Modern Stochastics: Theory and Applications 7 (2) (2020) 113–134
https://doi.org/10.15559/20-VMSTA152

A pure-jump mean-reverting short rate model

Markus Hess

Independent

Markus-Hess@gmx.net (M. Hess)

Received: 23 April 2019, Revised: 27 March 2020, Accepted: 27 March 2020,
Published online: 20 April 2020

Abstract A new multi-factor short rate model is presented which is bounded from below by a real-valued function of time. The mean-reverting short rate process is modeled by a sum of pure-jump Ornstein–Uhlenbeck processes such that the related bond prices possess affine representations. Also the dynamics of the associated instantaneous forward rate is provided and a condition is derived under which the model can be market-consistently calibrated. The analytical tractability of this model is illustrated by the derivation of an explicit plain vanilla option price formula. With view on practical applications, suitable probability distributions are proposed for the driving jump processes. The paper is concluded by presenting a post-crisis extension of the proposed short and forward rate model.

Keywords Short rate, forward rate, zero-coupon bond, option pricing, market-consistent calibration, post-crisis model, Lévy process, multi-factor model, Ornstein–Uhlenbeck process, stochastic differential equation

2010 MSC 91G30, 60G51, 60H10, 60H30, 91B30, 91B70

JEL classification G12, D52

1 Introduction

Stochastic interest rate models play an important role in the modeling of financial markets. The literature essentially distinguishes between short rate models, forward rate models and market models. In the sequel, we give a brief survey on the different classes of term structure models. For more detailed information, the reader is referred to the respective research articles or the textbooks [7, 21] and [26].

Widely applied short rate models are for example the Vasicek model [38], the Hull–White model [29] or the Cox–Ingersoll–Ross (CIR) model [10]. In [38] and

[29] the short rate process is modeled by a stochastic differential equation (SDE) of Ornstein–Uhlenbeck (OU) type driven by a Brownian motion (BM). As a consequence, the short rate process is normally distributed in these models and may become arbitrarily negative. Both features embody severe disadvantages with view on real-world market behavior, as the distribution of interest rate data frequently deviates from the normal distribution, while interest rates do not take arbitrarily large negative values in practice. In the recent years, there indeed appeared negative interest rates from time to time, but the negative values usually were small and stayed above some lower bound. However, in [10] the short rate is modeled by a so-called square-root process. This approach leads to a mean-reverting, strictly positive and chi-square distributed short rate process. In [6] the authors propose a time-homogeneous short rate model which is extended by a deterministic shift function in order to allow for negative rates and a perfect fit to the initially observed term structure. A very detailed overview on short rate models and their properties can be found in [7, 16] and [21]. In [26] and [33] short rate models in an extended multiple-curve framework are presented. The probably most famous forward rate model is the Heath–Jarrow–Morton (HJM) model proposed in [27]. Therein, the instantaneous forward rate process is modeled directly by an arithmetic SDE driven by a BM. In [4] the HJM model is extended to a jump-diffusion setup where the forward rate process is affected by both diffusion and random jump noise. HJM type models are also treated in [7] and Chapter 7 in [28]. In [12] and [26] HJM forward rate models in an extended multi-curve framework are discussed. The class of the so-called market models was introduced in [5]. For example, the popular LIBOR model belongs to this modeling class. In most cases, market models involve geometric SDEs such that the modeled interest rates usually turn out to be strictly positive. In order to allow the modeled rates also to take small negative values, shifted market model approaches have been proposed recently. Numerous properties of affine LIBOR models are provided in [31]. Market models are also presented in [7]. In [26] LIBOR models in an extended multi-curve framework are discussed. In [17] the authors propose a Lévy forward price model in a multi-curve setup which is able to generate negative interest rates. Term structure models which are driven by Lévy processes have also been proposed in [18–20].

In the present paper, we introduce a new pure-jump multi-factor short rate model which is bounded from below by a real-valued function of time which can be chosen arbitrarily. The short rate process is modeled by a deterministic function plus a sum of pure-jump zero-reverting Ornstein–Uhlenbeck processes. It turns out that the short rate is mean-reverting and that the related bond price formula possesses an affine representation. We also provide the dynamics of the related instantaneous forward rate, the latter being of HJM type. We further derive a condition under which the forward rate model can be market-consistently calibrated. The analytical tractability of our model is illustrated by the derivation of a plain-vanilla option price formula with Fourier transform methods. With view on practical applications, we make concrete assumptions on the distribution of the jump noises and show how explicit formulas can be deduced in these cases. We conclude the paper by presenting a post-crisis extension of our short and forward rate model.

The outline of the paper is as follows: In Section 2 we introduce our new pure-jump multi-factor short rate model which is bounded from below. Section 3 is dedicated to the derivation of related bond price and forward rate representations. Section 4 is devoted to option pricing. Section 5 contains guidelines for a practical application, while putting a special focus on possible distributional choices for the modeling of the involved jump noises. In Section 6 we consider a post-crisis extension of the proposed short and forward rate model.

2 A pure-jump multi-factor short rate model

Let $\left(\Omega, \mathbb{F}, \mathcal{F} = (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{Q}\right)$ be a filtered probability space satisfying the usual hypotheses, i.e. $\mathcal{F}_t = \mathcal{F}_{t+} := \cap_{s>t}\mathcal{F}_s$ constitutes a right-continuous filtration and $\mathbb{F}$ denotes the sigma-algebra augmented by all $\mathbb{Q}$-null sets (cf. [30, 35]). Here, $\mathbb{Q}$ is a risk-neutral probability measure and $T > 0$ denotes a fixed finite time horizon. In this setup, for arbitrary $n \in \mathbb{N}$ we define the stochastic short rate process $r = (r_t)_{t\in[0,T]}$ via

$$r_t := \mu(t) + \sum_{k=1}^{n} X_t^k \tag{2.1}$$

where $\mu(t)$ is a differentiable real-valued deterministic $\mathcal{L}^1$-function and X_t^k constitute pure-jump zero-reverting Ornstein–Uhlenbeck (OU) processes satisfying the SDE

$$dX_t^k = -\lambda_k X_t^k dt + \sigma_k dL_t^k \tag{2.2}$$

with deterministic initial values $X_0^k := x_k \geq 0$, constant mean-reversion velocities $\lambda_k > 0$ and constant volatility coefficients $\sigma_k > 0$. Herein, the independent, càdlàg, increasing, pure-jump, compound Poisson Lévy processes L_t^k are defined by

$$L_t^k := \int_0^t \int_{D_k} z dN_k(s, z) \tag{2.3}$$

where $D_k \subseteq \mathbb{R}^+ :=]0, \infty[\subset \mathbb{R}$ denote jump amplitude sets and N_k constitute Poisson random measures (PRMs). Note that the processes X_t^k and L_t^k always jump simultaneously, while X_t^k decays exponentially between its jumps due to the dampening linear drift term appearing in (2.2). A typical trajectory of a Lévy-driven OU process is shown in Figure 15.1 in [9]. Further note that the background-driving time-homogeneous Lévy processes L_t^k are increasing and thus, constitute so-called subordinators. Moreover, for all $k \in \{1, \dots, n\}$ and $(s, z) \in [0, T] \times D_k$ we define the $\mathbb{Q}$-compensated PRMs

$$d\tilde{N}_k^{\mathbb{Q}}(s, z) := dN_k(s, z) - d\nu_k(z)\, ds \tag{2.4}$$

which constitute $(\mathcal{F}, \mathbb{Q})$-martingale integrators. Herein, the positive and σ-finite Lévy measures ν_k satisfy the integrability conditions

$$\int_{D_k} (1 \wedge z)\, d\nu_k(z) < \infty, \qquad \int_{z>1} e^{\varpi z} d\nu_k(z) < \infty \tag{2.5}$$

for an arbitrary constant $\varpi \in \mathbb{R}$ (cf. [9, 17]). For all $k \in \{1, \dots, n\}$ and $t \in [0, T]$ we obtain

$$\mathbb{E}_{\mathbb{Q}}\left[L_t^k\right] = t \int_{D_k} z d\nu_k(z), \qquad \mathbb{Var}_{\mathbb{Q}}\left[L_t^k\right] = t \int_{D_k} z^2 d\nu_k(z)$$

both being finite entities due to (2.5) (cf. Section 1 in [17]). We remark that the currently proposed multi-factor short rate model (2.1) has been inspired by the electricity spot price model introduced in [2]. Arithmetic multi-factor models of this type have also been investigated in [28] and Section 3.2.2 in [3].

