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Alternative probabilistic representations of Barenblatt-type solutions

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Abstract A general class of probability density functions

$$u(x,t) = Ct^{-\alpha d}\left(1-\left(\frac{\|x\|}{ct^{\alpha}}\right)^{\beta}\right)_{+}^{\gamma}, \quad x \in \mathbb{R}^d, t>0,$$

is considered, containing as particular case the Barenblatt solutions arising, for instance, in the study of nonlinear heat equations. Alternative probabilistic representations of the Barenblatt-type solutions $u(x,t)$ are proposed. In the one-dimensional case, by means of this approach, $u(x,t)$ can be connected with the wave propagation.

Keywords Anomalous diffusion, Beta random variable, Euler–Poisson–Darboux equation, Fourier transform, nonlinear diffusion equation, random velocity
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1 Introduction

The Brownian motion is a stochastic process of interest for pure and applied mathematicians. The probability distribution of the Brownian motion is given by the Gaussian kernel

$$G(x,t) = \frac{1}{(4\pi t)^{d/2}} \exp(-\|x\|^2/4t), \quad x \in \mathbb{R}^d, t>0,$$

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which is the source-type solution (that is $G(x,0) = \delta(x)$, where $\delta(x)$ represents Dirac's delta function) to the parabolic heat equation

$$\frac{\partial u}{\partial t} = \Delta u.$$

The main unwanted feature of this solution is that it is inevitably positive everywhere in its domain of definition; i.e., the Brownian motion scatters with unbounded velocity. A way to overcome this feature is to consider the porous medium equation which is a nonlinear diffusion equation

$$\frac{\partial u}{\partial t} = \Delta(u^m), \quad m > 1, \tag{1.1}$$

having the source-type solution given by

$$U(x,t) = Ct^{-\alpha d}\left(1 - \frac{\|x\|^2}{c^2 t^{2\alpha}}\right)_+^{\frac{1}{m-1}}, \quad \alpha > 0, c > 0,$$

where $(x)_+ := \max(x,0)$ and C is a suitable constant such that $\int U(x,t)\mathrm{d}x = 1$. The solution $U(\cdot,t)$ is a compactly supported function and it is called the *Barenblatt solution*. For a complete description of the mathematical analysis related to the partial differential equation (1.1) the reader can consult [33]. The connection between the porous medium equation and the theory of stochastic processes has been investigated, for instance, in [16–18, 2, 11, 19, 29] and [9].

The aim of this paper is to study a class of functions generalizing the Barenblatt solution $U(x,t)$. For fixed $t > 0$, we consider the map

$$\mathbb{R}^d \ni x \mapsto u := u(x,t) := Ct^{-\alpha d}\left(1 - \left(\frac{\|x\|}{ct^\alpha}\right)^\beta\right)_+^\gamma, \tag{1.2}$$

where $\alpha > 0$, $\beta > 0$ and $\gamma > 0$, and

$$C := C(\beta,\gamma,d) := \frac{\beta}{c^d \sigma(\mathbb{S}^{d-1})\mathrm{Beta}(\frac{d}{\beta}, \gamma+1)}$$

is a positive constant determined by the condition $\|u(x,t)\|_{L^1(\mathbb{R}^d,\mathrm{d}x)} = 1$ (the property of the mass conservation), $\mathrm{d}x$ denoting the Lebesgue measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, $\sigma(\mathbb{S}^{d-1}) := 2\pi^{d/2}/\Gamma(d/2)$ represents the surface area of the $(d-1)$-dimensional sphere $\mathbb{S}^{d-1}$ with radius one and $\mathrm{Beta}(a,b) := \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$, $a, b > 0$. We observe that u is a probability density function with an associated absolutely continuous probability measure given by

$$\mu_t(\mathrm{d}x) = u(x,t)\mathrm{d}x, \tag{1.3}$$

having the following features:

- The density function u is compactly supported; i.e., for every $t > 0$, the support of $u(\cdot,t)$ is given by

$$\operatorname{supp} u(\cdot,t) = B_{r(t)} := \{x \in \mathbb{R}^d : \|x\| \le r(t)\},$$

representing a closed ball with radius $r(t) := ct^\alpha$. This property implies the finite speed of propagation of u; i.e., the $(d-1)$-dimensional sphere with radius $r(t)$ denoted by $\mathbb{S}^{d-1}_{r(t)}$ provides the free boundary separating the regions $\{(x,t) \in \mathbb{R}^d \times (0,\infty) : u(x,t) > 0\}$ and $\{(x,t) \in \mathbb{R}^d \times (0,\infty) : u(x,t) = 0\}$.

- The probability measure μ_t is rotationally invariant; that is, $\mu_t(\mathrm{d}x) = u(\|x\|, t)\mathrm{d}x$, or equivalently, $\mu(MA) = \mu(A)$ for $A \in \mathcal{B}(\mathbb{R}^d)$ and $M \in O(d)$, where $O(d)$ is the group of $d \times d$ orthogonal matrices acting in $\mathbb{R}^d$, where $d \geq 2$. As a direct consequence of this property we derive

$$\mu_t(B_a) = \sigma(\mathbb{S}^{d-1}) \int_0^a r^{d-1} u(r,t)\mathrm{d}r$$
$$= \frac{\mathrm{Beta}\left((a/ct^\alpha)^\beta; \frac{d}{\beta}, \gamma+1\right)}{\mathrm{Beta}(\frac{d}{\beta}, \gamma+1)}, \quad 0 < a < ct^\alpha, \tag{1.4}$$

where $\mathrm{Beta}(x; a, b) = \int_0^x y^{a-1}(1-y)^{b-1}\mathrm{d}y$, $x \in \mathbb{R}$, is the incomplete Gamma function.

