On a linear functional for infinitely divisible moving average random fields

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Abstract   Given a low-frequency sample of the infinitely divisible moving average random field $\{\int_{\mathbb{R}^d} f(t - x)\Lambda(dx), \ t \in \mathbb{R}^d\}$, in [13] we proposed an estimator $\widehat{uv_0}$ for the function $\mathbb{R} \ni x \mapsto u(x)v_0(x) = (uv_0)(x)$, with $u(x) = x$ and v_0 being the Lévy density of the integrator random measure Λ. In this paper, we study asymptotic properties of the linear functional $L^2(\mathbb{R}) \ni v \mapsto \langle v, \widehat{uv_0} \rangle_{L^2(\mathbb{R})}$, if the (known) kernel function f has a compact support. We provide conditions that ensure consistency (in mean) and prove a central limit theorem for it.

Keywords   Infinitely divisible random measure, stationary random field, Lévy process; moving average, Lévy density, Fourier transform, central limit theorem

1   Introduction

Consider a stationary infinitely divisible indepently scattered random measure Λ whose Lévy density is denoted by v_0. For some (known) Λ-integrable function $f : \mathbb{R}^d \to \mathbb{R}$ with a compact support, let

$$X = \{X(t); \ t \in \mathbb{R}^d\}, \qquad X(t) = \int_{\mathbb{R}^d} f(t - x)\Lambda(dx) \qquad (1.1)$$

be the corresponding moving average random field. In our recent preprint [13], we proposed an estimator $\widehat{uv_0}$ for the function $\mathbb{R} \ni x \mapsto u(x)v_0(x) = (uv_0)(x)$, $u(x) = x$, based on low frequency observations $(X(j\Delta))_{j \in W}$ of X, with $\Delta > 0$ and W a finite subset of $\mathbb{Z}^d$.

A wide class of spatio-temporal processes with the spectral representation (1.1) is provided by the so-called ambit random fields, where a space-time Lévy process

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serves as integrator. Such processes are, e.g., used to model the growth rate of tumours, where the spatial component describes the angle between the center of the tumour cell and the nearest point at its boundary (cf. [3, 16]). Ambit fields cover quite a number of different processes and fields including Ornstein–Uhlenbeck type and mixed moving average random fields (cf. [1, 2]). A further interesting application of (1.1) is given in [17], where the author uses infinitely divisible moving average random fields in order to model claims of natural disaster insurance within different postal code areas.

We point out that there is a large number of literature concerning estimation of the Lévy density v_1 (its Lévy measure, respectively) in the case when X is a Lévy process (cf. [5, 9, 10, 14, 19]). Moreover, in the recent paper [4] the authors provide an estimator for the Lévy density v_0 of the integrator Lévy process $\{L_s\}$ of a moving average process $X(t) = \int_{\mathbb{R}} f(t-s) dL_s, t \in \mathbb{R}$, which covers the case $d = 1$ in (1.1). For a discussion on the differences between our approach and the method provided in [4], we refer to [13] and [18].

In this paper, we investigate asymptotic properties of the linear functional $L^2(\mathbb{R}) \ni v \mapsto \hat{\mathcal{L}}_W v = \langle v, \widehat{uv_0} \rangle_{L^2(\mathbb{R})}$ as the sample size $|W|$ tends to infinity. It is motivated by the paper of Nickl and Reiss [20], where the authors provide a Donsker type theorem for the Lévy measure of pure jump Lévy processes. Since our observations are m-dependent, the classical i.i.d. theory does not apply here. Instead, we combine results of Chen and Shao [8] for m-dependent random fields and ideas of Bulinski and Shashkin [7] with exponential inequalities for weakly dependent random fields (see e.g. [15, 11]) in order to prove our limit theorems.

It turns out that under certain regularity assumptions on uv_0, $\hat{\mathcal{L}}_W v$ is a mean consistent estimator for $\mathcal{L}v = \langle v, uv_0 \rangle_{L^2(\mathbb{R})}$ with a rate of convergence given by $\mathcal{O}(|W|^{-1/2})$, for any v that belongs to a subspace $\mathcal{U}$ of $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Moreover, we give conditions such that finite dimensional distributions of the process $\{|W|^{1/2}(\hat{\mathcal{L}}_W - \mathcal{L})v; \ v \in \mathcal{U}\}$ are asymptotically Gaussian as $|W|$ is regularly growing to infinity.

From a practical point of view, a naturally arising question is wether a proposed model for v_0 (or equivalently uv_0) is suitable. Knowledge of the asymptotic distribution of $|W|^{1/2}(\hat{\mathcal{L}}_W - \mathcal{L})$ can be used in order to construct tests for different hypotheses, e.g., on regularity assumptions of the model for v_0. Indeed, the scalar product $\langle \cdot, \cdot \rangle_{L^2(\mathbb{R})}$ naturally induces that the class $\mathcal{U}$ of test functions is growing, when uv_0 becomes more regular.

This paper is organized as follows. In Section 2, we give a brief overview of regularly growing sets and infinitely divisible moving average random fields. We further recall some notation and the most frequently used results from [13]. Section 3 is devoted to asymptotic properties of $\hat{\mathcal{L}}_W$. Here we discuss our regularity assumptions and state the main results of this paper (Theorems 3.7 and 3.12). Section 4 is dedicated to the proofs of our limit theorems. Some of the shorter proofs as well as external results that will frequently be used in Section 3 are moved to Appendix.

2 Preliminaries

2.1 Notation

Throughout this paper, we use the following notation.

By $\mathcal{B}(\mathbb{R}^d)$ we denote the Borel σ-field on the Euclidean space $\mathbb{R}^d$. The Lebesgue measure on $\mathbb{R}^d$ is denoted by ν_d and we shortly write $\nu_d(dx) = dx$ when we integrate w.r.t. ν_d. For any measurable space $(M, \mathcal{M}, \mu)$ we denote by $L^\alpha(M)$, $1 \le \alpha < \infty$, the space of all $\mathcal{M}|\mathcal{B}(\mathbb{R})$-measurable functions $f : M \to \mathbb{R}$ with $\int_M |f|^\alpha(x)\mu(dx) < \infty$. Equipped with the norm $||f||_{L^\alpha(M)} = \left(\int_M |f|^\alpha(x)\mu(dx)\right)^{1/\alpha}$, $L^\alpha(M)$ becomes a Banach space and, in the case $\alpha = 2$, even a Hilbert space with scalar product $\langle f, g \rangle_{L^\alpha(M)} = \int_M f(x)g(x)\mu(dx)$, for any $f, g \in L^2(M)$. With $L^\infty(M)$ (i.e. if $\alpha = \infty$) we denote the space of all real-valued bounded functions on M. In the case $(M, \mathcal{M}, \mu) = (\mathbb{R}, \mathcal{B}(\mathbb{R}), \nu_1)$ we denote by

$$H^\delta(\mathbb{R}) = \left\{ f \in L^2(\mathbb{R}) : \int_\mathbb{R} |\mathcal{F}_+ f|^2(x)(1+x^2)^\delta dx < \infty \right\}$$

the Sobolev space of order $\delta > 0$ equipped with the Sobolev norm $||f||_{H^\delta(\mathbb{R})} = ||\mathcal{F}_+ f(\cdot)(1 + \cdot^2)^{\delta/2}||_{L^2(\mathbb{R})}$, where $\mathcal{F}_+$ is the Fourier transform on $L^2(\mathbb{R})$. For $f \in L^1(\mathbb{R})$, $\mathcal{F}_+ f$ is defined by $\mathcal{F}_+ f(x) = \int_\mathbb{R} e^{itx} f(t)dt$, $x \in \mathbb{R}$. Throughout the rest of this paper $(\Omega, \mathcal{A}, \mathbb{P})$ denotes a probability space. Note that in this case $L^\alpha(\Omega)$ is the space of all random variables with finite α-th moment. For an arbitrary set A we introduce the notation $\mathrm{card}(A)$ or briefly $|A|$ for its cardinality. Let $\mathrm{supp}(f) = \{x \in \mathbb{R}^d : f(x) \ne 0\}$ be the support set of a function $f : \mathbb{R}^d \to \mathbb{R}$. Denote by $\mathrm{diam}(A) = \sup\{||x - y||_\infty : x, y \in A\}$ the diameter of a bounded set $A \subset \mathbb{R}^d$.

2.2 Regularly growing sets

In this section, we briefly recall some basic facts about *regularly growing sets*. For a more detailed investigation on this topic, see, e.g., [7].

Let $a = (a_1, \ldots, a_d) \in \mathbb{R}^d$ be a vector with positive components. In the sequel, we shortly write $a > 0$ in this situation. Moreover, let

$$\Pi_0(a) = \{x \in \mathbb{R}^d, \ 0 < x_i \le a_i, \ i = 1, \ldots, d\}$$

and define for any $j \in \mathbb{Z}^d$ the *shifted block* $\Pi_j(a)$ by

$$\Pi_j(a) = \Pi_0(a) + ja = \{x \in \mathbb{R}^d, \ j_i a_i < x_i \le j_i(a_i + 1), \ i = 1, \ldots, d\}.$$

Clearly $\{\Pi_j, \ j \in \mathbb{Z}^d\}$ forms a partition of $\mathbb{R}^d$. For any $U \subset \mathbb{Z}^d$, introduce the sets

$$J_-(U, a) = \{j \in \mathbb{Z}^d, \ \Pi_j(a) \subset U\}, \qquad J_+(U, a) = \{j \in \mathbb{Z}^d, \ \Pi_j(a) \cap U \ne \emptyset\}$$

$$U^-(a) = \bigcup_{j \in J_-(U,a)} \Pi_j(a), \qquad U^+(a) = \bigcup_{j \in J_+(U,a)} \Pi_j(a).$$

A sequence of sets $U_n \subset \mathbb{R}^d$ ($n \in \mathbb{N}$) *tends to infinity in Van Hove sense* or shortly is *VH-growing*, if for any $a > 0$

$$\nu_d(U_n^-) \to \infty \quad \text{and} \quad \frac{\nu_d(U_n^-)}{\nu_d(U_n^+)} \to 1 \quad \text{as } n \to \infty.$$

For a finite set $A \subset \mathbb{Z}^d$, define by $\partial A = \{j \in \mathbb{Z}^d \backslash A, \ \mathrm{dist}(j, A) = 1\}$ its boundary, where $\mathrm{dist}(j, A) = \inf\{||j - x||_\infty, \ x \in A\}$.

A sequence of finite sets $A_n \in \mathbb{Z}^d$ ($n \in \mathbb{N}$) is called *regularly growing (to infinity)*, if

$$|A_n| \to \infty, \quad \text{and} \quad \frac{|\partial A_n|}{|A_n|} \to 0, \quad \text{as } n \to \infty.$$

Remark 2.1. Regular growth of a family $A_n \subset \mathbb{Z}^d$ means that the number of points in the boundary of A_n grows significantly slower than the number of its interior points.

The following result that connects regularly and VH-growing sequences can be found in [7, p.174].

Lemma 2.2.   *1. Let $U_n \subset \mathbb{R}^d$ ($n \in \mathbb{N}$) be VH-growing. Then $V_n = U_n \cap \mathbb{Z}^d$ ($n \in \mathbb{N}$) is regularly growing to infinity.*

2. *If $(U_n)_{n \in \mathbb{N}}$ is a sequence of finite subsets of $\mathbb{Z}^d$, regularly growing to infinity, then $V_n = \cup_{j \in U_n}[j, j+1)$ is VH-growing, where $[j, j+1) = \{x \in \mathbb{R}^d : j_k \le x_k < j_k + 1, \ k = 1, \ldots, d\}$.*

2.3   Infinitely divisible random measures

In what follows, denote by $\mathcal{E}_0(\mathbb{R}^d)$ the collection of all bounded Borel sets in $\mathbb{R}^d$.

Suppose that $\Lambda = \{\Lambda(A); \ A \in \mathcal{E}_0(\mathbb{R}^d)\}$ is an infinitely divisible random measure on some probability space $(\Omega, \mathcal{A}, P)$, i.e. a random measure with the following properties:

(a) Let $(E_m)_{m \in \mathbb{N}}$ be a sequence of disjoint sets in $\mathcal{E}_0(\mathbb{R}^d)$. Then the sequence $(\Lambda(E_m))_{m \in \mathbb{N}}$ consists of independent random variables; if, in addition, $\cup_{m=1}^\infty E_m \in \mathcal{E}_0(\mathbb{R}^d)$, then we have $\Lambda(\cup_{m=1}^\infty E_m) = \sum_{m=1}^\infty \Lambda(E_m)$ almost surely.

(b) The random variable $\Lambda(A)$ has an infinitely divisible distribution for any choice of $A \in \mathcal{E}_0(\mathbb{R}^d)$.

For every $A \in \mathcal{E}_0(\mathbb{R}^d)$, let $\varphi_{\Lambda(A)}$ denote the characteristic function of the random variable $\Lambda(A)$. Due to the infinite divisibility of the random variable $\Lambda(A)$, the characteristic function $\varphi_{\Lambda(A)}$ has a Lévy–Khintchin representation which can, in its most general form, be found in [21, p. 456]. Throughout the rest of the paper we make the additional assumption that the Lévy–Khintchin representation of $\Lambda(A)$ is of a special form, namely

$$\varphi_{\Lambda(A)}(t) = \exp\{v_d(A) K(t)\}, \quad A \in \mathcal{E}_0(\mathbb{R}^d),$$

with

$$K(t) = ita_0 - \frac{1}{2}t^2 b_0 + \int_{\mathbb{R}} \left(e^{itx} - 1 - itx \mathbb{1}_{[-1,1]}(x) \right) v_0(x) dx, \quad (2.1)$$

where v_d denotes the Lebesgue measure on $\mathbb{R}^d$, a_0 and b_0 are real numbers with $0 \le b_0 < \infty$ and $v_0 : \mathbb{R} \to \mathbb{R}$ is a Lévy density, i.e. a measurable function which fulfills $\int_{\mathbb{R}} \min\{1, x^2\} v_0(x) dx < \infty$. The triplet (a_0, b_0, v_0) will be referred to as the *Lévy characteristic* of Λ. It uniquely determines the distribution of Λ. This particular structure of the characteristic functions $\varphi_{\Lambda(A)}$ means that the random measure Λ is

stationary with the control measure $\lambda : \mathcal{B}(\mathbb{R}) \to [0, \infty)$ given by

$$\lambda(A) = v_d(A) \left[|a_0| + b_0 + \int\limits_{\mathbb{R}} \min\{1, x^2\} v_0(x) dx \right] \quad \text{for all } A \in \mathcal{E}_0(\mathbb{R}^d).$$

Now one can define the stochastic integral with respect to the infinitely divisible random measure Λ in the following way:

1. Let $f = \sum_{j=1}^n x_j \mathbb{1}_{A_j}$ be a real simple function on $\mathbb{R}^d$, where $A_j \in \mathcal{E}_0(\mathbb{R}^d)$ are pairwise disjoint. Then for every $A \in \mathcal{B}(\mathbb{R}^d)$ we define

$$\int\limits_A f(x) \Lambda(dx) = \sum_{j=1}^n x_j \Lambda(A \cap A_j).$$

2. A measurable function $f : (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is said to be Λ-integrable if there exists a sequence $(f^{(m)})_{m \in \mathbb{N}}$ of simple functions as in (1) such that $f^{(m)} \to f$ holds λ-almost everywhere and such that, for each $A \in \mathcal{B}(\mathbb{R}^d)$, the sequence $\left(\int_A f^{(m)}(x) \Lambda(dx) \right)_{m \in \mathbb{N}}$ converges in probability as $m \to \infty$. In this case we set

$$\int\limits_A f(x) \Lambda(dx) = \underset{m \to \infty}{\text{P-lim}} \int\limits_A f^{(m)}(x) \Lambda(dx).$$

A useful characterization of the Λ-integrability of a function f is given in [21, Theorem 2.7]. Now let $f : \mathbb{R}^d \to \mathbb{R}$ be Λ-integrable; then the function $f(t - \cdot)$ is Λ-integrable for every $t \in \mathbb{R}^d$ as well. We define the moving average random field $X = \{X(t), \ t \in \mathbb{R}^d\}$ by

$$X(t) = \int\limits_{\mathbb{R}^d} f(t - x) \Lambda(dx), \quad t \in \mathbb{R}^d. \tag{2.2}$$

Recall that a random field is called *infinitely divisible* if its finite dimensional distributions are infinitely divisible. The random field X above is (strictly) stationary and infinitely divisible and the characteristic function $\varphi_{X(0)}$ of $X(0)$ is given by

$$\varphi_{X(0)}(u) = \exp \left(\int_{\mathbb{R}^d} K(uf(s)) \, ds \right),$$

where K is the function from (2.1). The argument $\int_{\mathbb{R}^d} K(uf(s)) \, ds$ in the above exponential function can be shown to have a similar structure as $K(t)$; more precisely, we have

$$\int_{\mathbb{R}^d} K(uf(s)) \, ds = iua_1 - \frac{1}{2} u^2 b_1 + \int\limits_{\mathbb{R}} \left(e^{iux} - 1 - iux \mathbb{1}_{[-1,1]}(x) \right) v_1(x) \, dx \tag{2.3}$$

where a_1 and b_1 are real numbers with $b_0 \geq 0$ and the function v_1 is the Lévy density of $X(0)$. The triplet (a_1, b_1, v_1) is again referred to as the *Lévy characteristic* (of $X(0)$) and determines the distribution of $X(0)$ uniquely. A simple computation shows that the triplet (a_1, b_1, v_1) is given by the formulas

$$a_1 = \int_{\mathbb{R}^d} U(f(s)) \, ds, \qquad b_1 = b_0 \int_{\mathbb{R}^d} f^2(s) \, ds,$$

$$v_1(x) = \int_{\text{supp}(f)} \frac{1}{|f(s)|} v_0 \left(\frac{x}{f(s)} \right) \, ds, \tag{2.4}$$

where $\text{supp}(f) := \{s \in \mathbb{R}^d : f(s) \neq 0\}$ denotes the support of f and the function U is defined via

$$U(u) = u \left(a_0 + \int_{\mathbb{R}} x \left[\mathbb{1}_{[-1,1]}(ux) - \mathbb{1}_{[-1,1]}(x) \right] v_0(x) \, dx \right).$$

Note that the Λ-integrability of f immediately implies that $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. Hence, all integrals above are finite.