Remark 2.1. **(a)** Since L_t^k is increasing and X_t^k is zero-reverting from above, the function $\mu(t)$ is the mean-reversion floor or lower bound of the short rate process r_t, i.e. it holds $r_t \geq \mu(t)$ $\mathbb{Q}$-a.s. for all $t \in [0, T]$, while r_t is mean-reverting from above to $\mu(t)$. Also note that the presence of a Brownian motion (BM) as driving noise in one of the processes $X_t^1, \ldots, X_t^n$ would destroy the lower boundedness of r_t. In contrast to the presented pure-jump approach, it appears difficult to set up (lower-) bounded processes in arithmetic BM approaches. Moreover, we recall that negative rates have been observed in real-world post-crisis interest rate markets. Such scenarios can easily be captured by our model by choosing, e.g. $\mu(t) \equiv c$, where $c < 0$ is an arbitrary constant. (In practical applications, it may happen that the floor function $\mu(t)$ needs to be readjusted, if interest rates evolve lower than anticipated. This issue has been discussed in [1] in the context of the SABR model.)

(b) Our pure-jump model (2.1) is able to generate short rate trajectories which closely resemble those stemming from common Brownian motion approaches, if we allow for small jump sizes only, i.e. $D_k = \left[\epsilon_1^k, \epsilon_2^k\right]$ with small constants $0 < \epsilon_1^k < \epsilon_2^k$. In this context, we emphasize that the well-established pure-jump variance gamma model is likewise able to generate suitable price trajectories, although there is neither any diffusion component involved (cf. Section 2.6.3 in [3], Table 4.5 in [9], Section 5.3.7 in [37]). On top of that, our pure-jump model might even provide more flexibility concerning the modeling of distributional properties than common BM approaches, since we are able to implement tailor-made distributions via an appropriate choice of the Lévy measures ν_k which fit the empirical behavior of the rates in a best possible manner. This topic is further discussed in Section 5 below. For instance, (generalized) inverse Gaussian, tempered stable or gamma distributions might embody suitable choices (recall Appendix B.1.2 on p. 151 in [37]). We finally recall that a model of the type (2.1) has been fitted to real market data in [2] (yet in an electricity market context).

For a time partition $0 \leq t \leq s \leq T$ the solution of (2.2) under $\mathbb{Q}$ can be expressed as

$$X_s^k = X_t^k e^{-\lambda_k (s-t)} + \sigma_k \int_t^s \int_{D_k} e^{-\lambda_k (s-u)} z dN_k(u, z) \tag{2.6}$$

where we used (2.3). The representation (2.6) implies

$$X_t^k = x_k e^{-\lambda_k t} + \sigma_k \int_0^t \int_{D_k} e^{-\lambda_k (t-s)} z dN_k(s, z) \tag{2.7}$$

where $0 \leq t \leq T$. For all $t \in [0, T]$ we next define the historical filtration

$$\mathcal{F}_t := \sigma\{L_s^1, \ldots, L_s^n : 0 \leq s \leq t\}.$$

Proposition 2.2. *For* $0 \leq u \leq t \leq T$ *we have*

$$\mathbb{E}_{\mathbb{Q}}(r_t | \mathcal{F}_u) = \mu(t) + \sum_{k=1}^n \left(X_u^k e^{-\lambda_k (t-u)} + \sigma_k \frac{1 - e^{-\lambda_k (t-u)}}{\lambda_k} \int_{D_k} z d\nu_k(z) \right),$$

$$\mathbb{V}\mathrm{ar}_{\mathbb{Q}}\left(r_t|\mathcal{F}_u\right)=\sum_{k=1}^{n}\sigma_k^2\frac{1-e^{-2\lambda_k(t-u)}}{2\lambda_k}\int_{D_k}z^2d\nu_k\left(z\right)$$

where the short rate process r_t satisfies (2.1). Both entities are finite due to (2.5).

(Here and in what follows, we omit all proofs which are straightforward.) Taking $u=0$ in Proposition 2.2, we find for all $t\in[0,T]$

$$\mathbb{E}_{\mathbb{Q}}\left[r_t\right]=\mu\left(t\right)+\sum_{k=1}^{n}\left(x_ke^{-\lambda_kt}+\sigma_k\frac{1-e^{-\lambda_kt}}{\lambda_k}\int_{D_k}zd\nu_k\left(z\right)\right),$$
$$\mathbb{V}\mathrm{ar}_{\mathbb{Q}}\left[r_t\right]=\sum_{k=1}^{n}\sigma_k^2\frac{1-e^{-2\lambda_kt}}{2\lambda_k}\int_{D_k}z^2d\nu_k\left(z\right).$$

Note that it is possible to identify the entities $\mathbb{E}_{\mathbb{Q}}\left[X_t^k\right]$ and $\mathbb{V}\mathrm{ar}_{\mathbb{Q}}\left[X_t^k\right]$ inside the latter equations due to (2.1). Moreover, suppose that $\mu\left(t\right)\to\tilde{\mu}$ for $t\to\infty$ where $\tilde{\mu}\in\mathbb{R}$ is a finite constant. Then we observe

$$\lim_{t\to\infty}\mathbb{E}_{\mathbb{Q}}\left[r_t\right]=\tilde{\mu}+\sum_{k=1}^{n}\frac{\sigma_k}{\lambda_k}\int_{D_k}zd\nu_k\left(z\right),$$
$$\lim_{t\to\infty}\mathbb{V}\mathrm{ar}_{\mathbb{Q}}\left[r_t\right]=\sum_{k=1}^{n}\frac{\sigma_k^2}{2\lambda_k}\int_{D_k}z^2d\nu_k\left(z\right)$$

which both constitute finite constants. This limit behavior entirely stands in line with the requirements imposed on short rate models claimed on p. 46 in [7]. In the next step, we investigate the characteristic function of r_t which is defined via

$$\Phi_{r_t}\left(u\right):=\mathbb{E}_{\mathbb{Q}}\left[e^{iur_t}\right]$$

where $u\in\mathbb{R}$ and $t\in[0,T]$.

Proposition 2.3. *For $k\in\{1,\dots,n\}$ we define the deterministic functions*

$$\Lambda_k\left(s,z\right):=\sigma_ke^{-\lambda_k(t-s)}z,\qquad\psi_k\left(t,u\right):=iue^{-\lambda_kt},$$
$$\rho_k\left(t,u\right):=\int_0^t\int_{D_k}\left[e^{iu\Lambda_k(s,z)}-1\right]d\nu_k\left(z\right)ds.$$

Then for any $u\in\mathbb{R}$ and $t\in[0,T]$ the characteristic function of r_t can be decomposed as

$$\Phi_{r_t}\left(u\right)=e^{iu\mu(t)}\prod_{k=1}^{n}\Phi_{X_t^k}\left(u\right)$$

where the characteristic function of X_t^k is given by

$$\Phi_{X_t^k}\left(u\right)=e^{\psi_k(t,u)x_k+\rho_k(t,u)}$$

with deterministic and affine characteristic exponent.

An immediate consequence of Proposition 2.3 is the subsequent affine representation

$$\Phi_{r_t}(u) = \prod_{k=1}^{n} e^{\psi_k(t,u)x_k+\phi_k(t,u)} \tag{2.8}$$

where we introduced the deterministic functions

$$\phi_k(t,u) := \rho_k(t,u) + iu\mu(t)/n.$$

We emphasize that r_t is an affine function of the factors $X_t^1, \ldots, X_t^n$ such that our model turns out to be a special case of the affine short rate models considered in Section 3.3 in [14]. To read more on affine processes we refer to [7, 14, 16, 26] and [31]. We next define the moment generating function of r_t via

$$\kappa_{r_t}(v) := \mathbb{E}_{\mathbb{Q}}\left[e^{vr_t}\right] \tag{2.9}$$

which implies the well-known equalities $\Phi_{r_t}(u) = \kappa_{r_t}(iu)$ and $\kappa_{r_t}(v) = \Phi_{r_t}(-iv)$. Note that the moment generating function $\kappa_{r_t}(v)$ is well-defined due to (2.5). In the sequel, we derive the time dynamics of the short rate process.

Proposition 2.4. *For all $t \in [0,T]$ the short rate process follows the dynamics*

$$dr_t = \left(\mu'(t) - \sum_{k=1}^{n}\lambda_k X_t^k\right)dt + \sum_{k=1}^{n}\sigma_k\int_{D_k} z dN_k(t,z). \tag{2.10}$$

Remark 2.5. We recall that our model constitutes an extension of the short rate model proposed in [6], whereas we work with multiple pure-jump processes $L_t^1, \ldots, L_t^n$ as driving noises instead of the single Brownian motion W_t appearing in Eq. (1) in [6]. Moreover, comparing Eq. (3) in [6] with Eq. (2.1) above, we see that x_t and $\varphi(t;\alpha)$ in [6] correspond in our setup to $\sum_{k=1}^{n} X_t^k$ and $\mu(t)$, respectively.

