Modern Stochastics: Theory and Applications 7 (2) (2020) 135–156
https://doi.org/10.15559/20-VMSTA153

Single jump filtrations and local martingales

Alexander A. Gushchin[a,b]

[a]*Steklov Mathematical Institute, Gubkina 8, 119991 Moscow, Russia*
[b]*National Research University Higher School of Economics, Pokrovsky Boulevard 11, 109028 Moscow, Russia*

gushchin@mi-ras.ru (A. A. Gushchin)

Received: 24 September 2019, Revised: 30 April 2020, Accepted: 1 May 2020,
Published online: 25 May 2020

Abstract A single jump filtration $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$ generated by a random variable γ with values in $\overline{\mathbb{R}}_+$ on a probability space $(\Omega, \mathscr{F}, \mathsf{P})$ is defined as follows: a set $A \in \mathscr{F}$ belongs to $\mathscr{F}_t$ if $A\cap\{\gamma > t\}$ is either $\varnothing$ or $\{\gamma > t\}$. A process M is proved to be a local martingale with respect to this filtration if and only if it has a representation $M_t = F(t)\mathbb{1}_{\{t<\gamma\}} + L\mathbb{1}_{\{t\geqslant\gamma\}}$, where F is a deterministic function and L is a random variable such that $\mathsf{E}|M_t| < \infty$ and $\mathsf{E}(M_t) = \mathsf{E}(M_0)$ for every $t \in \{t \in \mathbb{R}_+ : \mathsf{P}(\gamma \geqslant t) > 0\}$. This result seems to be new even in a special case that has been studied in the literature, namely, where $\mathscr{F}$ is the smallest σ-field with respect to which γ is measurable (and then the filtration is the smallest one with respect to which γ is a stopping time). As a consequence, a full description of all local martingales is given and they are classified according to their global behaviour.

Keywords Filtration, local martingale, processes with finite variation, σ-martingale, stopping time
2010 MSC 60G44, 60G07

1 Introduction

Starting with Dellacherie [4], the following simple model has been studied and intensively used in applications. Given a random variable γ with positive values on a probability space $(\Omega, \mathscr{F}, \mathsf{P})$, one considers the smallest filtration with respect to which γ is a stopping time (or, equivalently, the process $\mathbb{1}_{\{t\geqslant\gamma\}}$ is adapted). In particular, Dellacherie gives a formula for the compensator of this single jump process $\mathbb{1}_{\{t\geqslant\gamma\}}$.

Chou and Meyer [2] describe all local martingales with respect to this filtration and prove a martingale representation theorem. A significant contribution is done in a recent paper by Herdegen and Herrmann [13], where a classification, whether a local martingale in this model is a strict local martingale, or a uniformly integrable martingale, etc., is given. Let us also mention some related papers [1, 15, 16, 3, 8, 21, 12], where, in particular, local martingales with respect to the filtrations generated by jump processes or measures of certain kind are studied.

Let us clarify that in the above model every local martingale has the form

$$M_t = F(t)\mathbb{1}_{\{t<\gamma\}} + H(\gamma)\mathbb{1}_{\{t\geqslant\gamma\}}, \tag{1}$$

or

$$M_t = F(t\wedge\gamma) - K(\gamma)\mathbb{1}_{\{t\geqslant\gamma\}},$$

where γ is a random variable with values in, say, $(0,+\infty)$, F, H, and $K = F - H$ are deterministic functions. Denote by G the distribution function of γ, $\overline{G}(t) = 1 - G(t)$, $t_G = \sup\{t\colon G(t) < 1\}$ is the right endpoint of the distribution of γ. Assume that $\mathsf{E}|M_t| < \infty$, then

$$\mathsf{E}(M_t) = F(t)\overline{G}(t) + \int_{[0,t]} H(s)\,dG(s),$$

where the corresponding Lebesgue–Stieltjes integral is finite. If (M_t) is a martingale, then $\mathsf{E}(M_t) = \mathsf{E}(M_0)$, and this equality can be written as