For details on the theory of infinitely divisible measures and fields we refer the interested reader to [21].

2.4 A plug-in estimation approach for v_0

Let the random field $X = \{X(t), t \in \mathbb{R}^d\}$ be given as in Section 2.3 and define the function $u : \mathbb{R} \to \mathbb{R}$ by $u(x) = x$. Suppose further that an estimator $\widehat{uv_1}$ for uv_1 is given. In our recent preprint [13], we provided an estimation approach for uv_0 based on relation (2.4) which we briefly recall in this section. Therefore, quite a number of notations are required.

Assume that f satisfies the integrability condition

$$\int_{\text{supp}(f(s))} |f(s)|^{1/2} ds < \infty, \tag{2.5}$$

and define the operator $\mathcal{G} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by

$$\mathcal{G}v = \int_{\text{supp}(f)} \text{sgn}(f(s)) v \left(\frac{\cdot}{f(s)} \right) ds, \quad v \in L^2(\mathbb{R}).$$

Moreover, define the isometry $\mathcal{M} : L^2(\mathbb{R}) \to L^2(\mathbb{R}^\times, \frac{dx}{|x|})$ by

$$(\mathcal{M}v)(x) = |x|^{1/2} v(x), \quad v \in L^2(\mathbb{R}),$$

and let the functions $m_{f,\pm} : \mathbb{R}^\times \to \mathbb{C}$ and $\mu_f : \mathbb{R}^\times \to \mathbb{C}$ be given by

$$m_{f,+}(x) = \int_{\text{supp}(f)} \text{sgn}(f(s)) |f(s)|^{1/2} e^{-ix \log |f(s)|} ds,$$

$$m_{f,-}(x) = \int_{\text{supp}(f)} |f(s)|^{1/2} e^{-ix \log |f(s)|} ds,$$

$$\mu_f(y) = \begin{cases} m_{f,+}(\log |y|) & \text{if } y > 0, \\ m_{f,-}(\log |y|) & \text{if } y < 0. \end{cases}$$

Multiplying both sides in (2.4) by u leads to the equivalent relation

$$uv_1 = \mathcal{G} u v_0. \tag{2.6}$$

Suppose $uv_1 \in L^2(\mathbb{R})$ and assume that for some $\beta \geq 0$,

$$|m_{f,\pm}(x)| \gtrsim \frac{1}{1 + |x|^\beta}, \quad \text{for all } x \in \mathbb{R}. \tag{$\mathbf{U}_\beta$}$$

Then the unique solution $uv_0 \in L^2(\mathbb{R})$ to equation (2.6) is given by

$$uv_0 = \mathcal{G}^{-1} uv_1 = \mathcal{M}^{-1} \mathcal{F}_\times^{-1} \left(\frac{1}{\mu_f} \mathcal{F}_\times \mathcal{M} u v_1 \right),$$

cf. [13, Theorem 3.1]. Based on this relation, the paper [13] provides the estimator

$$\widehat{uv_0} = \mathcal{M}^{-1} \mathcal{F}_\times^{-1} \left(\frac{1}{\mu_{f,n}} \mathcal{F}_\times \mathcal{M} \widehat{uv_1} \right) =: \mathcal{G}_n^{-1} \widehat{uv_1} \tag{2.7}$$

for uv_0, where $(a_n)_{n \in \mathbb{N}} \subseteq (0, \infty)$ is an arbitrary sequence, depending on the sample size n, that tends to 0 as $n \to \infty$, and the mapping $\frac{1}{\mu_{f,n}} : \mathbb{R} \to \mathbb{C}$ is defined by $\frac{1}{\mu_{f,n}} := \frac{1}{\mu_f} \mathbb{1}_{\{|\mu_f| > a_n\}}$. Here $\mathcal{F}_\times : L^2(\mathbb{R}^\times, \frac{dx}{|x|}) \to L^2(\mathbb{R}^\times, \frac{dx}{|x|})$ denotes the Fourier transform on the multiplicative group $\mathbb{R}^\times$ which is defined by

$$(\mathcal{F}_\times u)(y) = \int_{\mathbb{R}^\times} u(x) \, e^{-i \log|x| \cdot \log|y|} \cdot e^{i\pi \delta(x) \delta(y)} \, \frac{dx}{|x|},$$

for all $u \in L^1(\mathbb{R}^\times, \frac{dx}{|x|}) \cap L^2(\mathbb{R}^\times, \frac{dx}{|x|})$, with $\delta : \mathbb{R}^\times \to \mathbb{R}$ given by $\delta(x) = \mathbb{1}_{(-\infty,0)}(x)$ (cf. [13, Section 2.2]). A more detailed introduction to harmonic analysis on locally compact abelian groups can be found, e.g., in [12].

Remark 2.3. The linear operator $\mathcal{G}_n^{-1} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ defined in (2.7) is bounded in the operator norm $\|\mathcal{G}_n^{-1}\| \leq \frac{1}{a_n}$, whereas $\mathcal{G}^{-1}$ is unbounded in general.

2.5 *m-dependent random fields*

A random field $X = \{X(t), \ t \in T\}$, $T \subseteq \mathbb{R}^d$, defined on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$ is called *m-dependent* if for some $m \in \mathbb{N}$ and any finite subsets U and V of T the random vectors $(X(u))_{u \in U}$ and $(X(v))_{v \in V}$ are independent whenever

$$\|u - v\|_\infty = \max\{|u_i - v_i|, \ i = 1, \dots, d\} > m,$$

for all $u = (u_1, \dots, u_d)^\top \in U$ and $v = (v_1, \dots, v_d)^\top \in V$.

Lemma 2.4. *Let the random field X be given in* (2.2) *and suppose that f has a compact support. Then X is m-dependent with $m > \text{diam}(\text{supp}(f))$.*

Proof. Compactness of $\text{supp}(f)$ implies that $\text{supp}(f(t - \cdot))$ and $\text{supp}(f(s - \cdot))$ are disjoint whenever $\|t - s\|_\infty > \text{diam}(\text{supp}(f))$. Since further Λ is independently scattered and integration in (2.2) is done only on $\text{supp}(f(t - \cdot))$, $X(t)$ and $X(s)$ are independent for $\|t - s\|_\infty > \text{diam}(\text{supp}(f))$. $\qquad\square$

3 A linear functional for infinitely divisible moving averages

3.1 The setting

Let $\Lambda = \{\Lambda(A),\ A \in \mathcal{E}_0(\mathbb{R}^d)\}$ be a stationary infinitely divisible random measure defined on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with characteristic triplet $(a_0, 0, v_0)$, i.e. Λ is purely non-Gaussian. For a known Λ-integrable function $f : \mathbb{R}^d \to \mathbb{R}$ let $X = \{X(t) = \int_{\mathbb{R}^d} f(t-x)\Lambda(dx),\ t \in \mathbb{R}^d\}$ be the infinitely divisible moving average random field defined in Section 2.3.

Fix $\Delta > 0$ and suppose X is observed on a regular grid $\Delta\mathbb{Z}^d = \{j\Delta,\ j \in \mathbb{Z}^d\}$ with the mesh size Δ, i.e. consider the random field Y given by

$$Y = (Y_j)_{j \in \mathbb{Z}^d}, \quad Y_j = X(j\Delta),\ j \in \mathbb{Z}^d. \tag{3.1}$$

For a finite subset $W \subset \mathbb{Z}^d$ let $(Y_j)_{j \in W}$ be a sample drawn from Y and denote by n the cardinality of W.

Throughout this paper, for any numbers $a, b \geq 0$, we use the notation $a \lesssim b$ if $a \leq cb$ for some constant $c > 0$.

Assumption 3.1. Let the function $u : \mathbb{R} \to \mathbb{R}$ be given by $u(x) = x$. We make the following assumptions: for some $\tau > 0$

1. $f \in L^{2+\tau}(\mathbb{R}^d)$ has compact support;

2. $uv_0 \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is bounded;

3. $\int_{\mathbb{R}} |x|^{1+\tau} |(uv_0)(x)| dx < \infty$;

4. $|\int_{\text{supp}(f)} f(s)\mathcal{F}_+[uv_0](f(s)x)ds| \lesssim (1 + x^2)^{-1/2}$ for all $x \in \mathbb{R}$;

5. $\exists\, \varepsilon > 0$ such that the function

$$\mathbb{R} \ni x \mapsto \exp\left(\int_{\text{supp}(f)} \int_0^{f(s)x} \text{Im}\Big(\mathcal{F}_+[uv_0](y)\Big) dy\, ds \right) \tag{3.2}$$

   is contained in $H^{-1+\varepsilon}(\mathbb{R})$.

Suppose that $\widehat{uv_1}$ is an estimator for uv_1 (which we will precisely define in the next section) based on the sample $(Y_j)_{j \in W}$. Then, using the notation in Section 2.4, we introduce the linear functional

$$\hat{\mathcal{L}}_W : L^2(\mathbb{R}) \to \mathbb{R}, \quad \hat{\mathcal{L}}_W v := \langle v, \widehat{uv_0} \rangle_{L^2(\mathbb{R})} = \Big\langle v, \mathcal{G}_n^{-1}\widehat{uv_1} \Big\rangle_{L^2(\mathbb{R})}.$$

The purpose of this paper is to investigate asymptotic properties of $\hat{\mathcal{L}}_W$ as the sample size $|W| = n$ tends to infinity.

3.2 An estimator for **uv$_1$**

In this section we introduce an estimator for the function uv_1. Therefore, let ψ denote the characteristic function of $X(0)$. Then, by Assumption 3.1, (2), together with

formula (2.3), we find that ψ can be rewritten as

$$\psi(t) = \mathbb{E}e^{itY_0} = \exp\left(i\gamma t + \int_{\mathbb{R}} (e^{itx} - 1)v_1(x)dx\right), \quad t \in \mathbb{R}, \tag{3.3}$$

for some $\gamma \in \mathbb{R}$ and the Lévy density v_1 given in (2.4). We call γ the *drift parameter* or shortly *drift* of X. As a consequence of representation (3.3), the random field X is purely non-Gaussian. It is subsequently assumed that the drift γ is known.

Taking derivatives in (3.3) leads to the identity

$$-i\frac{\psi'(t)}{\psi(t)} = \gamma + \mathcal{F}_+[uv_1](t), \quad t \in \mathbb{R}.$$

Neglecting γ for the moment, this relation suggests that a natural estimator $\widehat{\mathcal{F}_+[uv_1]}$ for $\mathcal{F}_+[uv_1]$ is given by

$$\widehat{\mathcal{F}_+[uv_1]}(t) = \frac{\hat{\theta}(t)}{\tilde{\psi}(t)}, \quad t \in \mathbb{R},$$

with

$$\tilde{\psi}(t) = \frac{1}{\hat{\psi}(t)} \mathbb{1}_{\{|(\hat{\psi})(t)| > n^{-1/2}\}}, \quad t \in \mathbb{R},$$

and $\hat{\psi}(t) = \sum_{j \in W} e^{itY_j}$, $\hat{\theta}(t) = \sum_{j \in W} Y_j e^{itY_j}$ being the empirical counterparts of ψ and $\theta = -i\psi'$.

Now, consider for any $b > 0$ a function $K_b : \mathbb{R} \to \mathbb{R}$ with the following properties:

(K1) $K_b \in L^2(\mathbb{R})$;

(K2) $\text{supp}(\mathcal{F}_+[K_b]) \subseteq [-b^{-1}, b^{-1}]$;

(K3) $|1 - \mathcal{F}_+[K_b](x)| \lesssim \min\{1, b|x|\}$ for all $x \in \mathbb{R}$.

Then, for any $b > 0$, we define the estimator $\widehat{uv_1}$ for uv_1 by

$$\widehat{uv_1}(t) = \mathcal{F}_+^{-1}\left[\widehat{\mathcal{F}_+[uv_1]}\mathcal{F}_+[K_b]\right](t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-itx}\frac{\hat{\theta}(x)}{\tilde{\psi}(x)}\mathcal{F}_+[K_b](x)dx, \quad t \in \mathbb{R}. \tag{3.4}$$

Remark 3.2. (a) If $\widehat{uv_1}$ is a consistent estimator for uv_1, it is reasonable to assume that $\gamma = 0$ (cf. [18]). Indeed, for the asymptotic results below, the value of γ is irrelevant. Even if $\gamma \neq 0$, the functional $\hat{\mathcal{L}}_W$ estimates the intended quantity with $\widehat{uv_1}$ given in (3.4) (cf. Section 4.3).

(b) Choosing $K_b(x) = \frac{\sin(b^{-1}x)}{\pi x}$ yields the estimator $\widehat{uv_1}$ that we introduced in [18] and [13], originally designed by Comte and Genon-Catalot [10] in the case when X is a pure jump Lévy process.

3.3 Discussion and examples

In order to explain Assumption 3.1, we prepend the following proposition whose proof can be found in Appendix.

Proposition 3.3. *Let the infinitely divisible moving average random field $X = \{X(t),$ $t \in \mathbb{R}^d\}$ be given as above and suppose $u(x) = x$.*

(a) *Let Assumption 3.1, (1) and (2) be satisfied. Then $uv_1 \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is bounded. Moreover,*

$$\mathcal{F}_+[uv_1](x) = \int_{\text{supp}(f)} f(s)\mathcal{F}_+[uv_0](f(s)x)ds, \quad \text{for all } x \in \mathbb{R}, \qquad (3.5)$$

that is, the expression in Assumption 3.1, (4) is valid.

(b) *Let Assumption 3.1, (1) and (3) hold true. Then $\int_{\mathbb{R}} |x|^{2+\tau}|(uv_1)(x)|dx < \infty$ (also in the case when $\tau = 0$).*

(c) *Assumption 3.1, (5) is satisfied if and only if the function $\mathbb{R} \ni x \mapsto (1 + x^2)^{-\frac{1}{2}+\varepsilon} \frac{1}{\psi(x)}$, with ψ given in (3.3), for some $\varepsilon > 0$ belongs to $L^2(\mathbb{R})$.*

The compact support property in Assumption 3.1, (1) ensures that the random field $(Y_j)_{j \in \mathbb{Z}^d}$ is m-dependent with $m > \Delta^{-1}\text{diam}(\text{supp}(f))$ (cf. Lemma 2.4). In particular, m increases when the grid size Δ of the sample is decreasing. Moreover, compact support of f together with $f \in L^{2+\tau}(\mathbb{R})$ implies that $f \in L^q(\mathbb{R})$ for all $0 < q \leq 2 + \tau$. Consequently, f fulfills the integrability condition (2.5). In contrast, if f does not have compact support, the Λ-integrability only ensures $f \in L^2(\mathbb{R})$.

Assumption 3.1, (3) is a moment assumption on Λ. More precisely, it is satisfied if and only if

$$\mathbb{E}|\Lambda(A)|^{2+\tau} < \infty$$

for all $A \in \mathcal{E}_0(\mathbb{R}^d)$, cf. [22]. By Proposition 3.3, (b), this assumption also implies $\mathbb{E}|X(0)|^{2+\tau} < \infty$ in our setting.

As a consequence of Proposition 3.3, (a) and (c), Assumption 3.1, (4) ensures regularity of uv_1 whereas (5) yields the polynomial decay of ψ. It was shown in [13, Theorem 3.10] that ψ and uv_1 are connected via the relation

$$|\psi(x)| = \exp\left(-\int_0^x \text{Im}\big(\mathcal{F}_+[uv_1](y)\big)dy\right), \quad x \in \mathbb{R};$$

hence, more regularity of uv_1 ensures slower decay rates for $|\psi(x)|$ as $x \to \pm\infty$. Further results on the polynomial decay of infinitely divisible characteristic functions as well as sufficient conditions for this property to hold can be found in [23].

Let us give some examples of Λ and f satisfying Assumption 3.1, (1)–(5).