3 Bond prices and forward rates

In this section, we derive representations for zero-coupon bond prices, forward rates and the interest rate curve related to the short rate model introduced in Section 2. To begin with, we introduce a bank account with stochastic interest rate r_t satisfying

$$d\beta_t = r_t\beta_t dt \tag{3.1}$$

with normalized initial capital $\beta_0 = 1$. The solution of (3.1) reads as

$$\beta_t = \exp\left\{\int_0^t r_s ds\right\} \tag{3.2}$$

where $t \in [0,T]$. In this setup, the (zero-coupon) bond price at time $t \leq T$ with maturity T is given by

$$P(t,T) := \beta_t \mathbb{E}_{\mathbb{Q}}\left(\beta_T^{-1}|\mathcal{F}_t\right) = \mathbb{E}_{\mathbb{Q}}\left(\exp\left\{-\int_t^T r_s ds\right\}\Big|\mathcal{F}_t\right) \tag{3.3}$$

where $t \in [0, T]$ (cf. [6, 7, 26]). Note that $P(t, T) > 0$ $\mathbb{Q}$-a.s. $\forall\, t \in [0, T]$ by construction. Since $r_t \geq \mu(t)$ $\mathbb{Q}$-a.s. $\forall\, t \in [0, T]$ [recall Remark 2.1 (a)], we observe

$$P(t, T) \leq M_{t,T} := \exp\left\{-\int_t^T \mu(s)\, ds\right\} \tag{3.4}$$

$\mathbb{Q}$-a.s. $\forall\, t \in [0, T]$ due to (3.3) and the monotonicity of conditional expectations. The upper bound $M_{t,T}$ appearing in (3.4) is deterministic and strictly positive for all $0 \leq t \leq T$. If $\mu(t) \geq 0$, then it holds $P(t, T) \leq 1$ $\mathbb{Q}$-a.s. $\forall\, t \in [0, T]$ (similar to, e.g., the CIR model [10]; also see [7, 21]). On the other hand, if $\mu(t) < 0$, then we only know that $M_{t,T} > 1$.

Proposition 3.1. *For $k \in \{1, \ldots, n\}$ and $t \in [0, T]$ we define the deterministic functions*

$$\begin{aligned} A_k(t, T) &:= \int_t^T \left(-\frac{\mu(s)}{n} + \int_{D_k} \left[e^{\sigma_k B_k(s,T) z} - 1\right] d\nu_k(z)\right) ds, \\ B_k(t, T) &:= \frac{e^{-\lambda_k (T-t)} - 1}{\lambda_k} \leq 0. \end{aligned} \tag{3.5}$$

Then the bond price at time $t \leq T$ with maturity T possesses the affine representation

$$P(t, T) = \prod_{k=1}^{n} e^{A_k(t,T) + B_k(t,T) X_t^k} \tag{3.6}$$

where the factors X_t^k satisfy (2.7).

Proof. First of all, we put (2.6) into (2.1) and obtain

$$r_s = \mu(s) + \sum_{k=1}^{n} X_t^k e^{-\lambda_k (s-t)} + \sum_{k=1}^{n} \int_t^s \int_{D_k} \sigma_k e^{-\lambda_k (s-u)} z\, dN_k(u, z)$$

where $0 \leq t \leq s \leq T$. We next substitute the latter equation into (3.3), hereafter apply Fubini's theorem and identify the functions $B_k(\cdot, T)$. This procedure yields

$$\begin{aligned} P(t, T) = \exp&\left\{-\int_t^T \mu(s)\, ds + \sum_{k=1}^{n} B_k(t, T)\, X_t^k\right\} \\ &\times \mathbb{E}_{\mathbb{Q}}\left(\exp\left\{\sum_{k=1}^{n} \int_t^T \int_{D_k} \sigma_k B_k(s, T)\, z\, dN_k(s, z)\right\} \,\Bigg|\, \mathcal{F}_t\right). \end{aligned}$$

Taking the independent increment property of the ($\mathbb{Q}$-independent) Lévy processes $L^1, \ldots, L^n$ into account, we obtain

$$\mathbb{E}_{\mathbb{Q}}\left(\exp\left\{\sum_{k=1}^{n} \int_t^T \int_{D_k} \sigma_k B_k(s, T)\, z\, dN_k(s, z)\right\} \,\Bigg|\, \mathcal{F}_t\right)$$

$$= \prod_{k=1}^{n} \mathbb{E}_{\mathbb{Q}} \left[\exp \left\{ \int_t^T \int_{D_k} \sigma_k B_k (s, T) \, z dN_k (s, z) \right\} \right]$$

where $t \in [0, T]$. The usual expectations appearing here can be handled by the Lévy–Khinchin formula for additive processes (see, e.g., [9, 30, 36]) which leads us to

$$\mathbb{E}_{\mathbb{Q}} \left[\exp \left\{ \int_t^T \int_{D_k} \sigma_k B_k (s, T) \, z dN_k (s, z) \right\} \right]$$
$$= \exp \left\{ \int_t^T \int_{D_k} \left[e^{\sigma_k B_k (s,T) z} - 1 \right] d\nu_k (z) \, ds \right\}.$$

Putting the latter equations together and identifying the functions $A_k (\cdot, T)$, we end up with the asserted representation (3.6). □

Recall that the bond price in (3.6) is the product of exponential affine functions of the factors $X_t^1, \ldots, X_t^n$ (but not of r_t). Also note that for all $k \in \{1, \ldots, n\}$ and $t \in [0, T]$ it holds

$$A_k (t, t) = B_k (t, t) = 0. \tag{3.7}$$

We remark that the functions $B_k (t, T)$ in (3.5) possess the same structure as the corresponding ones in the Vasicek model (cf. [38], or [7, 16, 21]). For all $t \in [0, T]$ Eq. (3.6) can be rewritten as

$$P (t, T) = \exp \left\{ \sum_{k=1}^{n} \left[A_k (t, T) + B_k (t, T) \, X_t^k \right] \right\} \tag{3.8}$$

which implies $P (T, T) = 1$ due to (3.7). Moreover, from (3.5) we infer the time derivatives

$$A'_k (t, T) = \frac{\mu (t)}{n} - \int_{D_k} \left[e^{\sigma_k B_k (t,T) z} - 1 \right] d\nu_k (z), \qquad B'_k (t, T) = e^{-\lambda_k (T-t)} > 0 \tag{3.9}$$

where $A'_k := \partial_t A_k$ and $B'_k := \partial_t B_k$. Hence, the functions $B_k (t, T) \leq 0$ are strictly monotone increasing in t. Also note that the formulas found in (3.9) entirely stand in line with those claimed in (4.4)–(4.5) in [31]. From (3.5), (3.7) and (3.9) we deduce the following system of ordinary differential equations (ODEs)

$$A_k (t, T) = - \int_t^T A'_k (s, T) \, ds, \qquad B'_k (t, T) = 1 + \lambda_k B_k (t, T),$$
$$A_k (T, T) = B_k (T, T) = 0$$

where $t \in [0, T]$ and $k \in \{1, \ldots, n\}$. We are now prepared to derive the time dynamics of the bond price process $(P (t, T))_{t \in [0,T]}$.

Proposition 3.2. *For $k \in \{1, \ldots, n\}$, $t \in [0, T]$ and $z \in D_k$ we define the deterministic functions*

$$\zeta_k (t, T, z) := e^{\sigma_k B_k (t,T) z} - 1 \tag{3.10}$$

with $B_k(t,T)$ as in (3.5). Then the bond price satisfies the t-dynamics under $\mathbb{Q}$

$$\frac{dP(t,T)}{P(t-,T)} = r_t dt + \sum_{k=1}^{n} \int_{D_k} \zeta_k(t,T,z)\, d\tilde{N}_k^{\mathbb{Q}}(t,z). \tag{3.11}$$

Recall that it holds $\zeta_k(t,T,z) \leq 0$, since $\sigma_k\ B_k(t,T)\ z \leq 0$ for all k, t and z. We stress that (3.11) possesses the same structure as the corresponding Eq. (10.9) in [37], whereas the latter stems from a Brownian motion model without jumps. In the next step, we provide the solution of the SDE (3.11).

Proposition 3.3. *For all $t \in [0,T]$ the solution of (3.11) under $\mathbb{Q}$ reads as*

$$\begin{aligned} P(t,T) = P(0,T)\exp\Bigg\{ &\int_0^t r_s ds - \sum_{k=1}^{n}\int_0^t \int_{D_k} \zeta_k(s,T,z)\, d\nu_k(z)\, ds \\ &+ \sum_{k=1}^{n}\int_0^t \int_{D_k} \sigma_k B_k(s,T)\, z dN_k(s,z)\Bigg\} \end{aligned} \tag{3.12}$$

where the initial value $P(0,T)$ is deterministic and fulfills $P(0,T) > 0$.

Furthermore, for all $t \in [0,T]$ let us introduce the discontinuous Doléans-Dade exponential

$$\begin{aligned} \Xi_t^k := \mathcal{E}\big(h_k * \tilde{N}_k^{\mathbb{Q}}\big)_t := \exp\Bigg\{ &\int_0^t \int_{D_k} h_k(s,z)\, d\tilde{N}_k^{\mathbb{Q}}(s,z) \\ &- \int_0^t \int_{D_k} \big[e^{h_k(s,z)} - 1 - h_k(s,z)\big] d\nu_k(z)\, ds \Bigg\} \end{aligned} \tag{3.13}$$

where $h_k(s,z)$ is an arbitrary integrable deterministic function (which may also depend on T). We recall that $\Xi_0^k = 1$ and that $\big(\Xi_t^k\big)_{t\in[0,T]}$ constitutes a local $\mathbb{Q}$-martingale. With definition (3.13) at hand, we can express Eq. (3.12) as follows.

