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Stochastic two-species mutualism model with jumps

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Abstract The existence and uniqueness are proved for the global positive solution to the system of stochastic differential equations describing a two-species mutualism model disturbed by the white noise, the centered and non-centered Poisson noises. We obtain sufficient conditions for stochastic ultimate boundedness, stochastic permanence, nonpersistence in the mean, strong persistence in the mean and extinction of the solution to the considered system.

Keywords Stochastic mutualism model, global solution, stochastic ultimate boundedness, stochastic permanence, extinction, nonpersistence in the mean, strong persistence in the mean

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1 Introduction

The construction of the mutualism model and its properties are presented in K. Gopalsamy [4]. Mutualism occurs when one species provides some benefit in exchange for another benefit. A deterministic two-species mutualism model is described by the system

$$\frac{dN_1(t)}{dt} = r_1(t)N_1(t)\left[\frac{K_1(t)+\alpha_1(t)N_2(t)}{1+N_2(t)} - N_1(t)\right],$$

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$$\frac{dN_2(t)}{dt} = r_2(t)N_2(t)\left[\frac{K_2(t)+\alpha_2(t)N_1(t)}{1+N_1(t)} - N_2(t)\right],$$

where $N_1(t)$ and $N_2(t)$ denote the population densities of each species at time t, $r_i(t) > 0, i = 1, 2$, denotes the intrinsic growth rate of species $N_i, i = 1, 2$, and $\alpha_i(t) > K_i(t) > 0, i = 1, 2$. The carrying capacity of species $N_i(t)$ is $K_i(t), i = 1, 2$, in the absence of other species. In the paper by Hong Qiu, Jingliang Lv and Ke Wang [9] the stochastic mutualism model of the form

$$dx(t) = x(t)\left[\frac{a_1(t)+a_2(t)y(t)}{1+y(t)} - c_1(t)x(t)\right] + \sigma_1(t)x(t)dw_1(t),$$
$$dy(t) = y(t)\left[\frac{b_1(t)+b_2(t)x(t)}{1+x(t)} - c_2(t)y(t)\right] + \sigma_2(t)y(t)dw_2(t) \qquad (1)$$

is considered, where $a_i(t), b_i(t), c_i(t), \sigma_i(t), i = 1, 2$, are all positive, continuous and bounded functions on $[0, +\infty)$, and $w_1(t), w_2(t)$ are independent Wiener processes. The authors show that the stochastic system (1) has a unique global (no explosion in a finite time) solution for any positive initial value and that this stochastic model is stochastically ultimately bounded. The sufficient conditions for stochastic permanence and persistence in the mean of the solution to the system (1) are obtained.

Population systems may suffer abrupt environmental perturbations, such as epidemics, fires, earthquakes, etc. So it is natural to introduce Poisson noises into the population model for describing such discontinuous systems. In the paper by Mei Li, Hongjun Gao and Binjun Wang [5] the authors consider the stochastic mutualism model with the white and centered Poisson noises:

$$dx(t) = x(t^-)\left[\left(r_1(t) - \frac{b_1(t)x(t)}{K_1(t)+y(t)} - \varepsilon_1(t)x(t)\right)dt\right.$$
$$\left. + \alpha_1(t)dw_1(t) + \int_{\mathbb{Y}} \gamma_1(t,z)\tilde{\nu}(dt,dz)\right],$$
$$dy(t) = y(t^-)\left[\left(r_2(t) - \frac{b_2(t)y(t)}{K_2(t)+x(t)} - \varepsilon_2(t)y(t)\right)dt\right.$$
$$\left. + \alpha_2(t)dw_2(t) + \int_{\mathbb{Y}} \gamma_2(t,z)\tilde{\nu}(dt,dz)\right],$$

where $x(t^-), y(t^-)$ are the left limit of $x(t)$ and $y(t)$ respectively, $r_i(t), b_i(t), K_i(t), \alpha_i(t), i = 1, 2$, are all positive, continuous and bounded functions, $\mathbb{Y}$ is measurable subset of $(0, +\infty)$, $w_i(t), i = 1, 2$, are independent standard one-dimensional Wiener processes, $\tilde{\nu}(t, A) = \nu(t, A) - t\Pi(A)$ is the centered Poisson measure independent on $w_i(t), i = 1, 2$, $E[\nu(t, A)] = t\Pi(A)$, $\Pi(\mathbb{Y}) < \infty$, $\gamma_i(t, z), i = 1, 2$, are random, measurable, bounded, continuous in t. The global existence and uniqueness of the positive solution to this problem are proved. The sufficient conditions of stochastic boundedness, stochastic permanence, persistence in the mean and extinction of the solution are obtained.

In this paper, we consider the stochastic mutualism model with jumps generated by centered and noncentered Poisson measures. So, we take into account not only "small" jumps, corresponding to the centered Poisson measure, but also "large" jumps, corresponding to the noncentered Poisson measure. This model is driven by the system of stochastic differential equations

$$dx_i(t) = x_i(t)\left[\frac{a_{i1}(t) + a_{i2}(t)x_{3-i}(t)}{1 + x_{3-i}(t)} - c_i(t)x_i(t)\right]dt + \sigma_i(t)x_i(t)dw_i(t) + \int_{\mathbb{R}} \gamma_i(t,z)x_i(t)\tilde{\nu}_1(dt,dz) + \int_{\mathbb{R}} \delta_i(t,z)x_i(t)\nu_2(dt,dz),$$

$$x_i(0) = x_{i0} > 0, \quad i = 1, 2, \tag{2}$$

where $w_i(t), i = 1, 2$, are independent standard one-dimensional Wiener processes, $\tilde{\nu}_1(t, A) = \nu_1(t, A) - t\Pi_1(A)$, $\nu_i(t, A), i = 1, 2$, are independent Poisson measures, which are independent on $w_i(t), i = 1, 2$, $E[\nu_i(t, A)] = t\Pi_i(A)$, $i = 1, 2$, $\Pi_i(A), i = 1, 2$, are finite measures on the Borel sets A in $\mathbb{R}$.