- The density u is a self-similar function. Indeed

$$u(x,t) = L^{d\alpha} u(L^\alpha x, Lt), \quad L > 0.$$

We refer to (1.2) as the class of the *Barenblatt-type solutions*. The family of functions (1.2) contains as particular cases, for instance, weak solutions of several nonlinear and linear diffusion equations (see Section 2). Furthermore, in [13] it is proved that the mean exit time of a symmetric Lévy stable process from a ball admits a representation belonging to the Barenblatt-type solution class. In this paper we provide alternative probabilistic representations of Barenblatt-type density functions in terms of mean value of delta functions containing random terms (see Section 3). At least in the case $d = 1$, our approach permits to shed light on the connection of the nonlinear diffusion with the propagation of waves and spherical waves (which are described by means of linear partial differential equations). The main novelty of this interpretation is that a wave performs random displacements nonlinearly with respect to time. It is worth to mention that solutions belonging to family (1.2) emerge from different frameworks (linear hyperbolic, nonlinear parabolic and nonlocal). In this way, objects requiring different mathematical tools have common features.

It is worth to note that the connections between stochastic processes and nonlinear Fokker–Planck equations have also been analyzed in [15, 25] and in references therein.

2 Barenblatt-type solutions to diffusion equations

The aim of this section is to highlight that the class of density functions of the form (1.2) is very general. Solutions belonging to family (1.2) appear in different frameworks (linear hyperbolic, nonlinear parabolic and nonlocal). Therefore, in what follows, we list some diffusion equations studied by means of different approaches. Nevertheless, their solutions share the same analytic form.

The Fourier transform $\mathcal{F}$ and the inverse transform $\mathcal{F}^{-1}$ of a function $v \in L^1(\mathbb{R}^d, \mathrm{d}x)$ are defined by

$$\mathcal{F}v(\xi) = \int_{\mathbb{R}^d} v(x)e^{ix\cdot\xi}\mathrm{d}x, \qquad \mathcal{F}^{-1}v(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} v(\xi)e^{-ix\cdot\xi}\mathrm{d}\xi,$$

with $\xi \in \mathbb{R}^d$.

2.1 Nonlinear diffusions: p-Laplacian equation

We mean here the p-Laplacian equation (PLE for short) studied, for instance, in [20] and [23], which is the following nonlinear degenerate parabolic evolution equation

$$\frac{\partial u}{\partial t} = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right), \quad p > 2, \quad t > 0, \tag{2.1}$$

subject to the initial condition

$$u(x, 0) = \delta(x), \tag{2.2}$$

where $u := u(x, t)$, with $x \in \mathbb{R}^d$, $d \geq 1$, is a scalar function defined on $\mathbb{R}^d \times \mathbb{R}^+$. The Cauchy problem (2.1)–(2.2) admits a unique nonnegative fundamental solution: it is a function $u \geq 0$ solving (2.1)–(2.5) in a weak sense (see [20] and [23] for the detailed definition of weak solution to PLE). This solution is given by the following probability density function

$$u(x, t) = t^{-k}\left(\mathfrak{c} - q\left(\frac{\|x\|}{t^{k/d}}\right)^{\frac{p}{p-1}}\right)_+^{\frac{p-1}{p-2}}, \tag{2.3}$$

where

$$k := \left(p - 2 + \frac{p}{d}\right)^{-1}, \qquad q := \frac{p-2}{p}\left(\frac{k}{d}\right)^{\frac{1}{p-1}}$$

and $\mathfrak{c} := \mathfrak{c}(p, d)$ is a constant determined by the condition $\int u(x, t)\mathrm{d}x = 1$. By setting $\beta = \frac{p}{p-1}$, $\gamma = \frac{p-1}{p-2}$, $\alpha = \frac{k}{d}$, $C = \mathfrak{c}^{\frac{p-1}{p-2}}$ and $c = (\mathfrak{c}/q)^{\frac{p-1}{p}}$, the Barenblatt-type solution (1.2) coincides with (2.3).

2.2 Nonlinear diffusions: nonlocal porous medium equation

The Nonlocal Porous Medium Equation (NPME), studied in [3] and [4], is the following degenerate nonlinear and nonlocal evolution equation

$$\frac{\partial u}{\partial t} = \operatorname{div}\left(|u|\nabla^{\nu-1}(|u|^{m-2}u)\right), \quad m > 1, \ \nu \in (0, 2], \ t > 0, \tag{2.4}$$

subject to the initial condition

$$u(x, 0) = u_0(x). \tag{2.5}$$

The pseudo-differential operator $\nabla^{\nu-1}$ is the fractional gradient denoting the nonlocal operator defined as $\nabla^{\nu-1}u := \mathcal{F}^{-1}(i\xi\|\xi\|^{\nu-2}\mathcal{F}u)$. This notation highlights that

$\nabla^{\nu-1}$ is a pseudo-differential (vector-valued) operator of order $\nu-1$. Equivalently, we can define $\nabla^{\nu-1}$ as $\nabla(-\Delta)^{\frac{\nu}{2}-1}$, where $(-\Delta)^{\frac{\nu}{2}}u = \mathcal{F}^{-1}(\|\xi\|^{\nu}\mathcal{F}u)$ is the fractional Laplace operator, i.e., a Fourier multiplier with the symbol $\|\xi\|^{\nu}$. For $\nu = 2$, (2.4) becomes the classical nonlinear porous medium equation

$$\frac{\partial u}{\partial t} = \operatorname{div}\left(|u|\nabla(|u|^{m-2}u)\right) = \operatorname{div}\left((m-1)|u|^{m-1}\nabla u\right). \tag{2.6}$$

If we restrict our attention to nonnegative solution $u(x,t)$, equation (2.6) becomes

$$\frac{\partial u}{\partial t} = \frac{m-1}{m}\Delta(u^m), \tag{2.7}$$

which is usually adopted to model the flow of a gas through a porous medium.