$$F(t)\overline{G}(t) + \int_{[0,t]} H(s)\,dG(s) = F(0) \tag{2}$$

and can be viewed as a functional equation concerning one of functions in (F, G, H) or (F, G, K), where other two functions are assumed to be given. In fact, this equation takes place for $t < t_G$ or $t \leqslant t_G$, the latter in the case where $t_G < \infty$ and $\mathsf{P}(\gamma = t_G) > 0$. Moreover, it turns out that this is not only the necessary condition but also the sufficient one for $(M_t)_{t\in\mathbb{R}_+}$ given by (1) to be a local martingale. This consideration allows us to reduce problems to solving this functional equation. For example, to find the compensator $F(t\wedge\gamma)$ of $\mathbb{1}_{\{t\geqslant\gamma\}}$ as in [4] one needs to find a solution F given G and $K \equiv 1$. A possible way to explain the idea in [2] is the following: The terminal value M_∞ of any local martingale M in this model is represented as $H(\gamma)$, and to find a representation (1) for M it is enough to solve the equation for F given G and H; the linear dependence between H and F results in a representation theorem. Contrariwise, in [13] the authors suggest to find H from the equation for given F and G. This allows them to study global properties of M.

In this paper we consider a more general model, where all randomness appears "at time γ" but it may contain much more information than γ does. We start with a random variable γ on a probability space $(\Omega, \mathscr{F}, \mathsf{P})$, and define a single jump filtration $(\mathscr{F}_t)$ in such way that nothing happens strictly before γ, γ is a stopping time with respect to it, and the σ-field $\mathscr{F}_\gamma$ of events that occur before or at time γ coincides with $\mathscr{F}$ (in fact, on the set $\{\gamma < \infty\}$). We prove that every local martingale with respect to this filtration has the representation

$$M_t = F(t)\mathbb{1}_{\{t<\gamma\}} + L\mathbb{1}_{\{t\geqslant\gamma\}}, \tag{3}$$

where now L is a random variable which is not necessarily a function of γ. However, denoting $H(t) = \mathsf{E}[L|\gamma = t]$, we come to the same functional equation of type (2).

Some results of the paper can be deduced from known results for marked point processes, at least if $\mathscr{F}$ is countably generated; this applies, for example, to Theorem 5 about the compensator of a single jump process. Another example is Corollary 1 which says that every local martingale is the sum of a local martingale of form (1) and an "orthogonal" local martingale, the latter being characterised, essentially, by the property $F(t) \equiv 0$. The reader can recognize in this decomposition the representation of a local martingale as the sum of two stochastic integrals with respect to random measures, see [16] and [17]. However, our direct proofs are simpler due to the key feature of our paper. Namely, we obtain a simple necessary and sufficient condition for a process to be a local martingale and later exploit it. A description of *all* local martingales via a full description of *all* possible solutions to a functional equation of type (2) is a simple consequence of this necessary and sufficient condition. In particular, an absolute continuity type property of F with respect to G, considered as an assumption in [13], is proved to be a necessary condition. An elementary analysis of a functional equation of type (2) shows that, if γ has no atom at its right endpoint, there are different F satisfying the equation for given H and G. In particular, there is a local martingale M such that $M_0 = 1$ and $M_\infty = 0$; M is necessarily a closed supermartingale.

Another important feature of our model, in contrast to Dellacherie's model, is that it admits σ-martingales which are not local martingales.

Let us also mention some other papers where processes of form (1) or (3) are considered. Processes of form (1) with $t_G = \infty$ are typical in the modelling of credit risk, see, e.g., [18] and [19, Chapter 7], where usually F is expressed via G and one needs to find H. Since $t_G = \infty$, such a process is a martingale. For example, in the simplest case $F = 1/\overline{G}$ and hence $H = 0$. This process is the same that is mentioned in two paragraphs above. Single jump filtrations and processes of form (3) appear in [10] and [11]. It is interesting to note that, in [11], the random "time" γ is, in fact, the global maximum of a random process, say, a convergent continuous local martingale.

Section 2 contains our main results. In Theorem 1 we establish a necessary and sufficient condition for a process of type (3) to be a local martingale. This allows us to obtain a full description of all local martingales through a functional equation of type (2) in Theorem 2. A similar description is available for σ-martingales, see Theorem 3. Finally, Theorem 4 classifies local martingales in accordance with their global behaviour up to ∞. Section 3 contains the proofs of these results. In Section 4 we consider complementary questions. Namely, we find the compensator of a single jump process. We also consider submartingales of class (Σ), see [22], and show that their transformation via a change of time leads to processes of type (3). As a consequence, we reprove Theorem 4.1 of [22].