To the best of our knowledge, there are no papers devoted to the dynamical properties of the stochastic mutualism model (2), even in the case of the centered Poisson noise. It is worth noting that the impact of the centered and noncentered Poisson noises to the stochastic nonautonomous logistic model is studied in the papers by O.D. Borysenko and D.O. Borysenko [1, 2].

In the following we will use the notations $X(t) = (x_1(t), x_2(t))$, $X_0 = (x_{10}, x_{20})$, $|X(t)| = \sqrt{x_1^2(t) + x_2^2(t)}$, $\mathbb{R}_+^2 = \{X \in \mathbb{R}^2 : x_1 > 0, x_2 > 0\}$,

$$\beta_i(t) = \sigma_i^2(t)/2 + \int_{\mathbb{R}} [\gamma_i(t,z) - \ln(1 + \gamma_i(t,z))]\Pi_1(dz) - \int_{\mathbb{R}} \ln(1 + \delta_i(t,z))]\Pi_2(dz),$$

$i = 1, 2$. For the bounded, continuous functions $f_i(t), t \in [0, +\infty)$, $i = 1, 2$, let us denote

$$f_{i\,\sup} = \sup_{t\ge 0} f_i(t), \qquad f_{i\,\inf} = \inf_{t\ge 0} f_i(t), \quad i = 1, 2,$$

$$f_{\max} = \max\{f_{1\,\sup}, f_{2\,\sup}\}, \qquad f_{\min} = \min\{f_{1\,\inf}, f_{2\,\inf}\}.$$

We will prove that system (2) has a unique, positive, global (no explosion in a finite time) solution for any positive initial value, and that this solution is stochastically ultimate bounded. The sufficient conditions for stochastic permanence, nonpersistence in the mean, strong persistence in the mean and extinction of solution are derived.

The rest of this paper is organized as follows. In Section 2, we prove the existence of the unique global positive solution to the system (2). In Section 3, we prove the stochastic ultimate boundedness of the solution to the system (2). In Section 4, we obtain conditions under which the solution to the system (2) is stochastically permanent, and in Section 5 the sufficient conditions for nonpersistence in the mean, strong persistence in the mean and extinction of the solution are obtained.

2 Existence of the global solution

Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a probability space, $w_i(t), i = 1, 2, t \geq 0$, are independent standard one-dimensional Wiener processes on $(\Omega, \mathcal{F}, \mathrm{P})$, and $\nu_i(t, A), i = 1, 2$, are independent Poisson measures defined on $(\Omega, \mathcal{F}, \mathrm{P})$ independent on $w_i(t), i = 1, 2$. Here $\mathrm{E}[\nu_i(t, A)] = t\Pi_i(A)$, $i = 1, 2$, $\tilde{\nu}_i(t, A) = \nu_i(t, A) - t\Pi_i(A)$, $i = 1, 2$, $\Pi_i(\cdot), i = 1, 2$, are finite measures on the Borel sets in $\mathbb{R}$. On the probability space $(\Omega, \mathcal{F}, \mathrm{P})$ we consider an increasing, right-continuous family of complete sub-σ-algebras $\{\mathcal{F}_t\}_{t\geq 0}$, where $\mathcal{F}_t = \sigma\{w_i(s), \nu_i(s, A), s \leq t, i = 1, 2\}$.

We need the following assumption.

Assumption 1. It is assumed that $a_{ij}(t), i, j = 1, 2$, $c_i(t), \sigma_i(t), i = 1, 2$, are bounded, continuous in t functions, $a_{ij}(t) > 0, i, j = 1, 2, c_{\min} > 0, \gamma_i(t, z), \delta_i(t, z)$, $i = 1, 2$, are continuous in t functions and $\ln(1 + \gamma_i(t, z)), \ln(1 + \delta_i(t, z)), i = 1, 2$, are bounded, $\Pi_i(\mathbb{R}) < \infty, i = 1, 2$.

Theorem 1. *Let Assumption 1 be fulfilled. Then there exists a unique global solution $X(t)$ to the system (2) for any initial value $X(0) = X_0 > 0$, and* $\mathrm{P}\{X(t) \in \mathbb{R}^2_+\} = 1$.

Proof. Let us consider the system of stochastic differential equations

$$dv_i(t) = \left[\frac{a_{i1}(t) + a_{i2}(t)\exp\{v_{3-i}(t)\}}{1 + \exp\{v_{3-i}(t)\}} - c_i(t)\exp\{v_i(t)\} - \beta_i(t)\right]dt$$

$$+\sigma_i(t)dw_i(t) + \int_{\mathbb{R}} \ln(1 + \gamma_i(t, z))\tilde{\nu}_1(dt, dz) + \int_{\mathbb{R}} \ln(1 + \delta_i(t, z))\tilde{\nu}_2(dt, dz),$$

$$v_i(0) = \ln x_{i0}, \quad i = 1, 2. \quad (3)$$

The coefficients of equation (3) are locally Lipschitz continuous. Therefore, for any initial value $(v_1(0), v_2(0))$ there exists a unique local solution $V(t) = (v_1(t), v_2(t))$ on $[0, \tau_e)$, where $\sup_{t<\tau_e} |V(t)| = +\infty$ (cf. Theorem 6, p. 246, [3]). So, from the Itô formula we derive that the process $X(t) = (\exp\{v_1(t)\}, \exp\{v_2(t)\})$ is a unique, positive local solution to the system (2). To show this solution is global, we need to show that $\tau_e = +\infty$ a.s. Let $n_0 \in \mathbb{N}$ be sufficiently large for $x_{i0} \in [1/n_0, n_0]$, $i = 1, 2$. For any $n \geq n_0$ we define the stopping time

$$\tau_n = \inf\left\{t \in [0, \tau_e) : \ X(t) \notin \left(\frac{1}{n}, n\right) \times \left(\frac{1}{n}, n\right)\right\}.$$