Corollary 3.4. *For all $0 \leq t \leq T$ the bond price satisfies*

$$P(t,T) = P(0,T)\, \beta_t \prod_{k=1}^{n} \mathcal{E}\big(\xi_k * \tilde{N}_k^{\mathbb{Q}}\big)_t \tag{3.14}$$

where β_t is the bank account process given in (3.2), $\mathcal{E}$ denotes the Doléans-Dade exponential defined in (3.13) and $\xi_k(s,z) := \sigma_k\ B_k(s,T)\ z = \log(1 + \zeta_k(s,T,z))$ is a deterministic function.

Moreover, for all $0 \leq t \leq T$ we define the discounted bond price

$$\hat{P}(t,T) := \frac{P(t,T)}{\beta_t} \tag{3.15}$$

where $\hat{P}(0,T) = P(0,T)$. From (3.3) we deduce $\hat{P}(t,T) = \mathbb{E}_{\mathbb{Q}}(\beta_T^{-1} |\ \mathcal{F}_t)$ such that $\hat{P}(t,T)$ constitutes an $\mathcal{F}_t$-adapted (true) martingale under $\mathbb{Q}$, as required by the risk-neutral pricing theory. Plugging (3.14) into (3.15), for all $t \in [0,T]$ we obtain

$$\hat{P}(t,T) = P(0,T) \prod_{k=1}^{n} \mathcal{E}\big(\xi_k * \tilde{N}_k^{\mathbb{Q}}\big)_t \tag{3.16}$$

where $P(0,T)$ is deterministic and ξ_k is such as defined in Corollary 3.4. We obtain the following result.

Proposition 3.5. *For all $t \in [0,T]$ the discounted bond price satisfies the $\mathbb{Q}$-martingale dynamics*

$$\frac{d\hat{P}(t,T)}{\hat{P}(t-,T)} = \sum_{k=1}^{n} \int_{D_k} \zeta_k(t,T,z)\, d\tilde{N}_k^{\mathbb{Q}}(t,z)$$

where the deterministic functions $\zeta_k(t,T,z)$ are such as defined in (3.10).

With reference to [7], we define the instantaneous forward rate at time t with maturity T via

$$f(t,T) := -\partial_T \log P(t,T) \tag{3.17}$$

where $t \in [0,T]$ and ∂_T denotes the differential operator with respect to T. Equation (3.17) is equivalent to the representation

$$P(t,T) = \exp\left\{-\int_t^T f(t,u)\, du\right\}. \tag{3.18}$$

Lemma 3.6. *For all $k \in \{1,\dots,n\}$ and $t \in [0,T]$ it holds*

$$\partial_T A_k(t,T) = -\frac{\mu(T)}{n} - \sigma_k \int_t^T \int_{D_k} z e^{\sigma_k B_k(s,T)z - \lambda_k(T-s)} d\nu_k(z)\, ds,$$
$$\partial_T B_k(t,T) = -e^{-\lambda_k(T-t)}.$$

Proof. By the definition of $B_k(t,T)$ claimed in (3.5) we find

$$\partial_T B_k(t,T) = -e^{-\lambda_k(T-t)} \tag{3.19}$$

so that the functions $B_k(t,T)$ are strictly monotone decreasing in T. From (3.5) and (3.10) we further deduce

$$\partial_T A_k(t,T) = -\frac{\mu(T)}{n} + \partial_T \left(\int_t^T \int_{D_k} \zeta_k(s,T,z)\, d\nu_k(z)\, ds \right)$$

whereas Fubini's theorem (see, e.g., Theorem 2.2 in [3]) leads us to

$$\partial_T \left(\int_t^T \int_{D_k} \zeta_k(s,T,z)\, d\nu_k(z)\, ds \right) = \int_{D_k} \partial_T \left(\int_t^T \zeta_k(s,T,z)\, ds \right) d\nu_k(z).$$

(We are able to apply Fubini's theorem here, since the deterministic function $\zeta_k(s,T,z)$ is measurable and square-integrable with respect to $s \in [0,T]$ and $z \in D_k$.) The Leibniz formula for parameter integrals (see Lemma 2.4.1 on p. 13 in [28]) yields

$$\begin{aligned} \partial_T \left(\int_t^T \zeta_k(s,T,z)\, ds \right) &= \zeta_k(T,T,z) + \int_t^T \partial_T \zeta_k(s,T,z)\, ds \\ &= -\sigma_k \int_t^T z e^{\sigma_k B_k(s,T)z - \lambda_k(T-s)} ds \end{aligned}$$

where we used (3.10), (3.7) and (3.19). Putting these formulas together, the proof is complete. □

Proposition 3.7. *For all $t \in [0, T]$ the instantaneous forward rate can be represented as*

$$f(t, T) = \mu(T) + \sum_{k=1}^{n} \int_t^T \int_{D_k} \sigma_k z e^{\sigma_k B_k(s,T) z - \lambda_k (T-s)} d\nu_k(z)\, ds + \sum_{k=1}^{n} X_t^k e^{-\lambda_k (T-t)} \tag{3.20}$$

where the factor processes X_t^k satisfy (2.7) and $B_k(s, T)$ is like defined in (3.5).

Proof. We substitute (3.8) into (3.17) and obtain

$$f(t, T) = -\sum_{k=1}^{n} \left[\partial_T A_k(t, T) + X_t^k \partial_T B_k(t, T) \right].$$

Combining this equality with Lemma 3.6, we derive the claimed representation (3.20). □

Replacing T by t in (3.20), we immediately find $f(t, t) = r_t$ due to (2.1). This equality stands in line with the usual conventions in interest rate theory (see, e.g., [7, 16, 21]).

Proposition 3.8. *For all $t \in [0, T]$ the instantaneous forward rate fulfills the pure-jump multi-factor HJM type equation*

$$f(t, T) = f(0, T) + \sum_{k=1}^{n} \int_0^t \int_{D_k} \sigma_k z e^{-\lambda_k (T-s)} \left\{ dN_k(s, z) - e^{\sigma_k B_k(s,T) z} d\nu_k(z)\, ds \right\} \tag{3.21}$$

where the deterministic initial value is given by $f(0, T) = -\partial_T \log P(0, T)$.

In what follows, we illustrate how our forward rate model can be fitted to the initially observed term structure. This procedure is often called *market-consistent calibration* in the literature. For this purpose, we denote by $f^M(0, T)$ the deterministic initial forward rate. If $f(0, T) = f^M(0, T)$ and hence, if $P(0, T) = P^M(0, T)$ for all maturity times $T > 0$, then the underlying model is called *market-consistent*.

Proposition 3.9. *The forward rate model (3.20)–(3.21) can be market-consistently calibrated to a given term structure $f^M(0, T)$ by choosing the floor function $\mu(\cdot)$ in (3.20) according to*

$$\mu(T) = f^M(0, T) - \sum_{k=1}^{n} \left(x_k e^{-\lambda_k T} + \int_0^T \int_{D_k} \sigma_k z e^{\sigma_k B_k(s,T) z - \lambda_k (T-s)} d\nu_k(z)\, ds \right) \tag{3.22}$$

for all maturity times $T > 0$.

Note that the floor function $\mu(t)$ for all $t \in [0, T]$ can be obtained from (3.22) by replacing T by t therein. Moreover, we define the interest rate curve at time $t < T$ with maturity T via

$$R(t, T) := \frac{\log P(t, T)}{t - T}. \tag{3.23}$$

This object is called continuously-compounded spot rate on p. 60 in [7]. It obviously holds

$$P(t,T) = e^{-(T-t)R(t,T)} \tag{3.24}$$

where $t \in [0, T[$. Comparing the exponent in (3.24) with that in (3.8), we infer

$$R(t,T) = \frac{1}{t-T}\sum_{k=1}^{n}\left[A_k(t,T) + B_k(t,T)\,X_t^k\right]$$

where A_k and B_k are such as defined in (3.5). Hence, it turns out that the interest rate curve $R(t,T)$ can be represented as a sum of affine functions of the pure-jump OU factors $X_t^1, \ldots, X_t^n$. In this sense, our short rate model possesses an *affine term structure* (cf. Section 3.2.4 in [7], or [14, 16, 21]). The latter observation entirely stands in line with (3.8).

Proposition 3.10. *For all $t \in [0, T[$ the interest rate curve possesses the representation*

$$R(t,T) = \frac{1}{t-T}\left(\log P(0,T) + \int_0^t r_s ds - \sum_{k=1}^{n}\int_0^t\int_{D_k}\zeta_k(s,T,z)\,d\nu_k(z)\,ds \right.$$
$$\left. + \sum_{k=1}^{n}\int_0^t\int_{D_k}\sigma_k B_k(s,T)\,z dN_k(s,z)\right)$$

where ζ_k and B_k are such as defined in (3.10) and (3.5), respectively.