We use the following notation: $\mathbb{R}_+ = [0, +\infty)$, $\overline{\mathbb{R}}_+ = [0, +\infty]$, $a \wedge b = \min\{a, b\}$. The arrows $\uparrow$ and $\downarrow$ indicate monotone convergence, while $\lim_{s\uparrow\uparrow t}$ stands for $\lim_{s\to t, s<t}$.

A real-valued function $Z(t)$ defined at least for $t \in [0, s)$ is called càdlàg on $[0, s)$ if it is right-continuous at every $t \in [0, s)$ and has a finite left-hand limit at every $t \in (0, s)$; it is not assumed that it has a limit as $t \uparrow\uparrow s$. If, additionally, a finite limit $\lim_{t\uparrow\uparrow s} Z(t)$ exists, then $Z(t)$ is called càdlàg on $[0, s]$. Functions Z of finite variation on compact intervals are understood as usually and are assumed to be càdlàg. The variation at 0 includes $|Z(0)|$ as if Z is extended by 0 on negative axis. The total variation of Z over $[0, t]$ is denoted by $\mathrm{Var}(Z)_t$. We say that Z has a finite variation over $[0, s)$, $s \leqslant \infty$, if $\lim_{t\uparrow\uparrow s} \mathrm{Var}(Z)_t < \infty$. We denote $\mathrm{Var}(Z)_\infty := \lim_{t\to\infty} \mathrm{Var}(Z)_t$.

A filtration on a probability space $(\Omega, \mathscr{F}, \mathsf{P})$ is an increasing right-continuous family $\mathbb{F} = (\mathscr{F}_t)_{t\in\mathbb{R}_+}$ of sub-σ-fields of $\mathscr{F}$. No completeness assumption is made. As usual, we define $\mathscr{F}_\infty = \sigma\big(\cup_{t\in\mathbb{R}_+}\mathscr{F}_t\big)$ and, for a stopping time τ the σ-field $\mathscr{F}_\tau$ is defined by

$$\mathscr{F}_\tau = \big\{A \in \mathscr{F}_\infty \colon A \cap \{\tau \leqslant t\} \in \mathscr{F}_t \text{ for every } t \geqslant 0\big\}.$$

A set $B \subset \Omega \times \mathbb{R}_+$ is *evanescent* if $B \subseteq A \times \mathbb{R}_+$, where $A \in \mathscr{F}$ and $\mathsf{P}(A) = 0$. We say that two stochastic processes X and Y are *indistinguishable* if $\{X \neq Y\}$ is an evanescent set.

Since we do not suppose completeness of the filtration $\mathbb{F}$, we cannot expect that processes that we consider have all paths càdlàg. Instead we consider processes whose almost all paths are càdlàg. Obviously, for any càdlàg process X adapted with respect to the completed filtration, there is an a.s. càdlàg $\mathbb{F}$-adapted process indistinguishable from X. Furthermore, any $\mathbb{F}$-adapted process X with a.s. càdlàg paths is indistinguishable from an $\mathbb{F}$-optional process Y whose paths are right-continuous everywhere and have finite left-hand limits for $t < \rho(\omega)$ and $t > \rho(\omega)$, where ρ is a $\mathbb{F}$-stopping time with $\mathsf{P}(\rho < \infty) = 0$; let us call such Y *regular* and ρ a *moment of irregularity* for Y. Dellacherie and Meyer [6, VI.5 (a), p. 70] prove that, if the filtration is not complete, every supermartingale X (with right-continuous expectation) has a modification Y with the above regularity property. If we are given just an adapted process X with almost all paths càdlàg, we define ρ and Y from values of X on a countable set exactly as is done in [6] in the case where X is a supermartingale. Using [5, Theorem IV.22, p. 94], we obtain that $\rho(\omega) = \infty$ and paths $X_\cdot(\omega)$ and $Y_\cdot(\omega)$ coincide for those ω for which $X_\cdot(\omega)$ is càdlàg everywhere. Moreover, if $\rho(\omega) < \infty$, then $Y_t(\omega)$ is càdlàg for $t < \rho(\omega)$ and one may put $Y_t(\omega) = 0$ for $t \geqslant \rho(\omega)$.

