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Estimation of the drift parameter for the fractional stochastic heat equation via power variation

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Abstract We define power variation estimators for the drift parameter of the stochastic heat equation with the fractional Laplacian and an additive Gaussian noise which is white in time and white or correlated in space. We prove that these estimators are consistent and asymptotically normal and we derive their rate of convergence under the Wasserstein metric.

Keywords Stochastic heat equation, fractional Brownian motion, fractional Laplacian, q variation, drift parameter estimation

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1 Introduction

The purpose of this work is to estimate the drift parameter $\theta > 0$ of the fractional stochastic heat equation

$$\frac{\partial u_\theta}{\partial t}(t,x) = -\theta(-\Delta)^{\frac{\alpha}{2}} u_\theta(t,x) + \dot{W}(t,x), \quad t \geq 0, x \in \mathbb{R}, \tag{1}$$

with vanishing initial conditions, where $(-\Delta)^{\frac{\alpha}{2}}$ denotes the fractional Laplacian of order $\alpha \in (1,2]$, $\theta > 0$ and W is a Gaussian noise which is white in time and white or correlated in space.

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The parameter estimation for stochastic partial differential equations (SPDEs in the sequel) constitutes a research direction of wide interest in probability theory and mathematical statistics. We refer, among many others, to the recents surveys [16] and [4]. On the other side, there are relatively few works that consider the solution to a SPDE observed at discrete points in time and/or in space. Among the first works in this direction, we refer to [18] and [17] for the maximum likelihood and least square estimators for parabolic, respectively elliptic type SPDEs driven by a space-time white noise. The study in [17] has been then extended in [2], by adding a time-varying volatility in the noise term and by using power variation techniques to estimate the parameter of the model. Other recent works on parameter estimates for discretely sampled SPDEs via power variations are [5, 3, 1, 21] and [24].

In this paper, we extend the above results into two directions. Firstly, we replace the standard Laplacian operator used in all the above references by a fractional Laplacian. On the other hand, we consider a simpler form, comparing to [2, 17], of the differential operator. Secondly, we also consider a noise term which is correlated in space. Our purpose is to propose power variation type estimators for the drift parameter in the stochastic model (1), based on discrete observations of the solution in time or in space, and to analyze the consistency and the limit distribution of the estimators by taking advantage of the link between the solution and the fractional Brownian motion. Our approach to construct and analyze the estimators for the drift parameter is based on the asymptotic behavior of the q-variations of the mild solution to (1). It is well known (see, e.g., [7, 13, 23]) that there exists a strong link between the law of this mild solution with $\theta = 1$ and the fractional Brownian motion and related processes. We will use this connection in order to deduce the behavior of the q-variations (of suitable order q) of the solutions to (1) and to prove the consistency, asymptotic normality and Berry–Esséen bounds under the Wasserstein distance for the associated estimators. For the situation when W is a space-time white noise, we will obtain two estimators for the drift parameter: one based on the temporal variations and one based of the spatial variations of the mild solution u_θ. Similarly, two estimators are defined when the Gaussian noise W is white in time and colored in space (with the spatial covariance given by the Riesz kernel). Even if the order of the variations which appear in the definition of the estimator is different in the four cases (this order may depend on the parameter α of the fractional Laplacian and/or on the spatial correlation), all the estimators are asymptotically normal, they have the same rate of convergence of order $n^{-\frac{1}{2}}$ and they have the same distance to the Gaussian distribution. The case of the standard Laplacian (i.e., $\alpha = 2$) has been studied in [21].

We organize the paper as follows. In Section 2 we present general facts on the stochastic heat equation with the fractional Laplacian and the behavior of the variations of the perturbed fractional Brownian motion. In Section 3 we discuss the drift parameter estimation for the fractional heat equation with a space-time white noise while in Section 4 we treat the case when the noise is correlated in space.

We will denote by c, C a generic positive constant that may change from line to line (or even inside of the the same line). By $\to^{(d)}$ we denote the convergence in distribution while $\equiv^{(d)}$ stands for the equivalence of finite dimensional distributions.

2 The fractional heat equation driven by a space-time white noise

We start by treating the fractional stochastic heat equation with a space-time white noise. We recall the basic properties of the solution, its relation with the fractional Brownian motion and then we discuss the estimation of the drift parameter θ via the q-variations.

2.1 General properties of the solution

On the standard probability space $(\Omega, \mathcal{F}, P)$, we consider a centered Gaussian field $(W(t, A), t \geq 0, A \in \mathcal{B}_b(\mathbb{R}))$ with covariance

$$\mathbf{E}W(t, A)W(s, B) = (s \wedge t)\lambda(A \cap B) \quad \text{for every } s, t \geq 0, A, B \in \mathcal{B}_b(\mathbb{R}), \tag{2}$$

where λ denotes the Lebesgue measure on $\mathbb{R}$ and $\mathcal{B}_b(\mathbb{R})$ is the class of bounded Borel subsets of $\mathbb{R}$. The Gaussian field W is usually called the space-time white noise.

We will consider the stochastic heat equation

$$\frac{\partial u_\theta}{\partial t}(t, x) = -\theta(-\Delta)^{\frac{\alpha}{2}} u_\theta(t, x) + \dot{W}(t, x), \quad t \geq 0, x \in \mathbb{R}, \tag{3}$$

with vanishing initial condition $u(0, x) = 0$ for every $x \in \mathbb{R}$. In the above equation, $(-\Delta)^{\frac{\alpha}{2}}$ represents the fractional Laplacian of order α. We will assume in the sequel that $\alpha \in (1, 2]$. We refer to [6, 11, 10, 12] for the precise definition and other properties of the fractional Laplacian operator. We will denote its Green kernel (or the fundamental solution) by G_α, which represents the deterministic kernel that solves the heat equation without noise $\frac{\partial}{\partial t}u(t, x) = -(-\Delta)^{\frac{\alpha}{2}} u(t, x)$. It is know from the above references that for $t > 0, x \in \mathbb{R}$

$$G_\alpha(t, x) = \int_{\mathbb{R}} e^{it\xi - t|\xi|^\alpha} d\xi. \tag{4}$$

It is an immediate conclusion that the fundamental solution associated to the operator $-\theta(-\Delta)^{\frac{\alpha}{2}} u_\theta(t, x)$ is $G_\alpha(\theta t, x)$.

The solution to (3) is understood in the mild sense, i.e.,

$$u_\theta(t, x) = \int_0^t \int_{\mathbb{R}} G_\alpha(\theta(t-s), x-y) W(ds, dy), \tag{5}$$

where the stochastic integral $W(ds, dy)$ is the usual Wiener integral with respect to the space-time white noise, which satisfies the isometry

$$\mathbf{E}\left(\int_0^T \int_{\mathbb{R}} H(s, y) W(ds, dy)\right)^2 = \int_0^T \int_{\mathbb{R}} H(s, y)^2 dy ds$$

for every $T > 0$ and for every measurable square integrable function H.

For $\theta = 1$, the solution to the heat equation (3) has been studied in [13]. This solution exists only if the spatial dimension is $d = 1$, and it is connected to the bifractional Brownian motion. Recall that (see [9, 22]), given constants $H \in (0, 1)$

and $K \in (0, 1]$, the bifractional Brownian motion (bi-fBm for short) $(B_t^{H,K})_{t\geq 0}$ is a centered Gaussian process with covariance

$$R^{H,K}(t,s) := R(t,s) = \frac{1}{2^K}\left(\left(t^{2H}+s^{2H}\right)^K - |t-s|^{2HK}\right), \qquad s,t \geq 0. \quad (6)$$

In particular, for $K = 1$, $B^{H,} := B^{H,1}$ is the fractional Brownian motion (fBm in the sequel) with the Hurst parameter $H \in (0, 1)$.

Let us recall some of the results in [13] which will be needed in the sequel.

- The mild solution (5) is well-defined. For every $x \in \mathbb{R}$, the process $(u_1(t,x), t \geq 0)$ coincides in distribution, modulo a constant, with the bifractional Brownian motion, i.e.,

$$(u_1(t,x), t \geq 0) \equiv^{(d)} \left(c_{2,\alpha} B_t^{\frac{1}{2}, 1-\frac{1}{\alpha}}, t \geq 0\right),$$

  where $B^{\frac{1}{2}, 1-\frac{1}{\alpha}}$ is a bifractional Brownian motion with the Hurst parameters $H = \frac{1}{2}$ and $K = 1 - \frac{1}{\alpha}$ and

$$c_{2,\alpha}^2 = c_{1,\alpha} 2^{1-\frac{1}{\alpha}} \text{ with } c_{1,\alpha} = \frac{1}{2\pi(\alpha-1)}\Gamma\left(\frac{1}{\alpha}\right). \quad (7)$$

- For every $t \geq 0$, we have (see Proposition 3.1 in [7])

$$(u_1(t,x), x \in \mathbb{R}) \equiv^{(d)} \left(m_\alpha B^{\frac{\alpha-1}{2}}(x) + S_t(x), x \in \mathbb{R}\right), \quad (8)$$

  where $B^{\frac{\alpha-1}{2}}$ is a fractional Brownian motion with the Hurst parameter $\frac{\alpha-1}{2} \in [0, \frac{1}{2}]$, $(S_t(x))_{x\in\mathbb{R}}$ is a centered Gaussian process with C^∞ sample paths and m_α is an explicit numerical constant.

