfies Conditions $H_i([0, 1])$, for $i = 1, 2, 3$. Consider $x, y \in [0, 1]$; $x < y$ and $t \in [0, T]$.

4.2.1 Proof of $H_1([0, 1])$

We have

$$\begin{aligned}&\left\|\Delta_{y-x} G(t - s, x, .)\right\|^2_{L^2([0,t]\times\mathbb{R})} \\ = &\int_0^t \frac{1}{2\pi(t - s)} \left\{\int_{\mathbb{R}} \frac{1}{A(z)}\left[\left(A^-(t - s, y, z) - A^-(t - s, x, z)\right) \mathbf{1}_{\{z\leqslant 0\}}\right.\right. \\ &\left.\left. + \left(A^+(t - s, y, z) - A^+(t - s, x, z)\right) \mathbf{1}_{\{z>0\}}\right]^2 dz\right\} ds \\ = &\int_0^t \frac{1}{2\pi(t - s)} \left\{\int_{\mathbb{R}} \frac{1}{A(z)}\left[\left(E^-(t - s, y, z) - E^-(t - s, x, z)\right)\right.\right. \\ &\left.\left. + \beta\, sign(z)\left(E^+(t - s, y, z) - E^+(t - s, x, z)\right)\right]^2 dz\right\} ds \\ \geqslant &\min\left(\frac{1}{a_1}, \frac{1}{a_2}\right) \int_0^t \frac{1}{2\pi(t - s)} \left\{\int_{\mathbb{R}} \left|\left|E^-(t - s, y, z) - E^-(t - s, x, z)\right|\right.\right. \\ &\left.\left. - |\beta|\left|E^+(t - s, y, z) - E^+(t - s, x, z)\right|\right|^2 dz\right\} ds.\end{aligned}$$

According to [6, page 54], we know that

$$\|f-g\|^2_{L^2(\mathbb{R})} \geqslant \frac{1}{4}\left(\|f\|_{L^2(\mathbb{R})}+\|g\|_{L^2(\mathbb{R})}\right)^2 \left\| \frac{f}{\|f\|_{L^2(\mathbb{R})}} - \frac{g}{\|g\|_{L^2(\mathbb{R})}} \right\|^2_{L^2(\mathbb{R})}$$

for every $f, g \in L^2(\mathbb{R})$; $f \neq 0$ and $g \neq 0$ a.e. Thus,

$$\begin{aligned}
&\left\|\Delta_{y-x} G(t-s,x,.)\right\|^2_{L^2([0,t]\times\mathbb{R})} \\
\geqslant\ & \frac{1}{4}\min\left(\frac{1}{a_1},\frac{1}{a_2}\right)\int_0^t \frac{I(s)}{2\pi(t-s)} \\
&\times\left(\left\|E^-(t-s,y,.)-E^-(t-s,x,.)\right\|\right. \\
&\quad \left.+\ |\beta|\left\|E^+(t-s,y,.)-E^+(t-s,x,.)\right\|\right)^2 ds,
\end{aligned}$$

where

$$\begin{aligned}
I(s) = & \left\| \frac{|E^-(t-s,y,.)-E^-(t-s,x,.)|}{\|E^-(t-s,y,.)-E^-(t-s,x,.)\|} \right. \\
& \left. -|\beta|\frac{|E^+(t-s,y,.)-E^+(t-s,x,.)|}{\|E^+(t-s,y,.)-E^+(t-s,x,.)\|}\right\|^2,
\end{aligned}$$

and $\|.\|$ denotes $\|.\|_{L^2(\mathbb{R})}$. On the one hand we have

$$\begin{aligned}
& I(s) = 1+\beta^2 \\
& -\frac{2|\beta|}{\|E^-(t-s,y,z)-E^-(t-s,x,z)\|\,\|E^+(t-s,y,z)-E^+(t-s,x,z)\|} \\
& \times\int_{\mathbb{R}}\left|E^-(t-s,y,z)-E^-(t-s,x,z)\right|\,\left|E^+(t-s,y,z)-E^+(t-s,x,z)\right| dz.
\end{aligned}$$