Example 3.4 (Gamma random measure). Fix $b > 0$ and let for any $x \in \mathbb{R}$, $v_0(x) = x^{-1}e^{-bx}\mathbb{1}_{(0,\infty)}(x)$. Clearly, Assumption 3.1, (2) and (3) are satisfied for any $\tau > 0$. The Fourier transform of uv_0 is given by $\mathcal{F}_+[uv_0](x) = (b - ix)^{-1}$, $x \in \mathbb{R}$; hence

$$\int_{\text{supp}(f)} f(s)\mathcal{F}_+[uv_0](f(s)x)ds = \int_{\text{supp}(f)} \frac{f(s)}{b - if(s)x}ds, \quad x \in \mathbb{R}.$$

The latter identity shows that Assumption 3.1, (4) holds true for any integrable f with a compact support. Moreover, a simple calculation yields that for any $x \in \mathbb{R}$,

Assumption 3.1, (5) becomes

$$\int_{\mathbb{R}} (1 + x^2)^{-1+\varepsilon} \exp\left(\int_{\text{supp}(f)} \log\left(1 + \frac{x^2 f^2(s)}{b}\right) ds\right) dx < \infty. \qquad (3.6)$$

This condition is fulfilled for any $\varepsilon < \frac{1}{2} - \alpha$ if

$$\alpha := \int_{\text{supp}(f)} \max\left\{1, \frac{f^2(s)}{b}\right\} ds < \frac{1}{2}.$$

3.4 Consistency of $\hat{\mathcal{L}}_W$

In this section, we give an upper bound for the estimation error $\mathbb{E}|\hat{\mathcal{L}}_W v - \mathcal{L}v|$ that allows to derive conditions under which $\hat{\mathcal{L}}_W$ is consistent for the linear functional $\mathcal{L}: L^2(\mathbb{R}) \to \mathbb{R}$ given by

$$\mathcal{L}v = \langle v, uv_0 \rangle, \quad v \in L^2(\mathbb{R}).$$

With the notations from Section 2.4, we have that the adjoint operator $\mathcal{G}^{-1*}$: $\text{Image}(\mathcal{G}) \to L^2(\mathbb{R})$ of $\mathcal{G}^{-1}$ is given by

$$\mathcal{G}^{-1*}v = \mathcal{M}^{-1}\mathcal{F}_\times^{-1}\left(\frac{1}{\bar{\mu}_f}\mathcal{F}_\times \mathcal{M}v\right), \quad v \in \text{Image}(\mathcal{G}), \qquad (3.7)$$

where $\bar{\mu}_f$ denotes the complex conjugate function of μ_f. Moreover, the adjoint $\mathcal{G}_n^{-1*}: L^2(\mathbb{R}) \to L^2(\mathbb{R})$ of $\mathcal{G}_n^{-1}$ writes as

$$\mathcal{G}_n^{-1*}v = \mathcal{M}^{-1}\mathcal{F}_\times^{-1}\left(\frac{1}{\bar{\mu}_{f,n}}\mathcal{F}_\times \mathcal{M}v\right), \quad v \in L^2(\mathbb{R}),$$

with $\frac{1}{\bar{\mu}_{f,n}} = \frac{1}{\bar{\mu}_f}\mathbb{1}_{\{|\bar{\mu}_f| > a_n\}}$. Notice that $\mathcal{G}_n^{-1*}$ is a bounded operator whereas $\mathcal{G}^{-1*}$ is unbounded in general.

Remark 3.5. Notice that $\mathcal{G}_n^{-1*} = \mathcal{G}^{-1*}$ if $a_n = 0$ for any $n \in \mathbb{N}$. Hence, $\mathcal{G}_n^{-1*}\widehat{uv_1}$ in this case only is well-defined if $\widehat{uv_1}$ is an element of $\text{Image}(\mathcal{G}^*)$ what is indeed a very restrictive assumption. For a detailed discussion we refer to [13].

With the previous notations we now derive an upper bound for $\mathbb{E}|\hat{\mathcal{L}}_W v - \mathcal{L}v|$. Therefore, recall condition $(\mathbf{U}_\beta)$ from Section 2.4.

Lemma 3.6. *Let $\gamma = 0$ and suppose Assumption 3.1, (1)–(3) hold true for some $\tau \geq 0$. Moreover, let condition $(\mathbf{U}_\beta)$ be satisfied for some $\beta \geq 0$ and assume that $K_b: \mathbb{R} \to \mathbb{R}$ is a function with properties (K1)–(K3). Then*

$$\mathbb{E}|\hat{\mathcal{L}}_W v - \mathcal{L}v| \leq \frac{S}{\sqrt{\pi}}\mathbb{E}|Y_0|\left(\frac{n}{b}\right)^{1/2}\|(\mathcal{G}_n^{-1*} - \mathcal{G}^{-1*})v\|_{L^2(\mathbb{R})}$$

$$+ \frac{1}{2\pi}\left\langle|\mathcal{F}_+[\mathcal{G}^{-1*}v]|, |\mathcal{F}_+[uv_1]||1 - \mathcal{F}_+[K_b]|\right\rangle_{L^2(\mathbb{R})}$$

$$+ \frac{c \cdot S}{2\pi\sqrt{n}}\left(\sqrt{\mathbb{E}|Y_0|^2} + \|uv_1\|_{L^1(\mathbb{R})}\right)\int_{\mathbb{R}} \frac{|\mathcal{F}_+[\mathcal{G}^{-1*}v]|(x)}{|\psi(x)|} dx \quad (3.8)$$

for any $v \in \text{Image}(\mathcal{G})$ such that $\int_{\mathbb{R}} \frac{|\mathcal{F}_+[\mathcal{G}^{-1*}v](x)|}{|\psi(x)|} dx < \infty$, where $c > 0$ is some constant and $S := \sup_{b>0, x\in\mathbb{R}} |\mathcal{F}_+[K_b](x)|.$

A proof of Lemma 3.6 as well as of Theorem 3.7 below can be found in Appendix.

Theorem 3.7. *Fix $\gamma \in \mathbb{R}$. Suppose that condition $(\mathbf{U}_{\beta_1})$ is satisfied for some $\beta_1 \geq 0$ and let $v \in L^2(\mathbb{R})$ be such that $\mathcal{G}^{-1*}v \in H^1(\mathbb{R})$, $\frac{\mathcal{F}_+[\mathcal{G}^{-1*}v]}{\psi} \in L^1(\mathbb{R})$ and*

$$(\mathcal{M}v)(\exp(\cdot)), \quad (\mathcal{M}v)(-\exp(\cdot)) \in H^{\beta_2}(\mathbb{R}) \tag{3.9}$$

for some $\beta_2 > \beta_1$. Moreover, let $a = a_n$ and $b = b_n$ be sequences with the properties

$$a_n \to 0, \quad b_n \to 0 \quad and \quad a_n = o\left(\left(\frac{n}{b_n}\right)^{\frac{\beta_1}{2(\beta_1-\beta_2)}}\right), \quad as \ n \to \infty,$$

and assume that conditions (K1)–(K3) are fulfilled. Then, under Assumption 3.1, (1)–(4), $\mathbb{E}|\hat{\mathcal{L}}_W v - \mathcal{L}v| \to 0$ as $n \to \infty$ with the order of convergence given by

$$\mathbb{E}|\hat{\mathcal{L}}_W v - \mathcal{L}v| = \mathcal{O}\left(a_n^{\frac{\beta_2}{\beta_1}-1}\sqrt{\frac{n}{b_n}} + b_n + \frac{1}{\sqrt{n}}\right).$$

Remark 3.8. (a) Notice that condition $(\mathbf{U}_\beta)$ ensures uniqueness of $uv_0 \in L^2(\mathbb{R})$ as a solution of $\mathcal{G}uv_0 = uv_1$. In Lemma 3.6, it can be replaced by the more (and most) general assumption $m_{f,\pm} \neq 0$ almost everywhere on $\mathbb{R}$. Moreover, condition **(K3)** can be replaced by $\sup_{b>0, \, x \in \mathbb{R}} |\mathcal{F}_+[K_b](x)| < \infty$ in Lemma 3.6.

(b) In order to deduce the convergence rate in Theorem 3.7 explicitly, condition (3.9) is essential. Moreover, it ensures that the function v belongs to the range of $\mathcal{G}$ (cf. [13, Theorem 3.1]); hence the expression $\mathcal{G}^{-1*}v$ is well-defined.

(c) The condition $\mathcal{G}^{-1*}v \in H^1(\mathbb{R})$ in Theorem 3.7 can be dropped if $\gamma = 0$.

(d) Under the conditions of Theorem 3.7, the convergence rate of $\mathbb{E}|\hat{\mathcal{L}}_W v - \mathcal{L}v| \to 0$ is at least $\mathcal{O}(n^{-1/2})$ as $n \to \infty$, provided that

$$a_n = o\left(\left(\frac{n}{\sqrt{b_n}}\right)^{\frac{\beta_1}{\beta_1-\beta_2}}\right) \quad and \quad b_n = \mathcal{O}\left(\frac{1}{\sqrt{n}}\right), \quad as \ n \to \infty.$$

We close this section with the following example, showing that the functions g_t considered in [20, p. 3309] may belong to the range of $\mathcal{G}^{-1*}$.

Example 3.9. Fix $t > 0$ and let $v(x) = \frac{1}{x}\mathbb{1}_{\mathbb{R}\setminus[-t,t]}(x)$, $x \in \mathbb{R}$. Apparently, $v \in L^2(\mathbb{R})$ fulfills condition (3.9) for any $\beta_2 > 0$. Let for some fixed λ, $\theta > 0$, $f(s) = e^{-\lambda s}\mathbb{1}_{(0,\theta)}(s)$, $s \in \mathbb{R}$. Then a simple computation yields that $(\mathbf{U}_{\beta_1})$ is satisfied with $\beta_1 = 1$. Moreover, for all $x \neq 0$

$$(\mathcal{G}^{-1*}v)(x) = \frac{1}{2x}\log\left(\frac{|x|}{t}\right)\mathbb{1}_{(t,te^{\lambda\theta}]}(|x|) + \frac{\lambda\theta}{2x}\mathbb{1}_{(te^{\lambda\theta},\infty)}(|x|);$$

hence, $\mathcal{G}^{-1*}v \in H^1(\mathbb{R})$. Since

$$\left\|\frac{\mathcal{F}_+[\mathcal{G}^{-1*}v]}{\psi}\right\|_{L^1(\mathbb{R})} \leq \|\mathcal{G}^{-1*}v\|_{H^1(\mathbb{R})}\left\|\frac{(1+\cdot^2)^{-\frac{1+\varepsilon}{2}}}{\psi}\right\|_{L^2(\mathbb{R})},$$

any random measure Λ satisfying Assumption 3.1, (5) yields $\frac{\mathcal{F}_+[\mathcal{G}^{-1*}v]}{\psi} \in L^1(\mathbb{R})$ (cf. Proposition 3.3, (c)).

3.5　A central limit theorem for $\hat{\mathcal{L}}_W$

Provided the assumptions of Theorem 3.7 are satisfied,

$$\text{err}_W(v) := \sqrt{n}\,(\hat{\mathcal{L}}_W v - \mathcal{L} v)$$

is bounded in mean. In this section, we give conditions under which $\text{err}_W(v)$ is asymptotically Gaussian. For this purpose, introduce the following notation.

Definition 3.10. Let Assumption 3.1 be satisfied and suppose that condition $(\mathbf{U}_{\beta_1})$ is fulfilled for some $\beta_1 > 0$. A function $v \in L^2(\mathbb{R})$ is called *admissible* of index (ξ, β_2) if

(i) $\mathcal{G}^{-1*}v \in H^{\frac{3}{2}-\varepsilon}(\mathbb{R})$,

(ii) $(\mathcal{M}v)(\exp(\cdot))$, $(\mathcal{M}v)(-\exp(\cdot)) \in H^{\beta_2}(\mathbb{R})$ for some $\beta_2 > \beta_1$ and

(iii) $|\mathcal{F}_+[\mathcal{G}^{-1*}v](x)| \lesssim (1+x^2)^{-\xi/2}$ for some $\xi > 2(1-\varepsilon) - \left(\frac{1}{2}-\varepsilon\right)\frac{1+\tau}{2+\tau}$.

The linear subspace of all admissible functions of index (ξ, β_2) is denoted by $\mathcal{U}(\xi, \beta_2)$.

Remark 3.11. (a) The parameters ε and τ describe the size of $\mathcal{U}(\xi, \beta_2)$. In particular, for larger values of ε and τ, the set of admissible functions is increasing and vice versa.

(b) Assumption 3.1, (5) implies $\varepsilon < \frac{1}{2}$ (otherwise $\frac{1}{|\psi(x)|} \to 0$ as $|x| \to \infty$); hence Definition 3.10, (i) yields that $\mathcal{F}_+[\mathcal{G}^{-1*}v] \in L^1(\mathbb{R})$.

(c) Clearly, the lower bound for ξ in Definition 3.10, (iii) can be replaced by $\xi > \frac{7}{4} - \frac{3}{2}\varepsilon$. Nevertheless, since our purpose is to point out the influence of τ on the set of admissible functions, we do not use this simplification.

(d) It immediately follows from formula (3.7) that $\mathcal{G}^{-1*}v \in H^{\delta}(\mathbb{R})$ if and only if $\mathcal{G}^{-1}v \in H^{\delta}(\mathbb{R})$.

For any $j \in W$ and any admissible function $v \in \mathcal{U}(\xi, \beta_2)$, introduce the random variables

$$Z_{j,v}^{(1)} = \frac{1}{2\pi} Y_j \mathcal{F}_+\left[\frac{\mathcal{F}_+[\mathcal{G}^{-1*}v](-\cdot)}{\psi(\cdot)}\right](Y_j) \quad \text{and}$$

$$Z_{j,v}^{(2)} = \frac{i}{2\pi} \mathcal{F}_+\left[\mathcal{F}_+[\mathcal{G}^{-1*}v](-\cdot)\left(\frac{1}{\psi}\right)'\right](Y_j).$$

In the sequel, it is assumed that the random field Y introduced in (3.1) is observed on a sequence $(W_k)_{k\in\mathbb{N}}$ of regularly growing observation windows (cf. Section 2.2). To avoid longer notations, we drop the index k in this notation and shortly write W instead of W_k. Moreover, we denote by $n\ (= n(k))$ the cardinality of W.

With the previous notation, we now can formulate the main result of this section.

Theorem 3.12. *Fix $m \in \mathbb{N}$, $m > \Delta^{-1}\text{diam}(\text{supp}(f))$. Let Assumption 3.1 be satisfied and suppose that conditions (**K1**)–(**K3**) are fulfilled. Moreover, let for some $\eta > 0$ the*

sequences a_n and b_n be given by

$$a_n = o\left(\left(\frac{n}{\sqrt{b_n}}\right)^{\frac{\beta_1}{\beta_1 - \beta_2}}\right) \quad and \quad b_n \approx n^{-\frac{1}{1-2\varepsilon}}(\log n)^{\eta + \frac{1}{1-2\varepsilon}}, \quad as \ n \to \infty.$$

Then, as W is regularly growing to infinity,

$$\mathrm{err}_W(v) \xrightarrow{d} N_v,$$

for any admissible function $v \in \mathcal{U}(\xi, \beta_2)$, where N_v is a Gaussian random variable with zero expectation and variance given by

$$\sigma_v^2 = \sum_{j \in \mathbb{Z}^d : \|j\|_\infty \le m} \mathbb{E}\left[\left(Z_{j,v}^{(1)} - Z_{j,v}^{(2)}\right)\left(Z_{0,v}^{(1)} - Z_{0,v}^{(2)}\right)\right]. \tag{3.10}$$

A proof of Theorem 3.12 can be found in Section 4.

Remark 3.13. Unfortunately, we could not provide a rate for the convergence $\mathrm{err}_W(v) \xrightarrow{d} N_v$ in Theorem 3.12. Therefore, it would be sufficient to provide, e.g., $L^1(\Omega, \mathbb{P})$-rates for the convergence $\sup_x |\hat{\psi}(x) - \psi(x)|$, $\sup_x |\hat{\theta}(x) - \theta(x)| \to 0$ (as $|W| \to \infty$), that seems to be a hard problem in the dependent observations setting.

Corollary 3.14. *Let the assumptions of Theorem 3.12 hold true. Then, as W is regularly growing to infinity,*

$$(\mathrm{err}_W(v_1), \ldots, \mathrm{err}_W(v_k))^\top \xrightarrow{d} N_{v_1, \ldots, v_k},$$

for any $v_1 \in \mathcal{U}(\xi_1, \beta_2^{(1)}), \ldots, v_k \in \mathcal{U}(\xi_k, \beta_2^{(k)})$, where $N_{v_1, \ldots, v_k}$ is a centered Gaussian random vector with covariance matrix $(\Sigma_{s,t})_{s,t=1,\ldots,k}$ given by

$$\Sigma_{s,t} = \sum_{j \in \mathbb{Z}^d : \|j\|_\infty \le m} \mathbb{E}\left[\left(Z_{j,v_s}^{(1)} - Z_{j,v_s}^{(2)}\right)\left(Z_{0,v_t}^{(1)} - Z_{0,v_t}^{(2)}\right)\right].$$

Proof. Suppose $v_1 \in \mathcal{U}(\xi_1, \beta_2^{(1)}), \ldots, v_k \in \mathcal{U}(\xi_k, \beta_2^{(k)})$ and, for arbitrary numbers $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$, let $v = \sum_{l=1}^k \lambda_l v_l$. Then a simple calculation yields

$$\sum_{l=1}^k \lambda_l \, \mathrm{err}_W(v_l) = \sqrt{n} \, (\hat{\mathcal{L}}_W v - \mathcal{L}v).$$

Since $v \in \mathcal{U}(\min_l \xi_l, \min_l \beta_2^{(l)})$, by Theorem 3.12, $\sqrt{n} \, (\hat{\mathcal{L}}_W v - \mathcal{L}v) \xrightarrow{d} N_v$, where N_v is a Gaussian random variable with zero expectation and variance given in (3.10). Now, let $(T_1, \ldots, T_k)^\top$ be a zero mean Gaussian random vector with covariance given by $(\Sigma_{s,t})_{s,t=1,\ldots,k}$. Using linearity of $\mathcal{F}_+$ and $\mathcal{G}^{-1*}$, a short computation shows that

$$N_v \overset{d}{=} \sum_{l=1}^k \lambda_l T_l;$$

hence, the assertion follows by the Cramér–Wold theorem (cf. [6]).   □

4 Proof of Theorem 3.12

In order to prove Theorem 4, we adopt the strategy of the proof of [20, Theorem 2]. Nevertheless, the main difficulty in our setting is that the observations $(Y_j)_{j \in W}$ are not independent; hence the classical theory cannot be applied here. Instead, we use asymptotic results for partial sums of m-dependent random fields (cf. [8]) in combination with the theory developed by Bulinski and Shashkin in [7] for weakly dependent random fields.

We start with the following lemma.