It is clear that τ_n is increasing as $n \to +\infty$. Set $\tau_\infty = \lim_{n\to\infty} \tau_n$, whence $\tau_\infty \leq \tau_e$ a.s. If we prove that $\tau_\infty = \infty$ a.s., then $\tau_e = \infty$ a.s. and $X(t) \in \mathbb{R}^2_+$ a.s. for all $t \in [0, +\infty)$. So we need to show that $\tau_\infty = \infty$ a.s. If this statement is false, there are constants $T > 0$ and $\varepsilon \in (0, 1)$, such that $\mathrm{P}\{\tau_\infty < T\} > \varepsilon$. Hence, there is $n_1 \geq n_0$ such that

$$\mathrm{P}\{\tau_n < T\} > \varepsilon, \quad \forall n \geq n_1. \quad (4)$$

For the nonnegative function $V(X) = \sum_{i=1}^{2}(x_i - 1 - \ln x_i)$, $x_i > 0$, $i = 1, 2$, by the Itô formula we have

$$V(X(T \wedge \tau_n)) = V(X_0) + \sum_{i=1}^{2} \left\{ \int_0^{T\wedge\tau_n} \left[(x_i(t) - 1) \left(\frac{a_{i1}(t) + a_{i2}(t)x_{3-i}(t)}{1 + x_{3-i}(t)} - c_i(t)x_i(t) \right) + \beta_i(t) + \int_{\mathbb{R}} \delta_i(t, z)x_i(t)\Pi_2(dz) \right] dt + \int_0^{T\wedge\tau_n} (x_i(t) - 1)\sigma_i(t)dw_i(t) + \int_0^{T\wedge\tau_n}\int_{\mathbb{R}} [\gamma_i(t, z)x_i(t) - \ln(1 + \gamma_i(t, z))]\tilde{\nu}_1(dt, dz) + \int_0^{T\wedge\tau_n}\int_{\mathbb{R}} [\delta_i(t, z)x_i(t) - \ln(1 + \delta_i(t, z))]\tilde{\nu}_2(dt, dz) \right\}. \quad (5)$$

Under the conditions of the theorem there is a constant $K > 0$ such that

$$\sum_{i=1}^{2} \left[(x_i - 1) \left(\frac{a_{i1}(t) + a_{i2}(t)x_{3-i}}{1 + x_{3-i}} - c_i(t)x_i \right) + \beta_i(t) + \int_{\mathbb{R}} \delta_i(t, z)x_i\Pi_2(dz) \right] \le \sum_{i=1}^{2} \left[(a_{\max} + c_{\max})x_i - c_{\min}x_i^2 + \beta_{\max} + x_i\delta_{\max}\Pi_2(\mathbb{R}) \right] \le K, \quad (6)$$

where $a_{\max} = \max_{i,j=1,2}\{\sup_{t\ge0} a_{ij}(t)\}$. From (5) and (6) we have

$$V(X(T \wedge \tau_n)) \le V(X_0) + K(T \wedge \tau_n) + \sum_{i=1}^{2} \left\{ \int_0^{T\wedge\tau_n} (x_i(t) - 1)\sigma_i(t)dw_i(t) + \int_0^{T\wedge\tau_n}\int_{\mathbb{R}} [\gamma_i(t, z)x_i(t) - \ln(1 + \gamma_i(t, z))]\tilde{\nu}_1(dt, dz) + \int_0^{T\wedge\tau_n}\int_{\mathbb{R}} [\delta_i(t, z)x_i(t) - \ln(1 + \delta_i(t, z))]\tilde{\nu}_2(dt, dz) \right\}.$$

Whence, taking expectations yields

$$\mathrm{E}[V(X(T \wedge \tau_n))] \le V(X_0) + KT. \quad (7)$$

Set $\Omega_n = \{\tau_n \le T\}$ for $n \ge n_1$. Then by (4), $\mathrm{P}(\Omega_n) = \mathrm{P}\{\tau_n \le t\} > \varepsilon$, $\forall n \ge n_1$. Note that for every $\omega \in \Omega_n$ there is some i such that $x_i(\tau_n, \omega)$ equals either n or $1/n$.

So

$$V(X(\tau_n)) \geq \min\{n - 1 - \ln n, \frac{1}{n} - 1 + \ln n\}.$$

It then follows from (7) that

$$V(X_0) + KT \geq \mathrm{E}[\mathbf{1}_{\Omega_n} V(X(\tau_n))] \geq \varepsilon \min\{n - 1 - \ln n, \frac{1}{n} - 1 + \ln n\},$$

where $\mathbf{1}_{\Omega_n}$ is the indicator function of Ω_n. Letting $n \to \infty$ leads to the contradiction $\infty > V(X_0) + KT = \infty$. This completes the proof of the theorem. □

3 Stochastically ultimate boundedness

Definition 1. ([6]) The solution $X(t)$ to the system (2) is said to be stochastically ultimately bounded, if for any $\varepsilon \in (0, 1)$ there is a positive constant $\chi = \chi(\varepsilon) > 0$ such that for any initial value $X_0 \in \mathbb{R}^2_+$ this solution has the property

$$\limsup_{t\to\infty} \mathrm{P}\{|X(t)| > \chi\} < \varepsilon.$$

Theorem 2. *Under Assumption 1 the solution $X(t)$ to the system (2) is stochastically ultimately bounded for any initial value $X_0 \in \mathbb{R}^2_+$.*