Processes with finite variation are adapted and not assumed to start from 0. A moment of irregularity for them has additionally the property that their paths have finite variation over $[0, t]$ for all $t < \rho(\omega)$.

It is instructive to mention that, in our model, there is no need to use general results on the existence of (a.s.) càdlàg modifications for martingales since they can be proved directly. For example, if L is an integrable random variable with $\mathsf{E}L = 0$, then the process M given by (3) with $F(t) = \mathsf{E}[L|\gamma > t]\mathbb{1}_{\{t<t_G\}}$ satisfies $M_t = \mathsf{E}[L|\mathscr{F}_t]$ a.s. for an arbitrary t. It is trivial to check that this function F has finite variation over any $[0, t]$ with $\mathsf{P}(\gamma > t) > 0$ (and over $[0, t_G)$ if $\mathsf{P}(\gamma = t_G < \infty) > 0$). Thus M is regular. It may be that, if $t_G < \infty$ and $\mathsf{P}(\gamma = t_G) = 0$, the function F has not a finite limit as $t \uparrow\uparrow t_G$, or, more generally, has unbounded variation over $[0, t_G)$. Then a moment of irregularity is given by

$$\rho(\omega) = \begin{cases} t_G, & \text{if } \gamma \geqslant t_G; \\ +\infty, & \text{otherwise.} \end{cases}$$

It takes a finite value only on the set $\{\gamma \geqslant t_G\}$ of zero measure. In all other cases we may put $\rho \equiv +\infty$. See Remark 2 in Section 2 for more details.

2 Main results

Let γ be a random variable with values in $\overline{\mathbb{R}}_+$ on a probability space $(\Omega, \mathscr{F}, \mathsf{P})$. We tacitly assume that $\mathsf{P}(\gamma > 0) > 0$. $G(t) = \mathsf{P}(\gamma \leqslant t)$, $t \in \mathbb{R}_+$, stands for the distribution function of γ and $\overline{G}(t) = 1 - G(t)$. Put also $t_G = \sup\{t \in \mathbb{R}_+ : G(t) < 1\}$ and $\mathcal{T} = \{t \in \mathbb{R}_+ : \mathsf{P}(\gamma \geqslant t) > 0\}$. Note that $\mathsf{P}(\gamma \notin \mathcal{T}) = 0$. We will often distinguish between the following two cases:

Case A $\mathsf{P}(\gamma = t_G < \infty) = 0$.

Case B $\mathsf{P}(\gamma = t_G < \infty) > 0$.

It is clear that $\mathcal{T} = [0, t_G)$ in Case A and $\mathcal{T} = [0, t_G]$ in Case B.

We define $\mathscr{F}_t$, $t \in \mathbb{R}_+$, as the collection of subsets A of Ω such that $A \in \mathscr{F}$ and $A \cap \{t < \gamma\}$ is either $\varnothing$ or coincides with $\{t < \gamma\}$.

It is shown in Proposition 1 that $\mathscr{F}_t$ is a σ-field for every $t \in \mathbb{R}_+$ and the family $\mathbb{F} = (\mathscr{F}_t)_{t\in\mathbb{R}_+}$ is a filtration. We call this filtration a *single jump filtration*. It is determined by generating elements γ and $\mathscr{F}$. In this paper we consider only single jump filtrations and, if necessary to indicate generating elements, we use the notation $\mathbb{F}(\gamma, \mathscr{F})$ for the single jump filtration generated by γ and $\mathscr{F}$.

In this section a single jump filtration $\mathbb{F} = \mathbb{F}(\gamma, \mathscr{F})$ is fixed. All notions depending on filtration (stopping times, martingales, local martingales, etc.) refer to this filtration $\mathbb{F}$, unless otherwise specified.