The above facts, combined with the decomposition (18) of the bifractional Brownian motion, show that the solution to the heat equation can be expressed as the sum of a fBm and a smooth process (we will call this sum as a perturbed fractional Brownian motion).

2.2 *Variations of the perturbed fractional Brownian motion*

Since the process (5) is connected to the perturbed fBm (i.e., the sum of a fBm and a smooth Gaussian process), let us recall some facts concerning the asymptotic behavior of the variation of the perturbed fBm. Some of the below results are directly taken from [13] while those concerning the rate of convergence under the Wasserstein distance are deduced from [19].

We first define the notion of *(exact) q-variation* for stochastic processes.

Definition 1. Let $A_1 < A_2$, and for $n \geq 1$, let $t_i = A_1 + \frac{i}{n}(A_2 - A_1)$ for $i = 0, \ldots, n$. A continuous stochastic process $(X_t)_{t\geq 0}$ admits a q-variation (or a variation of order q) over the interval $[A_1, A_2]$ if the sequence

$$S_{[A_1,A_2]}^{n,q}(X) := \sum_{i=0}^{n-1} \left|X_{t_{i+1}} - X_{t_i}\right|^q$$

converges in probability as $n \to \infty$. The limit, when it exists, is called the exact q-variation of X over the interval $[A_1, A_2]$.

If $[A_1, A_2] = [0, t]$, we will simply denote $S_t^{n,q}(X) := S_{[0,t]}^{n,q}(X)$. Moreover, if $t = 1$, we denote $S^{q,n}(X) := S_t^{n,q}(X)$. In the case $q = 2$ the limit of $S^{2,n}$ is called the quadratic variation, while for $q = 3$ we have the cubic variation.

Let us recall the following result (see [13]) concerning the exact variation of the perturbed fractional Brownian motion, i.e., the sum of a fBm and a smooth Gaussian process. In the rest of this section, we will fix an interval $[A_1, A_2]$ with $A_1 < A_2$ and a partition $t_j = A_1 + \frac{j}{n}(A_2 - A_1)$, $n \geq 1$, $j = 0, \ldots, n$, of this interval. Also, we denote by Z a standard normal random variable, and $\mu_q = \mathbf{E}Z^q$ for $q \geq 1$. Define $\sigma_{H,q}^2 = q! \sum_{v \in \mathbb{Z}} \rho_H(v)^q$, with $\rho_H(v) = \frac{1}{2}\left(|v+1|^{2H} + |v-1|^{2H} - 2|v|^{2H}\right)$ for $v \in \mathbb{Z}$.

Lemma 1. *Let $(B_t^H)_{t\geq 0}$ be a fBm with $H \in (0, \frac{1}{2}]$ and consider a centered Gaussian process $(X_t)_{t\geq 0}$ such that*

$$\mathbf{E}\,|X_t - X_s|^2 \leq C|t-s|^2 \quad \textit{for every } s, t \geq 0. \tag{9}$$

Define

$$Y_t^H = aB_t^H + X_t \quad \textit{for every } t \geq 0$$

with $a \neq 0$.

1. *The process Y has $\frac{1}{H}$-variation over the interval $[A_1, A_2]$ which is equal to*

$$a^{-\frac{1}{H}}\mathbf{E}|Z|^{1/H}(A_2 - A_1).$$

2. *Let*

$$V_{q,n}(Y^H) := \sum_{i=0}^{n-1}\left[\frac{n^{Hq}}{(A_2-A_1)^{qH}a^q}(Y_{t_{i+1}}^H - Y_{t_i}^H)^q - \mu_q\right]. \tag{10}$$

Then, if $H \in (0, \frac{1}{2})$ and $q \geq 2$ is an integer,

$$\frac{1}{\sqrt{n}}V_{q,n}(Y^H) = \frac{1}{\sqrt{n}}\sum_{i=0}^{n-1}\left[\frac{n^{Hq}}{(A_2-A_1)^{qH}a^q}(Y_{t_{i+1}}^H - Y_{t_i}^H)^q - \mu_q\right] \to^{(d)} N(0, \sigma_{H,q}^2). \tag{11}$$

If $H = \frac{1}{2}$, $q = 2$ and the process $(X_t)_{t\geq 0}$ is adapted to the filtration generated by B, then

$$\frac{1}{\sqrt{n}}V_{2,n}(Y^H) = \frac{1}{\sqrt{n}}\sum_{i=0}^{n-1}\left[\frac{n}{(A_2-A_1)a^2}(Y_{t_{i+1}}^{\frac{1}{2}} - Y_{t_i}^{\frac{1}{2}})^2 - 1\right] \to^{(d)} N(0, \sigma_{\frac{1}{2},2}^2). \tag{12}$$

Using the recent Stein–Malliavin theory, it is also possible to deduce the rate of convergence in the above Central Limit Theorem (CLT in the sequel) under the

Wasserstein distance. Before stating and proving the result, let us briefly recall the definition of the Wasserstein distance. The Wasserstein distance between the laws of two $\mathbb{R}^d$-valued random variables F and G is defined as

$$d_W(F,G) = \sup_{h\in\mathcal{A}} |\mathbf{E}h(F) - \mathbf{E}h(G)| \tag{13}$$

where $\mathcal{A}$ is the class of Lipschitz continuous function $h : \mathbb{R}^d \to \mathbb{R}$ such that $\|h\|_{Lip} \leq 1$, where

$$\|h\|_{Lip} = \sup_{x,y\in\mathbb{R}^d, x\neq y} \frac{|h(x)-h(y)|}{\|x-y\|_{\mathbb{R}^d}}.$$

Proposition 1. *Assume $H \leq \frac{1}{2}$. Let Y^H be as in Lemma 1 and let $V_{q,n}(Y^H)$ be given by (10). Then for n large and with $\sigma_{H,q}$ from (11),*

$$d_W\left(\frac{1}{\sqrt{n}}V_{q,n}(Y^H), N(0,\sigma^2_{H,q})\right) \leq C\frac{1}{\sqrt{n}}.$$

Proof. From the proof of Lemma 2.1 in [13], we can express the variation of Y^H and the variation of the fBm B^H plus a rest term, i.e.,

$$\frac{1}{\sqrt{n}}V_{q,n}(Y^H) = \frac{1}{\sqrt{n}}V_{q,n}(B^H) + R_n,$$

where R_n satisfies, for every $n \geq 1$,

$$\mathbf{E}|R_n| \leq cn^{H-1}. \tag{14}$$

By the definition of the Wasserstein distance, we can write

$$\begin{aligned} & d_W\left(\frac{1}{\sqrt{n}}V_{q,n}(Y^H), N(0,\sigma^2_{H,q})\right) \\ \leq\ & d_W\left(\frac{1}{\sqrt{n}}V_{q,n}(B^H), N(0,\sigma^2_{H,q})\right) + d_W\left(\frac{1}{\sqrt{n}}V_{q,n}(Y^H), \frac{1}{\sqrt{n}}V_{q,n}(B^H)\right) \\ \leq\ & d_W\left(\frac{1}{\sqrt{n}}V_{q,n}(B^H), N(0,\sigma^2_{H,q})\right) + \mathbf{E}|R_n|. \end{aligned}$$

In order to estimate $d_W(\frac{1}{\sqrt{n}}V_{q,n}(B^H), N(0,\sigma^2_{H,q}))$, we will use the chaos expansion of the random variable $V_{q,n}(B^H)$ and several results in [19]. Notice that (see, e.g., the proof of Corollary 3 in [20]),

$$V_{q,n}(B^H) = \sum_{k=1}^{q} k! C_q^k \mu_{q-k} \sum_{i=0}^{n-1} H_k\left(\frac{n^{HK}}{(A_2-A_1)^{HK}}\left(B^H_{t_{i+1}} - B^H_{t_i}\right)\right),$$

where H_k is the k-th probabilists' Hermite polynomial

$$H_k(x) = (-1)^k e^{-\frac{x^2}{2}} \frac{d^n}{dx^n}\left(e^{-\frac{x^2}{2}}\right)$$

for $k \geq 1$ with $H_0(x) = 1$. We know from [19] that the vector

$$(F_{1,n}, F_{2,n}, \dots, F_{q,n}) := \left(\frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} H_k \left(\frac{n^{HK}}{(A_2 - A_1)^{HK}} \left(B^H_{t_{i+1}} - B^H_{t_i} \right) \right) \right)_{k=1,\dots,q}$$

converges in distribution to a centered Gaussian vector with diagonal covariance matrix C (the explicit expression of C can be found in [19], it is not needed in our work). Moreover, Proposition 6.2.2 and Corollary 7.4.3 in [19] imply that

$$d_W \left((F_{k,n})_{k=1,\dots,q}, N(0, C) \right) \leq c \frac{1}{\sqrt{n}}.$$

This will easily lead to

$$d_W \left(\frac{1}{\sqrt{n}} V_{q,n}(B^H), N(0, \sigma^2_{H,q}) \right) \leq c \frac{1}{\sqrt{n}}. \tag{15}$$

Since $H \leq \frac{1}{2}$, we obtain the conclusion via (14) and (15). □

2.3 *Estimators for the drift parameter*

Our purpose is to estimate the parameter $\theta > 0$ based on the observations of the process u_θ. We will define two estimators: the first is based on the temporal variations of the process u_θ while the second is constructed via its variation in space. Their behavior is strongly related to the law of the process u_θ, therefore we start by analyzing the distribution of this Gaussian process.