Proof. Let τ_n be the stopping time defined in Theorem 1. Applying the Itô formula to the process $V(t, x_i(t)) = e^t x_i^p(t)$, $i = 1, 2$, $p > 0$, we obtain for $i = 1, 2$ that

$$\begin{aligned} V(t \wedge \tau_n, x_i(t \wedge \tau_n)) = x_{i0}^p + \int_0^{t\wedge\tau_n} e^s x_i^p(s) \Bigg\{ 1 + p \left[\frac{a_{i1}(s) + a_{i2}(s) x_{3-i}(s)}{1 + x_{3-i}(s)} \right. \\ \left. - c_i(s) x_i(s) \right] + \frac{p(p-1)\sigma_i^2(s)}{2} + \int_{\mathbb{R}} \left[(1 + \gamma_i(s, z))^p - 1 - p\gamma_i(s, z) \right] \Pi_1(dz) \\ + \int_{\mathbb{R}} \left[(1 + \delta_i(s, z))^p - 1 \right] \Pi_2(dz) \Bigg\} ds + \int_0^{t\wedge\tau_n} p e^s x_i^p(s) \sigma_i(s) dw_i(s) \\ + \int_0^{t\wedge\tau_n} \int_{\mathbb{R}} e^s x_i^p(s) \left[(1 + \gamma_i(s, z))^p - 1 \right] \tilde{\nu}_1(ds, dz) \\ + \int_0^{t\wedge\tau_n} \int_{\mathbb{R}} e^s x_i^p(s) \left[(1 + \delta_i(s, z))^p - 1 \right] \tilde{\nu}_2(ds, dz). \quad (8) \end{aligned}$$

Under Assumption 1 there is a constant $K_i(p) > 0$ such that

$$e^s x_i^p \left\{ 1 + p \left[\frac{a_{i1}(s) + a_{i2}(s) x_{3-i}}{1 + x_{3-i}} - c_i(s) x_i \right] + \frac{p(p-1)\sigma_i^2(s)}{2} + \right.$$

$$+\int_{\mathbb{R}}\big[(1+\gamma_i(s,z))^p-1-p\gamma_i(s,z)\big]\,\Pi_1(dz)+\int_{\mathbb{R}}\big[(1+\delta_i(s,z))^p-1\big]\,\Pi_2(dz)\Bigg\}$$
$$\le K_i(p)e^s. \quad (9)$$

From (8) and (9), taking expectations, we obtain

$$\mathrm{E}[V(t\wedge\tau_n, x_i(t\wedge\tau_n))] \le x_{i0}^p + K(p)\mathrm{E}[e^{t\wedge\tau_n}] \le x_{i0}^p + K_i(p)e^t.$$

Letting $n\to\infty$ leads to the estimate

$$e^t\mathrm{E}[x_i^p(t)] \le x_{i0}^p + e^t K_i(p). \quad (10)$$

So we have for $i = 1, 2$ that

$$\limsup_{t\to\infty} E[x_i^p(t)] \le K_i(p). \quad (11)$$

For $X = (x_1, x_2) \in \mathbb{R}_+^2$ we have $|X|^p \le 2^{p/2}(x_1^p + x_2^p)$, therefore, from (11) $\limsup_{t\to\infty} E[|X(t)|^p] \le L(p) = 2^{p/2}(K_1(p) + K_2(p))$. Let $\chi > (L(p)/\varepsilon)^{1/p}$, $p > 0$, $\forall\varepsilon \in (0, 1)$. Then applying the Chebyshev inequality yields

$$\limsup_{t\to\infty} \mathrm{P}\{|X(t)| > \chi\} \le \frac{1}{\chi^p}\limsup_{t\to\infty} E[|X(t)|^p] \le \frac{L(p)}{\chi^p} < \varepsilon. \qquad \square$$

4 Stochastic permanence

The property of stochastic permanence is important since it means the long-time survival in a population dynamics.

Definition 2. ([5]) The solution $X(t)$ to the system (2) is said to be stochastically permanent if for any $\varepsilon > 0$ there are positive constants $H = H(\varepsilon)$, $h = h(\varepsilon)$ such that

$$\liminf_{t\to\infty} \mathrm{P}\{x_i(t) \le H\} \ge 1-\varepsilon, \qquad \liminf_{t\to\infty} \mathrm{P}\{x_i(t) \ge h\} \ge 1-\varepsilon,$$

for $i = 1, 2$ and for any inial value $X_0 \in \mathbb{R}_+^2$.

Theorem 3. *Let Assumption 1 be fulfilled. If*

$$\min_{i=1,2}\inf_{t\ge 0}(a_{i\min}(t)-\beta_i(t)) > 0, \quad \textit{where } a_{i\min}(t) = \min_{j=1,2} a_{ij}(t), \quad i = 1, 2,$$

then the solution $X(t)$ to the system (2) with the initial condition $X_0 \in \mathbb{R}_+^2$ is stochastically permanent.