Let $\nu \in (0,2]$ and $m > 1$. A weak solution, in the sense of Definition 1 in [4], is given by

$$u(x,t) = Ct^{-d\alpha}\left(1 - k^{\frac{2}{\nu}}\frac{\|x\|^2}{t^{2\alpha}}\right)_+^{\frac{\nu}{2(m-1)}}, \tag{2.8}$$

where $\alpha := \frac{1}{d(m-1)+\nu}$, $k := \frac{d\Gamma(d/2)}{(d(m-1)+\nu)2^{\nu}\Gamma(1+\frac{\nu}{2})\Gamma(\frac{d+\nu}{2})}$ and

$$C := \frac{\Gamma(\frac{d}{2} + \frac{\nu}{2(m-1)} + 1)k^{\frac{d}{\nu}}}{\pi^{\frac{d}{2}}\Gamma(\frac{\nu}{2(m-1)} + 1)}.$$

Furthermore, $u(x,t)$ is the pointwise solution of equation (2.4) for $\|x\| \neq k^{-\frac{1}{\nu}}t^{\alpha}$. The link between (2.8) and random flights has been investigated in [8]. For $\nu = 2$, the solution (2.3) becomes the Barenblatt–Kompanets–Zel'dovich–Pattle solution of the porous medium equation (2.7) supplemented with the initial condition $u(x,0) = \delta(x)$ (see, for instance, [33]).

The function (1.2) reduces to (2.8) for $\gamma = \frac{\nu}{2(m-1)}$, $\beta = 2$ and $c = 1/k^{\frac{2}{\nu}}$.

2.3 *Euler–Poisson–Darboux equation*

It is well known that the fundamental solution of the Euler–Poisson–Darboux (EPD) equation

$$\frac{\partial^2 u}{\partial t^2} + \frac{d+2\nu-1}{t}\frac{\partial u}{\partial t} = c^2\Delta u, \quad \nu > 0,\ t > 0,\ c > 0, \tag{2.9}$$

has the form

$$u(x,t) = \frac{\Gamma(\nu + \frac{d}{2})}{\pi^{d/2}\Gamma(\nu)}\frac{1}{(ct)^d}\left(1 - \frac{\|x\|^2}{(ct)^2}\right)_+^{\nu-1} \tag{2.10}$$

and therefore it belongs to the family of probability density functions with compact support (1.2) with $\beta = 2$, $\alpha = 1$, $k = d$ and $\gamma = \nu - 1$. There is a wide literature about the EPD equation and its applications. We refer to Bresters [5] for the construction of weak solutions of the initial value problem for the EPD equation based on distributional methods. We recall that, in the one-dimensional case, the solution to the Cauchy problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} + \dfrac{2\xi}{t}\dfrac{\partial u}{\partial t} = c^2 \dfrac{\partial^2 u}{\partial x^2}, \\ u(x,0) = f(x), \\ \dfrac{\partial u}{\partial t}(x,t)\Big|_{t=0} = 0, \end{cases} \tag{2.11}$$

can be represented as the Erdélyi–Kober fractional integral (see definition (3.4) below) of the D'Alembert solution of the wave equation (see [10])

$$\begin{cases} \dfrac{\partial^2 w}{\partial t^2} = c^2 \dfrac{\partial^2 w}{\partial x^2}, \\ w(x,0) = f(x), \\ \dfrac{\partial w}{\partial t}(x,t)\Big|_{t=0} = 0. \end{cases} \tag{2.12}$$

This means that

$$u(x,t) = \frac{2}{\mathrm{Beta}(\xi, \frac{1}{2})} \int_0^1 (1-y^2)^{\xi-1} \left[\frac{f(x+yct) + f(x-yct)}{2} \right] \mathrm{d}y. \tag{2.13}$$

The first probabilistic interpretation of this analytic representation, discussed by Rosencrans [31] and more recently by Garra and Orsingher [12], is the following one: solution (2.13) can be written as

$$u(x,t) = \mathbf{E}\left[\frac{f(x+\mathcal{U}(t)) + f(x-\mathcal{U}(t))}{2} \right], \tag{2.14}$$

where

$$\mathcal{U}(t) = U(0) \int_0^t (-1)^{N(s)} \mathrm{d}s, \tag{2.15}$$

and $(N(t))_{t\geq 0}$ is the nonhomogeneous Poisson process with rate $\lambda(t) = \frac{\xi}{t}$, $U(0)$ is a uniformly distributed r.v. on $\{-c, c\}$ (furthermore $(N(t))_{t\geq 0}$ and $U(0)$ are supposed independent). By means of the general Proposition 1 (see the next section), we here obtain a new interesting probabilistic interpretation of the fundamental solution of the EPD equation.