Proposition 3.11. *For all $t \in [0, T]$ the integrated short rate process can be represented as*

$$I_t := \int_0^t r_s ds = \int_0^t \mu(s)\,ds - \sum_{k=1}^{n} x_k B_k(0,t) - \sum_{k=1}^{n}\int_0^t\int_{D_k}\sigma_k B_k(s,t)\,z dN_k(s,z) \tag{3.25}$$

where the deterministic functions B_k are such as defined in (3.5).

Proof. We substitute (2.1) and (2.7) into the definition of I_t and obtain

$$I_t = \int_0^t \mu(s)\,ds - \sum_{k=1}^{n} x_k B_k(0,t) + \sum_{k=1}^{n}\int_0^t\int_0^s\int_{D_k}\sigma_k e^{-\lambda_k(s-u)} z dN_k(u,z)\,ds$$

where B_k is like defined in (3.5). An application of Fubini's theorem (see Theorem 2.2 in [3]) yields

$$\int_0^t\int_0^s\int_{D_k}\sigma_k e^{-\lambda_k(s-u)} z dN_k(u,z)\,ds = -\int_0^t\int_{D_k}\sigma_k B_k(u,t)\,z dN_k(u,z),$$

so that the proof is complete. □

Recall that the last jump integral in (3.25) constitutes a so-called Volterra integral, as the time parameter t appears both inside the integrand and inside the upper integration bound. Also note that it holds $I_t = \log\beta_t$ with $I_0 = 0$ due to (3.2).

4 Option pricing

In this section, we investigate the evaluation of a plain vanilla option written on the zero-coupon bond price $P(\cdot, T)$. With reference to the risk-neutral pricing theory, the price at time $t \leq \tau$ of an option with payoff H_τ at the maturity time τ reads as

$$C_t = \beta_t \mathbb{E}_{\mathbb{Q}}\big(\beta_\tau^{-1} H_\tau | \mathcal{F}_t\big) = \mathbb{E}_{\mathbb{Q}}\big(e^{-\int_t^\tau r_s ds} H_\tau | \mathcal{F}_t\big) \tag{4.1}$$

where β is the bank account process given in (3.2) and $\mathbb{Q}$ denotes a risk-neutral pricing measure (cf. Eq. (3.1) in [7]). We now consider a call option written on the bond price $P(\cdot, T)$ with maturity time T satisfying $T \geq \tau$. The payoff of the call option written on $P(\tau, T)$ with deterministic strike price $K > 0$ and maturity time τ is then given by

$$H_\tau = [P(\tau, T) - K]^+ := \max\{0, P(\tau, T) - K\}. \tag{4.2}$$

In what follows, we define the Fourier transform, respectively inverse Fourier transform, of a real-valued deterministic function $p(\cdot) \in \mathcal{L}^1(\mathbb{R})$ via

$$\hat{p}(y) := \frac{1}{2\pi}\int_{\mathbb{R}} p(u)\, e^{-iyu} du,\ p(u) = \int_{\mathbb{R}} \hat{p}(y)\, e^{iyu} dy.$$

Proposition 4.1. *[call option on bond price] For all $0 \leq t \leq \tau \leq T$ the price of a call option with payoff H_τ given in (4.2), strike price $K > 0$ and maturity time τ can be expressed as*

$$\begin{aligned} C_t = \int_{\mathbb{R}} \hat{q}(y) \exp\Bigg\{ & I_t + \theta(\tau, y) + \sum_{k=1}^n \overline{\psi}_k(t, \tau, y) \\ & + \sum_{k=1}^n \int_0^t \int_{D_k} \eta_k(s, z, y)\, dN_k(s, z)\Bigg\} dy \end{aligned} \tag{4.3}$$

where the integrated short rate process I_t is such as defined in (3.25) and

$$\begin{aligned} \eta_k(s, z, y) &:= \eta_k(s, z, T, \tau, y) \\ &:= \sigma_k\left[(a + iy) B_k(s, T) - (a + iy - 1) B_k(s, \tau)\right] z, \\ \hat{q}(y) &:= \frac{P(0, T)^{a+iy}}{2\pi (a + iy)(a + iy - 1) K^{a+iy-1}}, \\ \overline{\psi}_k(t, \tau, y) &:= \int_t^\tau \int_{D_k} \left[e^{\eta_k(s,z,y)} - 1\right] d\nu_k(z)\, ds, \\ \theta(\tau, y) &:= (a + iy - 1)\left(\int_0^\tau \mu(s)\, ds - \sum_{k=1}^n x_k B_k(0, \tau)\right) \\ &\quad - (a + iy) \sum_{k=1}^n \int_0^\tau \int_{D_k} \zeta_k(s, T, z)\, d\nu_k(z)\, ds \end{aligned} \tag{4.4}$$

constitute deterministic functions, while $a > 1$ is an arbitrary real-valued constant. Herein, the functions ζ_k and B_k are such as defined in (3.10) and (3.5), respectively, while $P(0, T)$ denotes the deterministic initial bond price.

Proof. We substitute (4.2) and (3.12) into (4.1) and obtain

$$C_t = \mathbb{E}_{\mathbb{Q}}\left(e^{I_t - I_\tau}\left[P\left(0, T\right) e^{G_\tau} - K\right]^+ | \mathcal{F}_t\right)$$

where I_t denotes the integrated short rate process defined in (3.25) and

$$G_\tau := I_\tau - \sum_{k=1}^{n} \int_0^\tau \int_{D_k} \zeta_k\left(s, T, z\right) d\nu_k\left(z\right) ds + \sum_{k=1}^{n} \int_0^\tau \int_{D_k} \sigma_k B_k\left(s, T\right) z dN_k\left(s, z\right)$$

is a real-valued stochastic process. For $u \in \mathbb{R}$ we introduce the deterministic function

$$q\left(u\right) := e^{-au}\left[P\left(0, T\right) e^u - K\right]^+$$

where $a > 1$ is a constant real-valued dampening parameter ensuring the integrability of the payoff function. Indeed, it holds $q\left(\cdot\right) \in \mathcal{L}^1\left(\mathbb{R}\right)$. With the latter definition at hand, we obtain

$$C_t = \mathbb{E}_{\mathbb{Q}}\left(e^{I_t - I_\tau + aG_\tau} q\left(G_\tau\right) | \mathcal{F}_t\right).$$

With reference to [8] (also see [18]), we apply the inverse Fourier transform on the latter equation and hereafter, use Fubini's theorem which leads us to

$$C_t = \int_{\mathbb{R}} \hat{q}(y) \mathbb{E}_{\mathbb{Q}}\left(e^{Z_{t,\tau}} | \mathcal{F}_t\right) dy$$

where we have set

$$Z_{t,\tau} := I_t - I_\tau + (a + iy)\, G_\tau$$

for all $0 \le t \le \tau$. By merging the definition of G_τ and (3.25) into the definition of $Z_{t,\tau}$ we deduce

$$Z_{t,\tau} = I_t + \theta\left(\tau, y\right) + \sum_{k=1}^{n} \int_0^\tau \int_{D_k} \eta_k\left(s, z, y\right) dN_k\left(s, z\right)$$

where we identified the deterministic functions $\theta\left(\tau, y\right)$ and $\eta_k\left(s, z, y\right)$ defined in (4.4). Hence,

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}\left(e^{Z_{t,\tau}} | \mathcal{F}_t\right) &= \exp\left\{I_t + \theta\left(\tau, y\right) + \sum_{k=1}^{n} \int_0^t \int_{D_k} \eta_k\left(s, z, y\right) dN_k\left(s, z\right)\right\} \\ &\quad \times \mathbb{E}_{\mathbb{Q}}\left[\exp\left\{\sum_{k=1}^{n} \int_t^\tau \int_{D_k} \eta_k\left(s, z, y\right) dN_k\left(s, z\right)\right\}\right] \end{aligned}$$

since I_t is $\mathcal{F}_t$-adapted and $\theta\left(\tau, y\right)$ is deterministic. In the derivation of the latter equation, we used the independent increment property under $\mathbb{Q}$ of the involved pure-jump integrals. We next apply the Lévy–Khinchin formula for additive processes (see, e.g., [9, 30, 36]) and derive

$$\mathbb{E}_{\mathbb{Q}}\left[\exp\left\{\sum_{k=1}^{n} \int_t^\tau \int_{D_k} \eta_k\left(s, z, y\right) dN_k\left(s, z\right)\right\}\right] = \exp\left\{\sum_{k=1}^{n} \overline{\psi}_k\left(t, \tau, y\right)\right\}$$

where the characteristic exponents $\overline{\psi}_k(t, \tau, y)$ are such as defined in (4.4). Putting the latter equations together, we eventually end up with (4.3). The expression for the Fourier transform $\hat{q}(y)$ is obtained by straightforward calculations using the definition of the function $q(u)$. □

Corollary 4.2. *In the special case $t = 0$, the call option price formula (4.3) simplifies to*

$$C_0 = \int_{\mathbb{R}} \hat{q}(y) \exp\left\{\theta(\tau, y) + \sum_{k=1}^{n} \overline{\psi}_k(0, \tau, y)\right\} dy$$

which is deterministic.