Lemma 4.1. *Let* $\gamma = 0$ *and suppose that* $v \in \mathcal{U}(\xi, \beta_2)$ *is an admissible function. Then Assumption 3.1 implies:*

1. xP *has a bounded Lebesgue density on* $\mathbb{R}$*, where* P *denotes the distribution of* $X(0)$.

2. $\left(\frac{1}{\psi}\right)' \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ *and* $\frac{1}{|\psi(x)|} \lesssim (1 + |x|)^{\frac{1}{2} - \varepsilon}$ *for all* $x \in \mathbb{R}$.

3. $\mathcal{F}_+[\mathcal{G}^{-1} * v], \frac{\mathcal{F}_+[\mathcal{G}^{-1} * v]}{\psi} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.

Proof.   1. Let $\mu(dx) = (uv_1)(x)dx$. By Proposition 3.3, (a), $uv_1 \in L^1(\mathbb{R})$; hence, μ defines a finite signed measure on $\mathbb{R}$. Since $\theta = \psi\mathcal{F}_+[uv_1]$, we conclude that

$$\mathcal{F}_+[xP](t) = \theta(t) = \mathcal{F}_+[P](t)\mathcal{F}_+[uv_1](t) = \mathcal{F}_+[\mu * P](t),$$

i.e. $xP(dx) = (\mu * P)(dx)$; thus, xP has the density given by $\frac{d[xP]}{dx} = \int_{\mathbb{R}}(uv_1)(x - y)P(dy)$ and consequently $\|\frac{d[xP]}{dx}\|_{L^\infty(\mathbb{R})} \leq \|uv_1\|_{L^\infty(\mathbb{R})}$.

2. By Assumption 3.1, (4), (5), Proposition 3.3, (a), (c) and the Cauchy–Schwarz inequality, we obtain for any $x \in \mathbb{R}$,

$$\begin{aligned}
\frac{1}{|\psi(x)|} &= 1 + \int_0^x \left(\frac{1}{\psi}\right)'(t)dt = 1 + \int_0^x \frac{|\theta(t)|}{|\psi(t)|^2}dt \\
&= 1 + \int_0^x \frac{|\mathcal{F}_+[uv_1](t)|}{|\psi(t)|}dt \\
&\lesssim 1 + \int_0^x (1 + t^2)^{-\frac{\varepsilon}{2}} \frac{(1 + t^2)^{-\frac{1-\varepsilon}{2}}}{|\psi(t)|}dt \\
&\leq 1 + \left\|\frac{(1 + \cdot^2)^{-\frac{1-\varepsilon}{2}}}{\psi}\right\|_{L^2(\mathbb{R})} \left(\int_0^x (1 + t^2)^{-\varepsilon}dt\right)^{1/2} \\
&\lesssim (1 + |x|)^{\frac{1}{2} - \varepsilon}.
\end{aligned}$$

Further, we have for any $x \in \mathbb{R}$,

$$\left|\left(\frac{1}{\psi}\right)'(x)\right| = \frac{|\mathcal{F}_+[uv_1](x)|}{|\psi(x)|} \lesssim (1 + |x|)^{-\frac{1}{2} - \varepsilon}.$$

The last expression is bounded and square integrable, hence $\left(\frac{1}{\psi}\right)' \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

3. $\mathcal{F}_+[\mathcal{G}^{-1*}v] \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ immediately follows from Definition 3.10, (i) (cf. Remark 3.11, (b)). Moreover, by Proposition 3.3, (a), we find that

$$\int_{\mathbb{R}} \frac{|\mathcal{F}_+[\mathcal{G}^{-1*}v](x)|}{|\psi(x)|} dx \le \|\mathcal{G}^{-1*}v\|_{H^{1-\varepsilon}(\mathbb{R})} \left\| \frac{(1+\cdot^2)^{-\frac{1-\varepsilon}{2}}}{\psi} \right\|_{L^2(\mathbb{R})},$$

where the latter is finite due to Definition 3.10, (i). The bound in part (2) finally yields

$$\int_{\mathbb{R}} \frac{|\mathcal{F}_+[\mathcal{G}^{-1*}v](x)|^2}{|\psi(x)|^2} dx \lesssim \int_{\mathbb{R}} |\mathcal{F}_+[\mathcal{G}^{-1*}v](x)|^2 (1+|x|^2)^{\frac{1}{2}-\varepsilon} dx$$

$$= \|\mathcal{G}^{-1*}v\|_{H^{1-2\varepsilon}(\mathbb{R})} < \infty. \qquad \square$$

In order to prove Theorem 3.12, consider the following decomposition that can be obtained by the isometry property of $\mathcal{F}_+$:

$$\mathrm{err}_W(v) = \sqrt{n}(\hat{\mathcal{L}}_W v - \mathcal{L}v) = \frac{1}{2\pi}\Big[E_1 + E_2 + E_3 + E_4\Big] + E_5,$$

with $E_1, \ldots, E_5$ given by

$$E_1 = \sqrt{n}\Big\langle \mathcal{F}_+[\mathcal{G}^{-1*}v], \Big\{\frac{\hat{\theta}-\theta}{\psi} - i\Big(\frac{1}{\psi}\Big)'(\hat{\psi}-\psi)\Big\}\mathcal{F}_+[K_b]\Big\rangle_{L^2(\mathbb{R})}$$

$$E_2 = \sqrt{n}\Big\langle \mathcal{F}_+[\mathcal{G}^{-1*}v], \Big\{R_n + \theta\frac{\psi-\hat{\psi}}{\psi^2}\mathbb{1}_{\{|\hat{\psi}|\le n^{-1/2}\}}\Big\}\mathcal{F}_+[K_b]\Big\rangle_{L^2(\mathbb{R})}$$

$$E_3 = \sqrt{n}\Big\langle \mathcal{F}_+[\mathcal{G}^{-1*}v], \frac{\theta}{\psi}(\mathcal{F}_+[K_b]-1)\Big\rangle_{L^2(\mathbb{R})}$$

$$E_4 = \sqrt{n}\Big\langle \mathcal{F}_+[\mathcal{G}^{-1*}v], \mathcal{F}_+[K_b]\Big\{\theta\frac{\hat{\psi}-\psi}{\psi^2} - \frac{\hat{\theta}}{\psi}\Big\}\mathbb{1}_{\{|\hat{\psi}|\le n^{-1/2}\}}\Big\rangle_{L^2(\mathbb{R})}$$

$$E_5 = \sqrt{n}\Big\langle (\mathcal{G}_n^{-1*} - \mathcal{G}^{-1*})v, \widehat{uv_1}\Big\rangle_{L^2(\mathbb{R})},$$

and $R_n = \big(1 - \frac{\hat{\psi}}{\psi}\big)\big(\frac{\hat{\theta}}{\psi} - \frac{\theta}{\psi}\big)$. We call the expression E_1 *main stochastic term* and the expression E_2 *remainder term*.

Subsequently, we give a step by step proof for Theorem 3.12 by considering each of the above terms $E_1, \ldots, E_5$ seperately.

We first show that the deterministic term E_3 tends to zero as the sample size n tends to infinity.

Lemma 4.2. *Suppose $\gamma = 0$. Then, under the conditions of Theorem 3.12,*

$$E_3 = \sqrt{n}\Big\langle \mathcal{F}_+[\mathcal{G}^{-1*}v], \frac{\theta}{\psi}(\mathcal{F}_+[K_b]-1)\Big\rangle_{L^2(\mathbb{R}^\times)} \to 0, \quad as\ n \to \infty,$$

for any admissible function $v \in \mathcal{U}(\xi, \beta_2)$.

Proof. Taking into account that $\left|\frac{\theta}{\psi}\right| = |\mathcal{F}_+[uv_1]|$, Assumption 3.1, (4), together with Proposition 3.3, (a) and condition **(K3)** yield

$$|E_3| \leq \sqrt{n} \int_{\mathbb{R}} |\mathcal{F}_+[\mathcal{G}^{-1*}v](x)||\mathcal{F}_+[uv_1](x)||1 - \mathcal{F}[K_{b_n}](x)|dx$$

$$\lesssim b_n \sqrt{n} \int_{\mathbb{R}} |\mathcal{F}_+[\mathcal{G}^{-1*}v](x)|dx,$$

where the last line is finite due to Lemma 4.1. Moreover, since, $b_n = o(n^{-1/2})$ it tends to 0 as $n \to \infty$. $\square$

Next, we observe that E_5 is asymptotically negligible in mean.

Lemma 4.3. *Let the assumptions of Theorem 3.12 be satisfied. Then*

$$\mathbb{E}|E_5| = n^{1/2}\mathbb{E}\left|\left\langle (\mathcal{G}_n^{-1*} - \mathcal{G}^{-1*})v, \widehat{uv_1}\right\rangle_{L^2(\mathbb{R})}\right| \to 0, \quad as \ n \to \infty,$$

for any $v \in \mathcal{U}(\xi, \beta_2)$.

Proof. From the proofs of Theorem 3.6 and Corollary 3.7 we conclude that

$$\sqrt{n}\,\mathbb{E}\left|\left\langle (\mathcal{G}_n^{-1*} - \mathcal{G}^{-1*})v, \widehat{uv_1}\right\rangle_{L^2(\mathbb{R})}\right| \lesssim \sqrt{n}\,a_n^{\frac{\beta_2}{\beta_1}-1}\left(\frac{n}{b_n}\right)^{1/2};$$

hence, $\mathbb{E}|E_5| \to 0$ as $n \to \infty$, since $a_n = o\left(\left(\frac{n}{\sqrt{b_n}}\right)^{\frac{\beta_1}{\beta_1-\beta_2}}\right)$. $\square$

Lemma 4.4. *Suppose $\gamma = 0$ and let the assumptions of Theorem 3.12 be satisfied. Then*

$$\mathbb{E}|E_4| = \sqrt{n}\,\mathbb{E}\left|\left\langle \mathcal{F}_+[\mathcal{G}^{-1*}v], \mathcal{F}_+[K_b]\left\{\theta\frac{\hat{\psi}-\psi}{\psi^2} - \frac{\hat{\theta}}{\psi}\right\}\mathbb{1}_{\{|\hat{\psi}|\leq n^{-1/2}\}}\right\rangle_{L^2(\mathbb{R})}\right| \to 0,$$

as $n \to \infty$, for any admissible function $v \in \mathcal{U}(\xi, \beta_2)$.

Proof. Since $\frac{\theta}{\psi^2} = i\left(\frac{1}{\psi}\right)'$, we obtain by conditions **(K2)** and **(K3)**,

$$\mathbb{E}|E_4| \leq S \int_{-b_n^{-1}}^{b_n^{-1}} |\mathcal{F}_+[\mathcal{G}^{-1*}v](x)|\left|\left(\frac{1}{\psi}\right)'(x)\right|\sqrt{n}\,\mathbb{E}\left[|\hat{\psi}(x) - \psi(x)|\mathbb{1}_{\{|\hat{\psi}(x)|\leq n^{-1/2}\}}\right]dx$$

$$+ S \int_{-b_n^{-1}}^{b_n^{-1}} \frac{|\mathcal{F}_+[\mathcal{G}^{-1*}v](x)|}{|\psi(x)|}\sqrt{n}\,\mathbb{E}\left[|\hat{\theta}(x)|\mathbb{1}_{\{|\hat{\psi}(x)|\leq n^{-1/2}\}}\right]dx,$$

with $S := \sup_{x \in \mathbb{R}, \, b > 0} |\mathcal{F}_+[K_b](x)|$. In order to bound the summands on the right-hand side of the latter inequality, we start with the following observation: $\exists\, n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$x \in [-b_n^{-1}, b_n^{-1}] \quad \Rightarrow \quad |\psi(x)| > 2n^{-1/2}. \tag{4.1}$$

Indeed, by Lemma 4.1, (2), there is a constant $c > 0$ such that $\frac{1}{|\psi(x)|} \leq c(1 + |x|)^{\frac{1}{2}-\varepsilon}$, for all $x \in \mathbb{R}$. Hence, if $|\psi(x)| \leq 2n^{-1/2}$, then $|x| \geq \left(\frac{1}{2c}\right)^{2/(1-2\varepsilon)} n^{1/(1-2\varepsilon)} - 1$. Since $b_n^{-1} = o(n^{1/(1-2\varepsilon)})$ as $n \to \infty$, there exists $n_0 \in \mathbb{N}$, such that $b_n^{-1} < \left(\frac{1}{2c}\right)^{2/(1-2\varepsilon)} n^{1/(1-2\varepsilon)} - 1$ for all $n \geq n_0$, i.e. $x \notin [-b_n^{-1}, b_n^{-1}]$. This shows (4.1).

In the sequel, we assume that $n \geq n_0$ and consider each summand in the above inequality seperately:

1. Using the m-dependence of $(Y_j)_{j \in \mathbb{Z}^d}$, we conclude as in the first part of the proof of [18, Lemma 8.3] that

$$\mathbb{P}(|\hat{\psi}(x)| \leq n^{-1/2}) \lesssim \frac{n^{-p}}{|\psi(x)|^{2p}}, \tag{4.2}$$

for any $p \geq 1/2$ and all $x \in \mathbb{R}$ with $|\psi(x)| > 2n^{-1/2}$. Taking $p = 1/2$ in (4.2), by the Cauchy–Schwarz inequality, [18, Lemma 8.2] and Lemma 4.1, (2), (3), we find that

$$\int_{-b_n^{-1}}^{b_n^{-1}} |\mathcal{F}_+[\mathcal{G}^{-1*}v](x)| \left|\left(\frac{1}{\psi}\right)'(x)\right| \sqrt{n}\, \mathbb{E}\left[|\hat{\psi}(x) - \psi(x)| \mathbb{1}_{\{|\hat{\psi}(x)| \leq n^{-1/2}\}}\right] dx$$

$$\lesssim \int_{-b_n^{-1}}^{b_n^{-1}} |\mathcal{F}_+[\mathcal{G}^{-1*}v](x)| \left|\left(\frac{1}{\psi}\right)'(x)\right| \mathbb{P}\left(|\hat{\psi}(x)| \leq n^{-1/2}\right)^{1/2} \mathbb{1}_{\{|\psi(x)| > 2n^{-1/2}\}} dx$$

$$\leq n^{-1/4} \int_{-b_n^{-1}}^{b_n^{-1}} \frac{|\mathcal{F}_+[\mathcal{G}^{-1*}v](x)|}{|\psi(x)|^{1/2}} \left|\left(\frac{1}{\psi}\right)'(x)\right| \mathbb{1}_{\{|\psi(x)| > 2n^{-1/2}\}} dx$$

$$\leq n^{-1/4} \left\|\frac{\mathcal{F}_+[\mathcal{G}^{-1*}v]}{\psi}\right\|_{L^1(\mathbb{R})} \left\|\left(\frac{1}{\psi}\right)'\right\|_{L^\infty(\mathbb{R})},$$

for all $n \geq n_0$, where the last inequality uses the fact that $|\psi(x)| \leq 1$. Hence, the first integral tends to zero as $n \to \infty$.

2. For the second integral, by the triangle inequality we observe that for any $n \geq n_0$,

$$\int_{-b_n^{-1}}^{b_n^{-1}} \frac{|\mathcal{F}_+[\mathcal{G}^{-1*}v](x)|}{|\psi(x)|} \sqrt{n}\, \mathbb{E}\left[|\hat{\theta}(x)| \mathbb{1}_{\{|\hat{\psi}(x)| \leq n^{-1/2}\}}\right] dx$$

$$\leq \int_{-b_n^{-1}}^{b_n^{-1}} \left(I_1(x) + I_2(x)\right) dx,$$

where

$$I_1(x) = \frac{|\mathcal{F}_+[\mathcal{G}^{-1*}v](x)|}{|\psi(x)|} \sqrt{n}\, \mathbb{E}\left[|\hat{\theta}(x) - \theta(x)| \mathbb{1}_{|\hat{\psi}(x)| \leq n^{-1/2}}\right] \mathbb{1}_{\{|\psi(x)| > 2n^{-1/2}\}}$$

and

$$I_2(x) = |\mathcal{F}_+[\mathcal{G}^{-1*}v](x)| \sqrt{n}\, \frac{|\theta(x)|}{|\psi(x)|} \mathbb{P}\left(|\hat{\psi}(x)| \leq n^{-1/2}\right) \mathbb{1}_{\{|\psi(x)| > 2n^{-1/2}\}}.$$

Applying Lemma A.2 with $q = 1/2$ (cf. Appendix A.4), we find that

$$I_1(x) \lesssim \sqrt{\mathbb{E}|Y_0|^2} \, \frac{|\mathcal{F}_+[\mathcal{G}^{-1*}v](x)|}{|\psi(x)|}, \qquad \text{for all } x \in \mathbb{R};$$

hence, by Lemma 4.1, (3) and the finite $(2 + \tau)$-moment condition, I_1 is majorized by an integrable function. Moreover, applying the Cauchy–Schwarz inequality, (4.2) and again Lemma A.2 (with $q = 1$) yields

$$I_1(x) \lesssim \frac{|\mathcal{F}_+[\mathcal{G}^{-1*}v](x)|}{|\psi(x)|} \mathbb{P}\Big(|\hat{\psi}(x)| \leq n^{-1/2}\Big)^{1/2} \mathbb{1}_{\{|\psi(x)|>2n^{-1/2}\}}$$
$$\leq \frac{|\mathcal{F}_+[\mathcal{G}^{-1*}v](x)|}{|\psi(x)|} \frac{n^{-1/4}}{|\psi(x)|^{1/2}} \to 0, \qquad \text{as } n \to \infty,$$

for all $x \in \mathbb{R}$. By dominated convergence, $\lim_{n\to\infty} \int_{-b_n^{-1}}^{b_n^{-1}} I_1(x)dx = 0$ follows. Further,

$$\int_{-b_n^{-1}}^{b_n^{-1}} I_2(x)dx \leq n^{-1/2} \int_{-b_n^{-1}}^{b_n^{-1}} \frac{|\mathcal{F}_+[\mathcal{G}^{-1*}v](x)|}{|\psi(x)|} \frac{|\theta(x)|}{|\psi(x)|^2} \mathbb{1}_{\{|\psi(x)|>2n^{-1/2}\}} dx$$
$$\leq n^{-1/2} \left\| \frac{\mathcal{F}_+[\mathcal{G}^{-1*}v]}{\psi} \right\|_{L^1(\mathbb{R})} \left\| \left(\frac{1}{\psi}\right)' \right\|_{L^\infty(\mathbb{R})},$$

by (4.2) (with $p = 1$), Lemma 4.1, (2), (3) and the Cauchy–Schwarz inequality; hence, also $\lim_{n\to\infty} \int_{-b_n^{-1}}^{b_n^{-1}} I_2(x)dx = 0$.