Moreover, we have the following interesting picture that underlines the role of the Barenblatt-type solution as a bridge between nonlinear and linear PDEs:

- The Erdélyi–Kober fractional integral of the solution of the D'Alembert equation leads to the solution of the Euler–Poisson–Darboux equation under the same initial conditions.
- As recently pointed out in [9] and [7], the time-rescaled Kompanets–Zel'dovich–Barenblatt solution of the Porous Medium Equation (PME) coincides with the fundamental solution of the Euler–Poisson–Darboux equation.

Therefore, we have found, by means of new analytical representations, a direct connection between nonlinear parabolic equations and linear hyperbolic equations. From the probabilistic point of view this connection could be expected because in both the cases we have generalizations of the diffusion equation leading to a finite speed propagation.

2.4 Nonlinear time-fractional diffusive equations admitting Barenblatt-type solutions

There is a wide literature about the probabilistic interpretation of linear space and time-fractional diffusive equations (see, e.g., [24, 1, 6, 27] and the references therein). On the other hand, a probabilistic approach to time-fractional nonlinear diffusive-type equations is still completly missing. Recently, the existence and uniqueness of compactly supported solutions for time-fractional porous medium equations has been investigated (see, e.g., [30]). However, up to our knowledge, it is not possible to find an explicit form of the Barenblatt-type solution. On the other hand, time-fractional diffusive equations are actracting an increasing interest in the literature. Explicit Barenblatt-type solutions for nonlinear time-fractional equations can play a relevant role for future studies in this context. We here consider a new family of nonlinear time-fractional diffusive equations admitting a Barenblatt-type solution of the form (1.2).

Let us consider the following nonlinear diffusive equation

$$\frac{1}{t^{2\nu}}\frac{\partial^\nu u}{\partial t^\nu}+\frac{1}{t^\nu}\frac{\partial^2 u}{\partial x^2}+\left(\frac{\partial u}{\partial x}\right)^2=0, \tag{2.16}$$

where $\frac{\partial^\nu}{\partial t^\nu}$ denotes the Riemann–Liouville derivative

$$\frac{\partial^\nu f(x,t)}{\partial t^\nu}=\frac{1}{\Gamma(1-\nu)}\frac{\partial}{\partial t}\int_0^t (t-s)^{-\nu} f(x,s)\mathrm{d}s, \qquad \nu\in(0,1),$$

and $f(x,\cdot)$ is a suitable well-behaved function (see [22] for details about the functional setting).

We start our analysis from a simple ansatz: equation (2.16) admits a solution in the form

$$u(x,t)=\frac{C_1}{t^\nu}-C_2\frac{x^2}{t^{3\nu}}, \qquad \nu\in(0,1/3)\setminus\{1/4\},\ (x,t)\in\mathbb{R}\times\mathbb{R}^+, \tag{2.17}$$

where C_1 and C_2 are real constants that we are going to find. We now directly check the correctness of this conjecture. We first recall that, for $\nu>0$ and $\beta>-1$,

$$\frac{\partial^\nu}{\partial t^\nu}t^\beta=\frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\nu)}t^{\beta-\nu}. \tag{2.18}$$

Then, by substituting (2.17) in (2.16) and by applying (2.18), we have

$$\frac{\Gamma(1-\nu)}{\Gamma(1-2\nu)}\frac{C_1}{t^{4\nu}}-\frac{\Gamma(1-3\nu)}{\Gamma(1-4\nu)}\frac{C_2\,x^2}{t^{6\nu}}-\frac{2C_2}{t^{4\nu}}+\frac{4C_2^2x^2}{t^{6\nu}}=0 \tag{2.19}$$

and balancing similar terms we have that

$$\begin{cases}\dfrac{\Gamma(1-\nu)}{\Gamma(1-2\nu)}C_1-2C_2=0,\\ 4C_2^2-\dfrac{\Gamma(1-3\nu)}{\Gamma(1-4\nu)}C_2=0.\end{cases} \tag{2.20}$$

Therefore

$$\begin{cases} C_1 = \dfrac{1}{2}\dfrac{\Gamma(1-3\nu)}{\Gamma(1-4\nu)}\dfrac{\Gamma(1-2\nu)}{\Gamma(1-\nu)}, \\ C_2 = \dfrac{1}{4}\dfrac{\Gamma(1-3\nu)}{\Gamma(1-4\nu)}. \end{cases} \tag{2.21}$$

Observe that the constraint on the real order of derivation $\nu \in \left(0, \frac{1}{3}\right)$ in (2.17) is due to the application of (2.18).

Moreover, $\nu \neq 1/4$ because of the coefficients appearing in the solution (depending on the Euler Gamma functions). We can conclude that equation (2.16) admits a solution of the form (2.17) if $\nu \in (0, 1/3) \setminus \{1/4\}$. Therefore, in particular, under the same constraints on ν, we can say that the time-fractional equation (2.16) admits a solution of the form

$$u(x,t) = \frac{C_1}{t^\nu}\left(1 - \frac{C_2}{C_1}\frac{x^2}{t^{2\nu}}\right)_+, \tag{2.22}$$

where C_1 and C_2 are given by (2.21).