Proof. For the process $U_i(t) = 1/x_i(t)$, $i = 1, 2$, by the Itô formula we have

$$U_i(t) = U_i(0) + \int_0^t U_i(s)\Bigg[-\frac{a_{i1}(s)+a_{i2}(s)x_{3-i}(s)}{1+x_{3-i}(s)} + c_i(s)x_i(s) + \sigma_i^2(s)$$

$$+\int_{\mathbb{R}} \frac{\gamma_i^2(s,z)}{1+\gamma_i(s,z)}\Pi_1(dz)\Bigg] ds - \int_0^t U_i(s)\sigma_i(s)dw_i(s)$$

$$-\int_0^t\int_{\mathbb{R}} U_i(s)\frac{\gamma_i(s,z)}{1+\gamma_i(s,z)}\tilde{\nu}_1(ds,dz) - \int_0^t\int_{\mathbb{R}} U_i(s)\frac{\delta_i(s,z)}{1+\delta_i(s,z)}\nu_2(ds,dz).$$

Then by the Itô formula we derive for $0 < \theta < 1$:

$$(1+U_i(t))^\theta = (1+U_i(0))^\theta + \int_0^t \theta(1+U_i(s))^{\theta-2}\Bigg\{(1+U_i(s))U_i(s)$$

$$\times\Bigg[-\frac{a_{i1}(s)+a_{i2}(s)x_{3-i}(s)}{1+x_{3-i}(s)} + c_i(s)x_i(s) + \sigma_i^2(s)$$

$$+\int_{\mathbb{R}} \frac{\gamma_i^2(s,z)}{1+\gamma_i(s,z)}\Pi_1(dz)\Bigg] + \frac{\theta-1}{2}U_i^2(s)\sigma_i^2(s)$$

$$+\frac{1}{\theta}\int_{\mathbb{R}}\Bigg[(1+U_i(s))^2\left(\left(\frac{1+U_i(s)+\gamma_i(s,z)}{(1+\gamma_i(s,z))(1+U_i(s))}\right)^\theta - 1\right)$$

$$+\theta(1+U_i(s))\frac{U_i(s)\gamma_i(s,z)}{1+\gamma_i(s,z)}\Bigg]\Pi_1(dz)$$

$$+\frac{1}{\theta}\int_{\mathbb{R}}(1+U_i(s))^2\left[\left(\frac{1+U_i(s)+\delta_i(s,z)}{(1+\delta_i(s,z))(1+U_i(s))}\right)^\theta - 1\right]\Pi_2(dz)\Bigg\} ds$$

$$-\int_0^t \theta(1+U_i(s))^{\theta-1}U_i(s)\sigma_i(s)dw_i(s)$$

$$+\int_0^t\int_{\mathbb{R}}\left[\left(1+\frac{U_i(s)}{1+\gamma_i(s,z)}\right)^\theta - (1+U_i(s))^\theta\right]\tilde{\nu}_1(ds,dz)$$

$$+\int_0^t\int_{\mathbb{R}}\left[\left(1+\frac{U_i(s)}{1+\delta_i(s,z)}\right)^\theta - (1+U_i(s))^\theta\right]\tilde{\nu}_2(ds,dz) = (1+U_i(0))^\theta$$

$$+\int_0^t \theta(1+U_i(s))^{\theta-2}J(s)ds - I_{1,\text{stoch}}(t) + I_{2,\text{stoch}}(t) + I_{3,\text{stoch}}(t), \quad (12)$$

where $I_{i,\text{stoch}}(t), i = \overline{1,3}$, are corresponding stochastic integrals in (12). Under the Assumption 1 there exist constants $|K_1(\theta)| < \infty$, $|K_2(\theta)| < \infty$ such that for the process $J(t)$ we have the estimate

$$J(t) \le (1+U_i(t))U_i(t)\Bigg[-a_{i\min}(t) + c_{\max}U_i^{-1}(t) + \sigma_i^2(t) + \int_{\mathbb{R}} \frac{\gamma_i^2(s,z)}{1+\gamma_i(s,z)}\Pi_1(dz)\Bigg] + \frac{\theta-1}{2}U_i^2(s)\sigma_i^2(s)$$
$$+\frac{1}{\theta}\int_{\mathbb{R}}\Bigg[(1+U_i(s))^2\left(\left(\frac{1}{1+\gamma_i(s,z)}+\frac{1}{1+U_i(s)}\right)^{\theta}-1\right) + \theta(1+U_i(s))\frac{U_i(s)\gamma_i(s,z)}{1+\gamma_i(s,z)}\Bigg]\Pi_1(dz)$$
$$+\frac{1}{\theta}\int_{\mathbb{R}}(1+U_i(s))^2\left[\left(\frac{1}{1+\delta_i(s,z)}+\frac{1}{1+U_i(s)}\right)^{\theta}-1\right]\Pi_2(dz)$$
$$\le U_i^2(t)\Bigg[-a_{i\min}(t)+\frac{\sigma_i^2(t)}{2}+\int_{\mathbb{R}}\gamma_i(t,z)\Pi_1(dz)+\frac{\theta}{2}\sigma_i^2(t)$$
$$+\frac{1}{\theta}\int_{\mathbb{R}}[(1+\gamma_i(t,z))^{-\theta}-1]\Pi_1(dz)+\frac{1}{\theta}\int_{\mathbb{R}}[(1+\delta_i(t,z))^{-\theta}-1]\Pi_2(dz)\Bigg]$$
$$+K_1(\theta)U_i(t)+K_2(\theta).$$

Here we used the inequality $(x+y)^\theta \le x^\theta + \theta x^{\theta-1}y$, $0<\theta<1$, $x,y>0$. Due to

$$\lim_{\theta\to 0+}\Bigg[\frac{\theta}{2}\sigma_i^2(t)+\frac{1}{\theta}\int_{\mathbb{R}}[(1+\gamma_i(t,z))^{-\theta}-1]\Pi_1(dz) +\frac{1}{\theta}\int_{\mathbb{R}}[(1+\delta_i(t,z))^{-\theta}-1]\Pi_2(dz)\Bigg]$$
$$=-\int_{\mathbb{R}}\ln(1+\gamma_i(t,z))\Pi_1(dz)-\int_{\mathbb{R}}\ln(1+\delta_i(t,z))\Pi_2(dz),$$

and the condition $\min_{i=1,2}\inf_{t\ge 0}(a_{i\min}(t)-\beta_i(t))>0$ we can choose a sufficiently small $0<\theta<1$ to satisfy

$$K_0(\theta)=\min_{i=1,2}\inf_{t\ge 0}\Bigg\{a_{i\min}(t)-\frac{\sigma_i^2(t)}{2}-\int_{\mathbb{R}}\gamma_i(t,z)\Pi_1(dz)-\frac{\theta}{2}\sigma_i^2(t)$$
$$-\frac{1}{\theta}\int_{\mathbb{R}}[(1+\gamma_i(t,z))^{-\theta}-1]\Pi_1(dz)-\frac{1}{\theta}\int_{\mathbb{R}}[(1+\delta_i(t,z))^{-\theta}-1]\Pi_2(dz)\Bigg\}>0.$$