All in all, this shows the assertion of the lemma. ☐

4.1 Main stochastic term

In this section we show the asymptotic normality of the main stochastic term. For this purpose, let $P_n : \mathcal{B}(\mathbb{R}) \to [0, 1]$ be the empirical measure given by

$$P_n = \frac{1}{n} \sum_{j \in W} \delta_{Y_j},$$

where $\delta_x : \mathcal{B}(\mathbb{R}) \to \{0, 1\}$ denotes the Dirac measure concentrated in $x \in \mathbb{R}$. Further, for any $v \in \mathcal{U}(\xi, \beta_2)$, define the random fields $(Z_{j,v,n}^{(k)})_{j \in \mathbb{Z}^d}$, $k = 1, 2$, by

$$Z_{j,v,n}^{(1)} = \frac{1}{2\pi} Y_j \mathcal{F}_+\left[\frac{\mathcal{F}_+[\mathcal{G}^{-1*}v](-\cdot)}{\psi} \mathcal{F}_+[K_{b_n}]\right](Y_j) \tag{4.3}$$

and

$$Z_{j,v,n}^{(2)} = \frac{i}{2\pi} \mathcal{F}_+\left[\mathcal{F}_+[\mathcal{G}^{-1*}v](-\cdot)\left(\frac{1}{\psi}\right)' \mathcal{F}_+[K_{b_n}]\right](Y_j). \tag{4.4}$$

The following theorem is the main result of this section.

Theorem 4.5. *Let the assumptions of Theorem 3.12 be satisfied. Then, as W is regularly growing to infinity,*

$$\frac{1}{2\pi} E_1 = \frac{\sqrt{n}}{2\pi} \left\langle \mathcal{F}_+[\mathcal{G}^{-1*}v], \left\{ \frac{\hat{\theta} - \theta}{\psi} - i\left(\frac{1}{\psi}\right)'(\hat{\psi} - \psi)\right\} \mathcal{F}_+[K_b] \right\rangle_{L^2(\mathbb{R})} \xrightarrow{d} N_v,$$

for any $v \in \mathcal{U}(\xi, \beta_2)$, where N_v is a Gaussian random variable with zero expectation and variance σ^2 given in (3.10).

In order to prove Theorem 4.5, we first show some auxiliary statements. We begin with the following representation for the main stochastic term.

Lemma 4.6. *Let $v \in \mathcal{U}(\xi, \beta_2)$. Then, under the assumptions of Theorem 3.12, the main stochastic term can be represented by*

$$\frac{1}{2\pi} E_1 = \frac{1}{\sqrt{n}} \sum_{j \in W} \left(Z^{(1)}_{j,v,n} - Z^{(2)}_{j,v,n} \right),$$

with $Z^{(k)}_{j,v,n}$, $k = 1, 2$, given in (4.3) and (4.4).

Proof. Since $\theta = -i\psi'$,

$$i\left(\psi\left(\frac{1}{\psi}\right)' - \frac{\theta}{\psi}\right) = i\left(\psi\frac{1}{\psi}\right)' = 0;$$

hence,

$$\begin{aligned}
\frac{1}{2\pi} E_1 &= \frac{\sqrt{n}}{2\pi} \left\langle \mathcal{F}_+[\mathcal{G}^{-1*}v], \left\{ \frac{\hat{\theta}}{\psi} - i\left(\frac{1}{\psi}\right)'\hat{\psi} \right\} \mathcal{F}_+[K_{b_n}] \right\rangle_{L^2(\mathbb{R})} \\
&= \frac{\sqrt{n}}{2\pi} \int_{\mathbb{R}} \mathcal{F}_+[\mathcal{G}^{-1*}v](x) \left\{ \frac{\hat{\theta}(-x)}{\psi(-x)} - i\left(\frac{1}{\psi}\right)'(-x)\hat{\psi}(-x) \right\} \mathcal{F}_+[K_{b_n}](-x)dx \\
&= \frac{\sqrt{n}}{2\pi} \Bigg[\int_{\mathbb{R}} \mathcal{F}_+[\mathcal{G}^{-1*}v](x) \frac{\hat{\theta}(-x)}{\psi(-x)} \mathcal{F}_+[K_{b_n}](-x)dx \\
&\quad - i \int_{\mathbb{R}} \mathcal{F}_+[\mathcal{G}^{-1*}v](x) \left(\frac{1}{\psi}\right)'(-x)\hat{\psi}(-x)\mathcal{F}_+[K_{b_n}](-x)dx \Bigg].
\end{aligned}$$

Now, taking into account that $\hat{\psi}(x) = \int_{\mathbb{R}} e^{itx} P_n(dt)$ and $\hat{\theta}(x) = \int_{\mathbb{R}} e^{itx} t P_n(dt)$, Fubini's theorem yields the desired result. $\qquad\square$

The following lemma justifies the asymptotic variance σ^2 in Theorem 3.12.

Lemma 4.7. *Let the assumptions of Theorem 3.12 be satisfied and suppose functions $v_1 \in \mathcal{U}(\xi_1, \beta_2^{(1)})$ and $v_2 \in \mathcal{U}(\xi_2, \beta_2^{(2)})$. Then, as W is regularly growing to infinity,*

$$\text{Cov}\left(|W|^{-1/2} \sum_{j \in W} \left(Z^{(1)}_{j,v_1,n} - Z^{(2)}_{j,v_1,n} \right), |W|^{-1/2} \sum_{j \in W} \left(Z^{(1)}_{j,v_2,n} - Z^{(2)}_{j,v_2,n} \right)\right) \to \sigma_{v_1,v_2},$$

with $\sigma_{v_1,v_2} \in \mathbb{R}$ given by

$$\sigma_{v_1,v_2} = \sum_{\substack{t \in \mathbb{Z}^d: \\ \|t\|_\infty \leq m}} \mathbb{E}\left[\left(Z^{(1)}_{t,v_1} - Z^{(2)}_{t,v_1} \right) \left(Z^{(1)}_{0,v_2} - Z^{(2)}_{0,v_2} \right) \right].$$

Proof. Let $v_1 \in \mathcal{U}(\xi_1, \beta_2^{(1)})$, $v_2 \in \mathcal{U}(\xi_2, \beta_2^{(2)})$ and define the functions $g^{(k)}$, $g_n^{(k)}$: $\mathbb{R} \to \mathbb{R}$, $k = 1, 2$, by

$$g_n^{(k)}(x) = \frac{x}{2\pi}\mathcal{F}_+\left[\frac{\mathcal{F}_+[\mathcal{G}^{-1*}v_k](-\cdot)}{\psi}\mathcal{F}_+[K_{b_n}]\right](x)$$
$$- \frac{i}{2\pi}\mathcal{F}_+\left[\mathcal{F}_+[\mathcal{G}^{-1*}v_k](-\cdot)\left(\frac{1}{\psi}\right)'\mathcal{F}_+[K_{b_n}]\right](x), \quad x \in \mathbb{R},$$

and

$$g^{(k)}(x) = \frac{x}{2\pi}\mathcal{F}_+\left[\frac{\mathcal{F}_+[\mathcal{G}^{-1*}v_k](-\cdot)}{\psi}\right](x)$$
$$- \frac{i}{2\pi}\mathcal{F}_+\left[\mathcal{F}_+[\mathcal{G}^{-1*}v_k](-\cdot)\left(\frac{1}{\psi}\right)'\right](x), \quad x \in \mathbb{R}.$$

Then $(g^{(k)}(Y_j))_{j \in \mathbb{Z}^d}$ and $(g_n^{(k)}(Y_j))_{j \in \mathbb{Z}^d}$ fulfill properties (1)–(3) from Lemma A.4 (cf. Appendix A.5). Indeed, by Lemma 4.6, it follows that

$$\mathbb{E}[g_n^{(k)}(Y_0)] = \mathbb{E}\left[Z_{0,v_k,n}^{(1)} - Z_{0,v_k,n}^{(2)}\right] = \frac{1}{n}\mathbb{E}\left[\sum_{j \in W}\left(Z_{j,v_k,n}^{(1)} - Z_{j,v_k,n}^{(2)}\right)\right]$$

$$= \frac{1}{2\pi}\left\langle\mathcal{F}_+[\mathcal{G}^{-1*}v], \left\{\frac{\hat{\theta} - \theta}{\psi} - i\left(\frac{1}{\psi}\right)'(\hat{\psi} - \psi)\right\}\mathcal{F}_+[K_{b_n}]\right\rangle_{L^2(\mathbb{R})}.$$

Since $\mathbb{E}[\hat{\psi}(x) - \psi(x)] = \mathbb{E}[\hat{\theta}(x) - \theta(x)] = 0$ for all $x \in \mathbb{R}$, we conclude by Fubini's theorem that $\mathbb{E}[g_n^{(k)}(Y_0)] = \mathbb{E}[g^{(k)}(Y_0)] = 0$, for $k = 1, 2$. Moreover, since the Fourier transform of an integrable function is bounded, the finite $(2 + \tau)$-moment condition together with Lemma 4.1, (2), (3) and **(K3)** imply $\mathbb{E}|g^{(k)}(Y_0)|^2$, $\mathbb{E}|g_n^{(k)}(Y_0)|^2 < \infty$, $k = 1, 2$. The same arguments in combination with the dominated convergence yields

$$\mathbb{E}\left[g_n^{(1)}(Y_0)g_n^{(2)}(Y_j)\right] \to \mathbb{E}\left[g^{(1)}(Y_0)g^{(2)}(Y_j)\right],$$

as $|W| \to \infty$. Hence, Lemma A.4 yields the assertion of the lemma. □

We now can give a proof of Theorem 4.5.

Proof of Theorem 4.5. If $\sigma_v^2 = 0$, then Lemma 4.7 yields

$$\sigma_{v,n}^2 := \mathbb{E}\left[\left(\frac{1}{\sqrt{n}}\sum_{j \in W}\left(Z_{0,v,n}^{(1)} - Z_{0,v,n}^{(2)}\right)\right)^2\right] \to \sigma_v^2 = 0,$$

as W is regularly growing to infinity; hence, $n^{-1/2}\sum_{j \in W}\left(Z_{0,v,n}^{(1)} - Z_{0,v,n}^{(2)}\right) \to 0$ in probability.

Now, assume that $\sigma_v^2 > 0$ and choose $n_0 \in \mathbb{N}$ such that $\sigma_{v,n}^2 > 0$ for all $n \geq n_0$ (which is indeed possible, since $\sigma_{v,n}^2 \to \sigma_v^2 > 0$ as $n \to \infty$). For any $n \geq n_0$, let

$$X_{j,n} := \frac{1}{\sqrt{n}}\frac{Z_{j,v,n}^{(1)} - Z_{j,v,n}^{(2)}}{\sigma_{v,n}}, \quad j \in \mathbb{Z}^d,$$

and denote by F_n the distribution function of $\sum_{j\in W} X_{j,n}$. In the proof of Lemma 4.7 we have seen that $(X_{j,n})_{j\in\mathbb{Z}^d}$ is a centered m-dependent random field and $\mathbb{E}|X_{j,n}|^{2+\tau} \leq cn^{-1-\tau/2}\sigma_{v,n}^{-(2+\tau)}$ for any $n \in \mathbb{N}$ and a constant $c > 0$. Hence, applying [8, Theorem 2.6] with $p = 2 + \tau$ yields

$$\sup_{x\in\mathbb{R}} |F_n(x) - \phi(x)| \leq 75c(m + 1)^{(1+\tau)d}\sigma_{v,n}^{-(2+\tau)}n^{-\tau/2} \to 0,$$

as $n \to \infty$. This completes the proof. $\square$

4.2   Remainder term

In this section, we show that the remainder term E_2 is stochastically negligible as the sample size n tends to infinity.

Theorem 4.8. *Let $\gamma = 0$ and suppose that the assumptions of Theorem 3.12 are satisfied. Then, as $n \to \infty$,*

$$E_2 = \sqrt{n}\left\langle \mathcal{F}_+[\mathcal{G}^{-1*}v], \left\{R_n + \theta\frac{\psi - \hat{\psi}}{\psi^2}\mathbb{1}_{\{|\hat{\psi}|\leq n^{-1/2}\}}\right\}\mathcal{F}_+[K_b]\right\rangle_{L^2(\mathbb{R})} \xrightarrow{\mathbb{P}} 0,$$

for any $v \in \mathcal{U}(\xi, \beta_2)$.

In order to prove Theorem 4.8, some auxiliary statements are required. Therefore, we introduce the following notation.

For any $t \in \mathbb{R}$, $j \in \mathbb{Z}^d$, let the centered random variables $\xi_j^{(l)}(t), \tilde{\xi}_j^{(l)}(t), l = 1, 2$, be given by

$$\xi_j^{(1)}(t) = \cos(tY_j) - \mathbb{E}\Big[\cos(tY_0)\Big],$$

$$\xi_j^{(2)}(t) = \sin(tY_j) - \mathbb{E}\Big[\sin(tY_0)\Big],$$

$$\tilde{\xi}_j^{(1)}(t) = Y_j\cos(tY_j) - \mathbb{E}\Big[Y_0\cos(tY_0)\Big],$$

$$\tilde{\xi}_j^{(2)}(t) = Y_j\sin(tY_j) - \mathbb{E}\Big[Y_0\sin(tY_0)\Big].$$

Then $\hat{\psi} - \psi$ and $\hat{\theta} - \theta$ can be rewritten as

$$\hat{\psi}(t) - \psi(t) = \frac{1}{n}\sum_{j\in W}\left(\xi_j^{(1)}(t) + i\xi_j^{(2)}(t)\right) \quad \text{and}$$

$$\hat{\theta}(t) - \theta(t) = \frac{1}{n}\sum_{j\in W}\left(\tilde{\xi}_j^{(1)}(t) + i\tilde{\xi}_j^{(2)}(t)\right).$$

In the sequel, we shortly write $\xi^{(l)}(t), \tilde{\xi}^{(l)}(t)$ for the random fields $(\xi_j^{(l)}(t))_{j\in\mathbb{Z}^d}$ and $(\tilde{\xi}_j^{(l)}(t))_{j\in\mathbb{Z}^d}, l = 1, 2$. Moreover, for any $K > 0$, we define the random fields $\bar{\xi}_K^{(l)}(t) = (\bar{\xi}_{j,K}^{(l)}(t))_{j\in\mathbb{Z}^d}, l = 1, 2$, by

$$\bar{\xi}_{j,K}^{(1)}(t) = Y_j\cos(tY_j)\mathbb{1}_{[-K,K]}(Y_j) - \mathbb{E}\Big[Y_0\cos(tY_0)\mathbb{1}_{[-K,K]}(Y_0)\Big],$$

$$\bar{\xi}_{j,K}^{(2)}(t) = Y_j\sin(tY_j)\mathbb{1}_{[-K,K]}(Y_j) - \mathbb{E}\Big[Y_0\sin(tY_0)\mathbb{1}_{[-K,K]}(Y_0)\Big].$$

For any finite subset $V \subset \mathbb{Z}^d$ and any random field $Y = (Y_j)_{j \in \mathbb{Z}^d}$, let

$$S_V(Y) = \sum_{j \in V} Y_j.$$

Lemma 4.9. *Let the assumptions of Theorem 3.12 be satisfied and suppose $K \geq 1$. Then*

$$\mathbb{P}(|S_W(\xi^{(l)}(t))| \geq x) \leq 2 \exp\left(-\frac{1}{8(m+1)^d} \frac{x^2}{x+2|W|}\right) \tag{4.5}$$

and

$$\mathbb{P}(|S_W(\bar{\xi}_K^{(l)}(t))| \geq x) \leq 2 \exp\left(-\frac{1}{8(m+1)^d K^2} \frac{x^2}{x+2|W|}\right), \tag{4.6}$$

for any $t \in \mathbb{R}$, $x \geq 0$ and $l = 1, 2$.