Obviously, the solution (2.22) is not normalized but it is a Barenblatt-type solution belonging to the general family considered in this paper. A systematic study of equation (2.16) should be an object of further investigation, both from the physical and mathematical points of view. We conjecture that this is a source-type solution of the nonlinear time-fractional equation (2.16); nevertheless a full rigorous analysis should be developed, but this is beyond the aims of this paper. The fractional equation (2.16) can be viewed as a hybrid between a diffusive equation with singular time-dependent coefficients (in some way similar to the EPD equation) and a nonlinear time-fractional porous medium type equation.

3 Main results

Let us start with our first result concerning the case $d = 1$.

Proposition 1. *For $d = 1$, the density function (1.2) can be written as*

$$u(x,t) = \mathbf{E}_V\left[\frac{\delta(x - Vt^\alpha) + \delta(x + Vt^\alpha)}{2}\right], \tag{3.1}$$

where $\mathbf{E}_V[\cdot]$ stands for the mean value w.r.t. $V \overset{(law)}{=} cY^{1/\beta}$, where $Y \sim \mathrm{Beta}(\frac{1}{\beta}, \gamma+1)$.

Proof. Let $d = 1$. We have

$$\begin{aligned} \hat{u}(\xi,t) &= \mathcal{F}u(\xi,t) \\ &= Ct^{-\alpha}\int_{-ct^\alpha}^{ct^\alpha} e^{i\xi x}\left(1 - \left(\frac{|x|}{ct^\alpha}\right)^\beta\right)^\gamma \mathrm{d}x \\ &= 2Ct^{-\alpha}\int_0^{ct^\alpha} \cos(\xi x)\left(1 - \left(\frac{x}{ct^\alpha}\right)^\beta\right)^\gamma \mathrm{d}x \\ &= \frac{\beta/c}{\mathrm{Beta}(\frac{1}{\beta}, \gamma+1)}\int_0^c \cos(\xi v t^\alpha)\left(1 - (v/c)^\beta\right)^\gamma \mathrm{d}v \end{aligned}$$

$$= \mathbf{E}_V \left[\cos(\xi V t^\alpha)\right]$$
$$= \mathbf{E}_V \left[\frac{e^{i\xi V t^\alpha} + e^{-i\xi V t^\alpha}}{2}\right].$$

Hence, by Fubini's theorem we immediately obtain

$$u(x,t) = \mathcal{F}^{-1}\hat{u}(\xi,t) = \mathbf{E}_V \left[\frac{\delta(x - Vt^\alpha) + \delta(x + Vt^\alpha)}{2}\right],$$

where in the last step we used the result $\mathcal{F}^{-1}e^{\pm ia\xi} = \delta(x \mp a), a \in \mathbb{R}$ (see, e.g., [14]). □

Based on Proposition 1 we argue the following random model. Let D be a random variable uniformly distributed on $\{-1, 1\}$, which is independent from V. We deal with the stochastic process $X := (X(t))_{t\geq 0}$, where

$$X(t) := D\, V\, t^\alpha$$

represents the position, at time $t > 0$, of a particle starting from the origin of the real line, which initially chooses with the same probability to move leftward or rightward and performs a random displacement of length equal to Vt^α. Therefore V represents the random velocity of the particle which is initially fixed with the probability law f_V.

Corollary 1. *At time $t > 0$*

$$P(X(t) \in \mathrm{d}x) = \mathbf{E}_V \left[\frac{\delta(x - Vt^\alpha) + \delta(x + Vt^\alpha)}{2}\right] \mathrm{d}x$$

and the cumulative distribution function of $X(t)$ is given by

$$\mathbf{F}(x) = \frac{1}{2}\left[1 + sgn(x)\frac{\mathrm{Beta}(|x|/ct^\alpha; \frac{1}{\beta}, \gamma+1)}{\mathrm{Beta}(\frac{1}{\beta}, \gamma+1)}\right], \quad x \in [-ct^\alpha, ct^\alpha]. \tag{3.2}$$

Proof. Given $V = v$, one has that

$$P(X(t) \in A | V = v) = \int_A \nu_v(x,t)\mathrm{d}x, \quad A \in \mathcal{B}(\mathbb{R}),$$

where

$$\nu_v(x,t) := \frac{\delta(x - vt^\alpha) + \delta(x + vt^\alpha)}{2}$$

represents the singular probability measure of X_1 (w.r.t. the Lebesgue measure $\mathrm{d}x$), and then

$$\mu_t(\mathrm{d}x) = P(X(t) \in \mathrm{d}x) = \mathbf{E}_V\left[\nu_V(x,t)\right]\mathrm{d}x.$$

Some simple calculations and the result (1.4) lead to (3.2). □

The following corollary highlights the link between $u(x,t)$ and a model of nonlinear wave propagation.

Corollary 2. *Let $v > 0$. Then*

$$\nu_v(x,t) := \frac{\delta(x - vt^\alpha) + \delta(x + vt^\alpha)}{2}$$

is the fundamental solution to the hyperbolic EPD-type partial differential equation

$$\frac{\partial^2 u}{\partial t^2} + \frac{(1-\alpha)}{t}\frac{\partial u}{\partial t} = v^2\alpha^2 t^{2\alpha-2}\frac{\partial^2 u}{\partial x^2} \tag{3.3}$$

subject to the initial conditions $u(x,0) = \delta(x)$, $\frac{\partial u}{\partial t}(x,t)\big|_{t=0} = 0$.