Proposition 1. (i) *$\mathscr{F}_t$ is a σ-field and a random variable ξ is $\mathscr{F}_t$-measurable, $t \in \mathbb{R}_+$, if and only if ξ is constant on $\{t < \gamma\}$. ξ is $\mathscr{F}_\infty$-measurable if and only if ξ is constant on $\{\gamma = \infty\}$.*

(ii) *The family $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$ is increasing and right-continuous, i.e. $\mathbb{F} = (\mathscr{F}_t)_{t\in\mathbb{R}_+}$ is a filtration.*

(iii) *γ is a stopping time and $\mathscr{F}_\gamma = \mathscr{F}_\infty$.*

(iv) *A random variable T with values in $\overline{\mathbb{R}}_+$ is a stopping time if and only if it satisfies the following property: if the set $\{T < \gamma\}$ is not empty, then there is a number r such that*

$$\{T < \gamma\} = \{T = r < \gamma\} = \{r < \gamma\}. \tag{4}$$

Proposition 2. (i) *If $X = (X_t)_{t\in\mathbb{R}_+}$ is an adapted process, then there is a deterministic function $F(t)$, $0 \leqslant t < t_G$, such that $X_t = F(t)$ on $\{t < \gamma \wedge t_G\}$. If $Y = (Y_t)_{t\in\mathbb{R}_+}$ is an adapted process and $\mathsf{P}(X_t = Y_t) = 1$ for every $t \in \mathbb{R}_+$, then $X_t = Y_t$ identically on $\{t < \gamma \wedge t_G\}$.*

(ii) *If $Y = (Y_t)_{t\in\mathbb{R}_+}$ is a predictable process, then there is a measurable deterministic function $C(t)$, $t \in \mathcal{T}$, such that $Y_t = C(t)$ on $\{t \leqslant \gamma\}$, $t \in \mathcal{T}$.*

(iii) *If $X = (X_t)_{t\in\mathbb{R}_+}$ is a process with finite variation, then $F(t)$ in* (i) *has a finite variation over* $[0, t]$ *for every $t < t_G$ in Case A and over* $[0, t_G)$ *in Case B.*

(iv) *Every semimartingale is a process with finite variation.*

(v) *If $M = (M_t)_{t\in\mathbb{R}_+}$ is a σ-martingale then there are a deterministic function $F(t)$, $t \in \mathbb{R}_+$, and a finite random variable L such that, up to P-indistinguishability,*

$$M_t = F(t)\mathbb{1}_{\{t<\gamma\}} + L\mathbb{1}_{\{t\geqslant\gamma\}}. \tag{5}$$

Statement (iv) is not surprising. If the σ-field $\mathscr{F}$ is countably generated, then our filtration is a special case of a filtration generated by a marked point process, and it is known, see [17], that then all martingales are of finite variation. In general, a single jump filtration is a special case of a jumping filtration, see [14], where again all martingales are of finite variation.

Remark 1. If M is a σ-martingale, then it is a process with finite variation due to (iv) and, hence, the function $F(t)$ in (5) has a finite variation over $[0, t]$ for every $t < t_G$ in Case A and over $[0, t_G)$ in Case B according to (iii).

Remark 2. According to (i), the function $F(t)$ in (5) is uniquely determined for $t < t_G$. Since $\mathsf{P}(\gamma > t_G) = 0$, the stochastic interval $[\![t_G, \gamma[\![$ is an evanescent set. Hence, $F(t)$ can be defined arbitrarily for $t \geqslant t_G$. For example, we can put it equal to 0 for $t \geqslant t_G$. Then $F(t)$ has a finite variation on compact intervals if $t_G = +\infty$ or in Case B. In Case A, if t_G is finite, $F(t)$ may have infinite variation over $[0, t_G)$ (and even not have a finite limit as $t \upuparrows t_G$), see Theorem 2 and Example 3 below. All other points are regular for $F(t)$. Now put $\rho(\omega) = t_G < +\infty$ if we are in Case A, $t_G < +\infty$, $\lim_{t \upuparrows t_G} \mathrm{Var}(F)_t = \infty$, and $\gamma(\omega) \geqslant t_G$, and let $\rho(\omega) = +\infty$ in all other cases. It follows that ρ is a moment of irregularity for the process in the right-hand side of (5).