2.3.1 *The law of the solution*

Let $G_\alpha(t, x)$ be the Green kernel associated to the operator $-(-\Delta)^{\frac{\alpha}{2}}$. Then the Green kernel associated to the operator operator $-\theta(-\Delta)^{\frac{\alpha}{2}}$ is

$$G_\alpha(\theta t, x).$$

Lemma 2. *Suppose that the process $(u_\theta(t, x), t \geq 0, x \in \mathbb{R})$ satisfies (3). Define*

$$v_\theta(t, x) := u_\theta \left(\frac{t}{\theta}, x \right), \quad t \geq 0, x \in \mathbb{R}. \tag{16}$$

Then the process $(v_\theta(t, x), t \geq 0, x \in \mathbb{R})$ satisfies

$$\frac{\partial v_\theta}{\partial t}(t, x) = -(-\Delta)^{\frac{\alpha}{2}} v_\theta(t, x) + (\theta)^{-\frac{1}{2}} \dot{\tilde{W}}(t, x), \quad t \geq 0, x \in \mathbb{R}, \tag{17}$$

with $v_\theta(0, x) = 0$ for every $x \in \mathbb{R}$, where $\dot{\tilde{W}}$ is a space-time white noise, i.e., a centered Gaussian random field with covariance (2).

Proof. From (5), we have for every $t \geq 0, x \in \mathbb{R}$,

$$v_\theta(t, x) = u_\theta \left(\frac{t}{\theta}, x \right) = \int_0^{\frac{t}{\theta}} \int_{\mathbb{R}} G_\alpha(t - \theta s, x - y) W(ds, dy)$$

$$= \int_0^t \int_{\mathbb{R}} G_\alpha(t-s, x-y) W(d\frac{s}{\theta}, dy)$$
$$= \theta^{-\frac{1}{2}} \int_0^t \int_{\mathbb{R}} G_\alpha(t-s, x-y) \tilde{W}(ds, dy),$$

where, for $t \geq 0$, $A \in \mathcal{B}(\mathbb{R})$, we denoted $\tilde{W}(t, A) := \theta^{\frac{1}{2}} W\left(\frac{t}{\theta}, A\right)$. Notice that $\tilde{W}$ has the same finite dimensional distributions as W, due to the scaling property of the white noise. □

We can deduce the law of the process u_θ in time and space.

Proposition 2. *For every $x \in \mathbb{R}$ and $\theta > 0$, we have*

$$(u_\theta(t, x), t \geq 0) \equiv^{(d)} \left(\theta^{-\frac{1}{2\alpha}} c_{2,\alpha} B_t^{\frac{1}{2}, 1-\frac{1}{\alpha}}, t \geq 0\right),$$

where $B^{\frac{1}{2}, 1-\frac{1}{\alpha}}$ is a bifractional Brownian motion with parameters $H = \frac{1}{2}$ and $K = 1 - \frac{1}{\alpha}$ and $c_{2,\alpha}$ is given by (7).

Proof. Fix $x \in \mathbb{R}$ and $\theta > 0$. Then for every $s, t \geq 0$, we have

$$\mathbf{E} u_\theta(t, x) u_\theta(s, x) = \mathbf{E} v_\theta(\theta t, x) v_\theta(\theta s, x)$$
$$= \theta^{-1} \mathbf{E} u_1(\theta t, x) u_1(\theta s, x) = \theta^{-1} c_{1,\alpha} \left[(\theta t + \theta s)^{1-\frac{1}{\alpha}} - |\theta t - \theta s|^{1-\frac{1}{\alpha}}\right]$$
$$= \theta^{-\frac{1}{\alpha}} c_{2,\alpha}^2 \mathbf{E} B_t^{\frac{1}{2}, 1-\frac{1}{\alpha}} B_s^{\frac{1}{2}, 1-\frac{1}{\alpha}}.$$
□

Proposition 3. *For every $t \geq 0$, $\theta > 0$, we have the following equality in distribution*

$$(u_\theta(t, x), x \in \mathbb{R}) \equiv^{(d)} \left(\theta^{-\frac{1}{2}} m_\alpha B^{\frac{\alpha-1}{2}}(x) + S_{\theta t}(x), x \in \mathbb{R}\right),$$

where $B^{\frac{\alpha-1}{2}}$ is a fractional Brownian motion with the Hurst parameter $\frac{\alpha-1}{2} \in (0, \frac{1}{2}]$, $(S_{\theta t}(x))_{x \in \mathbb{R}}$ is a centered Gaussian process with C^∞ sample paths and m_α from (8).

Proof. The result is immediate since for every $t > 0, \theta > 0$

$$(u_\theta(t, x), x \in \mathbb{R}) = (v_\theta(\theta t, x), x \in \mathbb{R}) \equiv^{(d)} \theta^{-\frac{1}{2}} (u_1(\theta t, x), x \in \mathbb{R})$$

$$\equiv^{(d)} \left(\theta^{-\frac{1}{2}} m_\alpha B^{\frac{\alpha-1}{2}}(x) + S_{\theta t}(x), x \in \mathbb{R}\right),$$

where we used (8). □

Notice that the Hurst parameter of the fBm in Proposition 3 may be $\frac{1}{2}$ if $\alpha = 2$.

2.3.2 Estimators based on the temporal variation

Proposition 2 indicates that the process u_θ behaves as a bi-fBm in time. Recall the following connection between the fBm and the bi-fBm (see [14]): Let $H \in (0,1)$, $K \in (0,1]$. If $(B_t^{HK})_{t\geq 0}$ is a fBm with the Hurst parameter HK and $(B_t^{H,K})_{t\geq 0}$ is a bi-fBm, then

$$\left(C_1 X_t^{H,K} + B_t^{H,K}, t \geq 0\right) \equiv^{(d)} \left(C_2 B_t^{HK}, t \geq 0\right), \tag{18}$$

with $C_1 > 0$ and $C_2 = 2^{\frac{1-K}{2}}$. In (18), $X^{H,K}$ is a Gaussian process, independent of $B^{H,K}$ with C^∞ sample paths. In particular, it satisfies (9). Therefore, the bi-fBm is a perturbed fBm and the same holds true for the solution $(u_\theta(t,x), t \geq 0)$, by Proposition 2. Therefore, we obtain, by using the notation $t_j = A_1 + \frac{j}{n}(A_2 - A_1)$, $n \geq 1$, $j = 0, \ldots, n$, the following lemma.

Lemma 3. *Let u_θ be the solution to (3). Then for every $x \in \mathbb{R}$,*

$$\begin{aligned} S_{[A_1,A_2]}^{n,\frac{2\alpha}{\alpha-1}} &:= \sum_{i=0}^{n-1} \left|u_\theta(t_{j+1},x) - u_\theta(t_j,x)\right|^{\frac{2\alpha}{\alpha-1}} \\ &\to_{n\to\infty} c_{2,\alpha}^{\frac{2\alpha}{\alpha-1}} 2^{\frac{1}{\alpha-1}} \mu_{\frac{2\alpha}{\alpha-1}} (A_2 - A_1)|(\theta)|^{\frac{-1}{\alpha-1}} \end{aligned} \tag{19}$$

in probability.

Relation (19) motivates the definition of the following estimator for the parameter $\theta > 0$ of the model (3):

$$\begin{aligned} &\widehat{\theta}_{n,1} \\ &= \left(\left(c_{2,\alpha}^{\frac{2\alpha}{\alpha-1}} 2^{\frac{1}{\alpha-1}} \mu_{\frac{2\alpha}{\alpha-1}} (A_2 - A_1) \right)^{-1} \sum_{i=0}^{n-1} \left|u_\theta(t_{j+1},x) - u_\theta(t_j,x)\right|^{\frac{2\alpha}{\alpha-1}} \right)^{1-\alpha} \\ &= \left(c_{2,\alpha}^{\frac{2\alpha}{\alpha-1}} 2^{\frac{1}{\alpha-1}} \mu_{\frac{2\alpha}{\alpha-1}} (A_2 - A_1) \right)^{\alpha-1} \left(S^{n,\frac{2\alpha}{\alpha-1}}(u_\theta(\cdot,x)) \right)^{1-\alpha}, \end{aligned} \tag{20}$$

and so

$$\widehat{\theta}_{n,1}^{\frac{1}{1-\alpha}} = \frac{1}{c_{2,\alpha}^{\frac{2\alpha}{\alpha-1}} 2^{\frac{1}{\alpha-1}} \mu_{\frac{2\alpha}{\alpha-1}} (A_2 - A_1)} \sum_{i=0}^{n-1} \left|u_\theta(t_{j+1},x) - u_\theta(t_j,x)\right|^{\frac{2\alpha}{\alpha-1}}. \tag{21}$$

We will prove the consistency and the asymptotic normality of the above estimator.