Proof. We observe that $\frac{\delta(x-vs)+\delta(x+vs)}{2}$, $v > 0$, $s > 0$, is the fundamental solution to the wave equation

$$\frac{\partial^2 u}{\partial s^2} = v^2\frac{\partial^2 u}{\partial x^2},$$

subject to the initial condition $u(x,0) = \delta(x)$, $\frac{\partial u}{\partial s}(x,s)\big|_{s=0} = 0$. Therefore, the change of variable $s = t^\alpha$ and direct calculations permit to prove (3.3). □

Remark 3.1. The Erdélyi–Kober fractional integral is defined as (see, e.g., [28])

$$I_\eta^{\zeta,\mu} f(x) = \frac{\eta\, x^{-\eta(\mu+\zeta)}}{\Gamma(\mu)}\int_0^x \tau^{\eta(\zeta+1)-1}(x^\eta - \tau^\eta)^{\mu-1} f(\tau)\mathrm{d}\tau, \tag{3.4}$$

where $\mu > 0$, $\eta > 0$ and $\zeta \in \mathbb{R}$. Evidently, for $\zeta = 0$, $\eta = 1$, (3.4) reduces to the Riemann–Liouville integral with a power weight.

From Proposition 1 it is easy to check that the Fourier transform $\hat{u}(\xi,t)$ and $u(x,t)$ are Erdélyi–Kober integrals of the cosine function $f_\alpha(v;\xi,t) := \cos(\xi v t^\alpha)$ and $g_\alpha(v;x,t) := \frac{\delta(x-vt^\alpha)+\delta(x+vt^\alpha)}{2}$, respectively; i.e.,

$$\hat{u}(\xi,t) = \frac{\Gamma(\frac{1}{\beta}+\gamma+1)}{\Gamma(\frac{1}{\beta})} I_\beta^{\frac{1}{\beta}-1,\gamma+1} f_\alpha(c;\xi,t) \tag{3.5}$$

and

$$u(x,t) = \frac{\Gamma(\frac{1}{\beta}+\gamma+1)}{\Gamma(\frac{1}{\beta})} I_\beta^{\frac{1}{\beta}-1,\gamma+1} g_\alpha(c;x,t). \tag{3.6}$$

A recent interesting probabilistic interpretation of the Erdélyi–Kober integral is also discussed in [32].

Remark 3.2. The (centred) Wigner law is defined by the probability distribution

$$\mathfrak{m}(x,t) = \frac{1}{2\pi t}\sqrt{4t - x^2}, \quad |x| \le 2\sqrt{t}. \tag{3.7}$$

A simple calculation proves that the even moments are given by (scaled) Catalan numbers, that is,

$$\int_{-2\sqrt{t}}^{2\sqrt{t}} x^{2m}\mathfrak{m}(x,t)\mathrm{d}x = C_m t^m, \quad m \in \mathbb{N},$$

with $C_m = \frac{1}{m+1}\binom{2m}{m}$. The probability law (3.7) is the density function of the free Brownian motion $S := (S_t)_{t\geq 0}$, i.e. for $0 \leq t_1 < t_2 < \infty$, the law of $S_{t_2} - S_{t_1}$ is given by $\mathfrak{m}(x, t_2 - t_1)$ and $\mathbf{E}(S_{t_2} - S_{t_1}) = 0$, $\mathbf{E}(S_{t_2} - S_{t_1})^2 = t_2 - t_1$. For a, detailed introduction to the free probability and free Brownian motion the reader can consult, for instance, [34, 35] and [26].

By setting $d = 1, \alpha = 1/2, \beta = 2, \gamma = 1/2$ and $c = 2$, the function (1.2) coincides with $\mathfrak{m}(x,t)$. Furthermore, we observe that $\mathfrak{m} := \mathfrak{m}(x,t)$ is equal to the time-rescaled, with $t = s^2$, solution to the EPD equation (2.10), given by

$$\frac{\partial^2 u}{\partial s^2} + \frac{1}{s}\frac{\partial u}{\partial s} = 4\frac{\partial^2 u}{\partial s^2}, \quad s > 0. \tag{3.8}$$

Therefore some simple calculations allow to deduce

$$\frac{\partial^2 \mathfrak{m}}{\partial t^2} + \frac{1}{2t}\frac{\partial \mathfrak{m}}{\partial t} = \frac{1}{t}\frac{\partial^2 \mathfrak{m}}{\partial x^2}.$$

The study of the Barenblatt-type solutions for $d \geq 2$ leads to the following alternative representations of (1.2).

Proposition 2. *For $d \geq 2$, the probability density functions (1.2) have the representation*

$$u(x,t) = \frac{\mathrm{Beta}(\frac{1}{\beta}(\frac{d}{2}+1), \gamma+1)}{\sigma(\mathbb{S}^{d-1})(ct^\alpha\|x\|)^{\frac{d}{2}-1}\mathrm{Beta}(\frac{d}{\beta}, \gamma+1)}\mathbf{E}_Z\left[\delta(\|x\| - Zt^\alpha)\right] \tag{3.9}$$

where $\mathbf{E}_Z[\cdot]$ stands for the mean value w.r.t. $Z \overset{(law)}{=} cY_1^{1/\beta}$ where $Y_1 \sim \mathrm{Beta}(\frac{1}{\beta}(\frac{d}{2}+1), \gamma+1)$.