In what follows, when we write that the process M has the representation (5), this means that M and the right-hand side of (5) are indistinguishable. Moreover, we tacitly assume that $F(t)$ is right-continuous for $t \geqslant t_G$ to ensure that the right-hand side of (5) is right-continuous.

Propositions 1 and 2 explain why we call $\mathbb{F}$ a single jump filtration: all randomness appears at time γ. It is not so natural to describe local martingales with respect to $\mathbb{F}$ as single jump processes. As we will see, the function F in (5) need not be continuous, so local martingales may have several jumps.

Our main goal is to provide a complete description of all local martingales. According to Proposition 2 (v), a necessary condition is that it is represented in form (5). Thus, it is enough to study only processes of this form.

Theorem 1. *Let $F(t)$, $0 \leqslant t < t_G$, be a deterministic càdlàg function, L be a random variable, and a process $M = (M_t)_{t \in \mathbb{R}_+}$ be given by*

$$M_t = F(t)\mathbb{1}_{\{t<\gamma\}} + L\mathbb{1}_{\{t \geqslant \gamma\}}. \tag{6}$$

The following statements are equivalent:

(i) *$M = (M_t)_{t \in \mathbb{R}_+}$ is a local martingale.*

(ii) *$(M_t)_{t \in \mathfrak{T}}$ is a martingale.*

(iii)
$$\mathsf{E}\big(|M_t|\big) < \infty, \quad t \in \mathfrak{T}, \tag{7}$$
and
$$\mathsf{E}(M_t) = \mathsf{E}(M_0), \quad t \in \mathfrak{T}. \tag{8}$$

In the case where $\mathscr{F} = \sigma\{\gamma\}$, equivalence (i) and (ii) is proved in [2].

Concerning the last statement of the proposition, let us emphasize that if $t_G < \infty$ and $\mathsf{P}(\gamma = t_G) = 0$, a local martingale $M = (M_t)_{t\in\mathbb{R}_+}$ may not be a martingale on $[0, t_G]$; obviously, if it is a martingale, then it is uniformly integrable, and necessary and sufficient conditions for this are given in Theorem 4.

If (6) and (7) hold, then

$$\mathsf{E}\big(|L|\mathbb{1}_{\{\gamma\leqslant t\}}\big) < \infty, \quad t \in \mathcal{T}, \tag{9}$$

and one can define the conditional expectation $H(t)$ of L given that $\gamma = t$ for $t \in \mathcal{T}$:

$$H(t) = \mathsf{E}[L|\gamma = t]. \tag{10}$$

More precisely, $H(t)$ is a Borel function on $\mathcal{T}$ with finite values such that for any $t \in \mathcal{T}$

$$\mathsf{E}\big(L\mathbb{1}_{\{\gamma\leqslant t\}}\big) = \int_{[0,t]} H(s)\,dG(s).$$

Note that the function H is dG-a.s. unique and is dG-integrable over any closed interval in $\mathcal{T}$. It is convenient to introduce a notation for such functions.

Let $L^1_{\mathrm{loc}}(dG)$ be the set of all Borel functions z on $\mathcal{T}$ such that

$$\int_{[0,t]} |z(s)|\,dG(s) < \infty \quad \text{for all } t \in \mathcal{T}.$$

Given a function $Z\colon [0, t_G) \to \mathbb{R}$, let us write $Z \overset{\mathrm{loc}}{\ll} G$ if there is $z \in L^1_{\mathrm{loc}}(dG)$ such that $Z(t) = Z(0) + \int_{(0,t]} z(s)\,dG(s)$ for all $t < t_G$; in this case we put $\frac{dZ}{dG}(t) := z(t)$ for $0 < t < t_G$. Let us emphasize that in Case B this definition implies that z is dG-integrable over $[0, t_G]$ and, hence, the function Z has a finite variation over $[0, t_G)$ and there is a finite limit $\lim_{t\uparrow\uparrow t_G} Z(t) = Z(0) + \int_{(0,t_G)} z(s)\,dG(s)$. Note also that in this definition the value $z(0)$ can be chosen arbitrarily even if $G(0) > 0$; the same refers to the value $z(t_G)$ in Case B. Correspondingly, dZ/dG is defined only for $0 < t < t_G$.