On the other hand, applying Hölder's Inequality, we get

$$\begin{aligned}
&\int_{\mathbb{R}}\left|E^-(t-s,y,z)-E^-(t-s,x,z)\right|\,\left|E^+(t-s,y,z)-E^+(t-s,x,z)\right| dz \\
&\quad \leq \left\|E^-(t-s,y,.)-E^-(t-s,x,.)\right\|\,\left\|E^+(t-s,y,.)-E^+(t-s,x,.)\right\|.
\end{aligned}$$

Hence,

$$I(s) \geqslant 1+\beta^2-2|\beta| = \left(1-|\beta|\right)^2$$

and therefore,

$$\begin{aligned}
&\left\|\Delta_{y-x} G(t-s,x,.)\right\|^2_{L^2([0,t]\times\mathbb{R})} \\
\geqslant\ & \frac{(1-|\beta|)^2}{4}\min\left(\frac{1}{a_1},\frac{1}{a_2}\right)\int_0^t \frac{1}{2\pi(t-s)} \\
&\times\left(\left\|E^-(t-s,y,.)-E^-(t-s,x,.)\right\|\right. \\
&\left.+\ |\beta|\left\|E^+(t-s,y,.)-E^+(t-s,x,.)\right\|\right)^2 ds
\end{aligned}$$

$$\geqslant \frac{(1-|\beta|)^2}{4} \min\left(\frac{1}{a_1}, \frac{1}{a_2}\right) \int_0^t \frac{1}{2\pi u}$$
$$\times \left(\left\| E^-(u,y,.) - E^-(u,x,.)\right\|^2 + \beta^2 \left\| E^+(u,y,.) - E^+(u,x,.)\right\|^2\right) du,$$

where in the last inequality we used the fact that $x^2 + y^2 \le (x+y)^2$ for every non-negative real numbers x and y.

This and Lemma 5 show that Hypothesis $H_1([0,1])$ is satisfied.

4.2.2 *Proof of* $H_2([0,1])$

Using the expressions of A^- and A^+ given in (30) we get

$$\begin{aligned}
&\left\| \Delta_{y-x} G(t-.,x,.)\right\|^2_{L^2([0,t]\times\mathbb{R})} \\
=\ & \int_0^t \int_{\mathbb{R}} \left| G(t-s,y,z) - G(t-s,x,z)\right|^2 ds\, dz \\
=\ & \int_0^t \left[\frac{1}{2a_1\pi(t-s)} \int_{-\infty}^0 \left| A^-(t-s,y,z) - A^-(t-s,x,z)\right|^2 dz \right] ds \\
& + \int_0^t \left[\frac{1}{2a_2\pi(t-s)} \int_0^{\infty} \left| A^+(t-s,y,z) - A^+(t-s,x,z)\right|^2 dz \right] ds \\
\le\ & \int_0^t \frac{1}{2\pi(t-s)} \int_{-\infty}^0 \Delta_{max}(s,z) dz ds \\
& + \int_0^t \frac{1}{2\pi(t-s)} \int_0^{\infty} \Delta_{max}(s,z) dz\, ds,
\end{aligned} \tag{42}$$

where

$$\begin{aligned}
\Delta_{\max}(s,z) &= \max\Bigg(\frac{1}{a_1} \left| A^-(t-s,y,z) - A^-(t-s,x,z)\right|^2, \\
&\qquad \frac{1}{a_2} \left| A^+(t-s,y,z) - A^+(t-s,x,z)\right|^2 \Bigg) \\
&= \max\Bigg(\frac{1}{a_1} \Big(\big(E^-(t-s,y,z) - E^-(t-s,x,z)\big) \\
&\qquad - \beta\big(E^+(t-s,y,z) - E^+(t-s,x,z)\big)\Big)^2, \\
&\qquad \frac{1}{a_2} \Big(\big(E^-(t-s,y,z) - E^-(t-s,x,z)\big) \\
&\qquad + \beta\big(E^+(t-s,y,z) - E^+(t-s,x,z)\big)\Big)^2 \Bigg)
\end{aligned}$$

for every $t \in (0,T]$ and $(x,y) \in [0,1]^2;\ \ y > x$.