Proposition 4. *Assume $q := \frac{2\alpha}{\alpha-1}$ is an even integer and consider the estimator $\widehat{\theta}_{n,1}$ defined by (20). Then $\widehat{\theta}_{n,1} \to_{n\to\infty} \theta$ in probability and*

$$\sqrt{n}\left[\widehat{\theta}_{n,1}^{\frac{1}{1-\alpha}} - \theta^{\frac{1}{1-\alpha}}\right] \to^{(d)} N(0, s_{1,\theta,\alpha}^2) \text{ with } s_{1,\theta,\alpha}^2 = \sigma_{\frac{1}{q},q}^2 \theta^{\frac{2}{1-\alpha}} \mu_{\frac{2\alpha}{\alpha-1}}^{-2}. \tag{22}$$

Moreover, for n large enough

$$d_W\left(\sqrt{n}\left[\widehat{\theta}_{n,1}^{\frac{1}{1-\alpha}} - \theta^{\frac{1}{1-\alpha}}\right], N(0, s_{\theta,\alpha}^2)\right) \leq c\frac{1}{\sqrt{n}}.$$

Proof. From Proposition 2 and the relation between the fBm and the bi-fBm (18), we obtain that

$$\left(u_\theta(t,x)+c_{2,\alpha}\theta^{-\frac{1}{2\alpha}}X_t\right)\equiv^{(d)} c_{2,\alpha}\theta^{-\frac{1}{2\alpha}}2^{\frac{1}{2\alpha}}B^{\frac{\alpha-1}{2\alpha}},$$

where $B^{\frac{\alpha-1}{2\alpha}}$ is a fBm with the Hurst parameter $\frac{\alpha-1}{2\alpha}\in(0,\frac{1}{2})$. Therefore, u_θ is a perturbed fBm and we obtain, by taking $H=\frac{\alpha-1}{2\alpha}$ and $q=\frac{1}{H}=\frac{2\alpha}{\alpha-1}$,

$$\frac{1}{\sqrt{n}}\sum_{i=0}^{n-1}\left[\frac{n\theta^{\frac{1}{\alpha-1}}}{c_{2,\alpha}^{\frac{2\alpha}{\alpha-1}}2^{\frac{1}{\alpha-1}}(A_2-A_1)}\left(u_\theta(t_{j+1},x)-u_\theta(t_j,x)\right)^{\frac{2\alpha}{\alpha-1}}-\theta^{\frac{1}{1-\alpha}}\right]$$
$$\to^{(d)} N(0,\sigma^2_{\frac{1}{q},q}).$$

This means

$$\sqrt{n}\mu_{\frac{2\alpha}{\alpha-1}}\theta^{\frac{1}{\alpha-1}}\left[\widehat{\theta}_{n,1}^{\frac{1}{1-\alpha}}-\theta^{\frac{1}{1-\alpha}}\right]\to^{(d)} N(0,\sigma^2_{\frac{1}{q},q})$$

which is equivalent to (22). □

Using the so-called delta-method, we can get the asymptotic behavior of the estimator $\widehat{\theta}_n$. Recall that if $(X_n)_{n\geq 1}$ is a sequence of random variables such that

$$\sqrt{n}(X_n-\gamma_0)\to^{(d)} N(0,\sigma^2)$$

and g is a function such that $g'(\gamma_0)$ exists and does not vanish, then

$$\sqrt{n}(g(X_n)-g(\gamma_0))\to^{(d)} N(0,\sigma^2 g'(\gamma_0)^2). \tag{23}$$

Proposition 5. *Consider the estimator (20) and let $s_{1,\theta,\alpha}$ be given by (22). Then as $n\to\infty$,*

$$\sqrt{n}(\widehat{\theta}_{n,1}-\theta)\to N(0,s^2_{1,\theta,\alpha}(1-\alpha)^2\theta^{\frac{2\alpha}{\alpha-1}}), \tag{24}$$

and for n large enough,

$$d_W\left(\sqrt{n}(\widehat{\theta}_{n,1}-\theta),N(0,s^2_{1,\theta,\alpha}(1-\alpha)^2\theta^{\frac{2\alpha}{\alpha-1}})\right)\leq c\frac{1}{\sqrt{n}}.$$

Proof. By applying the delta-method for the function $g(x)=x^{1-\alpha}$, $X_n=\widehat{\theta}_{n,1}^{\frac{1}{1-\alpha}}$ and $\gamma_0=\theta^{\frac{1}{1-\alpha}}$, we immediately obtain the convergence (24). Concerning the rate of convergence, we can write, with $\widetilde{\gamma_0}$ a random point located between X_n and γ_0,

$$\begin{aligned}\sqrt{n}(g(X_n)-g(\gamma_0)) &= \sqrt{n}g'(\widetilde{\gamma_0})(X_n-\gamma_0)\\ &= g'(\gamma_0)\sqrt{n}(X_n-\gamma_0)+\sqrt{n}(X_n-\gamma_0)(g'(\widetilde{\gamma_0})-g'(\gamma_0))\\ &=: g'(\gamma_0)\sqrt{n}(X_n-\gamma_0)+T_n.\end{aligned}$$

We have, for n large,

$$\mathbf{E}|T_n| = \mathbf{E}\left|\sqrt{n}(X_n-\gamma_0)(g'(\widetilde{\gamma_0})-g'(\gamma_0))\right|$$

$$\leq \left(\mathbf{E}\left(\sqrt{n}(X_n-\gamma_0)\right)^2\right)^{\frac{1}{2}}\left(\mathbf{E}(g'(\widetilde{\gamma_0})-g'(\gamma_0))^2\right)^{\frac{1}{2}}$$
$$\leq c\left(\mathbf{E}(g'(\widetilde{\gamma_0})-g'(\gamma_0))^2\right)^{\frac{1}{2}} \leq c\left(\mathbf{E}\left(\widehat{\theta}_{n,1}^{\frac{\alpha}{\alpha-1}}-\theta^{\frac{\alpha}{\alpha-1}}\right)^2\right)^{\frac{1}{2}}$$
$$\leq c\left(\mathbf{E}\left(\widehat{\theta}_{n,1}^{\frac{1}{\alpha-1}}-\theta^{\frac{1}{\alpha-1}}\right)^2\right)^{\frac{1}{2}} \leq c\frac{1}{\sqrt{n}}$$

where we used the assumption $\alpha > 1$ for the first inequality of the line above and relation (22) (which gives in particular the $L^2(\Omega)$-convergence of $\widehat{\theta}_{n,1}^{\frac{1}{\alpha-1}}$ to $\theta^{\frac{\alpha}{\alpha-1}}$ as $n \to \infty$) for the second inequality on the same line. Therefore, by the triangle inequality and Proposition 4, for n large enough,

$$d_W\left(\sqrt{n}(\widehat{\theta}_{n,1}-\theta), N(0, s^2_{1,\theta,\alpha}(1-\alpha)^2\theta^{\frac{\alpha}{\alpha-1}})\right)$$
$$\leq c d_W\left(\sqrt{n}(X_n-\gamma_0), N(0, s^2_{1,\theta,\alpha})\right)+\mathbf{E}|T_n| \leq c\frac{1}{\sqrt{n}}. \qquad \square$$

2.3.3 Estimators based on the spatial variation

It is possible to define an estimator for the parameter θ based on the spatial variations of the solution (5). The result in Proposition 3 says that the process $(u_\theta(t,x), x \in \mathbb{R})$ is a perturbed fBm, so we know its exact variation in space. Below $x_j = A_1 + \frac{j}{n}(A_2 - A_1)$, $j = 0, \dots, n$, will denote a partition of the interval $[A_1, A_2]$.

Proposition 6. *Let u_θ be given by (3). Then*

$$\sum_{i=0}^{n-1}\left|u_\theta(t,x_{j+1})-u_\theta(t,x_j)\right|^{\frac{2}{\alpha-1}} \to_{n\to\infty} m_\alpha^{\frac{2}{\alpha-1}}\mu_{\frac{2}{\alpha-1}}(A_2-A_1)|\theta|^{\frac{-1}{\alpha-1}}$$

and if $q := \frac{2}{\alpha-1}$ is an integer,

$$\frac{1}{\sqrt{n}}\sum_{i=0}^{n-1}\left[\left(\frac{n}{m_\alpha^{\frac{2}{\alpha-1}}(A_2-A_1)}\right)\theta^{\frac{1}{\alpha-1}}(u_\theta(t,x_{i+1})-u_\theta(t,x_i))^{\frac{2}{\alpha-1}}-\mu_{\frac{2}{\alpha-1}}\right]$$
$$\to^{(d)} N(0, \sigma^2_{\frac{\alpha-1}{2},\frac{2}{\alpha-1}}).$$

Proposition 6 leads to the definition of the estimator

$$\widehat{\theta}_{n,2} = \left[(m_\alpha^{\frac{2}{\alpha-1}}\mu_{\frac{2}{\alpha-1}}(A_2-A_1))^{-1}\sum_{i=0}^{n-1}\left|u_\theta(t,x_{j+1})-u_\theta(t,x_j)\right|^{\frac{2}{\alpha-1}}\right]^{1-\alpha}, \qquad (25)$$

and we can immediately deduce from Proposition 3 its asymptotic proprieties.