So from (12) and the estimate for $J(t)$ we derive

$$d\left[(1+U_i(t))^{\theta}\right] \le \theta(1+U_i(t))^{\theta-2}[-K_0(\theta)U_i^2(t)+K_1(\theta)U_i(t)+K_2(\theta)]dt$$
$$-\theta(1+U_i(t))^{\theta-1}U_i(t)\sigma_i(t)dw_i(t)+\int_{\mathbb{R}}\left[\left(1+\frac{U_i(t)}{1+\gamma_i(t,z)}\right)^{\theta}\right.$$
$$\left.-(1+U_i(t))^{\theta}\right]\tilde{\nu}_1(dt,dz)+\int_{\mathbb{R}}\left[\left(1+\frac{U_i(t)}{1+\delta_i(t,z)}\right)^{\theta}-(1+U_i(t))^{\theta}\right]\tilde{\nu}_2(dt,dz). \tag{13}$$

By the Itô formula and (13) we have

$$d\left[e^{\lambda t}(1+U_i(t))^{\theta}\right]=\lambda e^{\lambda t}(1+U_i(t))^{\theta}dt+e^{\lambda t}d\left[(1+U_i(t))^{\theta}\right]$$
$$\le e^{\lambda t}\theta(1+U_i(t))^{\theta-2}\left[-\left(K_0(\theta)-\frac{\lambda}{\theta}\right)U_i^2(t)+\left(K_1(\theta)+\frac{2\lambda}{\theta}\right)U_i(t)\right.$$
$$\left.+K_2(\theta)+\frac{\lambda}{\theta}\right]dt-\theta e^{\lambda t}(1+U_i(t))^{\theta-1}U_i(t)\sigma_i(t)dw_i(t)$$
$$+e^{\lambda t}\int_{\mathbb{R}}\left[\left(1+\frac{U_i(t)}{1+\gamma_i(t,z)}\right)^{\theta}-(1+U_i(t))^{\theta}\right]\tilde{\nu}_1(dt,dz)$$
$$+e^{\lambda t}\int_{\mathbb{R}}\left[\left(1+\frac{U_i(t)}{1+\delta_i(t,z)}\right)^{\theta}-(1+U_i(t))^{\theta}\right]\tilde{\nu}_2(dt,dz). \tag{14}$$

Let us choose $\lambda>0$ such that $K_0(\theta)-\lambda/\theta>0$. Then the function

$$(1+U_i(t))^{\theta-2}\left[-\left(K_0(\theta)-\frac{\lambda}{\theta}\right)U_i^2(t)+\left(K_1(\theta)+\frac{2\lambda}{\theta}\right)U_i(t)+K_2(\theta)+\frac{\lambda}{\theta}\right]$$

is bounded from above by some constant $K>0$. So by integrating (14) and taking the expectation we obtain

$$e^{\lambda t}\mathrm{E}\left[(1+U_i(t))^{\theta}\right]\le\left(1+\frac{1}{x_{i0}}\right)^{\theta}+\frac{\lambda}{\theta}K\left(e^{\lambda t}-1\right), \tag{15}$$

because the expectation of stochastic integrals are equal zero by (10) and

$$\mathrm{E}\left[\int_0^t e^{2\lambda s}(1+U_i(s))^{2\theta-2}U_i^2(s)\sigma_i^2(s)ds\right]=\mathrm{E}\left[\int_0^t\frac{e^{2\lambda s}\sigma_i^2(s)}{(1+x_i^{-1}(s))^{2-2\theta}x_i^2(s)}ds\right]$$
$$\le\int_0^t e^{2\lambda s}\sigma_i^2(s)\mathrm{E}[x_i^{2\theta}]ds<\infty,$$

$$\mathrm{E}\left[\int_0^t\int_{\mathbb{R}}e^{2\lambda s}\left[\left(1+\frac{U_i(s)}{1+\gamma_i(s,z)}\right)^{\theta}-(1+U_i(s))^{\theta}\right]^2\Pi_1(dz)ds\right]$$

$$\leq L_1 \int_0^t e^{2\lambda s} \mathrm{E}\left[x_i^{2\theta}(s)\right] ds < \infty,$$

$$\mathrm{E}\left[\int_0^t\int_{\mathbb{R}} e^{2\lambda s}\left[\left(1+\frac{U_i(s)}{1+\delta_i(s,z)}\right)^{\theta} - (1+U_i(s))^{\theta}\right]^2 \Pi_2(dz)ds\right]$$
$$\leq L_2 \int_0^t e^{2\lambda s} \mathrm{E}\left[x_i^{2\theta}(s)\right] ds < \infty,$$

where

$$L_1 = \theta^2 \max\left\{\frac{(\gamma^2)_{\max}}{(1+\gamma_{\min})^2}, \frac{(\gamma^2)_{\max}}{(1+\gamma_{\min})^{2\theta}}\right\}\Pi_1(\mathbb{R}) < \infty,$$
$$L_2 = \theta^2 \max\left\{\frac{(\delta^2)_{\max}}{(1+\delta_{\min})^2}, \frac{(\delta^2)_{\max}}{(1+\delta_{\min})^{2\theta}}\right\}\Pi_2(\mathbb{R}) < \infty.$$