5 Practical applications

In this section, we show how the short rate model introduced in Section 2 can be implemented in practical applications. For this purpose, we now present more detailed expressions in order to prepare our model for a possible calibration of the involved parameters. First of all, let us recall that the increasing compound Poisson processes L_t^k defined in (2.3) for every $k \in \{1, \ldots, n\}$ and $t \in [0, T]$ can be expressed as

$$L_t^k = \sum_{j=1}^{N_t^k} Y_j^k \tag{5.1}$$

(cf. Section 5.3.2 in [37]) where N_t^k constitutes a standard Poisson process under $\mathbb{Q}$ with deterministic jump intensity $\alpha_k > 0$. That is, $N_t^k \sim Poi(\alpha_k t)$ such that for all $m \in \mathbb{N}_0$ it holds

$$\mathbb{Q}\left(N_t^k = m\right) = \frac{(\alpha_k t)^m}{m!} e^{-\alpha_k t}.$$

The strictly positive jump amplitudes of the Lévy process L_t^k are modeled by the i.i.d. random variables $Y_1^k, Y_2^k, \ldots$ which take values in the set $D_k \subseteq]0, \infty[$. We recall that the random variables $Y_1^k, Y_2^k, \ldots$ are independent of the Poisson processes $N_t^{\overline{k}}$ for all combinations of indices $k, \overline{k} \in \{1, \ldots, n\}$. We further put $c_k := \mathbb{E}_{\mathbb{Q}}[Y_1^k] \in D_k$ and recall that the compensated compound Poisson process $\left(L_t^k - c_k \alpha_k t\right)_{t \in [0,T]}$ constitutes an $(\mathcal{F}_t, \mathbb{Q})$-martingale for each k which implies

$$c_k \alpha_k = \int_{D_k} z d\nu_k(z)$$

due to (2.3) and (2.4). We stress that the Poisson processes N_t^k shall not be mixed up with the Poisson random measures $dN_k(t, z)$.

In the following, we propose a number of probability distributions living on the positive half-line (recall Section B.1.2 in [37]) which constitute suitable candidates for the modeling of the jump size distribution in our new short rate model. As a first example, we propose to work with the *gamma distribution* and thus, assume that each random variable Y_j^k is exponentially distributed under $\mathbb{Q}$ with parameter $\varepsilon_k > 0$ for all j and k. In this case, the related Lévy measure possesses the Lebesgue density

$$d\nu_k(z) = \alpha_k \varepsilon_k e^{-\varepsilon_k z} dz \tag{5.2}$$

where $z \in D_k =]0, \infty[$ and $k \in \{1, \dots, n\}$. We find $c_k = 1/\varepsilon_k$ and $Y_j^k \sim \Gamma(1, \varepsilon_k)$. Hence, following the notation used in Section 5.5.1 in [37], we state that we presently are in a $\Gamma(\alpha_k, \varepsilon_k)$-Ornstein–Uhlenbeck process setup (also see Section 8.2 in [31] and Example 15.1 in [9] in this context).

Proposition 5.1. *Suppose that the random variables Y_j^k in (5.1) are exponentially distributed (i.e. $\Gamma(1, \varepsilon_k)$-distributed) under $\mathbb{Q}$ with parameters $\varepsilon_k > 0$ for all j and k. Then, for $u \in \mathbb{R}$ and $t \in [0, T]$ the characteristic function of L_t^k is given by*

$$\Phi_{L_t^k}(u) = \exp\left\{\frac{iu\alpha_k t}{\varepsilon_k - iu}\right\}$$

where α_k denotes the jump intensity of the standard Poisson process N_t^k appearing in (5.1).

Proof. Successively applying the definition of the characteristic function, (2.3), the Lévy–Khinchin formula and (5.2), for $u \in \mathbb{R}$ and $t \in [0, T]$ we obtain

$$\Phi_{L_t^k}(u) = \mathbb{E}_{\mathbb{Q}}\left[e^{iuL_t^k}\right] = \exp\left\{\alpha_k \varepsilon_k t \int_0^\infty \left[e^{iuz} - 1\right] e^{-\varepsilon_k z} dz\right\}.$$

We eventually perform the integration and end up with the asserted equality. □

An immediate consequence of Proposition 5.1 is the following representation for the moment generating function of L_t^k being valid for all $v \in \mathbb{R} \setminus \{\varepsilon_k\}$

$$\kappa_{L_t^k}(v) = \Phi_{L_t^k}(-iv) = \exp\left\{\frac{v\alpha_k t}{\varepsilon_k - v}\right\}.$$

Proposition 5.2. *Assume that the random variables Y_j^k in (5.1) are exponentially distributed (i.e. $\Gamma(1, \varepsilon_k)$-distributed) under $\mathbb{Q}$ with parameters $\varepsilon_k > 0$ for all j and k. Then, for all $t \in [0, T]$, $k \in \{1, \dots, n\}$ and $x \in \mathbb{R}$ the probability density function of L_t^k under $\mathbb{Q}$ takes the form*

$$f_{L_t^k}(x) = \frac{1}{2\pi} \int_0^\infty \exp\left\{iu\left(x - \frac{\alpha_k t}{\varepsilon_k + iu}\right)\right\} du.$$

Proof. First, note that it holds

$$\Phi_{L_t^k}(-u) = \int_0^\infty e^{-iux} f_{L_t^k}(x)\, dx = 2\pi \hat{f}_{L_t^k}(u)$$

due to the definitions of the characteristic function and the Fourier transform claimed in the sequel of (4.2). We next apply the inverse Fourier transform which yields the density function

$$f_{L_t^k}(x) = \frac{1}{2\pi} \int_0^\infty \Phi_{L_t^k}(-u)\, e^{iux} du.$$

We finally plug the result of Proposition 5.1 into the latter equation which completes the proof. □

We stress that Eq. (5.2) can be substituted into the corresponding formulas appearing in the previous Propositions 2.2, 2.3, 3.1, 3.3, 3.7–3.10 and 4.1 yielding more explicit expressions for the involved entities, yet associated with gamma-distributed jump amplitudes in the underlying short rate model. We illustrate this statement by an application of Eq. (5.2) on Proposition 2.3. The precise result reads as follows.

Proposition 5.3. *Suppose that the random variables Y_j^k in (5.1) are exponentially distributed (i.e. $\Gamma(1, \varepsilon_k)$-distributed) under $\mathbb{Q}$ with parameters $\varepsilon_k > 0$ for all j and k. Let $\sigma_k > 0$ be the constant volatility coefficients introduced in (2.2). Then, for all $t \in [0, T]$ and $v \in \mathbb{R}$ with $v < \min_k \{\varepsilon_k/\sigma_k\}$, $k \in \{1, \dots, n\}$, the moment generating function under $\mathbb{Q}$ of the short rate process r_t reads as*

$$\kappa_{r_t}(v) = \Phi_{r_t}(-iv) = \exp\left\{ v\mu(t) + \sum_{k=1}^{n} \rho_k(t, -iv) + \sum_{k=1}^{n} \psi_k(t, -iv)\, x_k \right\}$$

with deterministic functions

$$\psi_k(t, -iv) = ve^{-\lambda_k t}, \qquad \rho_k(t, -iv) = \frac{\alpha_k}{\lambda_k} \log \left| \frac{\varepsilon_k - v\sigma_k e^{-\lambda_k t}}{\varepsilon_k - v\sigma_k} \right|.$$

Proof. For each $k \in \{1, \dots, n\}$ we define the deterministic functions $b_k(s, v) := v\,\sigma_k\, e^{-\lambda_k(t-s)} - \varepsilon_k$ which satisfy $b_k(s, v) < 0$ whenever $s \in [0, t]$ and $v < \min_k \{\varepsilon_k/\sigma_k\}$. In this setting, we combine Eq. (5.2) with the definitions of ρ_k and Λ_k given in Proposition 2.3 and obtain

$$\rho_k(t, -iv) = -\alpha_k \int_0^t \frac{\varepsilon_k + b_k(s, v)}{b_k(s, v)} ds.$$

We perform the integration and obtain the formula for ρ_k claimed in the proposition. The representation for the moment generating function $\kappa_{r_t}(v)$ finally follows from Proposition 2.3. $\square$

Other distributional choices for the random variables Y_j^k modeling the jump amplitudes would be, for example, the inverse Gaussian distribution (see Section 5.5.2 in [37]), the generalized inverse Gaussian distribution (see Section 5.3.5 in [37]) or the tempered stable distribution (see Section 5.3.6 in [37]). The related formulas for the Lebesgue density of the Lévy measure $d\nu_k(z)$ corresponding to Eq. (5.2) can be found in [37].