Proof. Let $d \geq 2$. Let σ be the measure on $\mathbb{S}^{d-1}$. We recall that (see (2.12), p. 690, [7]),

$$\int_{\mathbb{S}^{d-1}} e^{i\rho\xi\cdot\theta}\mathrm{d}\sigma(\theta) = (2\pi)^{d/2}\frac{J_{\frac{d}{2}-1}(\rho\|\xi\|)}{(\rho\|\xi\|)^{\frac{d}{2}-1}}. \tag{3.10}$$

One has that

$$\begin{aligned}\hat{u}(\xi,t) &= \mathcal{F}u(\xi,t)\\ &= C\int_0^{ct^\alpha}\rho^{d-1}t^{-\alpha d}\left(1-\left(\frac{\rho}{ct^\alpha}\right)^\beta\right)^\gamma \mathrm{d}\rho\int_{\mathbb{S}^{d-1}} e^{i\rho\xi\cdot\theta}\mathrm{d}\sigma(\theta)\\ &= \frac{t^{-\alpha d}C(2\pi)^{d/2}}{\|\xi\|^{\frac{d}{2}-1}}\int_0^{ct^\alpha}\rho^{\frac{d}{2}}\left(1-\left(\frac{\rho}{ct^\alpha}\right)^\beta\right)^\gamma J_{\frac{d}{2}-1}(\rho\|\xi\|)\mathrm{d}\rho\end{aligned}$$

$$= \left(\frac{2}{ct^\alpha\|\xi\|}\right)^{\frac{d}{2}-1} \frac{\Gamma(d/2)\beta/c}{\mathrm{Beta}(\frac{d}{\beta},\gamma+1)} \int_0^c (z/c)^{\frac{d}{2}}(1-(z/c)^\beta)^\gamma J_{\frac{d}{2}-1}\left(t^\alpha z\|\xi\|\right)\mathrm{d}z$$

$$= \left(\frac{2}{ct^\alpha\|\xi\|}\right)^{\frac{d}{2}-1} \frac{\Gamma(d/2)\mathrm{Beta}(\frac{1}{\beta}(\frac{d}{2}+1),\gamma+1)}{\mathrm{Beta}(\frac{d}{\beta},\gamma+1)}\mathbf{E}_Z\left[J_{\frac{d}{2}-1}\left(Zt^\alpha\|\xi\|\right)\right], \tag{3.11}$$

where $Z \stackrel{\text{(law)}}{=} cY^{1/\beta}$ is the random variable with the density function given by

$$f_Z(z) = \frac{\beta/c}{\mathrm{Beta}(\frac{1}{\beta}(\frac{d}{2}+1),\gamma+1)}(z/c)^{\frac{d}{2}}(1-(z/c)^\beta)^\gamma 1_{0<z<c}.$$

Hence, by Fubini's theorem we obtain

$$\begin{aligned}
u(x,t) &= \mathcal{F}^{-1}u(\xi,t)\\
&= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-ix\cdot\xi}u(\xi,t)\mathrm{d}\xi\\
&= \text{(by passing to spherical coordinates)}\\
&= \frac{1}{(2\pi)^d}\left(\frac{2}{ct^\alpha}\right)^{\frac{d}{2}-1}\frac{\Gamma(d/2)\beta/c}{\mathrm{Beta}(\frac{d}{\beta},\gamma+1)}\\
&\quad\times\int_0^c (z/c)^{\frac{d}{2}}(1-(z/c)^\beta)^\gamma \mathrm{d}z\\
&\quad\times\int_0^\infty \rho^{d/2}J_{\frac{d}{2}-1}\left(zt^\alpha\rho\right)\left(\int_{\mathbb{S}^{d-1}} e^{-i\rho x\cdot\theta}\mathrm{d}\sigma(\theta)\right)\mathrm{d}\rho\\
&= \text{(by explotinig the result (3.10))}\\
&= \frac{1}{2\pi^{d/2}}\frac{\Gamma(d/2)}{(ct^\alpha\|x\|)^{\frac{d}{2}-1}}\frac{\beta/c}{\mathrm{Beta}(\frac{d}{\beta},\gamma+1)}\\
&\quad\times\int_0^c (z/c)^{\frac{d}{2}}(1-(z/c)^\beta)^\gamma \mathrm{d}z\int_0^\infty \rho J_{\frac{d}{2}-1}\left(\rho t^\alpha u\right)J_{\frac{d}{2}-1}(\rho\|x\|)\mathrm{d}\rho\\
&= \frac{1}{2\pi^{d/2}}\frac{\Gamma(d/2)}{(ct^\alpha\|x\|)^{\frac{d}{2}-1}}\frac{\beta/c}{\mathrm{Beta}(\frac{d}{\beta},\gamma+1)}\\
&\quad\times\int_0^c (z/c)^{\frac{d}{2}}(1-(z/c)^\beta)^\gamma \delta(\|x\|-zt^\alpha)\mathrm{d}z\\
&= \text{(by Proposition 2, in [21])}\\
&= \frac{1}{2\pi^{d/2}}\frac{\Gamma(d/2)}{(ct^\alpha\|x\|)^{\frac{d}{2}-1}}\frac{\mathrm{Beta}(\frac{1}{\beta}(\frac{d}{2}+1),\gamma+1)}{\mathrm{Beta}(\frac{d}{\beta},\gamma+1)}\mathbf{E}_Z\left[\delta(\|x\|-Zt^\alpha)\right]
\end{aligned}$$

which coincides with (3.9). □

Remark 3.3. For $d = 2$, the representation (3.9) is particularly simple and reads as

$$u(x,t) = \frac{1}{2\pi}\mathbf{E}_Z\left[\delta(\|x\|-Zt^\alpha)\right].$$

Let

$$g_{uw}(x,t) := \mathcal{F}^{-1} e^{i\|\xi\|uwt} = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} e^{i\|\xi\|uwt} \mathrm{d}\xi_d.$$

We are able to provide an alternative representation of $u(x,t)$ with respect to (3.9).