Proposition 7. *The estimator (25) converges in probability as $n \to \infty$ to the parameter θ. Moreover, if $q := \frac{2}{\alpha-1}$ is an even integer,*

$$\sqrt{n}\left[\widehat{\theta}_{n,2}^{\frac{1}{1-\alpha}}-\theta^{\frac{1}{1-\alpha}}\right] \to^{(d)} N(0, s^2_{2,\theta,\alpha}) \text{ with } s^2_{2,\theta,\alpha} = \sigma^2_{\frac{\alpha-1}{2},\frac{2}{\alpha-1}}\mu^{-2}_{\frac{2}{\alpha-1}}\theta^{\frac{2}{1-\alpha}}. \qquad (26)$$

Moreover, for n large,

$$d_W\left(\sqrt{n}\left[\widehat{\theta}_{n,2}^{\frac{1}{1-\alpha}}-\theta^{\frac{1}{1-\alpha}}\right], N(0, s^2_{2,\theta,\alpha})\right)\leq c\frac{1}{\sqrt{n}}.$$

Proof. Using the law of the process $(u_\theta(t,x), x\in\mathbb{R})$ obtained in Proposition 3, we deduce that the Gaussian process $\left(\theta^{\frac{1}{2}}m_\alpha^{-1}u_\theta(t,x), x\in\mathbb{R}\right)$ is a perturbed fractional Brownian motion. Therefore, by relation (11) in Lemma 1,

$$\begin{aligned}&\frac{1}{\sqrt{n}}\sum_{i=0}^{n-1}\left(\frac{n\theta^{\frac{1}{\alpha-1}}}{(A_2-A_1)m_\alpha^{\frac{2}{\alpha-1}}}\left(u_\theta(t,x_{j+1})-u_\theta(t,x_j)\right)^{\frac{2}{\alpha-1}}-\mu_{\frac{2}{\alpha-1}}\right)\\ =\;&\sqrt{n}\mu_{\frac{2}{\alpha-1}}\theta^{\frac{1}{\alpha-1}}\left[\widehat{\theta}_{n,2}^{\frac{1}{1-\alpha}}-\theta^{\frac{1}{1-\alpha}}\right]\to^{(d)}_{n\to\infty}N\left(0,\sigma^2_{\frac{\alpha-1}{2},\frac{2}{\alpha-1}}\right).\end{aligned}$$

Moreover, Proposition 1 implies that

$$d_W\left(\sqrt{n}\mu_{\frac{2}{\alpha-1}}\theta^{\frac{1}{\alpha-1}}\left[\widehat{\theta}_{n,2}^{\frac{1}{1-\alpha}}-\theta^{\frac{1}{1-\alpha}}\right], N(0,\sigma^2_{\frac{\alpha-1}{2},\frac{2}{\alpha-1}})\right)\leq c\frac{1}{\sqrt{n}}$$

and this obviously leads to the desired conclusion. □

By using the delta-method, we can obtain the asymptotic distribution of $\widehat{\theta}_{n,2}$.

Proposition 8. *Let $\widehat{\theta}_{n,2}$ be given by (25). Then, with $s_{2,\theta,\alpha}$ from (26), as $n\to\infty$,*

$$\sqrt{n}(\widehat{\theta}_{n,2}-\theta)\to^{(d)}N\left(0, s^2_{2,\theta,\alpha}(1-\alpha)^2\theta^{\frac{2\alpha}{\alpha-1}}\right),$$

and for n large enough,

$$d_W\left(\sqrt{n}(\widehat{\theta}_{n,2}-\theta), N(0, s^2_{2,\theta,\alpha}(1-\alpha)^2\theta^{\frac{2\alpha}{\alpha-1}})\right)\leq c\frac{1}{\sqrt{n}}.$$

Proof. It suffices to apply (23) to the function $g(x)=x^{1-\alpha}$ and $\gamma_0=\theta^{\frac{1}{1-\alpha}}$ and to follow the proof of Proposition 5. □

Remark 1.

- The estimators (20) and (25) coincide with the estimators in [21] in the case of the standard Laplacian $\alpha=2$.
- The distance of the estimators (20) and (25) to their limit distribution is of the same order, although they involve q-variations with different q.

3 Heat equation with the fractional Laplacian and a white-colored noise

In this section, we will consider the stochastic heat equation with an additive Gaussian noise which behaves as a Wiener process in time and as a fractional Brownian motion in space, i.e. its spatial covariance is given by the so-called Riesz kernel. We will again study the distribution of the solution, its connection with the fractional and bifractional Brownian motion and we apply the q-variation method to obtain an asymptotically normal estimator for the drift parameter.

3.1 General properties of the solution

We will consider the stochastic heat equation

$$\frac{\partial}{\partial t}u_\theta(t,\mathbf{x}) = -\theta(-\Delta)^{\frac{\alpha}{2}}u_\theta(t,\mathbf{x}) + \dot{W}^\gamma(t,\mathbf{x}), \qquad t \geq 0, \mathbf{x} \in \mathbb{R}^d, \tag{27}$$

with $u_\theta(0,\mathbf{x}) = 0$ for every $\mathbf{x} \in \mathbb{R}^d$. In (27), $-(-\Delta)^{\frac{\alpha}{2}}$ denotes the fractional Laplacian with exponent $\frac{\alpha}{2}$, $\alpha \in (1,2]$, and W^γ is the so-called white-colored noise, i.e. $W^\gamma(t,A), t \geq 0, A \in \mathcal{B}(\mathbb{R}^d)$, is a centered Gaussian field with covariance

$$\mathbf{E}W^\gamma(t,A)W^\gamma(s,B) = (t \wedge s)\int_A\int_B f(\mathbf{x}-\mathbf{y})d\mathbf{x}d\mathbf{y}, \tag{28}$$

where f is the so-called Riesz kernel of order γ given by

$$f(\mathbf{x}) = R_\gamma(\mathbf{x}) := g_{\gamma,d}\|\mathbf{x}\|^{-d+\gamma}, \quad 0 < \gamma < d, \tag{29}$$

where $g_{\gamma,d} = 2^{d-\gamma}\pi^{d/2}\Gamma((d-\gamma)/2)/\Gamma(\gamma/2)$. As usual, the mild solution to (27) is given by

$$u_\theta(t,\mathbf{x}) = \int_0^t\int_{\mathbb{R}^d} G_\alpha(\theta(t-s),\mathbf{x}-\mathbf{z})W^\gamma(ds,d\mathbf{z}), \tag{30}$$

where the above integral $W^\gamma(ds,d\mathbf{z})$ is a Wiener integral with respect to the Gaussian noise W^γ.

We know the following facts concerning the mild solution (30) when $\theta = 1$.

- The mild solution (27) is well-defined as a square integrable process satisfying

  $$\sup_{t\in[0,T],\mathbf{x}\in\mathbb{R}^d} \mathbf{E}|u_1(t,\mathbf{x})|^2 < \infty$$

  if and only if

  $$d < \gamma + \alpha. \tag{31}$$

  In particular, condition (31) shows that the solution exists in any spatial dimension d, via suitable choice of the parameter γ.

- Assume (31) is satisfied. Then for every $\mathbf{x} \in \mathbb{R}^d$, we have the following equivalence in distribution

  $$(u_1(t,\mathbf{x}), t \geq 0) \equiv^{(d)} \left(c_{2,\alpha,\gamma}B_t^{\frac{1}{2},1-\frac{d-\gamma}{\alpha}}, t \geq 0\right), \tag{32}$$

  where $B^{\frac{1}{2},1-\frac{d-\gamma}{\alpha}}$ is a bifractional Brownian motion with the Hurst parameters $H = \frac{1}{2}$ and $K = 1 - \frac{d-\gamma}{\alpha}$ and

  $$c^2_{2,\alpha,\gamma} = c_{1,\alpha,\gamma}2^{1-\frac{d-\gamma}{\alpha}} \tag{33}$$

  with

  $$c_{1,\alpha,\gamma} = (2\pi)^{-d}\int_{\mathbb{R}^d} d\xi\|\xi\|^{-\gamma}e^{-\|\xi\|^\alpha}\frac{1}{2(1-\frac{d-\gamma}{\alpha})}.$$

- For every $t \geq 0$, we have (see Proposition 4.6 in [13])

$$\left(u(t,\mathbf{x}), \mathbf{x} \in \mathbb{R}^d\right) \equiv^{(d)} \left(m_{\alpha,\gamma} B^{\frac{\alpha+\gamma-d}{2}}(\mathbf{x}) + S_t(\mathbf{x}), \mathbf{x} \in \mathbb{R}^d\right), \tag{34}$$

where $B^{\frac{\alpha+\gamma-d}{2}}$ is an isotropic d-dimensional fractional Brownian motion (see the next section) with the Hurst parameter $\frac{\alpha+\gamma-d}{2}$, $(S_t(\mathbf{x}))_{\mathbf{x}\in\mathbb{R}^d}$ is a centered Gaussian process with C^∞ sample paths and $m^2_{\alpha,\gamma}$ is an explicit numerical constant.