Let G be a distribution function of a law on $[0, +\infty]$. We will say that a pair (F, H) satisfies Condition M if

$$F\colon [0, t_G) \to \mathbb{R}, \quad F \overset{\mathrm{loc}}{\ll} G, \tag{11}$$

$$H\colon \mathcal{T} \to \mathbb{R}, \quad H \in L^1_{\mathrm{loc}}(dG), \tag{12}$$

$$F(t)\overline{G}(t) + \int\limits_{(0,t]} H(s)\,dG(s) = F(0)\overline{G}(0), \quad t < t_G, \tag{13}$$

and, additionally in Case B,

$$\lim_{t\uparrow\uparrow t_G} F(t) = H(t_G). \tag{14}$$

Proposition 3. (a) *Let H be any function satisfying* (12). *Define*

$$F(t) = \overline{G}(t)^{-1}\Big[F(0)\overline{G}(0) - \int\limits_{(0,t]} H(s)\,dG(s)\Big], \quad 0 < t < t_G, \tag{15}$$

where $F(0)$ is an arbitrary real number in Case A and

$$F(0) = \overline{G}(0)^{-1} \int_{(0,t_G]} H(s)\,dG(s) \tag{16}$$

in Case B. Then the pair (F, H) satisfies Condition M. Conversely, if F is such that the pair (F, H) satisfies Condition M, then F satisfies (15) *and, in Case B,* (16) *holds.*

(b) *Let F be any function satisfying* (11)*. Define $H(0)$ arbitrarily,*

$$H(t) = F(t) - \overline{G}(t-)\frac{dF}{dG}(t), \quad 0 < t < t_G, \tag{17}$$

$H(t_G)$ arbitrarily in Case A and

$$H(t_G) = \lim_{t \uparrow\uparrow t_G} F(t) \tag{18}$$

in Case B. Then the pair (F, H) satisfies Condition M. Conversely, if H is such that the pair (F, H) satisfies Condition M, then H satisfies (17) *and, in Case B,* (18) *holds.*

Theorem 2. *In order that a right-continuous process $M = (M_t)_{t\in\mathbb{R}_+}$ be a local martingale it is necessary and sufficient that there be a pair (F, H) satisfying Condition M and a random variable L' satisfying*

$$\mathsf{E}\big(|L'|\mathbb{1}_{\{\gamma\leqslant t\}}\big) < \infty, \quad t \in \mathcal{T}, \qquad \textit{and} \qquad \mathsf{E}[L'|\gamma] = 0, \tag{19}$$

such that, up to P-indistinguishability,

$$M_t = F(t)\mathbb{1}_{\{t<\gamma\}} + \big(H(\gamma) + L'\big)\mathbb{1}_{\{t\geqslant\gamma\}}, \tag{20}$$

The statement that the process M given by (20) with $L' = 0$ is a local martingale if $F \overset{\text{loc}}{\ll} G$ and H is constructed as in part (b) of Proposition 3, is essentially due to Herdegen and Herrmann [13], though they formulate (17) in an equivalent form:

$$H(t) = F(t-) - \overline{G}(t)\frac{dF}{dG}(t), \quad 0 < t < t_G. \tag{21}$$

They also prove that, in Case B, if F has infinite variation on $[0, t_G)$ (and hence does not satisfy $F \overset{\text{loc}}{\ll} G$), then M given by (6) is not a semimartingale, see [13, Lemma B.6]. (Note that this follows also from our Proposition 2 (iv).) We add that, also in Case B, if H is dG-integrable over $(0, t_G)$, F satisfies (15), but $F(0)$ is greater or less than the right-hand side of (16), then M given by (20) with L' satisfying (19), is a supermartingale or a submartingale, respectively, cf. Theorem 4.

The fact that $H(0)$ can be chosen arbitrarily in Proposition 3 (b) says only that L can be an arbitrary integrable random variable on the set $\{\gamma = 0\}$, which is evident ab initio. On the contrary, the fact that $F(0)$ can be chosen arbitrarily in (a) in Case A is an interesting feature of this model. It says that, given the terminal value M_∞ of M (on $\{\gamma < \infty\}$), one can freely choose the initial value M_0 of M (on $\{\gamma > 0\}$) to keep the property of being a local martingale for M.