Remark 5.4. We recall that the time-homogeneous compound Poisson processes L_t^k introduced in (2.3) can be simulated according to Algorithms 6.1 and 6.2 in [9]. Further, in our model it is easily possible to calculate the moments of X_t^k and r_t (see the sequel of Proposition 2.2) so that our model can be fitted to any yield curve observed in the market by using the *moment estimation method* described in Section 7.2.2 in [9]. This procedure is also called *moment matching*, as the underlying idea is to make the empirical moments match with the theoretical moments of the model by finding a suitable parameter set.

6 A post-crisis model extension

In this section, we propose a post-crisis extension of the pure-jump lower-bounded short rate model introduced in Section 2. (To read more on post-crisis interest rate models, the reader is referred to [11–14, 17, 22–26, 32–34].) Inspired by the modeling setups presented in [33] and Chapter 2 in [26], for all $t \in [0, T]$ we define the short rate spread under $\mathbb{Q}$ by the stochastic process

$$s_t := \mu^*(t) + \sum_{k=n+1}^{l} X_t^k$$

showing a similar structure as (2.1). Herein, $\mu^*(t) \geq 0$ constitutes an integrable real-valued deterministic function and the factors X_t^k satisfy the SDE (2.2), but presently for indices $k \in \{n+1, \ldots, l\}$ where $l \in \mathbb{N}$ with $l > n$. Note that it holds $s_t \geq \mu^*(t)$ $\mathbb{Q}$-a.s. for all $t \in [0, T]$ such that the short rate spread is bounded from below – similar to the short rate itself [recall Remark 2.1 (a)]. We interpret s_t as an *additive* spread and therefore set for all $t \in [0, T]$

$$\overline{r}_t := r_t + s_t \tag{6.1}$$

(cf. [12, 33]) where r_t denotes the short rate process and $\overline{r}_t$ is called fictitious short rate, similarly to [26]. With reference to p. 46 in [26], we recall that the short rate spread s_t not only incorporates credit risks, but also various other risks in the interbank sector which affect the evolution of the LIBOR rates. Let us moreover mention that the short rate r_t defined in (2.1) and the short rate spread s_t can be 'correlated' by allowing for (at least) one common factor in their respective definitions. More precisely, if the sum in the definition of s_t started running from $k = n$ (instead of $k = n + 1$), then the factor X_t^n would appear both in the definition of r_t and in the definition of s_t such that the two latter stochastic processes would no longer be independent.

We next substitute (2.1) as well as the definition of s_t into (6.1) and deduce

$$\overline{r}_t = \overline{\mu}(t) + \sum_{k=1}^{l} X_t^k \tag{6.2}$$

where we introduced the real-valued deterministic function $\overline{\mu}(t) := \mu(t) + \mu^*(t)$. It obviously holds $\overline{r}_t \geq \overline{\mu}(t)$ $\mathbb{Q}$-a.s. for all $t \in [0, T]$. In accordance to Section 3.4.1 in [14], Eq. (2.35) in [26] and Section 1 in [33], we define the fictitious bond price in our post-crisis short rate model via

$$\overline{P}(t, T) := \mathbb{E}_{\mathbb{Q}}\left(\exp\left\{-\int_t^T \overline{r}_u du\right\} \Big| \mathcal{F}_t\right) \tag{6.3}$$

where $t \in [0, T]$. The object $\overline{P}(t, T)$ is sometimes called artificial bond price in the literature, as it is not physically traded in the market. Evidently, $\overline{P}(T, T) = 1$. Comparing (6.2) with (2.1) and (6.3) with (3.3), we see that all our single-curve results presented in the previous sections carry over to the currently considered post-crisis case. More precisely, the entities $\mu(t)$, r_t, $P(t,T)$ and n emerging in the previous single-curve equations presently have to be replaced by $\overline{\mu}(t)$, $\overline{r}_t$, $\overline{P}(t,T)$ and l, respectively. Moreover, in the present case, we observe $\mathbb{Q}$-a.s. for all $t \in [0,T]$

$$0 < \overline{P}(t,T) \le \exp\left\{-\int_t^T \overline{\mu}(u)\,du\right\}$$

due to (6.3), the lower boundedness of $\overline{r}_t$ and the monotonicity of conditional expectations.

Proposition 6.1. *It holds $\mathbb{Q}$-a.s. for all $t \in [0,T]$*

$$\overline{P}(t,T) \le P(t,T) \tag{6.4}$$

where $\overline{P}(t,T)$ constitutes the bond price defined in (6.3) and $P(t,T)$ is given in (3.3).

Proof. Note that taking $\mu^*(t) \ge 0\ \forall\, t \in [0,T]$ implies $s_t \ge 0$ $\mathbb{Q}$-a.s. for all $t \in [0,T]$. In this case, we deduce $\overline{r}_t \ge r_t$ $\mathbb{Q}$-a.s. for all $t \in [0,T]$ due to (6.1). Hence, we find

$$\exp\left\{-\int_t^T \overline{r}_u du\right\} \le \exp\left\{-\int_t^T r_u du\right\}$$

$\mathbb{Q}$-a.s. for all $t \in [0,T]$. We next take conditional expectations with respect to $\mathcal{F}_t$ and $\mathbb{Q}$ in the latter inequality, hereafter apply the monotonicity of conditional expectations and finally identify (6.3) and (3.3) in the resulting inequality. □

The result obtained in Proposition 6.1 possesses the following economical interpretation: The inequality (6.4) shows that nontraded bonds are cheaper than their nonfictitious counterparts which are physically traded in the market. This feature appears economically reasonable and stands in accordance with the modeling assumptions and statements in [12], Section 2.1.3 in [26] and Section 2.1 in [33]. Moreover, combining (6.3) and (3.18), we obtain (parallel to [12])

$$\overline{P}(t,T) = \exp\left\{-\int_t^T \overline{f}(t,u)\,du\right\}$$

where $\overline{f}$ is sometimes called fictitious/artificial forward rate in the literature. It holds $\overline{f}(t,t) = \overline{r}_t$ for all $t \in [0,T]$. With reference to [11] and [12], for all $t \in [0,T]$ we introduce the forward rate spread via

$$g(t,T) := \overline{f}(t,T) - f(t,T)$$

so that we have not only set up a new pure-jump post-crisis *short* rate model, but simultaneously a new pure-jump post-crisis *forward* rate model of HJM-type in the current section. Recall that (6.4) is equivalent to $f(t,u) \le \overline{f}(t,u)$ $\mathbb{Q}$-a.s. for all $0 \le t \le u \le T$. From this, we conclude that $g(t,T) \ge 0$ $\mathbb{Q}$-a.s. for all $t \in [0,T]$. It further holds $g(t,t) = s_t$ for all $t \in [0,T]$ due to (6.1).

Furthermore, in the present post-crisis setting, for a time partition $0 \le t \le T_1 < T_2$ we define the (forward) overnight indexed swap (OIS) rate via

$$F(t,T_1,T_2) := \frac{1}{\delta}\left(\frac{P(t,T_1)}{P(t,T_2)} - 1\right)$$

(cf. Eq. (4.1) in [26]) where P denotes the zero-coupon bond price defined in (3.3) and $\delta := \delta(T_1, T_2)$ is the year fraction with expiry date T_1 and maturity date T_2. With reference to [12], for $0 \le t \le T_1 < T_2$ we define the forward LIBOR rate via

$$L(t, T_1, T_2) := \frac{1}{\delta}\left(\frac{\overline{P}(t, T_1)}{\overline{P}(t, T_2)} - 1\right)$$

where δ is the year fraction and $\overline{P}$ denotes the bond price introduced in (6.3). Note that the LIBOR rate $L(t, T_1, T_2)$ shall not be mixed up with the Lévy processes L_t^k defined in (2.3). In a pre-crisis single-curve approach, it holds $\overline{P}(t, T) = P(t, T)$ which implies $F(t, T_1, T_2) = L(t, T_1, T_2)$ $\mathbb{Q}$-a.s. for all t.

We are now prepared to derive the dynamics of the short rate spread s_t, the fictitious short rate $\overline{r}_t$, the bond price $\overline{P}(t, T)$, the forward rate $\overline{f}(t, T)$, the forward rate spread $g(t, T)$ and the LIBOR rate $L(t, T_1, T_2)$. The associated computations can be accomplished by similar techniques as presented in Sections 2 and 3 and thus, are not worked out explicitly. We provide as an example two results without proofs in the sequel. For all $t \in [0, T]$ it holds

$$\frac{d\overline{P}(t, T)}{\overline{P}(t-, T)} = \overline{r}_t dt + \sum_{k=1}^{l} \int_{D_k} \zeta_k(t, T, z)\, d\tilde{N}_k^{\mathbb{Q}}(t, z)$$

where the functions ζ_k are such as claimed in (3.10). We further obtain in the post-crisis case

$$L(t, T_1, T_2) = \frac{1}{\delta}\Bigg(\frac{\overline{P}(0, T_1)}{\overline{P}(0, T_2)} \times \prod_{k=1}^{l} \exp\Bigg\{\int_0^t \int_{D_k} \Psi_k(s, z)\, d\nu_k(z)\, ds + \int_0^t \int_{D_k} \Sigma_k(s, z)\, dN_k(s, z)\Bigg\} - 1\Bigg)$$

with deterministic functions

$$\Psi_k(s, z) := e^{\sigma_k B_k(s, T_2) z} - e^{\sigma_k B_k(s, T_1) z} < 0,$$
$$\Sigma_k(s, z) := \sigma_k \left[B_k(s, T_1) - B_k(s, T_2)\right] z > 0.$$

References

[1] Antonov, A., Konikov, M., Spector, M.: The free boundary SABR: natural extension to negative rates. Risk Magazine (September 2015).