3.2 Perturbed isotropic fractional Brownian motion

Since the law of the solution (30) is related to the isotropic fBm, let us recall the definition of this process. The isotropic d-parameter fBm (also known as the Lévy fBm) $(B^H_d(\mathbf{x}), \mathbf{x} \in \mathbb{R}^d)$ with the Hurst parameter $H \in (0,1)$ is defined as a centered Gaussian process, starting from zero, with covariance function

$$\mathbf{E}(B^H_d(\mathbf{x})B^H_d(\mathbf{y})) = \frac{1}{2}\left(\|\mathbf{x}\|^{2H} + \|\mathbf{y}\|^{2H} - \|\mathbf{x}-\mathbf{y}\|^{2H}\right) \quad \text{for every } \mathbf{x}, \mathbf{y} \in \mathbb{R}^d, \tag{35}$$

where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^d$. It can be also represented as a Wiener integral with respect to the Wiener sheet, see [8, 15].

As in the one-parameter case, we define the q-variation of the isotropic fBm as the limit in probability as $n \to \infty$ of the sequence

$$S^{n,q}_{[A_1,A_2]}(B^H) = \sum_{i=0}^{n-1} \left|B^H_d(\mathbf{x}_{i+1}) - B^H_d(\mathbf{x}_i)\right|^q,$$

where $\mathbf{x}_i = (x_i^{(1)}, \ldots, x_i^{(d)})$ with $x_i^{(j)} = A_1 + \frac{i}{n}(A_2 - A_1)$ for $i = 0, \ldots, n$ and $j = 1, \ldots, d$. And from [13] we know that the isotropic fBm $(B^H(\mathbf{x}))_{\mathbf{x}\in\mathbb{R}^d}$ has $\frac{1}{H}$-variation over $[A_1, A_2]$ which is equal to

$$(A_2 - A_1)\mathbf{E}|B^H_d(\mathbf{1})|^{1/H} = (A_2 - A_1)\sqrt{d}\mathbf{E}|Z|^{1/H}.$$

The q-variation of the isotropic fBm perturbed by a regular multiparameter process has been obtained in [13], Lemma 4.1.

Lemma 4. *Let $(B^H(\mathbf{x}))_{\mathbf{x}\in\mathbb{R}^d}$ be a d-parameter isotropic fBm and consider a d-parameter stochastic process $(X(\mathbf{x}))_{\mathbf{x}\in\mathbb{R}^d}$, independent of B^H, that satisfies*

$$\mathbf{E}\left|X(\mathbf{x}) - X(\mathbf{y})\right|^2 \leq C\|\mathbf{x}-\mathbf{y}\|^2, \quad \textit{for every } \mathbf{x}, \mathbf{y} \in \mathbb{R}^d. \tag{36}$$

Define

$$Y(\mathbf{x}) = B^H_d(\mathbf{x}) + X(\mathbf{x}) \quad \textit{for every } \mathbf{x} \in \mathbb{R}^d.$$

Then:

1. *The process $(Y(\mathbf{x}))_{\mathbf{x}\in\mathbb{R}^d}$ has $\frac{1}{H}$-variation which is equal to*

$$(A_2 - A_1)\sqrt{d}\mathbf{E}|Z|^{1/H}.$$

2. *If* $H \in (0, \frac{1}{2})$ *and* $q \geq 2$,

$$\frac{1}{\sqrt{n}} V_{q,n}(Y^H) := \frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \left[\frac{n^{Hq} d^{-Hq/2}}{(A_2 - A_1)^{qH}} (Y^H(\mathbf{x}_{i+1}) - Y^H(\mathbf{x}_i))^q - \mu_q \right]$$
$$\to^{(d)} N(0, \sigma^2_{H,q}). \tag{37}$$

It is immediate to deduce the rate of convergence in the above central limit theorem. Recall that we denoted by d_W the Wasserstein distance.

Proposition 9. *Let* Y^H *be as in the statement of Lemma 4. Then for n large,*

$$d_W\left(\frac{1}{\sqrt{n}} V_{q,n}(Y^H), N(0, \sigma^2_{H,q})\right) \leq C \frac{1}{\sqrt{n}}.$$

Proof. We notice that the Gaussian vector $\left(B_d^H(\mathbf{x}_{i+1}) - B_d^H(\mathbf{x}_i)\right)_{0,1,\ldots,n-1}$ has the same law as $d^{H/2}(B^H(x_{j+1}) - B^H(x_j))_{0,1,\ldots,n-1}$ where B is a one-parameter fBm with the Hurst parameter H and we then apply Lemma 1. Therefore, the distribution of the sequence $\frac{1}{\sqrt{n}} V_{q,n}(B_d^H)$ is independent of $d \geq 1$ and we can use the same argument as in the proof of Proposition 1 above. □

3.3 Estimators for the drift paramater

Throughout this section we will assume (31). As in the previous section, we will construct and analyze estimators for the drift parameter θ by using the limit behavior of the variations (in time and in space) of the process (30).

3.3.1 The law of the solution

Let us start by analyzing the distribution of the solution to (27) and its link with the (bi)fractional Brownian motion.

Proposition 10. *For every* $\mathbf{x} \in \mathbb{R}^d$ *and* $\theta > 0$, *we have*

$$(u_\theta(t, \mathbf{x}), t \geq 0) \equiv^{(d)} \left(\theta^{-\frac{d-\gamma}{2\alpha}} c_{2,\alpha,\gamma} B_t^{\frac{1}{2}, 1-\frac{d-\gamma}{\alpha}}, t \geq 0\right),$$

where $B^{\frac{1}{2}, 1-\frac{d-\gamma}{\alpha}}$ *is a bifractional Brownian motion with parameters* $H = \frac{1}{2}$ *and* $K = 1 - \frac{d-\gamma}{\alpha}$ *and the constant* $c_{2,\alpha,\gamma}$ *is defined by (33).*

Proof. Denote

$$v_\theta(t, \mathbf{x}) = u_\theta\left(\frac{t}{\theta}, \mathbf{x}\right) \qquad \text{for every } t \geq 0, \mathbf{x} \in \mathbb{R}^d.$$

Then, as in Lemma 2, v_θ solves the equation

$$\frac{\partial v_\theta}{\partial t}(t, \mathbf{x}) = -(-\Delta)^{\frac{\alpha}{2}} v_\theta(t, \mathbf{x}) + (\theta)^{-\frac{1}{2}} \dot{\widetilde{W}}^\gamma(t, \mathbf{x}), \qquad t \geq 0, \mathbf{x} \in \mathbb{R}^d, \tag{38}$$

with $v_\theta(0, \mathbf{x}) = 0$ for every $\mathbf{x} \in \mathbb{R}^d$, where $\dot{\widetilde{W}}^\gamma$ is a white colored Gaussian noise (i.e. a Gaussian process with zero mean and covariance (28)).

Fix $\mathbf{x} \in \mathbb{R}^d$ and $\theta > 0$. For every $s, t \geq 0$, we have

$$\begin{aligned}\mathbf{E}u_\theta(t,\mathbf{x})u_\theta(s,\mathbf{x}) &= \mathbf{E}v_\theta(\theta t,\mathbf{x})v_\theta(\theta s,\mathbf{x}) \\ &= \theta^{-1}\mathbf{E}u_1(\theta t,\mathbf{x})u_1(\theta s,\mathbf{x}) \\ &= \theta^{-1}c_{1,\alpha,\gamma}\left[(\theta t+\theta s)^{1-\frac{d-\gamma}{\alpha}} - |\theta t-\theta s|^{1-\frac{d-\gamma}{\alpha}}\right] \\ &= \theta^{-\frac{d-\gamma}{\alpha}}c_{2,\alpha,\gamma}^2\mathbf{E}B_t^{\frac{1}{2},1-\frac{d-\gamma}{\alpha}}B_s^{\frac{1}{2},1-\frac{d-\gamma}{\alpha}}. \qquad \square\end{aligned}$$

For the behavior with respect to the space variable, we obtain the following result.

Proposition 11. *For every $t \geq 0$, $\theta > 0$, we have the following equality in distribution*

$$\left(u_\theta(t,\mathbf{x}),\mathbf{x}\in\mathbb{R}^d\right) \equiv^{(d)} \left(\theta^{-\frac{1}{2}}m_{\alpha,\gamma}B^{\frac{\alpha+\gamma-d}{2}}(\mathbf{x}) + S_{\theta t}(\mathbf{x}),\mathbf{x}\in\mathbb{R}^d\right),$$

where $B^{\frac{\alpha+\gamma--d}{2}}$ is a fractional Brownian motion with the Hurst parameter $\frac{\alpha+\gamma-d}{2} \in (0,\frac{1}{2}]$, $(S_{\theta t}(\mathbf{x}))_{\mathbf{x}\in\mathbb{R}^d}$ is a centered Gaussian process with C^∞ sample paths and $m_{\alpha,\gamma}$ from (34).