Proof. Since $|\xi_j^{(l)}(t)| \leq 2$ for all $t \in \mathbb{R}$, $j \in \mathbb{Z}^d$ and $l = 1, 2$, we have that

$$|\mathbb{E}[\xi_j^{(l)}(t)^p]| \leq \mathbb{E}[|\xi_j^{(l)}(t)|^{p-2} \xi_j^{(l)}(t)^2] \leq 2^{p-2} \mathbb{E}[\xi_j^{(l)}(t)^2], \quad p = 3, 4, \ldots;$$

hence, Theorem A.1 (with $H = 2$) implies (4.5). Next, we obtain

$$|\mathbb{E}[\bar{\xi}_{j,K}^{(l)}(t)^p]| \leq \mathbb{E}[|\bar{\xi}_{j,K}^{(l)}(t)|^{p-2} |\bar{\xi}_{j,K}^{(l)}(t)|^2] \leq (2K)^{p-2} \mathbb{E}[\bar{\xi}_{j,K}^{(l)}(t)^2], \quad p = 3, 4, \ldots.$$

Taking into account that $\sum_{j \in W} \mathbb{E}[\bar{\xi}_{j,K}^{(l)}(t)^2] \leq 4K^2|W|$, Theorem A.1 (with $H = 2K$) yields the bound in (4.6). $\square$

Lemma 4.10. *Let the assumptions of Theorem 3.12 be satisfied and let $n = |W|$. Moreover, for any n, suppose numbers $\varepsilon_n > 0$, $K_n \geq 1$, such that $\varepsilon_n \to 0$ and $K_n \to \infty$ as $n \to \infty$. Then, for any n with $\varepsilon_n < \min\{1, \frac{T}{4}\}$,*

$$\mathbb{P}\left(\sup_{t \in [-T,T]} n^{-1}|S_W(\xi^{(l)}(t))| \geq \varepsilon_n\right) \leq C_1 \sqrt{\frac{T}{\varepsilon_n}} \exp\left\{-\frac{n\varepsilon_n^2}{160(m+1)^d}\right\} \tag{4.7}$$

and

$$\mathbb{P}\left(\sup_{t \in [-T,T]} n^{-1}|S_W(\bar{\xi}^{(l)}(t))| \geq \varepsilon_n\right) \leq C_2 \sqrt{\frac{T}{\varepsilon_n}} \exp\left\{-\frac{n\varepsilon_n^2}{576(m+1)^d K_n^2}\right\}$$
$$+ \frac{C_3}{\varepsilon_n K_n^{1+\tau}}, \tag{4.8}$$

$l = 1, 2$, where $C_1 = 4(1 + 2\mathbb{E}|Y_0|)$, $C_2 = 4\sqrt{2}(1 + 2\mathbb{E}|Y_0|^2)$ and $C_3 = 8\mathbb{E}|Y_0|^{2+\tau}$.

Proof. We use the same idea as in the proof of [4, Theorem 2]: divide the interval $[-T, T]$ by $2J$ equidistant points $(t_k)_{k=1,\ldots,2J} = \mathcal{D}$, where $t_k = -T + k\frac{T}{J}$, $k = 1, \ldots, 2J$. Then, for any $t \in [-T, T]$ such that $|t - t_k| \leq \frac{T}{J}$, we have for any $j \in \mathbb{Z}^d$ that

$$|\xi_j^{(l)}(t) - \xi_j^{(l)}(t_k)| \leq |t - t_k|(|Y_j| + \mathbb{E}|Y_0|) \leq (|Y_j| + \mathbb{E}|Y_0|)\frac{T}{J}, \quad l = 1, 2.$$

Hence, by Markov's inequality and Lemma 4.9, for any $n \in \mathbb{N}$, we obtain that

$$\mathbb{P}\Big(\sup_{t\in[-T,T]} n^{-1}|S_W(\xi^{(l)}(t))| \geq \varepsilon_n\Big) = \mathbb{P}\Big(\sup_{t_k\in\mathcal{D}} \sup_{t:|t-t_k|\leq\frac{T}{J}} |S_W(\xi^{(l)}(t))| \geq n\varepsilon_n\Big)$$

$$\leq \mathbb{P}\Big(\sup_{t_k\in\mathcal{D}} |S_W(\xi^{(l)}(t_k))| \geq \frac{n\varepsilon_n}{2}\Big)$$

$$+ \mathbb{P}\Big(\sup_{t_k\in\mathcal{D}} \sup_{t:|t-t_k|\leq\frac{T}{J}} |S_W(\xi^{(l)}(t)) - S_W(\xi^{(l)}(t_k))| \geq \frac{n\varepsilon_n}{2}\Big)$$

$$\leq \sum_{t_k\in\mathcal{D}} \mathbb{P}\Big(|S_W(\xi^{(l)}(t_k))| \geq \frac{n\varepsilon_n}{2}\Big) + \mathbb{P}\Big(\sum_{j\in W}(|Y_j| + \mathbb{E}|Y_0|)\frac{T}{J} \geq \frac{n\varepsilon_n}{2}\Big)$$

$$\leq 4J \exp\Big\{-\frac{1}{16(m+1)^d}\frac{n\varepsilon_n^2}{\varepsilon_n+4}\Big\} + \frac{4T}{J\varepsilon_n}\mathbb{E}|Y_0|,$$

$l = 1, 2$. Now, let $n \in \mathbb{N}$ be such that $\varepsilon_n < \frac{T}{4}$ and choose

$$J = \left\lfloor \Big(\frac{T}{\varepsilon_n}\exp\Big\{\frac{1}{16(m+1)^d}\frac{n\varepsilon_n^2}{\varepsilon_n+4}\Big\}\Big)^{1/2} \right\rfloor,$$

where $\lfloor x \rfloor$ denotes the integer part of $x \in \mathbb{R}$. Then

$$\mathbb{P}\Big(\sup_{t\in[-T,T]} n^{-1}|S_W(\xi^{(l)}(t))| \geq \varepsilon_n\Big) \leq C_1\sqrt{\frac{T}{\varepsilon_n}}\exp\Big\{-\frac{1}{32(m+1)^d}\frac{n\varepsilon_n^2}{\varepsilon_n+4}\Big\}$$

with $C_1 = 4(1 + 2\mathbb{E}|Y_0|)$. Applying the same arguments to $\sup_{t\in[-T,T]} |S_W(\bar{\xi}^{(l)}_{K_n}(t))|$ yields

$$\mathbb{P}\Big(\sup_{t\in[-T,T]} n^{-1}|S_W(\bar{\xi}^{(l)}_{K_n}(t))| \geq \varepsilon_n\Big) \leq \tilde{C}\sqrt{\frac{T}{\varepsilon_n}}\exp\Big\{-\frac{1}{32(m+1)^d K_n^2}\frac{n\varepsilon_n^2}{\varepsilon_n+4}\Big\},$$

whenever $\varepsilon_n < \frac{T}{4}$, where $\tilde{C} = 4(1 + 2\mathbb{E}|Y_0|^2)$. Combining Markov's inequality, Hölder's inequality and the finite $(2 + \tau)$-moment property of Y_0 implies

$$\mathbb{P}\Big(\sup_{t\in[-T,T]} n^{-1}\Big|\sum_{j\in W}\big(\tilde{\xi}^{(l)}_j(t) - \bar{\xi}^{(l)}_{j,K_n}(t)\big)\Big| \geq \frac{\varepsilon_n}{2}\Big)$$

$$\leq \mathbb{P}\Big(\sum_{j\in W}(|Y_j|\mathbb{1}_{(K_n,\infty)}(|Y_j|) + \mathbb{E}|Y_0|\mathbb{1}_{(K_n,\infty)}(|Y_0|)) \geq \frac{n\varepsilon_n}{2}\Big)$$

$$\leq \frac{4}{\varepsilon_n}\Big(\mathbb{E}|Y_0|^{2+\tau}\Big)^{1/(2+\tau)}\mathbb{P}(|Y_0| > K_n)^{(1+\tau)/(2+\tau)}$$

$$\leq \frac{4}{K_n^{1+\tau}\varepsilon_n}\mathbb{E}|Y_0|^{2+\tau},$$

$l = 1, 2$. All in all, we have for any n such that $\varepsilon_n < \frac{T}{2}$,

$$\mathbb{P}\left(\sup_{t \in [-T,T]} n^{-1}|S_W(\tilde{\xi}^{(l)}(t))| \geq \varepsilon_n \right)$$

$$\leq \mathbb{P}\left(\sup_{t \in [-T,T]} n^{-1}|S_W(\tilde{\xi}^{(l)}_{K_n}(t))| \geq \frac{\varepsilon_n}{2} \right)$$

$$+ \mathbb{P}\left(\sup_{t \in [-T,T]} n^{-1}|S_W(\tilde{\xi}^{(l)}(t) - \tilde{\xi}^{(l)}_{K_n}(t))| \geq \frac{\varepsilon_n}{2} \right)$$

$$\leq \sqrt{2}\tilde{C}\sqrt{\frac{T}{\varepsilon_n}} \exp\left\{ -\frac{1}{64(m+1)^d K_n^2} \frac{n\varepsilon_n^2}{\varepsilon_n + 8} \right\} + \frac{8}{K_n^{1+\tau} \varepsilon_n} \mathbb{E}|Y_0|^{2+\tau}.$$

Hence, it follows for any n with $\varepsilon_n < \min\{1, \frac{T}{4}\}$ that

$$\mathbb{P}\left(\sup_{t \in [-T,T]} n^{-1}|S_W(\xi^{(l)}(t))| \geq \varepsilon_n \right) \leq C_1 \sqrt{\frac{T}{\varepsilon_n}} \exp\left\{ -\frac{n\varepsilon_n^2}{160(m+1)^d} \right\}$$

as well as

$$\mathbb{P}\left(\sup_{t \in [-T,T]} n^{-1}|S_W(\tilde{\xi}^{(l)}_j(t))| \geq \varepsilon_n \right) \leq C_2 \sqrt{\frac{T}{\varepsilon_n}} \exp\left\{ -\frac{n\varepsilon_n^2}{576(m+1)^d K_n^2} \right\}$$

$$+ \frac{C_3}{K_n^{1+\tau} \varepsilon_n},$$

where $C_2 = \sqrt{2}\tilde{C}$ and $C_3 = 8\mathbb{E}|Y_0|^{2+\tau}$. $\qquad\square$

Theorem 4.11. *For some $\zeta > 0$, suppose $\varepsilon_n \approx n^{-\frac{1+\tau}{2(2+\tau)}} \left[\log\left(T^{\frac{1}{2}} n^{\frac{1+\tau}{4(2+\tau)}} \right) \right]^{\zeta+\frac{1}{2}}$ and $K_n = n^{\frac{1}{2(2+\tau)}}$ in Lemma 4.10. Then, for n sufficiently large,*

$$\mathbb{P}\left(\max\left\{ \sup_{t \in [-T,T]} |\hat{\psi}(t) - \psi(t)|, \sup_{t \in [-T,T]} |\hat{\theta}(t) - \theta(t)| \right\} > \varepsilon_n \right) \leq \tilde{C} y_n,$$

where

$$y_n = \left[\log\left(T^{\frac{1}{2}} n^{\frac{1+\tau}{4(2+\tau)}} \right) \right]^{-\frac{\zeta}{2} - \frac{1}{4}}$$

and $\tilde{C} > 0$ is a constant (independent of T).

Proof. By Lemma 4.10 it follows that

$$\mathbb{P}\left(\max\left\{ \sup_{t \in [-T,T]} |\hat{\psi}(t) - \psi(t)|, \sup_{t \in [-T,T]} |\hat{\theta}(t) - \theta(t)| \right\} > \varepsilon_n \right)$$

$$\leq \sum_{l=1}^{2} \mathbb{P}\left(\sup_{t \in [-T,T]} n^{-1}|S_W(\xi^{(l)}(t))| \geq \varepsilon_n \right)$$

$$+ \sum_{l=1}^{2} \mathbb{P}\left(\sup_{t \in [-T,T]} n^{-1}|S_W(\tilde{\xi}^{(l)}(t))| \geq \varepsilon_n \right)$$

$$\leq C\left(\sqrt{\frac{T}{\varepsilon_n}}\exp\left\{-\frac{n\varepsilon_n^2}{576(m+1)^d K_n^2}\right\}+\frac{1}{\varepsilon_n K_n^{1+\tau}}\right),$$

for some constant $C>0$. Let $\varepsilon_n=n^{-\frac{1+\tau}{2(2+\tau)}}\left[\log\left(T^{\frac{1}{2}}n^{\frac{1+\tau}{4(2+\tau)}}\right)\right]^{\zeta+\frac{1}{2}}$, without loss of generality. Then we observe that

$$\frac{1}{\varepsilon_n K_n^{1+\tau}}=\left[\log\left(T^{\frac{1}{2}}n^{\frac{1+\tau}{4(2+\tau)}}\right)\right]^{-\zeta-\frac{1}{2}}.$$

Moreover,

$$\sqrt{\frac{T}{\varepsilon_n}}\exp\left\{-\frac{n\varepsilon_n^2}{576(m+1)^d K_n^2}\right\}=\left(T^{\frac{1}{2}}n^{\frac{1+\tau}{4(2+\tau)}}\right)^{1-\frac{r_n}{576(m+1)^d}}y_n,$$

with $r_n=\left[\log(T^{1/2}n^{\frac{1+\tau}{4(2+\tau)}})\right]^{2\zeta}$. Hence, the assertion of the theorem follows. $\qquad\square$

Remark 4.12. (a) Fix $T>0$. Then, provided the assumptions of Theorem 3.12 are satisfied, Theorem 4.11 states that

$$\max\left\{\sup_{t\in[-T,T]}|\hat{\psi}(t)-\psi(t)|,\quad\sup_{t\in[-T,T]}|\hat{\theta}(t)-\theta(t)|\right\}=\mathcal{O}_{\mathbb{P}}(\varepsilon_n),$$

as $n\to\infty$, where $\mathcal{O}_{\mathbb{P}}$ denotes the probabilistic order of convergence.

(b) *For large n* in Theorem 4.11 is understood in the following sense: for any fixed m, there exists $n_0=n_0(m)$ such that the bound holds for all $n\geq n_0$. Of course, the function $m\mapsto n_0(m)$ is increasing.

The following corollary is an immediate consequence of Theorem 4.11.

Corollary 4.13. *Let the assumptions of Theorem 3.12 be satisfied. Then*

$$\lim_{n\to\infty}\mathbb{P}\left(\sup_{t\in[-b_n^{-1},b_n^{-1}]}|\hat{\psi}(t)-\psi(t)|\geq cb_n^{\frac{1}{2}-\varepsilon}\right)=0$$

for any constant $c>0$.

Proof. Fix $c>0$ and assume that $b_n=n^{-\frac{1}{1-2\varepsilon}}\left(\log n\right)^{n+\frac{1}{1-2\varepsilon}}$, without loss of generality. Since $b_n\to0$ as $n\to\infty$, there exists $n_0\in\mathbb{N}$ such that $cb_n^{\frac{1}{2}-\varepsilon}<\min\{1,\frac{1}{4b_n}\}$ for all $n\geq n_0$. Taking $\varepsilon_n=cb_n^{\frac{1}{2}-\varepsilon}$ and $T=b_n^{-1}$ in Lemma 4.10, it follows that

$$\mathbb{P}\left(\sup_{t\in[-b_n^{-1},b_n^{-1}]}|\hat{\psi}(t)-\psi(t)|\geq cb_n^{\frac{1}{2}-\varepsilon}\right)$$

$$\leq\mathbb{P}\left(\sup_{t\in[-b_n^{-1},b_n^{-1}]}n^{-1}|S_W(\xi^{(1)}(t))|\geq cb_n^{\frac{1}{2}-\varepsilon}\right)$$

$$+\mathbb{P}\left(\sup_{t\in[-b_n^{-1},b_n^{-1}]}n^{-1}|S_W(\xi^{(2)}(t))|\geq cb_n^{\frac{1}{2}-\varepsilon}\right)$$

$$\leq 2\tilde{C}c^{-\frac{1}{2}}b_n^{\frac{2\varepsilon-3}{4}}\exp\left\{-\frac{c^2 n b_n^{1-2\varepsilon}}{160(m+1)^d}\right\}$$

for all $n \geq n_0$ and some constant $\tilde{C} > 0$. Hence,

$$\mathbb{P}\left(\sup_{t\in[-b_n^{-1},b_n^{-1}]}|\hat{\psi}(t)-\psi(t)| > cb_n^{\tilde{\delta}/2}\right)$$

$$\leq \check{C}n^{\frac{3-2\varepsilon}{4(1-2\varepsilon)}-\frac{c^2(\log n)^{\eta(1-2\varepsilon)}}{160(m+1)^d}}\left(\log n\right)^{-\frac{3-2\varepsilon}{4(1-2\varepsilon)}(1+\eta(1-2\varepsilon))} \to 0,$$

as $n \to \infty$, where $\check{C} = 2\tilde{C}c^{-\frac{1}{2}}$. $\qquad\square$

Corollary 4.14. *Let $\gamma = 0$ and suppose that the assumptions of Theorem 3.12 are satisfied. Moreover, let $\kappa_n = 2\left(\log n\right)^{\frac{1+\eta(1+2\varepsilon)}{2}}$. Then,*

$$\lim_{n\to\infty}\mathbb{P}\left(\frac{|\psi(t)|}{|\tilde{\psi}(t)|} \geq \kappa_n \text{ for some } t \in [-b_n^{-1},b_n^{-1}]\right) = 0.$$

Proof. By Lemma 4.1, (2), $\frac{1}{|\psi(x)|} \leq c(1+|x|)^{\frac{1}{2}-\varepsilon}$ for some constant $c > 0$; hence, there exists $n_0 \in \mathbb{N}$ such that

$$\inf_{t\in[-b_n^{-1},b_n^{-1}]}|\psi(t)| \geq c^{-1}(1+|b_n^{-1}|)^{\varepsilon-\frac{1}{2}} \geq c^{-1}b_n^{\frac{1}{2}-\varepsilon} \tag{4.9}$$

for all $n \geq n_0$. We first show that the probabilities of the events

$$A_n := \left\{|\hat{\psi}(t)| < n^{-1/2} \text{ for some } t \in [-b_n^{-1},b_n^{-1}]\right\}$$

tend to 0 as $n \to \infty$: By (4.1), $t \in [-b_n^{-1},b_n^{-1}]$ implies $|\psi(t)| > 2n^{-1/2}$, for all $n \geq n_1$ and some $n_1 \in \mathbb{N}$. Set $\tilde{n} = \max\{n_0, n_1\}$. Then

$$\mathbb{P}(A_n) \leq \mathbb{P}\left(|\hat{\psi}(t)-\psi(t)| > |\psi(t)|-n^{-1/2} \text{ for some } t \in [-b_n^{-1},b_n^{-1}]\right)$$

$$\leq \mathbb{P}\left(|\hat{\psi}(t)-\psi(t)| > \frac{1}{2}|\psi(t)| \text{ for some } t \in [-b_n^{-1},b_n^{-1}]\right)$$

$$\leq \mathbb{P}\left(|\hat{\psi}(t)-\psi(t)| > \frac{1}{2c}b_n^{\frac{1}{2}-\varepsilon} \text{ for some } t \in [-b_n^{-1},b_n^{-1}]\right),$$

for all $n \geq \tilde{n}$, where the last inequality follows from (4.9). Hence, by Corollary 4.13, $\lim_{n\to\infty}\mathbb{P}(A_n) = 0$.