Proposition 3. *For $d \geq 2$, the law (1.2) can be rewritten as*

$$u(x,t) = \mathbf{E}_{\mathfrak{U}\mathfrak{W}} \left[\frac{g_{\mathfrak{U}\mathfrak{W}}(x,t^\alpha) + g_{\mathfrak{U}\mathfrak{W}}(x,-t^\alpha)}{2} \right] \tag{3.12}$$

where $\mathfrak{U} \overset{(law)}{=} cY_2^{1/\beta}$, with $Y_2 \sim \mathrm{Beta}(\frac{d}{\beta}, \gamma+1)$, and $\mathfrak{W}$ admits the density function given by $f_{\mathfrak{W}}(w) = \frac{2}{\mathrm{Beta}(\frac{1}{2}, \frac{d-1}{2})}(1-w^2)_+^{\frac{d-1}{2}-1}$. Moreover $\mathfrak{U}$ and $\mathfrak{W}$ are independent.

Proof. From (3.11), one has that

$$\begin{aligned}
\hat{u}(\xi,t) &= \left(\frac{2}{ct^\alpha\|\xi\|}\right)^{\frac{d}{2}-1} \frac{\Gamma(d/2)\beta/c}{\mathrm{Beta}(\frac{d}{\beta},\gamma+1)} \int_0^c (z/c)^{\frac{d}{2}} (1-(z/c)^\beta)^\gamma J_{\frac{d}{2}-1}\left(t^\alpha z\|\xi\|\right) \mathrm{d}z \\
&= \frac{2\beta/c}{\mathrm{Beta}(\frac{d}{\beta},\gamma+1)\mathrm{Beta}(\frac{1}{2},\frac{d-1}{2})} \int_0^c (z/c)^{d-1}(1-(z/c)^\beta)^\gamma \mathrm{d}z \\
&\quad \times \int_0^1 (1-w^2)^{\frac{d-1}{2}-1} \cos(\|\xi\| zwt^\alpha) \mathrm{d}w \\
&= \mathbf{E}_{\mathfrak{U}\mathfrak{W}} \left[\cos(\|\xi\|\mathfrak{U}\mathfrak{W}t^\alpha)\right]
\end{aligned} \tag{3.13}$$

where we have used

$$J_\mu(z) = \frac{(z/2)^\mu}{\sqrt{\pi}\,\Gamma(\mu+\frac{1}{2})} \int_{-1}^{+1} (1-w^2)^{\mu-\frac{1}{2}} \cos(zw) \mathrm{d}w \tag{3.14}$$

valid for $\mu > -\frac{1}{2}$, $z \in \mathbb{R}$. Therefore, from (3.13) we get (3.12). Indeed,

$$\begin{aligned}
u(x,t) &= \mathcal{F}^{-1}\hat{u}(\xi,t) \\
&= \frac{1}{(2\pi)^d} \frac{2\beta/c}{\mathrm{Beta}(\frac{d}{\beta},\gamma+1)\mathrm{Beta}(\frac{1}{2},\frac{d-1}{2})} \int_0^c (z/c)^{d-1}(1-(z/c)^\beta)^\gamma \mathrm{d}z \\
&\quad \times \int_0^1 (1-w^2)^{\frac{d-1}{2}-1} \mathrm{d}w \int_{\mathbb{R}^d} e^{-ix\cdot\xi} \left[\frac{e^{i\|\xi\|zwt^\alpha} + e^{-i\|\xi\|zwt^\alpha}}{2}\right] \mathrm{d}\xi
\end{aligned}$$

which concludes the proof. □

Remark 3.4. We observe that:

- For $d=2$, the density function $f_{\mathfrak{W}}$ becomes the probability law of the square root of T, where

$$T := \mathrm{meas}\{t \in [0,1] : B(t) > 0\},$$

and $(B(t))_{t\geq 0}$ represents the standard one-dimensional Brownian motion. T leads to the well-known arcsin law of the Wiener process which is given by

$$\frac{1}{\pi\sqrt{w(1-w)}}1_{0<w<1}.$$

It is easy to prove that $\sqrt{T} \stackrel{\text{(law)}}{=} \mathfrak{W}$.

- For $d = 3$, the random variable $\mathfrak{W}$, $t > 0$, is uniformly distributed in $(0, 1)$.
- For $d \geq 4$, the density function $f_{\mathfrak{W}}$ represents a Wigner $(d-2)$-sphere law.

Remark 3.5. It is simple to prove by direct calculations that the mean squared displacement goes like $t^{2\alpha}$, recovering the entire range of behaviours from sub-diffusion ($\alpha < 1/2$) to super-diffusion ($\alpha > 1/2$); i.e.,

$$\int_{\mathbb{R}^d} \|x\|^2 u(x,t)\mathrm{d}x = \frac{\Gamma(\frac{1}{\beta}(d+2))\Gamma(\frac{d}{\beta}+\gamma+1)}{\Gamma(\frac{d}{\beta})\Gamma(\frac{1}{\beta}(d+2)+\gamma+1)}c^2 t^{2\alpha}.$$

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