Proof. The result is immediate since for a fixed time $t > 0$

$$\left(u_\theta(t,\mathbf{x}),\mathbf{x}\in\mathbb{R}^d\right) = \left(v_\theta(\theta t,\mathbf{x}),\mathbf{x}\in\mathbb{R}^d\right) \equiv^{(d)} \theta^{-\frac{1}{2}}\left(u_1(\theta t,\mathbf{x}),\mathbf{x}\in\mathbb{R}^d\right)$$

$$\equiv^{(d)} \left(\theta^{-\frac{1}{2}}m_{\alpha,\gamma}B^{\frac{\alpha+\gamma-d}{2}}(\mathbf{x}) + S_{\theta t}(\mathbf{x}),\mathbf{x}\in\mathbb{R}^d\right). \qquad \square$$

3.3.2 Estimators based on the temporal variation

Again $t_j = A_1 + \frac{j}{n}(A_2 - A_1)$, $j = 0,\dots,n$, will denote a partition of the interval $[A_1, A_2]$.

Lemma 5. *Assume (31). Let u_θ be the solution to (27). Then for every $\mathbf{x} \in \mathbb{R}^d$, the process $(u_\theta(t,\mathbf{x}), t \geq 0)$ admits $\frac{2\alpha}{\alpha+\gamma-d}$-variation over the interval $[A_1, A_2]$, i.e.*

$$\begin{aligned} S_{[A_1,A_2]}^{n,\frac{2\alpha}{\alpha+\gamma-d}} &:= \sum_{i=0}^{n-1}\left|u_\theta(t_{j+1},\mathbf{x}) - u_\theta(t_j,\mathbf{x})\right|^{\frac{2\alpha}{\alpha+\gamma-d}} \\ &\to_{n\to\infty} c_{2,\alpha,\gamma}^{\frac{2\alpha}{\alpha+\gamma-d}}2^{\frac{d-\gamma}{\alpha+\gamma-d}}\mu_{\frac{2\alpha}{\alpha+\gamma-d}}(A_2-A_1)|\theta|^{\frac{\gamma-d}{\alpha+\gamma-d}} \end{aligned} \tag{39}$$

in probability.

Proof. Clearly, for fixed $\mathbf{x} \in \mathbb{R}^d$,

$$\sum_{i=0}^{n-1}\left|u_\theta(t_{j+1},\mathbf{x}) - u_\theta(t_j,\mathbf{x})\right|^{\frac{2\alpha}{\alpha+\gamma-d}} = \sum_{i=0}^{n-1}\left|v(\theta t_{j+1},\mathbf{x}) - v(\theta t_j,\mathbf{x})\right|^{\frac{2\alpha}{\alpha+\gamma-d}},$$

where $(v_\theta(t,\mathbf{x}), t \geq 0) \equiv^{(d)} (\theta^{-\frac{1}{2}} u_1(t,\mathbf{x}), t \geq 0)$. And from Proposition 4.3 in [13] we know that u_1 admits a variation of order $\frac{2\alpha}{\alpha+\gamma-d}$ which is equal to $c_{2,\alpha,\gamma}^{\frac{2\alpha}{\alpha+\gamma-d}} C_{\frac{1}{2},1-\frac{d-\gamma}{\alpha}}(A_2 - A_1)$ with $C_{\frac{1}{2},1-\frac{d-\gamma}{\alpha}} = 2^{\frac{d-\gamma}{\alpha+\gamma-d}} \mu_{\frac{2\alpha}{\alpha+\gamma-d}}$ and it means that

$$\begin{aligned}
&\sum_{i=0}^{n-1} \left| u_\theta(t_{j+1},\mathbf{x}) - u_\theta(t_j,\mathbf{x}) \right|^{\frac{2\alpha}{\alpha+\gamma-d}} \\
&\to_{n\to\infty} c_{2,\alpha,\gamma}^{\frac{2\alpha}{\alpha+\gamma-d}} 2^{\frac{d-\gamma}{\alpha+\gamma-d}} \mu_{\frac{2\alpha}{\alpha+\gamma-d}} (\theta A_2 - \theta A_1) |\theta^{-\frac{1}{2}}|^{\frac{2\alpha}{\alpha+\gamma-d}} \\
&c_{2,\alpha,\gamma}^{\frac{2\alpha}{\alpha+\gamma-d}} 2^{\frac{d-\gamma}{\alpha+\gamma-d}} \mu_{\frac{2\alpha}{\alpha+\gamma-d}} (A_2 - A_1) |\theta|^{\frac{\gamma-d}{\alpha+\gamma-d}}.
\end{aligned}$$

□

From relation (39) we can naturally define the following estimator for the parameter $\theta > 0$ of the stochastic partial differential equation (27)

$$\begin{aligned}
\widehat{\theta}_{n,3} &= \left(\left(c_{2,\alpha,\gamma}^{\frac{2\alpha}{\alpha+\gamma-d}} 2^{\frac{d-\gamma}{\alpha+\gamma-d}} \mu_{\frac{2\alpha}{\alpha+\gamma-d}} (A_2 - A_1) \right)^{-1} \right. \\
&\quad \left. \times \sum_{i=0}^{n-1} \left| u_\theta(t_{j+1},\mathbf{x}) - u_\theta(t_j,\mathbf{x}) \right|^{\frac{2\alpha}{\alpha+\gamma-d}} \right)^{\frac{\alpha+\gamma-d}{\gamma-d}} \\
&= \left(c_{2,\alpha,\gamma}^{\frac{2\alpha}{\alpha+\gamma-d}} 2^{\frac{d-\gamma}{\alpha+\gamma-d}} \mu_{\frac{2\alpha}{\alpha+\gamma-d}} (A_2 - A_1) \right)^{\frac{d-\gamma}{\alpha+\gamma-d}} \\
&\quad \times \left(S^{n,\frac{2\alpha}{\alpha+\gamma-d}}(u_\theta(\cdot,\mathbf{x})) \right)^{\frac{\alpha+\gamma-d}{\gamma-d}},
\end{aligned} \tag{40}$$

and so

$$\widehat{\theta}_{n,3}^{\frac{\gamma-d}{\alpha+\gamma-d}} = \frac{1}{c_{2,\alpha,\gamma}^{\frac{2\alpha}{\alpha+\gamma-d}} 2^{\frac{d-\gamma}{\alpha+\gamma-d}} \mu_{\frac{2\alpha}{\alpha+\gamma-d}} (A_2 - A_1)} \sum_{i=0}^{n-1} \left| u_\theta(t_{j+1},\mathbf{x}) - u_\theta(t_j,\mathbf{x}) \right|^{\frac{2\alpha}{\alpha+\gamma-d}}. \tag{41}$$

We have the following asymptotic behavior.

Proposition 12. *Assume $\frac{2\alpha}{\alpha+\gamma-d} := q$ is an even integer and consider the estimator $\widehat{\theta}_{n,3}$ in (40). Then $\widehat{\theta}_{n,3} \to_{n\infty} \theta$ in probability and*

$$\sqrt{n} \left[\widehat{\theta}_{n,3}^{\frac{\gamma-d}{\alpha+\gamma-d}} - \theta^{\frac{\gamma-d}{\alpha+\gamma-d}} \right] \to^{(d)} N(0, s_{3,\theta,\alpha,\gamma}^2) \text{ with } s_{3,\theta,\alpha,\gamma}^2 = \sigma_{\frac{1}{q},q}^2 \theta^{\frac{2(\gamma-d)}{\alpha+\gamma-d}} \mu_{\frac{2\alpha}{\alpha+\gamma-d}}^{-2}, \tag{42}$$

and for n large enough,

$$d_W \left(\sqrt{n} \left[\widehat{\theta}_{n,3}^{\frac{\gamma-d}{\alpha+\gamma-d}} - \theta^{\frac{\gamma-d}{\alpha+\gamma-d}} \right], N(0, s_{\theta,\alpha}^2) \right) \leq c \frac{1}{\sqrt{n}}. \tag{43}$$

Proof. From Proposition 10 and the relation between the fractional and bifractional Brownian motion (see (18)), we can see that, as $n \to \infty$,

$$\left(c_{2,\alpha,\gamma}^{-1} 2^{\frac{d-\gamma}{2\alpha}} \theta^{\frac{d-\gamma}{2\alpha}} u_\theta(t,\mathbf{x}), t \geq 0 \right)$$

converges to a perturbed fBm with Hurst parameter $H = \frac{\alpha-d+\gamma}{2\alpha}$. By taking $H = \frac{\alpha+\gamma-d}{2\alpha}$ and $q = \frac{1}{H} = \frac{2\alpha}{\alpha+\gamma-d}$ in Lemma 1, we get

$$\frac{1}{\sqrt{n}}\sum_{i=0}^{n-1}\left[\frac{n\theta^{\frac{d-\gamma}{\alpha+\gamma-d}}}{c_{2,\alpha,\gamma}^{\frac{2\alpha}{\alpha+\gamma-d}}2^{\frac{d-\gamma}{\alpha+\gamma-d}}(A_2-A_1)}\left(u_\theta(t_{j+1},\mathbf{x})-u_\theta(t_j,\mathbf{x})\right)^{\frac{2\alpha}{\alpha+\gamma-d}}-\mu_{\frac{2\alpha}{\alpha+\gamma-d}}\right]$$
$$\to N(0,\sigma^2_{\frac{1}{q},q})$$

or, equivalently

$$\sqrt{n}\mu_{\frac{2\alpha}{\alpha+\gamma-d}}\theta^{\frac{d-\gamma}{\alpha+\gamma-d}}\left[\widehat{\theta}_{n,3}^{\frac{\gamma-d}{\alpha+\gamma-d}}-\theta^{\frac{\gamma-d}{\alpha+\gamma-d}}\right]\to N(0,\sigma^2_{\frac{1}{q},q}),$$

which is equivalent to (22). The bound (43) follows easily from Proposition 1. □

We finally obtain the asymptotic normality and the rate of convergence for the estimator $\widehat{\theta}_{n,3}$.