Since

$$\max\left(\gamma_1(a-b)^2, \gamma_2(a+b)^2\right) \leqslant 2\max(\gamma_1,\gamma_2)\left(a^2+b^2\right)$$

for any $(a, b) \in \mathbb{R}^2$ and any $\gamma_1, \gamma_2 > 0$, we have

$$\begin{aligned}\Delta_{max}(s,z) &\le 2\max\Big(\frac{1}{a_1},\frac{1}{a_2}\Big)\Big(\big|E^-(t-s,y,z)-E^-(t-s,x,z)\big|^2\\ &\quad + \beta^2\big|E^+(t-s,y,z)-E^+(t-s,x,z)\big|^2\Big).\end{aligned}$$

Thus,

$$\begin{aligned}&\big\|\Delta_{y-x}G(t-.,x,.)\big\|^2_{L^2([0,t]\times\mathbb{R})}\\ \le\ &\int_0^t \frac{1}{2\pi(t-s)}\int_{\mathbb{R}}\Delta_{max}(s,z)dzds\\ \le\ &2\max\Big(\frac{1}{a_1},\frac{1}{a_2}\Big)\Bigg[\int_0^t\Bigg[\frac{1}{2\pi(t-s)}\int_{\mathbb{R}}\big|E^-(t-s,y,z)\\ &-E^-(t-s,x,z)\big|^2dz\Bigg]ds\\ &+\beta^2\int_0^t\Bigg[\frac{1}{2\pi(t-s)}\int_{\mathbb{R}}\big|E^+(t-s,y,z)-E^+(t-s,x,z)\big|^2dz\Bigg]ds\Bigg]. \qquad (43)\end{aligned}$$

This and Lemma 4 show that Condition $H_2([0,1])$ is also satisfied.

4.2.3 Proof of $H_3([0,1])$

Consider $x, x' \in [0,1]$ and $h > 0$. We have

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}}\Delta_hG(t-s,x,z)\,\Delta_hG\big(t-s,x',z\big)\,dz\,ds\\ =\ &\int_0^t\int_{\mathbb{R}}\big(G(t-s,x+h,z)-G(t-s,x,z)\big)\\ &\big(G\big(t-s,x'+h,z\big)-G\big(t-s,x',z\big)\big)dz\,ds\\ =\ &\int_0^t\Bigg[\frac{m^2(t-s)}{a_1}\int_{-\infty}^0\big(A^-(t-s,x+h,z)-A^-(t-s,x,z)\big)\\ &\big(A^-\big(t-s,x'+h,z\big)-A^-\big(t-s,x',z\big)\big)dz\Bigg]ds\\ +\ &\int_0^t\Bigg[\frac{m^2(t-s)}{a_2}\int_0^{\infty}\big(A^+(t-s,x+h,z)-A^+(t-s,x,z)\big)\\ &\big(A^+\big(t-s,x'+h,z\big)-A^+\big(t-s,x',z\big)\big)dz\Bigg]ds\\ =\ &L+K. \qquad (44)\end{aligned}$$

Using the expression of A^- (30), denoting $\tilde{x} = \frac{x}{\sqrt{a_2}}$, $\tilde{x}' = \frac{x'}{\sqrt{a_2}}$ and $\tilde{h} = \frac{h}{\sqrt{a_2}}$, then making the change of variables $z' = \frac{z}{\sqrt{a_1}}$, we get

$$L = \int_0^t\Bigg[\frac{(1-\beta)^2}{2\pi a_1 u}\int_{-\infty}^0\big(E^-(u,x+h,z)-E^-(u,x,z)\big)$$

$$\left(E^-\big(u, x'+h, z\big) - E^-\big(u, x', z\big)\right)dz\Bigg]du$$

$$= \frac{1}{\sqrt{a_2}} \int_0^t \sqrt{\frac{a_2}{a_1}} \frac{(1-\beta)^2}{2\pi u} \Bigg[\int_{-\infty}^0 \Bigg(\exp\Bigg(-\frac{(z'-\tilde{x}-\tilde{h})^2}{2u}\Bigg) - \exp\Bigg(-\frac{(z'-\tilde{x})^2}{2u}\Bigg)\Bigg) \times \Bigg(\exp\Bigg(-\frac{(z'-\tilde{x}'-\tilde{h})^2}{2u}\Bigg) - \exp\Bigg(-\frac{(z'-\tilde{x}')^2}{2u}\Bigg)\Bigg)dz'\Bigg]du.$$