Suppose $\kappa_n = 2\left(\log n\right)^{\frac{1+\eta(1-2\varepsilon)}{2}}$. Then we find that

$$\mathbb{P}\left(\sup_{t\in[-b_n^{-1},b_n^{-1}]}\frac{|\psi(t)|}{|\tilde{\psi}(t)|} \geq \kappa_n\right)$$

$$= \mathbb{P}\left(\sup_{t\in[-b_n^{-1},b_n^{-1}]}\frac{|\psi(t)|}{|\tilde{\psi}(t)|} \geq \kappa_n, \inf_{t\in[-b_n^{-1},b_n^{-1}]}|\hat{\psi}(t)| \geq n^{-1/2}\right)$$

$$+ \mathbb{P}\Big(\sup_{t \in [-b_n^{-1}, b_n^{-1}]} \frac{|\psi(t)|}{|\tilde{\psi}(t)|} \geq \kappa_n, \quad \inf_{t \in [-b_n^{-1}, b_n^{-1}]} |\hat{\psi}(t)| < n^{-1/2} \Big)$$

$$\leq \mathbb{P}\Big(\sup_{t \in [-b_n^{-1}, b_n^{-1}]} \frac{|\psi(t) - \hat{\psi}(t)|}{|\hat{\psi}(t)|} \geq \kappa_n - 1, \quad \inf_{t \in [-b_n^{-1}, b_n^{-1}]} |\hat{\psi}(t)| \geq n^{-1/2} \Big)$$

$$+ \mathbb{P}\Big(\inf_{t \in [-b_n^{-1}, b_n^{-1}]} |\hat{\psi}(t)| < n^{-1/2} \Big)$$

$$\leq \mathbb{P}\Big(\sup_{t \in [-b_n^{-1}, b_n^{-1}]} |\psi(t) - \hat{\psi}(t)| \geq (\kappa_n - 1) n^{-1/2} \Big) + \mathbb{P}(A_n),$$

for all $n \geq \tilde{n}$. Taking into account that for large n, $(\kappa_n - 1) n^{-1/2} = 2 b_n^{\frac{1}{2} - \varepsilon} - n^{-1/2} \geq b_n^{\frac{1}{2} - \varepsilon}$, the assertion follows by Corollary 4.13. $\qquad\square$

Now we can give the proof for Theorem 4.8.

Proof of Theorem 4.8. First of all, observe that

$$R_n + \theta \frac{\psi - \hat{\psi}}{\psi^2} \mathbb{1}_{|\hat{\psi}| \leq n^{-1/2}} = \Big(1 - \frac{\hat{\psi}}{\psi} \Big) \Big(\frac{\hat{\theta} - \theta}{\tilde{\psi}} + \theta \frac{\psi - \hat{\psi}}{\psi \tilde{\psi}} \Big)$$

$$= \frac{\psi}{\tilde{\psi}} \Big(\frac{\psi - \hat{\psi}}{\psi} \Big) \Big(\frac{\hat{\theta} - \theta}{\psi} + \theta \frac{\psi - \hat{\psi}}{\psi^2} \Big)$$

$$= \frac{\psi}{\tilde{\psi}} \Big(\frac{\psi - \hat{\psi}}{\psi} \Big) \Big(\frac{\hat{\theta} - \theta}{\psi} + i \Big(\frac{1}{\psi} \Big)' \big(\psi - \hat{\psi} \big) \Big).$$

Now, fix $\tilde{\gamma} > 0$ and let $\kappa_n = 2 \big(\log n \big)^{\frac{1 + \eta(1 - 2\varepsilon)}{2}}$. Moreover, let

$$M_n = \max \Big\{ \sup_{t \in [-b_n^{-1}, b_n^{-1}]} |\hat{\psi}(t) - \psi(t)|, \quad \sup_{t \in [-b_n^{-1}, b_n^{-1}]} |\hat{\theta}(t) - \theta(t)| \Big\}.$$

By **(K3)**, $\sup_{x \in \mathbb{R}, \, n \in \mathbb{N}} |\mathcal{F}_+[K_{b_n}](x)| \leq 2$; hence

$$\mathbb{P}\Big(E_2 \geq \tilde{\gamma} \Big)$$

$$= \mathbb{P}\Big(\sqrt{n} \big\langle \mathcal{F}_+[\mathcal{G}^{-1*} v], \, \Big\{ R_n + \theta \frac{\psi - \hat{\psi}}{\psi^2} \mathbb{1}_{\{|\hat{\psi}| \leq n^{-1/2}\}} \Big\} \mathcal{F}_+[K_b] \big\rangle_{L^2(\mathbb{R}^\times)} \geq \tilde{\gamma} \Big)$$

$$\leq \mathbb{P}\Big(\int_{-b_n^{-1}}^{b_n^{-1}} |\mathcal{F}_+[\mathcal{G}^{-1*} v](x)| \frac{|\psi(x)|}{|\tilde{\psi}(x)|} \frac{|\hat{\psi}(x) - \psi(x)|}{|\psi(x)|}$$

$$\times \Big(\frac{|\hat{\theta}(x) - \theta(x)|}{|\psi(x)|} + \Big| \Big(\frac{1}{\psi} \Big)'(x) \Big| |\psi(x) - \hat{\psi}(x)| \Big) dx \geq \frac{\tilde{\gamma}}{2} n^{-\frac{1}{2}} \Big)$$

$$\leq \mathbb{P}\Big(M_n b_n^{\frac{1}{2} - \varepsilon} \int_{-b_n^{-1}}^{b_n^{-1}} \frac{|\mathcal{F}_+[\mathcal{G}^{-1*} v](x)|}{|\psi(x)|} \Big(\frac{1}{|\psi(x)|} + \Big| \Big(\frac{1}{\psi} \Big)'(x) \Big| \Big) dx \geq \frac{\tilde{\gamma}}{2} n^{-\frac{1}{2}} \kappa_n^{-1} \Big)$$

$$+ \mathbb{P}\Big(\sup_{t \in [-b_n^{-1}, b_n^{-1}]} \frac{|\psi(t)|}{|\tilde{\psi}(t)|} > \kappa_n \Big) + \mathbb{P}\Big(\sup_{t \in [-b_n^{-1}, b_n^{-1}]} |\psi(t) - \hat{\psi}(t)| > b_n^{\frac{1}{2} - \varepsilon} \Big).$$

Since by Lemma 4.1, (3), $\left(\frac{1}{\psi}\right)' \in L^\infty(\mathbb{R})$, there is a constant $\tilde{c} > 0$ such that

$$\mathbb{P}\left(M_n b_n^{\frac{1}{2}-\varepsilon} \int_{-b_n^{-1}}^{b_n^{-1}} \frac{|\mathcal{F}_+[\mathcal{G}^{-1}*v](x)|}{|\psi(x)|}\left(\frac{1}{|\psi(x)|} + \left|\left(\frac{1}{\psi}\right)'(x)\right|\right)dx \geq \frac{\tilde{\gamma}}{2}n^{-\frac{1}{2}}\kappa_n^{-1}\right)$$

$$\leq \mathbb{P}\left(M_n \int_{-b_n^{-1}}^{b_n^{-1}} \frac{|\mathcal{F}_+[\mathcal{G}^{-1}*v](x)|}{|\psi(x)|^2}dx \geq \frac{\tilde{\gamma}}{2\tilde{c}}n^{-\frac{1}{2}}\kappa_n^{-1}b_n^{\varepsilon-\frac{1}{2}}\right).$$

Moreover, by Definition 3.10, (iii), we have for some $\check{c} > 0$,

$$\mathbb{P}\left(M_n \int_{-b_n^{-1}}^{b_n^{-1}} \frac{|\mathcal{F}_+[\mathcal{G}^{-1}*v](x)|}{|\psi(x)|^2}dx \geq \frac{\tilde{\gamma}}{2\tilde{c}}n^{-\frac{1}{2}}\kappa_n^{-1}b_n^{\varepsilon-\frac{1}{2}}\right)$$

$$\leq \mathbb{P}\left(M_n \int_{-b_n^{-1}}^{b_n^{-1}} \frac{(1+x^2)^{-1+\varepsilon}}{|\psi(x)|^2}(1+x^2)^{1-\varepsilon-\xi/2}dx \geq \frac{\tilde{\gamma}}{2\tilde{c}\check{c}}n^{-\frac{1}{2}}\kappa_n^{-1}b_n^{\varepsilon-\frac{1}{2}}\right)$$

$$\leq \mathbb{P}\left(M_n \geq \frac{\tilde{\gamma}}{2\tilde{c}\check{c}}n^{-\frac{1}{2}}\kappa_n^{-1}b_n^{\frac{3}{2}-\varepsilon-\xi}\left\|\frac{(1+\cdot^2)^{-\frac{\varepsilon-1}{2}}}{\psi}\right\|_{L^2(\mathbb{R})}^{-1}\right),$$

where the last line follows because $\frac{(1+\cdot^2)^{-(\varepsilon-1)/2}}{\psi} \in L^2(\mathbb{R})$ by Proposition 3.3, (c). Now, for n sufficiently large, we obtain

$$n^{-\frac{1}{2}}\kappa_n^{-1}b_n^{\frac{3}{2}-\varepsilon-\xi} = \frac{1}{2}n^{\frac{\xi-1}{1-2\varepsilon}-1}(\log n)^{(1-\xi)\left(\eta+\frac{1}{1-2\varepsilon}\right)}$$

$$> n^{-\frac{1+\tau}{2(2+\tau)}}\left[\log\left(b_n^{-\frac{1}{2}}n^{\frac{1+\tau}{4(2+\tau)}}\right)\right]^{\eta+\frac{1}{2}}.$$

Indeed, by Definition 3.10, (iii), we have

$$\frac{\xi-1}{1-2\varepsilon} - 1 > \frac{1}{1-2\varepsilon} - \frac{1+\tau}{2(2+\tau)} > -\frac{1+\tau}{2(2+\tau)}.$$

Hence, we conclude by Theorem 4.11, Corollary 4.13 and Corollary 4.14 that

$$\mathbb{P}\left(E_2 \geq \tilde{\gamma}\right) \to 0, \quad \text{as } n \to \infty,$$

for any $\tilde{\gamma} > 0$. $\qquad\square$

4.3 Neglecting the drift γ

It remains to show that the result of Theorem 3.12 still holds true if γ is assumed to be arbitrary. For this purpose, consider the sample $(\tilde{Y}_j)_{j\in W}$ given by $\tilde{Y}_j = Y_j - \gamma$, $j \in W$. Moreover, let $\psi_*(t) = \mathbb{E}[e^{it\tilde{Y}_0}]$ be the characteristic function of $\tilde{Y}_0$ and write $\hat{\psi}_*$ for its empirical counterpart, i.e. $\hat{\psi}_*(t) = \frac{1}{n}\sum_{j\in W} e^{it\tilde{Y}_j}$. Then, with the notation

$$\frac{1}{\widehat{\widetilde{\psi}}_*(t)} := \frac{1}{\hat{\psi}_*(t)}\mathbb{1}_{\{|\hat{\psi}_*(t)|>n^{-1/2}\}} = e^{it\gamma}\frac{1}{\widehat{\widetilde{\psi}}(t)},$$

we have for any $t \in \mathbb{R}$,

$$\frac{\hat{\theta}_*(t)}{\hat{\psi}_*(t)} = \frac{\hat{\theta}(t)}{\hat{\psi}(t)} - \gamma \mathbb{1}_{\{|\hat{\psi}_*(t)| > n^{-1/2}\}} \quad \text{as well as} \quad \frac{\theta_*(t)}{\psi_*(t)} = \frac{\theta(t)}{\psi(t)} - \gamma, \qquad (4.10)$$

where $\theta_*(t) = \mathbb{E}[\tilde{Y}_0 e^{it\tilde{Y}_0}]$ and $\hat{\theta}_*(t) = \frac{1}{n}\sum_{j \in W} \tilde{Y}_j e^{it\tilde{Y}_j}$. For any $v \in \mathcal{U}(\xi, \beta_2)$, consider the decomposition

$$\sqrt{n}(\hat{L}_W v - \mathcal{L}v) = \frac{\sqrt{n}}{2\pi}\left\langle \mathcal{F}_+[\mathcal{G}^{-1*}v], \frac{\hat{\theta}}{\hat{\psi}}\mathcal{F}_+[K_{b_n}] - \frac{\theta}{\psi}\right\rangle_{L^2(\mathbb{R})}$$

$$= \frac{\sqrt{n}}{2\pi}\left\langle \mathcal{F}_+[\mathcal{G}^{-1*}v], \frac{\hat{\theta}_*}{\hat{\psi}_*}\mathcal{F}_+[K_{b_n}] - \frac{\theta_*}{\psi_*}\right\rangle_{L^2(\mathbb{R})}$$

$$+ \frac{\sqrt{n}}{2\pi}\left\langle \mathcal{F}_+[\mathcal{G}^{-1*}v], \gamma\mathbb{1}_{\{|\hat{\psi}_*| > n^{-1/2}\}}\mathcal{F}_+[K_{b_n}] - \gamma\right\rangle_{L^2(\mathbb{R})}.$$

As W is regularly growing to infinity, the first summand on the right-hand side of the last equation tends to a Gaussian random variable since ψ_* is an infinitely divisible characteristic function without drift component. For the second summand, we find that

$$\frac{\sqrt{n}}{2\pi}\left\langle \mathcal{F}_+[\mathcal{G}^{-1*}v], \gamma\mathbb{1}_{\{|\hat{\psi}_*| > n^{-1/2}\}}\mathcal{F}_+[K_{b_n}] - \gamma\right\rangle_{L^2(\mathbb{R})}$$

$$= \frac{\sqrt{n}\gamma}{2\pi}\left\langle \mathcal{F}_+[\mathcal{G}^{-1*}v], \mathcal{F}_+[K_{b_n}] - 1\right\rangle_{L^2(\mathbb{R})}$$

$$- \frac{\sqrt{n}\gamma}{2\pi}\left\langle \mathcal{F}_+[\mathcal{G}^{-1*}v], \mathbb{1}_{\{|\hat{\psi}_*| \le n^{-1/2}\}}\mathcal{F}_+[K_{b_n}]\right\rangle_{L^2(\mathbb{R})}.$$

Hence, by **(K3)** and Definition 3.10, (iii), we obtain

$$\sqrt{n}\mathbb{E}\left|\left\langle \mathcal{F}_+[\mathcal{G}^{-1*}v], \mathcal{F}_+[K_{b_n}] - 1\right\rangle_{L^2(\mathbb{R})}\right|$$

$$\le \int_{\mathbb{R}} \sqrt{n}|\mathcal{F}_+[\mathcal{G}^{-1*}v](x)||1 - \mathcal{F}_+[K_{b_n}](x)|dx \lesssim b_n\sqrt{n}\|\mathcal{G}^{-1*}v\|_{H^1(\mathbb{R})},$$

where the last term tends to zero as $n \to \infty$, since $b_n = o(n^{-1/2})$. Moreover,

$$\sqrt{n}\mathbb{E}\left|\left\langle \mathcal{F}_+[\mathcal{G}^{-1*}v], \mathbb{1}_{\{|\hat{\psi}_*| \le n^{-1/2}\}}\mathcal{F}_+[K_{b_n}]\right\rangle_{L^2(\mathbb{R})}\right|$$

$$\le 2\sqrt{n}\int_{-b_n^{-1}}^{b_n^{-1}} |\mathcal{F}_+[\mathcal{G}^{-1*}v](x)|\mathbb{P}\left(|\hat{\psi}_*(x)| \le n^{-\frac{1}{2}}\right)dx.$$

Taking into account that $|\psi(x)| = |\hat{\psi}_*(x)|$, relation (4.2) with $p = 1/2$ yields

$$\sqrt{n}|\mathcal{F}_+[\mathcal{G}^{-1*}v](x)|\mathbb{P}\left(|\hat{\psi}_*(x)| \le n^{-\frac{1}{2}}\right)\mathbb{1}_{[-b_n^{-1},b_n^{-1}]}(x) \le \frac{|\mathcal{F}_+[\mathcal{G}^{-1*}v](x)|}{|\psi(x)|}.$$

Applying again (4.2) with $p = 1$ implies

$$\sqrt{n}|\mathcal{F}_+[\mathcal{G}^{-1*}v](x)|\mathbb{P}\left(|\hat{\psi}_*(x)| \le n^{-\frac{1}{2}}\right)\mathbb{1}_{[-b_n^{-1},b_n^{-1}]}(x) \le n^{-\frac{1}{2}}\frac{|\mathcal{F}_+[\mathcal{G}^{-1*}v](x)|}{|\psi(x)|^2} \to 0,$$

as $n \to \infty$; thus, by dominated convergence, we have

$$\sqrt{n}\mathbb{E}\left|\left\langle \mathcal{F}_+[\mathcal{G}^{-1*}v], \mathbb{1}_{\{|\hat{\psi}_*|\leq n^{-1/2}\}}\mathcal{F}_+[K_{b_n}]\right\rangle_{L^2(\mathbb{R})}\right| \to 0, \quad \text{as } n \to \infty.$$

All in all, this shows that Theorem 3.12 holds for any fixed $\gamma \in \mathbb{R}$.