Proposition 13. *Let $\widehat{\theta}_{n,3}$ be given by (40) and $s_{3,\theta,\alpha,\gamma}$ be given by (42). Then as $n\to\infty$,*

$$\sqrt{n}\left(\widehat{\theta}_{n,3}-\theta\right)\to^{(d)} N\left(0, s_{3,\theta,\alpha,\gamma}\left(\frac{\alpha+\gamma-d}{\gamma-d}\right)^2\theta^{\frac{2\alpha}{\alpha+\gamma-d}}\right)$$

and

$$d_W\left(\sqrt{n}\left(\widehat{\theta}_{n,3}-\theta\right), N\left(0, s_{3,\theta,\alpha,\gamma}\left(\frac{\alpha+\gamma-d}{\gamma-d}\right)^2\theta^{\frac{2\alpha}{\alpha+\gamma-d}}\right)\right)\le c\frac{1}{\sqrt{n}}.$$

Proof. It suffices to apply (23) with $g(x) = x^{\frac{\alpha+\gamma-d}{\gamma-d}}$ and $\gamma_0 = \theta^{\frac{\gamma-d}{\gamma+\alpha-d}}$ and to follow the proof of Proposition 5. □

3.4 Estimators based on the spatial variation

We will repeat the method employed in the previous parts of our work in order to define an estimator expressed in terms of the variations in space of the process (30) for the parameter θ in (27).

Recall that we proved in Proposition 11 that for every fixed time $t>0$,

$$\left(\theta^{\frac{1}{2}}m_{\alpha,\gamma}^{-1}u_\theta(t,\mathbf{x}),\mathbf{x}\in\mathbb{R}^d\right)$$

is a perturbed multiparameter isotropic fractional Brownian motion as defined in Lemma 4. Then we can deduce the variation in space of u_θ recalling that $\mathbf{x}_i = (x_i^{(1)},\ldots,x_i^{(d)})$ with $x_i^{(j)} = A_1+\frac{i}{n}(A_2-A_1)$ for $i=0,\ldots,n$ and $j=1,\ldots,d$.

Proposition 14. *Let u_θ be given by (30). Then*

$$\sum_{i=0}^{n-1}\left|u_\theta(t,\mathbf{x}_{j+1})-u(_\theta t,\mathbf{x}_j)\right|^{\frac{2}{\alpha+\gamma-d}}\to_{n\to\infty} m_{\alpha,\gamma}^{\frac{2}{\alpha+\gamma-d}}(A_2-A_1)\sqrt{d}\mu_{\frac{2}{\alpha+\gamma-d}}|\theta|^{\frac{-1}{\alpha+\gamma-d}}$$

Proof. We use Lemma 4, point 1. □

For every $n \geq 1$, define

$$\widehat{\theta}_{n,4} = \Bigg[(m_{\alpha,\gamma}^{\frac{2}{\alpha+\gamma-d}} \mu_{\frac{2}{\alpha+\gamma-d}} \sqrt{d}(A_2 - A_1))^{-1} \times \sum_{i=0}^{n-1} \left| u_\theta(t, \mathbf{x}_{j+1}) - u_\theta(t, \mathbf{x}_j) \right|^{\frac{2}{\alpha+\gamma-d}} \Bigg]^{-(\alpha+\gamma-d)}, \tag{44}$$

and so

$$\widehat{\theta}_{n,4}^{\frac{-1}{\alpha+\gamma-d}} = \frac{1}{m_{\alpha,\gamma}^{\frac{2}{\alpha+\gamma-d}} \mu_{\frac{2}{\alpha+\gamma-d}} \sqrt{d}(A_2 - A_1)} \sum_{i=0}^{n-1} \left| u_\theta(t, \mathbf{x}_{j+1}) - u_\theta(t, \mathbf{x}_j) \right|^{\frac{2}{\alpha+\gamma-d}}. \tag{45}$$

We can deduce the asymptotic properties of the estimator by using Lemma 4 and Proposition 9.

Proposition 15. *The estimator (44) converges in probability as $n \to \infty$ to the parameter θ. Moreover, if $\frac{2}{\alpha+\gamma-d}$ is an even integer, then*

$$\sqrt{n}\left[\widehat{\theta}_{n,4}^{\frac{-1}{\alpha+\gamma-d}} - \theta^{\frac{-1}{\alpha+\gamma-d}}\right] \to N(0, s_{4,\theta,\alpha,\gamma}^2)$$

$$\text{with } s_{4,\theta,\alpha,\gamma}^2 = \sigma^2_{\frac{\alpha+\gamma-1}{2}, \frac{2}{\alpha+\gamma-d}} \mu^{-2}_{\frac{2}{\alpha+\gamma-d}} \theta^{\frac{-2}{\alpha+\gamma-d}}.$$

We also have, for n large enough,

$$d_W\left(\sqrt{n}\left[\widehat{\theta}_{n,4}^{\frac{-1}{\alpha+\gamma-d}} - \theta^{\frac{-1}{\alpha+\gamma-d}}\right], N(0, s_{4,\theta,\alpha,\gamma}^2)\right) \leq c\frac{1}{\sqrt{n}}.$$

Finally, we get the following proposition.

Proposition 16. *With $\widehat{\theta}_{n,4}$ from (44), as $n \to \infty$,*

$$\sqrt{n}\left(\widehat{\theta}_{n,4} - \theta\right) \to^{(d)} N\left(0, s_{4,\theta,\alpha,\gamma}\left(\frac{\alpha+\gamma-d}{\gamma-d}\right)^2 \theta^{\frac{2\alpha}{\alpha+\gamma-d}}\right)$$

and

$$d_W\left(\sqrt{n,4}\left(\widehat{\theta}_n - \theta\right), N\left(0, s_{4,\theta,\alpha,\gamma}\left(\frac{\alpha+\gamma-d}{\gamma-d}\right)^2 \theta^{\frac{2\alpha}{\alpha+\gamma-d}}\right)\right) \leq c\frac{1}{\sqrt{n}}.$$

Proof. Apply again (23) with $g(x) = x^{\frac{\alpha+\gamma-d}{\gamma-d}}$ and $\gamma_0 = \theta^{\frac{\gamma-d}{\gamma+\alpha-d}}$. □

Remark 2. Notice that in the case $\gamma = 1$ (i.e., there is no spatial correlation and in this case d has to be 1), we retrieve the results of Section 2. Observe, as in Section 2, that the distance of the estimators (40) and (44) to their limit distribution is of the same order, although they involve q-variations of different orders.

4 Conclusion

To conclude, in this paper we provide estimators based on power variation for the drift parameter θ of the solution to the fractional stochastic heat equation (3). The novelty of our approach is that it allows, comparing with the literature on statistical inference for SPDEs (see [4, 17, 2], etc.), to consider the case of a Gaussian noise with non-trivial spatial correlation and to treat the situation when the differential operator in the heat equation (3) is the fractional Laplacian instead of the standard Laplacian. The proofs of the asymptotic behavior of the estimators are relatively simple and they are based on the link between the law of the solution and the fractional Brownian motion, using known results on the behavior of the power variations of the fBm. Our approach also gives the rate of convergence of the estimators under the Wasserstein distance via some recent results in Stein–Malliavin calculus (see [19]). We assumed for simplicity a vanishing initial condition in (3) but the case of a notrivial initial value, whose power variations are dominated by those of the fBm, can be also treated by our approach. Another open problem of interest that could by treated via our techniques is adding an unknown volatility parameter in the disturbance term and jointly estimating the drift and the volatility parameters. The case of the fractional heat equation on bounded domains is also interesting but in this case the fundamental solution and implicitly the law of the mild solution changes. Consequently, the relation between the law of the solution and the fBm is not obvious and therefore new techniques are needed.

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