Now, using the expression of A^+ (30) and making the change of variable $z' = \frac{z}{\sqrt{a_2}}$, the integral K can be written in the form

$$K = \int_0^t \frac{1}{2\sqrt{a_2}\,\pi u}\big[K_1 + \beta\, K_2 + \beta\, K_3 + \beta^2\, K_4\big]\, du,$$

where

$$K_1 = \int_0^{+\infty} \Bigg(\exp\Bigg(-\frac{(z'-\tilde{x}-\tilde{h})^2}{2u}\Bigg) - \exp\Bigg(-\frac{(z'-\tilde{x})^2}{2u}\Bigg)\Bigg) \times \Bigg(\exp\Bigg(-\frac{(z'-\tilde{x}'-\tilde{h})^2}{2u}\Bigg) - \exp\Bigg(-\frac{(z'-\tilde{x}')^2}{2u}\Bigg)\Bigg)dz',$$

$$K_2 = \int_0^{+\infty} \Bigg(\exp\Bigg(-\frac{(z'-\tilde{x}-\tilde{h})^2}{2u}\Bigg) - \exp\Bigg(-\frac{(z'-\tilde{x})^2}{2u}\Bigg)\Bigg) \times \Bigg(\exp\Bigg(-\frac{(z'+\tilde{x}'+\tilde{h})^2}{2u}\Bigg) - \exp\Bigg(-\frac{(z'+\tilde{x}')^2}{2u}\Bigg)\Bigg)dz',$$

$$K_3 = \int_0^{+\infty} \Bigg(\exp\Bigg(-\frac{(z'+\tilde{x}+\tilde{h})^2}{2u}\Bigg) - \exp\Bigg(-\frac{(z'+\tilde{x})^2}{2u}\Bigg)\Bigg) \times \Bigg(\exp\Bigg(-\frac{(z'-\tilde{x}'-\tilde{h})^2}{2u}\Bigg) - \exp\Bigg(-\frac{(z'-\tilde{x}')^2}{2u}\Bigg)\Bigg)dz',$$

$$K_4 = \int_0^{+\infty} \Bigg(\exp\Bigg(-\frac{(z'+\tilde{x}+\tilde{h})^2}{2u}\Bigg) - \exp\Bigg(-\frac{(z'+\tilde{x})^2}{2u}\Bigg)\Bigg) \times \Bigg(\exp\Bigg(-\frac{(z'+\tilde{x}'+\tilde{h})^2}{2u}\Bigg) - \exp\Bigg(-\frac{(z'+\tilde{x}')^2}{2u}\Bigg)\Bigg)dz'.$$

By using the change of variable $z = -z'$, we get

$$K_2 + K_3 = \int_{\mathbb{R}} \Bigg(\exp\Bigg(-\frac{(z'+\tilde{x}+\tilde{h})^2}{2u}\Bigg) - \exp\Bigg(-\frac{(z'+\tilde{x})^2}{2u}\Bigg)\Bigg) \times \Bigg(\exp\Bigg(-\frac{(z'-\tilde{x}'-\tilde{h})^2}{2u}\Bigg) - \exp\Bigg(-\frac{(z'-\tilde{x}')^2}{2u}\Bigg)\Bigg)dz'$$

and

$$
\begin{aligned}
& K_1 + \beta^2 K_4 \\
= & \int_{\mathbb{R}} \left(\exp\left(-\frac{(z' - \tilde{x} - \tilde{h})^2}{2u} \right) - \exp\left(-\frac{(z' - \tilde{x})^2}{2u} \right) \right) \\
& \times \left(\exp\left(-\frac{(z' - \tilde{x}' - \tilde{h})^2}{2u} \right) - \exp\left(-\frac{(z' - \tilde{x}')^2}{2u} \right) \right) dz' \\
& + (\beta^2 - 1) \int_{-\infty}^{0} \left(\exp\left(-\frac{(z' - \tilde{x} - \tilde{h})^2}{2u} \right) - \exp\left(-\frac{(z' - \tilde{x})^2}{2u} \right) \right) \\
& \times \left(\exp\left(-\frac{(z' - \tilde{x}' - \tilde{h})^2}{2u} \right) - \exp\left(-\frac{(z' - \tilde{x}')^2}{2u} \right) \right) dz'.
\end{aligned}
$$