From (15) we obtain

$$\limsup_{t\to\infty} \mathrm{E}\left[\left(\frac{1}{x_i(t)}\right)^{\theta}\right] = \limsup_{t\to\infty} \mathrm{E}\left[U_i^{\theta}(t)\right]$$
$$\leq \limsup_{t\to\infty} \mathrm{E}\left[(1+U_i(t))^{\theta}\right] \leq \frac{\theta K}{\lambda}, \quad i=1,2. \tag{16}$$

From (11) and (16) by the Chebyshev inequality we can derive that for an arbitrary $\varepsilon \in (0,1)$ there are positive constants $H = H(\varepsilon)$ and $h = h(\varepsilon)$ such that

$$\liminf_{t\to\infty} \mathrm{P}\{x_i(t) \leq H\} \geq 1-\varepsilon, \qquad \liminf_{t\to\infty} \mathrm{P}\{x_i(t) \geq h\} \geq 1-\varepsilon, \quad i=1,2. \quad \square$$

5 Extinction, nonpersistence and strong persistence in the mean

The property of extinction in the stochastic models of population dynamics means that every species will become extinct with probability 1.

Definition 3. The solution $X(t) = (x_1(t), x_2(t))$, $t \geq 0$, to the system (2) is said to be extinct if for every initial data $X_0 > 0$ we have $\lim_{t\to\infty} x_i(t) = 0$ almost surely (a.s.), $i = 1, 2$.

Theorem 4. *Let Assumption 1 be fulfilled. If*

$$\bar{p}_i^* = \limsup_{t\to\infty} \frac{1}{t}\int_0^t p_{i\max}(s)ds < 0, \quad \textit{where } p_{i\max}(s) = a_{i\max}(s) - \beta_i(s),$$

$a_{i\max}(t) = \max_{j=1,2} a_{ij}(t)$, $i = 1, 2$, *then the solution* $X(t)$ *to the system* (2) *with the initial condition* $X_0 \in \mathbb{R}_+^2$ *will be extinct.*

Proof. By the Itô formula, we have

$$\ln x_i(t) = \ln x_{i0} + \int_0^t \Bigg[\frac{a_{i1}(s) + a_{i2}(s)x_{3-i}(s)}{1 + x_{3-i}(s)} - c_i(s)x_i(s) - \frac{\sigma_i^2(s)}{2}$$

$$+ \int_{\mathbb{R}} [\ln(1 + \gamma_i(s,z)) - \gamma_i(s,z)]\Pi_1(dz) + \int_{\mathbb{R}} \ln(1 + \delta_i(s,z))\Pi_2(dz) \Bigg] ds$$

$$+ \int_0^t \sigma_i(s)dw_i(s) + \int_0^t\int_{\mathbb{R}} \ln(1 + \gamma_i(s,z))\tilde{\nu}_1(ds,dz)$$

$$+ \int_0^t\int_{\mathbb{R}} \ln(1 + \delta_i(s,z))\tilde{\nu}_2(ds,dz) \le \ln x_{i0} + \int_0^t p_{i\,\max}(s)ds + M(t), \qquad (17)$$

where the martingale

$$M(t) = \int_0^t \sigma_i(s)dw_i(s) + \int_0^t\int_{\mathbb{R}} \ln(1 + \gamma_i(s,z))\tilde{\nu}_1(ds,dz)$$

$$+ \int_0^t\int_{\mathbb{R}} \ln(1 + \delta_i(s,z))\tilde{\nu}_2(ds,dz) \qquad (18)$$

has quadratic variation

$$\langle M, M\rangle(t) = \int_0^t \sigma_i^2(s)ds + \int_0^t\int_{\mathbb{R}} \ln^2(1 + \gamma_i(s,z))\Pi_1(dz)ds$$

$$+ \int_0^t\int_{\mathbb{R}} \ln^2(1 + \delta_i(s,z))\Pi_2(dz)ds \le Kt.$$

Then the strong law of large numbers for local martingales ([7]) yields $\lim_{t\to\infty} M(t)/t = 0$ a.s. Therefore, from (17) we have

$$\limsup_{t\to\infty} \frac{\ln x_i(t)}{t} \le \limsup_{t\to\infty} \frac{1}{t} \int_0^t p_{i\,\max}(s)ds < 0, \quad \text{a.s.}$$

So $\lim_{t\to\infty} x_i(t) = 0$, $i = 1, 2$, a.s. □

Definition 4. ([8]) The solution $X(t) = (x_1(t), x_2(t))$, $t \ge 0$, to the system (2) is said to be nonpersistence in the mean if for every initial data $X_0 > 0$ we have $\lim_{t\to\infty} \frac{1}{t}\int_0^t x_i(s)ds = 0$ a.s., $i = 1, 2$.

Theorem 5. *Let Assumption 1 be fulfilled. If* $\bar{p}_i^* = 0$*,* $i = 1, 2$*, then the solution* $X(t)$ *to the system (2) with the initial condition* $X_0 \in \mathbb{R}_+^2$ *will be nonpersistence in the mean.*

Proof. From the first equality in (17) we have

$$\ln x_i(t) \leq \ln x_{i0} + \int_0^t p_{i\,\max}(s)ds - c_{\min}\int_0^t x_i(s)ds + M(t), \tag{19}$$

where the martingale $M(t)$ is defined in (18). From the definition of $\bar{p}_i^*$ and the strong law of large numbers for $M(t)$ it follows that $\forall\varepsilon > 0$, $\exists t_0 \geq 0$ such that

$$\frac{1}{t}\int_0^t p_{i\,\max}(s)ds \leq \bar{p}_i^* + \frac{\varepsilon}{2}, \quad \frac{M(t)}{t} \leq \frac{\varepsilon}{2}, \quad \forall t \geq t_0, \text{ a.s.}$$