[2] Benth, F., Kallsen, J., Meyer-Brandis, T.: A non-Gaussian Ornstein-Uhlenbeck Process for Electricity Spot Price Modeling and Derivatives Pricing. Appl. Math. Finance **14**(2), 153–169 (2006). MR2323278. https://doi.org/10.1080/13504860600725031

[3] Benth, F., Saltyte-Benth, J., Koekebakker, S.: Stochastic Modeling of Electricity and Related Markets, 1st edn. World Scientific, (2008). MR2416686. https://doi.org/10.1142/9789812812315

[4] Björk, T., Kabanov, Y., Runggaldier, W.: Bond market structure in the presence of marked point processes. Math. Finance **7**, 211–239 (1997). MR1446647. https://doi.org/10.1111/1467-9965.00031

[5] Brace, A., Gatarek, D., Musiela, M.: The market model of interest rate dynamics. Math. Finance **7**, 127–155 (1997). MR1446645. https://doi.org/10.1111/1467-9965.00028

[6] Brigo, D., Mercurio, F.: A deterministic-shift extension of analytically-tractable and time-homogeneous short-rate models. Finance Stoch. **5**(3), 369–387 (2001). MR1849427. https://doi.org/10.1007/PL00013541

[7] Brigo, D., Mercurio, F.: Interest Rate Models – Theory and Practice, 1st edn. Springer, (2001). MR1846525. https://doi.org/10.1007/978-3-662-04553-4

[8] Carr, P., Madan, D.: Option valuation using the fast Fourier transform. J. Comput. Finance **2**(4), 61–73 (1999). https://doi.org/10.21314/JCF.1999.043

[9] Cont, R., Tankov, P.: Financial Modeling with Jump Processes, 1st edn. Chapman & Hall/CRC, (2004). MR2042661

[10] Cox, J., Ingersoll, J., Ross, S.: A theory of the term structure of interest rates. Econometrica **53**, 385–407 (1985). MR0785475. https://doi.org/10.2307/1911242

[11] Crépey, S., Grbac, Z., Ngor, N., Skovmand, D.: A Lévy HJM multiple-curve model with application to CVA computation. Quant. Finance **15**(3), 401–419 (2015). MR3313198. https://doi.org/10.1080/14697688.2014.942232

[12] Crépey, S., Grbac, Z., Nguyen, H.: A multiple-curve HJM model of interbank risk. Math. Financ. Econ. **6**, 155–190 (2013). MR2966729. https://doi.org/10.1007/s11579-012-0083-4

[13] Cuchiero, C., Fontana, C., Gnoatto, A.: A General HJM Framework for Multiple Yield Curve Modeling. Finance Stoch. **20**(2), 267–320 (2016). MR3479323. https://doi.org/10.1007/s00780-016-0291-5

[14] Cuchiero, C., Fontana, C., Gnoatto, A.: Affine Multiple Yield Curve Models. Math. Finance **29**(2), 568–611 (2019). preprint: arXiv:1603.00527v2. MR3925431. https://doi.org/10.1111/mafi.12183

[15] Di Nunno, G., Øksendal, B., Proske, F.: Malliavin Calculus for Lévy Processes with Applications to Finance, 1st edn. Springer, (2009). MR2460554. https://doi.org/10.1007/978-3-540-78572-9

[16] Duffie, D., Filipovic, D., Schachermayer, W.: Affine Processes and Applications in Finance. Ann. Appl. Probab. **13**, 984–1053 (2003). MR1994043. https://doi.org/10.1214/aoap/1060202833

[17] Eberlein, E., Gerhart, C., Grbac, Z.: Multiple curve Lévy forward price model allowing for negative interest rates. Mathematical Finance (2019). forthcoming, preprint: arXiv:1805.02605v1. https://doi.org/10.1007/978-3-030-26106-1

[18] Eberlein, E., Kluge, W.: Exact pricing formulae for caps and swaptions in a Lévy term structure model. J. Comput. Finance **9**(2), 99–125 (2006). MR2359367. https://doi.org/10.21314/JCF.2005.158

[19] Eberlein, E., Kluge, W.: Valuation of floating range notes in Lévy term structure models. Math. Finance **16**, 237–254 (2006). MR2212265. https://doi.org/10.1111/j.1467-9965.2006.00270.x

[20] Eberlein, E., Kluge, W., Papapantoleon, A.: Symmetries in Lévy Term Structure Models. Int. J. Theor. Appl. Finance **9**(6), 967–986 (2006). MR2260054. https://doi.org/10.1142/S0219024906003809

[21] Filipovic, D.: Term-Structure Models. Springer Finance, Springer, (2009). MR2553163. https://doi.org/10.1007/978-3-540-68015-4

[22] Filipovic, D., Trolle, A.: The Term Structure of Interbank Risk. J. Financ. Econ. **109**(3), 707–733 (2013). https://doi.org/10.1016/j.jfineco.2013.03.014

[23] Gallitschke, J., Seifried, S., Seifried, F.: Post-crisis Interest Rates: Xibor Mechanics and Basis Spreads. J. Bank. Finance **78**, 142–152 (2017). https://doi.org/10.1016/j.jbankfin.2017.01.002

[24] Grasselli, M., Miglietta, G.: A Flexible Spot Multiple-curve Model. Quant. Finance **6**(10), 1465–1477 (2016). MR3564921. https://doi.org/10.1080/14697688.2015.1108521

[25] Grbac, Z., Papapantoleon, A., Schoenmakers, J., Skovmand, D.: Affine LIBOR Models with Multiple Curves: Theory, Examples and Calibration. SIAM J. Financ. Math. **6**(1), 984–1025 (2015). MR3413586. https://doi.org/10.1137/15M1011731

[26] Grbac, Z., Runggaldier, W.: Interest Rate Modeling: Post-Crisis Challenges and Approaches, Springer Briefs in Quantitative Finance, 1st edn. Springer, (2015). MR3467166. https://doi.org/10.1007/978-3-319-25385-5

[27] Heath, D., Jarrow, R., Morton, A.: Bond pricing and the term structure of interest rates: a new methodology for contingent claims valuation. Econometrica **60**, 77–105 (1992). https://doi.org/10.2307/2951677

[28] Hess, M.: Pricing Energy, Weather and Emission Derivatives under Future Information. PhD Dissertation, University Duisburg-Essen, Germany (2013). http://duepublico.uni-duisburg-essen.de/servlets/DocumentServlet?id=31060

[29] Hull, J., White, A.: Pricing interest-rate-derivative securities. Rev. Financ. Stud. **3**(4), 573–592 (1990). https://doi.org/10.1093/rfs/3.4.573

[30] Jacod, J., Shiryaev, A.: Limit Theorems for Stochastic Processes, 2nd edn. Springer, (2003). MR1943877. https://doi.org/10.1007/978-3-662-05265-5

[31] Keller-Ressel, M., Papapantoleon, A., Teichmann, J.: The Affine LIBOR Models. Math. Finance **23**(4), 627–658 (2013). MR3094715. https://doi.org/10.1111/j.1467-9965.2012.00531.x

[32] Mercurio, F.: A LIBOR Market Model with a Stochastic Basis, Risk, pp. 96–101 (December 2013)

[33] Morino, L., Runggaldier, W.: On multicurve models for the term structure. In: Nonlinear Economic Dynamics and Financial Modeling, pp. 275–290. Springer, (2014). MR3329961

[34] Nguyen, T., Seifried, F.: The Multi-curve Potential Model. Int. J. Theor. Appl. Finance **18**(7), 1550049 (2015). MR3423186. https://doi.org/10.1142/S0219024915500491

[35] Protter, P.: Stochastic Integration and Differential Equations, 2nd edn. Springer, (2005). MR2020294

[36] Sato, K.: Lévy Processes and Infinitely Divisible Distributions. Cambridge studies in advanced mathematics, vol. 68 (1999). MR1739520

[37] Schoutens, W.: Lévy Processes in Finance: Pricing Financial Derivatives. John Wiley & Sons, Ltd., (2003)

[38] Vasicek, O.: An equilibrium characterization of the term structure. J. Financ. Econ. **5**(2), 177–188 (1977). https://doi.org/10.1016/0304-405X(77)90016-2