Therefore,

$$
L + K = L_1 + \beta L_2 + \left(\sqrt{\frac{a_2}{a_1}} (1 - \beta)^2 + \beta^2 - 1 \right) L_3, \tag{45}
$$

where

$$
\begin{aligned}
L_1 &= \int_0^t \frac{du}{2\sqrt{a_2}\pi u} \int_{\mathbb{R}} \left(\exp\left(-\frac{(z' - \tilde{x} - \tilde{h})^2}{2u} \right) - \exp\left(-\frac{(z' - \tilde{x})^2}{2u} \right) \right) \\
& \quad \times \left(\exp\left(-\frac{(z' - \tilde{x}' - \tilde{h})^2}{2u} \right) - \exp\left(-\frac{(z' - \tilde{x}')^2}{2u} \right) \right) dz', \\
L_2 &= \int_0^t \frac{du}{2\sqrt{a_2}\pi u} \int_{\mathbb{R}} \left(\exp\left(-\frac{(z' + \tilde{x} + \tilde{h})^2}{2u} \right) - \exp\left(-\frac{(z' + \tilde{x})^2}{2u} \right) \right) \\
& \quad \times \left(\exp\left(-\frac{(z' - \tilde{x}' - \tilde{h})^2}{2u} \right) - \exp\left(-\frac{(z' - \tilde{x}')^2}{2u} \right) \right) dz'
\end{aligned}
$$

and

$$
\begin{aligned}
L_3 &= \int_0^t \frac{du}{2\sqrt{a_2}\pi u} \int_{-\infty}^{0} \left(\exp\left(-\frac{(z' - \tilde{x} - \tilde{h})^2}{2u} \right) - \exp\left(-\frac{(z' - \tilde{x})^2}{2u} \right) \right) \\
& \quad \times \left(\exp\left(-\frac{(z' - \tilde{x}' - \tilde{h})^2}{2u} \right) - \exp\left(-\frac{(z' - \tilde{x}')^2}{2u} \right) \right) dz'.
\end{aligned}
$$

We first investigate the sign of the third integral. On the one hand, $z' \leq 0$, $\tilde{x} \geq 0$ and $\tilde{h} \geq 0$; thus, by virtue of the fact that the function $x \longmapsto \exp(-x^2)$ is increasing on the interval $(-\infty, 0]$, we see that $L_3 \geq 0$. On the other hand, using the expression of β, given in (32), by the fact that $\rho_2 \geq \rho_1$ (see Condition (33)), we get

$$
\sqrt{\frac{a_2}{a_1}} (1 - \beta)^2 + \beta^2 - 1 = \frac{-4\rho_1 \sqrt{a_1} \sqrt{a_2}}{(\rho_2 \sqrt{a_2} + \rho_1 \sqrt{a_1})^2} (\rho_2 - \rho_1) \leq 0.
$$

Therefore,

$$
\left(\sqrt{\frac{a_2}{a_1}} (1 - \beta)^2 + \beta^2 - 1 \right) L_3 \leq 0. \tag{46}
$$

Now we will calculate explicitely the integral L_1. With the notation

$$\mathcal{T}(x, y, u) := \int_{\mathbb{R}} \exp\left(-\frac{(v-y)^2}{2u}\right) \exp\left(-\frac{(v-x)^2}{2u}\right) dv, \tag{47}$$

for every $x, y \in \mathbb{R}$ and $u > 0$, L_1 can be written in the form

$$\begin{aligned} L_1 \quad = \quad & \int_0^t \frac{1}{2\sqrt{a_2}\pi u} \left\{ \mathcal{T}\big(\tilde{x} + \tilde{h}, \tilde{x}' + \tilde{h}, u\big) - \mathcal{T}\big(\tilde{x}, \tilde{x}' + \tilde{h}, u\big) \right. \\ & \left. - \mathcal{T}\big(\tilde{x} + \tilde{h}, \tilde{x}', u\big) + \mathcal{T}\big(\tilde{x}, \tilde{x}', u\big) \right\} du. \end{aligned}$$

By the changes of variables $V = v - x$ and $W = \frac{y-x-2v}{2\sqrt{u}}$, we get

$$\begin{aligned} \mathcal{T}(x, y, u) \quad &= \quad \int_{\mathbb{R}} \exp\left(-\frac{v^2}{2u}\right) \exp\left(\frac{-((y-x)-v)^2}{2u}\right) dv \\ &= \quad \exp\left(-\frac{(y-x)^2}{4u}\right) \int_{\mathbb{R}} \exp\left(-\frac{((y-x)-2v)^2}{4u}\right) dv \\ &= \quad \sqrt{\pi u}\, \exp\left(-\frac{(y-x)^2}{4u}\right). \end{aligned}$$