So, from (19) we derive

$$\ln x_i(t) - \ln x_{i0} \leq t(\bar{p}_i^* + \varepsilon) - c_{\min}\int_0^t x_i(s)ds = t\varepsilon - c_{\min}\int_0^t x_i(s)ds. \tag{20}$$

Let $y_i(t) = \int_0^t x_i(s)ds$, then from (20) we have

$$\ln\left(\frac{dy_i(t)}{dt}\right) \leq \varepsilon t - c_{\min}y_i(t) + \ln x_{i0} \Rightarrow e^{c_{\min}y_i(t)}\frac{dy_i(t)}{dt} \leq x_{i0}e^{\varepsilon t}.$$

Integrating the last inequality from t_0 to t yields

$$e^{c_{\min}y_i(t)} \leq \frac{c_{\min}x_{i0}}{\varepsilon}\left(e^{\varepsilon t} - e^{\varepsilon t_0}\right) + e^{c_{\min}y_i(t_0)}, \quad \forall t \geq t_0, \text{ a.s.}$$

So

$$y_i(t) \leq \frac{1}{c_{\min}}\ln\left[e^{c_{\min}y_i(t_0)} + \frac{c_{\min}x_{i0}}{\varepsilon}\left(e^{\varepsilon t} - e^{\varepsilon t_0}\right)\right], \quad \forall t \geq t_0, \text{ a.s.,}$$

and therefore

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t x_i(s)ds \leq \frac{\varepsilon}{c_{\min}}, \quad \text{a.s.}$$

Since $\varepsilon > 0$ is arbitrary and $X(t) \in \mathbb{R}_+^2$, we have

$$\lim_{t\to\infty}\frac{1}{t}\int_0^s x_i(s)ds = 0, \quad i = 1, 2, \text{ a.s.} \qquad \square$$

Definition 5. ([8]) The solution $X(t) = (x_1(t), x_2(t))$, $t \geq 0$, to the system (2) is said to be strongly persistence in the mean if for every initial data $X_0 > 0$ we have $\liminf_{t\to\infty} \frac{1}{t}\int_0^t x_i(s)ds > 0$ a.s., $i = 1, 2$.

Theorem 6. *Let Assumption 1 be fulfilled. If* $\bar{p}_{i*} = \liminf_{t\to\infty} \frac{1}{t}\int_0^t p_{i\,\min}(s)ds > 0$, *where* $p_{i\,\min}(s) = a_{i\,\min}(s) - \beta_i(s)$, $a_{i\,\min}(t) = \min_{j=1,2} a_{ij}(t)$, $i = 1, 2$, *then*

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t x_i(s)ds \geq \frac{\bar{p}_{i*}}{c_{i\,\sup}}.$$

Therefore, the solution $X(t)$ *to the system* (2) *with the initial condition* $X_0 \in \mathbb{R}^2_+$ *will be strongly persistence in the mean.*

Proof. From the first equality in (17) we have

$$\ln x_i(t) \geq \ln x_{i0} + \int_0^t p_{i\,\min}(s)ds - c_{i\,\sup}\int_0^t x_i(s)ds + M(t), \tag{21}$$

where the martingale $M(t)$ is defined in (18). From the definition of $\bar{p}_{i*}$ and the strong law of large numbers for $M(t)$ it follows that $\forall\varepsilon > 0$, $\exists t_0 \geq 0$ such that $\frac{1}{t}\int_0^t p_{i\,\min}(s)ds \geq \bar{p}_{i*} - \frac{\varepsilon}{2}$, $\frac{M(t)}{t} \geq -\frac{\varepsilon}{2}$, $\forall t \geq t_0$, a.s. So, from (21) we obtain

$$\ln x_i(t) \geq \ln x_{i0} + t(\bar{p}_{i*} - \varepsilon) - c_{i\,\sup}\int_0^t x_i(s)ds. \tag{22}$$

Let us choose sufficiently small $\varepsilon > 0$ such that $\bar{p}_{i*} - \varepsilon > 0$.

Let $y_i(t) = \int_0^t x_i(s)ds$, then from (22) we have

$$\ln\left(\frac{dy_i(t)}{dt}\right) \geq (\bar{p}_{i*} - \varepsilon)t - c_{i\,\sup}y_i(t) + \ln x_{i0}.$$

Hence $e^{c_{i\,\sup}y_i(t)}\frac{dy_i(t)}{dt} \geq x_{i0}e^{(\bar{p}_{i*}-\varepsilon)t}$. Integrating the last inequality from t_0 to t yields

$$e^{c_{i\,\sup}y_i(t)} \geq \frac{c_{i\,\sup}x_{i0}}{\bar{p}_{i*} - \varepsilon}\left(e^{(\bar{p}_{i*}-\varepsilon)t} - e^{(\bar{p}_{i*}-\varepsilon)t_0}\right) + e^{c_{i\,\sup}y_i(t_0)}, \quad \forall t \geq t_0, \text{ a.s.}$$

So

$$y_i(t) \geq \frac{1}{c_{i\,\sup}}\ln\left[e^{c_{i\,\sup}y_i(t_0)} + \frac{c_{i\,\sup}x_{i0}}{\bar{p}_{i*} - \varepsilon}\left(e^{(\bar{p}_{i*}-\varepsilon)t} - e^{(\bar{p}_{i*}-\varepsilon)t_0}\right)\right], \text{ a.s.,}$$

$\forall t \geq t_0$, and therefore

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t x_i(s)ds \geq \frac{(\bar{p}_{i*} - \varepsilon)}{c_{i\,\sup}}, \text{ a.s.}$$

Using the arbitrariness of $\varepsilon > 0$ we get the assertion of the theorem. □

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