A Appendix

A.1 Proof of Lemma 3.3

(a) Minkowski's integral inequality together with formula (2.4) yields

$$\|uv_1\|_{L^k(\mathbb{R})} \leq \|uv_0\|_{L^k(\mathbb{R})} \int_{\text{supp}(f)} |f(s)|^{1/k} ds, \quad k = 1, 2.$$

The right-hand side in the last inequality is finite by Assumption 3.1, (1) and (2); hence $uv_1 \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. In particular, $\mathbb{R} \ni x \mapsto \mathcal{F}_+[uv_1](x) = \int_{\mathbb{R}} e^{itx}(uv_1)(t)dt$ is well-defined. Using again formula (2.4) together with Fubini's theorem and a simple integral substitution yields (3.5).

(b) The triangle inequality followed by a simple integral substitution shows that

$$\int_{\mathbb{R}} |x|^{1+\tau}|(uv_1)|(x)dx \leq \|f\|_{L^{2+\tau}(\mathbb{R})} \int_{\mathbb{R}} |x|^{1+\tau}|(uv_0)(x)|dx < \infty.$$

(c) The proof of Theorem 3.10 in [13] yields that $|\psi(x)|$ coincides with the inverse of the right-hand side in (3.2). This shows part (c). $\qquad \square$

A.2 Proof of Lemma 3.6

Let $v \in \text{Image}(\mathcal{G})$ be such that $\int_{\mathbb{R}} \frac{|\mathcal{F}_+[\mathcal{G}^{-1*}v]|(x)}{|\psi(x)|} dx < \infty$. In order to prove the upper bound in Theorem 3.6, decompose $\mathbb{E}|\hat{\mathcal{L}}_W v - \mathcal{L}v|$ as follows:

$$\mathbb{E}|\hat{\mathcal{L}}_W v - \mathcal{L}v| \leq \underbrace{\mathbb{E}\left|\left\langle (\mathcal{G}_n^{-1*} - \mathcal{G}^{-1*})v, \widehat{uv_1}\right\rangle_{L^2(\mathbb{R})}\right|}_{\text{(I)}} + \underbrace{\mathbb{E}\left|\left\langle \mathcal{G}^{-1*}v, \widehat{uv_1} - uv_1\right\rangle_{L^2(\mathbb{R})}\right|}_{\text{(II)}}.$$

We estimate parts (I) and (II) seperately. Using the isometry property of $\mathcal{F}_+$, we obtain

$$\text{(I)} \leq \frac{1}{2\pi} \int_{\mathbb{R}} |\mathcal{F}_+[(\mathcal{G}_n^{-1*} - \mathcal{G}^{-1*})v](x)|\mathbb{E}|\mathcal{F}_+[\widehat{uv_1}](x)|dx.$$

Furthermore, since $\widehat{uv_1} = \mathcal{F}_+^{-1}[\frac{\hat{\theta}(x)}{\hat{\psi}(x)}\mathcal{F}_+[K_b]]$, stationarity of Y yields for any $x \in \mathbb{R}$,

$$\mathbb{E}|\mathcal{F}_+[\widehat{uv_1}](x)| = |\mathcal{F}_+[K_b](x)|\mathbb{E}\left|\frac{\sum_{t\in W} Y_t}{n\hat{\psi}(x)}\mathbb{1}_{\{|\hat{\psi}(x)|>n^{-1/2}\}}\right|$$

$$\leq n^{1/2}|\mathcal{F}_+[K_b](x)|\mathbb{E}|Y_0|.$$

Hence, by the Cauchy–Schwarz inequality we obtain that

$$
\begin{aligned}
\textbf{(I)} &\leq \frac{n^{1/2}\mathbb{E}|Y_0|}{2\pi}\left\|\mathcal{F}_+[(\mathcal{G}_n^{-1*}-\mathcal{G}^{-1*})v]\right\|_{L^2(\mathbb{R})}\left\|\mathcal{F}_+[K_b]\right\|_{L^2(\mathbb{R})}\\
&\leq \frac{S\mathbb{E}|Y_0|}{\sqrt{\pi}}\left(\frac{n}{b}\right)^{1/2}\left\|(\mathcal{G}_n^{-1*}-\mathcal{G}^{-1*})v\right\|_{L^2(\mathbb{R})},
\end{aligned}
$$

where the last line follows from **(K2)** and again by applying the isometry property of $\mathcal{F}_+$. For the second part, we find that

$$
\begin{aligned}
\textbf{(II)} &= \frac{1}{2\pi}\int_{\mathbb{R}}|\mathcal{F}_+[\mathcal{G}^{-1*}v](x)|\mathbb{E}\left|\frac{\hat{\theta}(x)}{\tilde{\psi}(x)}\mathcal{F}_+[K_b](x)-\frac{\theta(x)}{\psi(x)}\right|\,dx\\
&\leq \underbrace{\frac{1}{2\pi}\int_{\mathbb{R}}|\mathcal{F}_+[\mathcal{G}^{-1*}v](x)|\mathbb{E}\left|\frac{\hat{\theta}(x)}{\tilde{\psi}(x)}-\frac{\theta(x)}{\psi(x)}\right||\mathcal{F}_+[K_b](x)|\,dx}_{\textbf{(III)}}\\
&\quad +\frac{1}{2\pi}\left\langle|\mathcal{F}_+[\mathcal{G}^{-1*}v]|,|\mathcal{F}_+[uv_1]||1-\mathcal{F}_+[K_b]|\right\rangle_{L^2(\mathbb{R})},
\end{aligned}
$$

where the identity $|\mathcal{F}_+[uv_1]|(x)=\left|\frac{\theta(x)}{\psi(x)}\right|$ was used in the last line. Hence, it remains to bound expression **(III)**. Indeed, applying the triangle inequality followed by the Cauchy–Schwarz inequality and the bounds in [18, Lemma 8.1 and 8.3] yields

$$
\begin{aligned}
\textbf{(III)} &\leq \int_{\mathbb{R}}|\mathcal{F}_+[K_b](x)||\mathcal{F}_+[\mathcal{G}^{-1*}v](x)|\mathbb{E}|\hat{\theta}(x)-\theta(x)|\left|\frac{1}{\tilde{\psi}(x)}-\frac{1}{\psi(x)}\right|dx\\
&\quad +\int_{\mathbb{R}}|\mathcal{F}_+[K_b](x)||\mathcal{F}_+[\mathcal{G}^{-1*}v](x)||\theta(x)|\mathbb{E}\left|\frac{1}{\tilde{\psi}(x)}-\frac{1}{\psi(x)}\right|dx\\
&\quad +\int_{\mathbb{R}}|\mathcal{F}_+[K_b](x)||\mathcal{F}_+[\mathcal{G}^{-1*}v](x)|\frac{\mathbb{E}|\hat{\theta}(x)-\theta(x)|}{|\psi(x)|}dx\\
&\leq S\Bigg[\int_{\mathbb{R}}|\mathcal{F}_+[\mathcal{G}^{-1*}v](x)|\sqrt{\mathbb{E}|\hat{\theta}(x)-\theta(x)|^2}\sqrt{\mathbb{E}\left|\frac{1}{\tilde{\psi}(x)}-\frac{1}{\psi(x)}\right|^2}dx\\
&\quad +\int_{\mathbb{R}}\frac{|\mathcal{F}_+[\mathcal{G}^{-1*}v](x)|}{|\psi(x)|}|\mathcal{F}_+[uv_1](x)|\sqrt{\mathbb{E}\left|\frac{1}{\tilde{\psi}(x)}-\frac{1}{\psi(x)}\right|^2}dx\\
&\quad +\int_{\mathbb{R}}\frac{|\mathcal{F}_+[\mathcal{G}^{-1*}v](x)}{|\psi(x)|}\sqrt{\mathbb{E}|\hat{\theta}(x)-\theta(x)|^2}dx\Bigg]\\
&\leq S\Bigg[\frac{c_1}{n^{1/2}}\sqrt{\mathbb{E}|Y_0|^2}\int_{\mathbb{R}}\frac{|\mathcal{F}_+[\mathcal{G}^{-1*}v](x)|}{|\psi(x)|}dx\\
&\quad +\frac{c_2}{n^{1/2}}\int_{\mathbb{R}}\frac{|\mathcal{F}_+[\mathcal{G}^{-1*}v](x)|}{|\psi(x)|}|\mathcal{F}_+[uv_1](x)|dx\\
&\quad +\frac{c_3}{n^{1/2}}\sqrt{\mathbb{E}|Y_0|^2}\int_{\mathbb{R}}\frac{|\mathcal{F}_+[\mathcal{G}^{-1*}v](x)}{|\psi(x)|}dx\Bigg],
\end{aligned}
$$

with constants $c_1, c_2, c_3 > 0$. Hence, by integrability of uv_1 it follows

$$\textbf{(III)} \leq \frac{c \cdot S}{\sqrt{n}} \left(\sqrt{\mathbb{E}|Y_0|^2} + \|uv_1\|_{L^1(\mathbb{R}^\times)} \right) \int_{\mathbb{R}^\times} \frac{|\mathcal{F}_+[\mathcal{G}^{-1*}v](x)|}{|\psi(x)|} dx$$

for some constant $c > 0$. This finishes the proof. $\qquad\square$

A.3 Proof of Theorem 3.7

Using Assumption 3.1, (4), **(K3)** and $\mathcal{F}_+[\mathcal{G}^{-1*}] \in L^1(\mathbb{R})$ we find that

$$\left\langle |\mathcal{F}_+[\mathcal{G}^{-1*}v]|, |\mathcal{F}_+[uv_1]||1 - \mathcal{F}_+[K_b]| \right\rangle_{L^2(\mathbb{R})} \lesssim \min\{1, b_n\} \int_{\mathbb{R}} |\mathcal{F}_+[\mathcal{G}^{-1*}v](x)| dx$$
$$= \mathcal{O}(b_n), \quad \text{as } n \to \infty.$$

Moreover, applying the same arguments as in the proof of [13, Corollary 3.7], we observe that

$$\|(\mathcal{G}_n^{-1*} - \mathcal{G}^{-1*})v\|_{L^2(\mathbb{R})} \lesssim a_n^{\frac{\beta_2}{\beta_1} - 1}.$$

Hence, if $\gamma = 0$, the assertions of the theorem immediately follow by the upper bound in Lemma 3.6. Otherwise, if $\gamma \neq 0$, consider the sample $(\tilde{Y}_j)_{j \in W}$ defined in Section 4.3. Following the computations there, one finds that on the right-hand side of (3.8) the additional term

$$\frac{\gamma}{2\pi} \mathbb{E} \left| \left\langle \mathcal{F}_+[\mathcal{G}^{-1*}v], \mathcal{F}_+[K_{b_n}] - 1 \right\rangle_{L^2(\mathbb{R})} - \left\langle \mathcal{F}_+[\mathcal{G}^{-1*}v], \mathbb{1}_{\{|\hat{\psi}_*| \leq n^{-1/2}\}} \mathcal{F}_+[K_{b_n}] \right\rangle_{L^2(\mathbb{R})} \right|$$

arises. Using $\mathcal{G}^{-1*}v \in H^1(\mathbb{R})$, $\mathcal{F}_+[\mathcal{G}^{-1*}] \in L^1(\mathbb{R})$ and **(K3)** yields that the latter expression can be estimated from above by

$$\frac{\gamma}{2\pi} \left(b_n \|\mathcal{G}^{-1*}v\|_{H^1(\mathbb{R})} + \frac{1}{\sqrt{n}} S \left\| \frac{\mathcal{F}_+[\mathcal{G}^{-1*}v]}{\psi} \right\|_{L^1(\mathbb{R})} \right).$$

This completes the proof. $\qquad\square$

A.4 Moment inequalities for m-dependent random fields

In this section, we sum up some moment inequalities that are quite helpful for the proofs in Section 3.

We start with the following Bernstein type inequality that is due to [15, p. 316].

Theorem A.1. *Let* $(X_j)_{j \in V}$, $V \subset \mathbb{Z}^d$, *be a centered m-dependent random field satisfying* $0 < \mathbb{E}X_j^2 < \infty$ *and, for some* $H > 0$,

$$|\mathbb{E}X_j^p| \leq \frac{p!}{2} H^{p-2} \mathbb{E}X_j^2, \quad p \geq 3, \ j \in V. \tag{A.1}$$

Then

$$P(S_V(X) \geq xB_V) \leq \begin{cases} \exp\left(-\frac{x^2}{4(m+1)^d \rho_V}\right), & 0 \leq x \leq \rho_V B_V/H, \\ \exp\left(-\frac{xB_V}{4(m+1)^d H}\right), & x \geq \rho_V B_V/H, \end{cases}$$

where

$$S_V(X) = \sum_{j \in V} X_j, \qquad B_V^2 = \mathbb{E}S_V^2 \quad and \quad \rho_V = \sum_{j \in V} \mathbb{E}X_j^2 / B_V^2.$$

The following lemma generalizes Lemma 8.1 in [18]. It can be easily proven using the same arguments as there.

Lemma A.2. *Let* $(Y_j)_{j \in \mathbb{Z}^d}$ *be a stationary m-dependent random field satisfying* $\mathbb{E}|Y_0|^{2q} < \infty$. *Furthermore, let* $W \subset \mathbb{Z}^d$ *be a finite subset,* $n = \mathrm{card}(W)$, *and let* $\hat{\theta}(u) = \frac{1}{n} \sum_{j \in W} Y_j e^{iuY_j}$ *and* $\theta(u) = \mathbb{E}Y_0 e^{iuY_0}$. *Then*

$$\mathbb{E}|\hat{\theta}(u) - \theta(u)|^{2q} \leq \frac{C}{n^q} \mathbb{E}|Y_0|^{2q},$$

where $C > 0$ *is a constant.*

Remark A.3. Clearly, applying the Cauchy–Schwarz inequality, Lemma A.2 also yields a bound in case that $q = 1/2$.

A.5 Asymptotic covariance of m-dependent random field

Lemma A.4. *Let the sequence* $(B_n)_{n \in \mathbb{N}}$ *be regularly growing to infinity. Moreover, let* $(X_j)_{j \in \mathbb{Z}^d}$ *be a stationary m-dependent random field and suppose there are measurable functions* $g^{(1)}, g_n^{(1)}, g^{(2)}, g_n^{(2)} : \mathbb{R} \to \mathbb{R}, n \in \mathbb{N}$, *with the following properties:*

1. $\mathbb{E}[g_n^{(k)}(X_0)] = 0$ *for all* $n \in \mathbb{N}$, $k = 1, 2$;

2. $\mathbb{E}[g^{(k)}(X_0)^2], \mathbb{E}[g_n^{(k)}(X_0)^2] < \infty$, $k = 1, 2, n \in \mathbb{N}$;

3. $\lim_{n \to \infty} \mathbb{E}[g_n^{(1)}(X_0) g_n^{(2)}(X_k)] = \mathbb{E}[g^{(1)}(X_0) g^{(2)}(X_k)] =: \sigma_k$, *for any* $k \in \mathbb{Z}^d$.

Then

$$\lim_{n \to \infty} \mathrm{Cov}\left(|B_n|^{-1/2} \sum_{j \in B_n} g_n^{(1)}(X_j), |B_n|^{-1/2} \sum_{k \in B_n} g_n^{(2)}(X_k)\right) = \sum_{\substack{t \in \mathbb{Z}^d: \\ \|t\|_\infty \leq m}} \sigma_t.$$

Proof. We observe that

$$\mathrm{Cov}\left(|B_n|^{-1/2} \sum_{j \in B_n} g_n^{(1)}(X_j), |B_n|^{-1/2} \sum_{k \in B_n} g_n^{(2)}(X_k)\right)$$

$$= \underbrace{\frac{1}{|B_n|} \sum_{j \in B_n} \sum_{k \in B_n} \left(\mathbb{E}\left[g_n^{(1)}(X_j), g_n^{(2)}(X_k)\right] - \mathbb{E}\left[g^{(1)}(X_j), g^{(2)}(X_k)\right]\right)}_{=y_n}$$

$$+ \underbrace{\frac{1}{|B_n|} \sum_{j \in B_n} \sum_{k \in B_n} \mathbb{E}\left[g^{(1)}(X_j), g^{(2)}(X_k)\right]}_{=z_n}.$$

Since $(X_j)_{j \in \mathbb{Z}^d}$ is m-dependent and stationary, and, since $(B_n)_{n \in \mathbb{N}}$ is regularly growing to infinity, the same computation as in the proof of Theorem 1.8 in [7, p.175]

shows that

$$\lim_{n\to\infty} z_n = \sum_{\substack{t\in\mathbb{Z}^d:\\ \|t\|_\infty\leq m}} \sigma_t.$$

It remains to show that $\lim_{n\to\infty} y_n = 0$. Indeed, the m-dependence and the property 1. yield

$$|y_n| \leq \frac{1}{|B_n|} \sum_{j\in B_n}\sum_{k\in\mathbb{Z}^d} \left|\mathbb{E}\left[g_n^{(1)}(X_0), g_n^{(2)}(X_{k-j})\right] - \mathbb{E}\left[g^{(1)}(X_0), g^{(2)}(X_{k-j})\right]\right|$$

$$\leq \sum_{\substack{k\in\mathbb{Z}^d:\\ \|k\|_\infty\leq m}} \left|\mathbb{E}\left[g_n^{(1)}(X_0), g_n^{(2)}(X_k)\right] - \mathbb{E}\left[g^{(1)}(X_0), g^{(2)}(X_k)\right]\right|$$

$$\to 0, \quad \text{as } n\to\infty. \qquad\qquad \Box$$

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