Thus, applying an integration by parts then the change of variables $w = \frac{y-x}{2\sqrt{u}}$, we get

$$\begin{aligned} & \int_0^t \frac{1}{2\pi u} \mathcal{T}(x, y, u)\, du := \int_0^t \frac{1}{2\sqrt{\pi u}} \exp\left(-\frac{(y-x)^2}{4u}\right) du \\ = \quad & \sqrt{\frac{t}{\pi}} \exp\left(-\frac{(y-x)^2}{4t}\right) - \frac{(y-x)^2}{4\sqrt{\pi}} \int_0^t u^{-3/2} \exp\left(-\frac{(y-x)^2}{4u}\right) du \\ = \quad & \sqrt{\frac{t}{\pi}} \exp\left(-\frac{(y-x)^2}{4t}\right) - \frac{1}{2}(y-x)\, \mathbf{erfc}\left(\frac{y-x}{2\sqrt{t}}\right). \end{aligned}$$

Hence,

$$\begin{aligned} L_1 \quad = \quad & \sqrt{\frac{t}{a_2\pi}} \left\{ 2\exp\left(-\frac{(\tilde{x}' - \tilde{x})^2}{4t}\right) - \exp\left(-\frac{(\tilde{x}' - \tilde{x} + \tilde{h})^2}{4t}\right) \right. \\ & \left. - \exp\left(-\frac{(\tilde{x}' - \tilde{x} - \tilde{h})^2}{4t}\right) \right\} \\ & - \frac{1}{2\sqrt{a_2}} \left\{ 2\left(\tilde{x}' - \tilde{x}\right) \mathbf{erfc}\left(\frac{\tilde{x}' - \tilde{x}}{2\sqrt{t}}\right) - \left(\tilde{x}' - \tilde{x} + \tilde{h}\right) \mathbf{erfc}\left(\frac{\tilde{x}' - \tilde{x} + \tilde{h}}{2\sqrt{t}}\right) \right. \\ & \left. - \left(\tilde{x}' - \tilde{x} - \tilde{h}\right) \mathbf{erfc}\left(\frac{\tilde{x}' - \tilde{x} - \tilde{h}}{2\sqrt{t}}\right) \right\}. \end{aligned}$$

The function $\tilde{h} \longmapsto L_1(\tilde{h}) = L_1$ is clearly twice differentiable and via a simple calculation we get

$$L_1'(\tilde{h}) = -\frac{1}{2\sqrt{a_2}} \left\{ \mathbf{erfc}\left(\frac{\tilde{x}' - \tilde{x} - \tilde{h}}{2\sqrt{t}}\right) - \mathbf{erfc}\left(\frac{\tilde{x}' - \tilde{x} + \tilde{h}}{2\sqrt{t}}\right) \right\}$$

and

$$L_1''(\tilde{h}) = \frac{-1}{2\sqrt{a_2}\sqrt{\pi t}}\left[\exp\left(-\frac{(\tilde{x}'-\tilde{x}-h)^2}{4t}\right)+\exp\left(-\frac{(\tilde{x}'-\tilde{x}+h)^2}{4t}\right)\right].$$

It's easy to check that $L_1(0) = L_1'(0) = 0$ and that L_1'' is bounded. From all this and by Taylor's formula, we obtain

$$L_1(\tilde{h}) \leqslant C\,\tilde{h}^2 \leq Ch^2 \tag{48}$$

for every $h > 0$, where C denotes a positive universal constant.

Applying the same techniques used in the above argunents, and since $\beta \geq 0$ (see the expression of β given in (32) and Assumption (33)), we get

$$\beta L_2(\tilde{h}) \leqslant C\,\tilde{h}^2 \leq Ch^2 \tag{49}$$

for every $h > 0$. Combining (44), (31), (46), (48) and (49), the proof of $H_3([0, 1])$ is finished, and consequently, the proof of Theorem 6 is also finshed.

Remark 4. Considering an integer $d \geq 1$, one can extend Equation (5) to the d-dimensional case by introducing the following SPDE:

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = \mathcal{L}_d u(t,x) + \dot{W}_d(t,x); \quad t>0,\ x=(x_1,\ldots,x_d)\in\mathbb{R}^d, \\ u(0,.) := 0, \end{cases} \tag{50}$$

with

$$\mathcal{L}_d = \sum_{i,j=1}^{d} \frac{1}{r_{ij}(x)}\frac{\partial}{\partial x_i}\left(R_{ij}(x)\frac{\partial}{\partial x_j}\right), \tag{51}$$

where $x \longmapsto R_{ij}(x)$ and $x \longmapsto r_{ij}(x)$ are two measurable and bounded real-valued functions satisfying

$$r_{ij}(x) = r_{ji}(x), \qquad R_{ij}(x) = R_{ji}(x),$$

and there exists a constant $\nu > 0$ such that

$$r_{ij}(x)\xi_i\xi_j \geq \nu\|\xi\|_d^2 \quad \text{and} \quad R_{ij}(x)\xi_i\xi_j \geq \nu\|\xi\|_d^2 \tag{52}$$

for every $x \in \mathbb{R}^d$, $\xi = (\xi_1,\ldots,\xi_d) \in \mathbb{R}^d$ and $i, j \in \{1,\ldots,d\}$. In (52), $\|.\|_d$ denotes the Euclidean norm in $\mathbb{R}^d$, and in (51) $\frac{\partial}{\partial x_i}$ denotes the partial derivative in the distributional sense. The noise W_d is a centered Gaussian field $W_d = \{W_d(t,C); t \in [0,T], C \in B_b(\mathbb{R}^d)\}$ with covariance

$$\mathbb{E}\big(W_d(t,C)W_d(s,D)\big) = (t\wedge s)\lambda_d(C\cap D), \tag{53}$$

where λ_d denotes the Lebesgue measure on $\mathbb{R}^d$ and $B_b(\mathbb{R}^d)$ is the set of λ_d-bounded Borel sub-sets of $\mathbb{R}^d$. In the particular case where $d = 1$, SPDE (50) is clearly reduced to Equation (5).

According to [1], if we denote by G_d the fundamental solution associated to the operator $\mathcal{L}_d$, then there exist two constants $D_1 > 0$ and $D_2 > 0$ such that

$$G_d(t,x,y) \geq \frac{D_1}{t^{d/2}} \exp\Big(-\frac{D_2\|x-y\|_d^2}{t}\Big)$$

for every $t \in [0,T]$ and $(x,y) \in \mathbb{R}^d$. It follows then that

$$\int_0^t \int_{\mathbb{R}^d} G_d^2(t-s,x,y)dyds \geq \int_0^t \int_{\mathbb{R}^d} \frac{D_1^2}{(t-s)^d} \exp\Big(-\frac{2D_2\|x-y\|_d^2}{t-s}\Big)dyds. \quad (54)$$

Denoting by I the right-hand side of Inequality (54), we have

$$\begin{aligned} I &= \int_0^t \frac{D_1^2}{(t-s)^d} \prod_{i=1}^d \Big(\int_{\mathbb{R}} \exp\Big(-\frac{2D_2(x_i-y_i)^2}{t-s}\Big)dy_i\Big)ds \\ &= \int_0^t \frac{D_1^2}{(t-s)^d}\Big(\sqrt{\frac{\pi(t-s)}{2D_2}}\Big)^d ds \\ &= D_1^2\Big(\frac{\pi}{2D_2}\Big)^{d/2} \int_0^t \frac{ds}{(t-s)^{d/2}}, \end{aligned} \quad (55)$$

where the second equality in (55) is obtained by the change of variables $y_i' = (x_i - y_i)\sqrt{\frac{2D_2}{t-s}}$. Since the term $\int_0^t \frac{ds}{(t-s)^{d/2}}$ is finite if, and only if $d < 2$, from (54) and (55) we deduce that, for every $d \geq 2$ we have

$$\int_0^t \int_{\mathbb{R}^d} G_d^2(t-s,x,y)dyds = +\infty.$$

Therefore, for $d \geq 2$, the Wiener integral $\int_0^t \int_{\mathbb{R}^d} G_d(t-s,x,y)W_d(ds,dy)$ is not well-defined and consequently, the mild solution to Equation (50) exists if, and only if $d = 1$.

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