Corollary 1. *Every local martingale $M = (M_t)_{t\in\mathbb{R}_+}$ has a unique decomposition into the sum $M = M' + M''$ of two local martingales M' and M'', where M' is adapted with respect to the smallest filtration making γ a stopping time, and M'' which vanishes on $\{t < \gamma\}$ and satisfies $\mathsf{E}M''_0 = 0$.*

Remark 3. If $\mathsf{P}(\gamma = 0) = 0$, then it follows from the first property for M'' that $M''_0 = 0$ a.s. and thus the second property holds automatically.

Remark 4. The smallest filtration making γ a stopping time is a single jump filtration $\mathbb{F}(\gamma, \sigma\{\gamma\})$ generated by γ and the smallest σ-field $\sigma\{\gamma\}$ with respect to which γ is measurable. Let M be a $\mathbb{F}$-local martingale adapted to $\mathbb{F}(\gamma, \sigma\{\gamma\})$. It follows from Theorem 1 that M is a $\mathbb{F}(\gamma, \sigma\{\gamma\})$-local martingale.

As the next example shows, the product $M'M''$ of local martingales from the above decomposition may not be a local martingale because the first condition in (19) may fail. It will follow from Theorem 3 below that this product is always a σ-martingale.

Example 1. Let γ have an exponential distribution, e.g., $\overline{G}(t) = e^{-t}$, F is given by (15) with $H(t) = t^{-1/2}$ and an arbitrary $F(0)$, $M'_t = F(t)\mathbb{1}_{\{t<\gamma\}} + H(\gamma)\mathbb{1}_{\{t\geqslant\gamma\}}$, $M''_t = Y\gamma^{-1/2}\mathbb{1}_{\{t\geqslant\gamma\}}$, where Y takes values ±1 with probabilities 1/2 and is independent of γ. It follows that M' and M'' are local martingales but their product $M'_tM''_t = Y\gamma^{-1}\mathbb{1}_{\{t\geqslant\gamma\}}$ does not satisfy the integrability condition (7) and, hence, is not a local martingale. This process is a classical example (due to Émery) of a σ-martingale which is not a local martingale, see, e.g., [9, Example 2.3, p. 86].

The previous example shows that our model admits σ-martingales which are not local martingales. In the next theorem we describe all σ-martingales in our model. In particular, it implies that if $\mathscr{F} = \sigma\{\gamma\}$, then all σ-martingales that are integrable at 0 are local martingales.

Theorem 3. *In order that a right-continuous process $M = (M_t)_{t\in\mathbb{R}_+}$ be a σ-martingale it is necessary and sufficient that it have a representation* (20)*, where a pair (F, H) satisfies Condition M and a random variable L' satisfies*

$$\mathsf{E}\big[|L'|\mathbb{1}_{\{\gamma>0\}}\big|\gamma\big] < \infty \qquad \text{and} \qquad \mathsf{E}[L'|\gamma] = 0. \tag{22}$$

The next theorem complements the classification of the limit behaviour of local martingales that was considered in Herdegen and Herrmann [13] in the case where $\mathscr{F} = \sigma\{\gamma\}$. Let us say that a local martingale $M = (M_t)_{t\in\mathbb{R}_+}$ has

type 1 if the limit $M_\infty = \lim_{t\to\infty} M_t$ does not exist with positive probability or exists with probability one but is not integrable: $\mathsf{E}|M_\infty| = \infty$;

type 2a if M is a closed supermartingale (in particular, $\mathsf{E}|M_\infty| < \infty$) and $\mathsf{E}(M_\infty) < \mathsf{E}(M_0)$;

type 2b if M is a closed submartingale (in particular, $\mathsf{E}|M_\infty| < \infty$) and $\mathsf{E}(M_\infty) > \mathsf{E}(M_0)$;

type 3 if M is a uniformly integrable martingale (in particular, $\mathsf{E}|M_\infty| < \infty$ and $\mathsf{E}(M_\infty) = \mathsf{E}(M_0)$) and $\mathsf{E}(\sup_t |M_t|) = \infty$;

type 4 if M has an integrable variation: $\mathsf{E}\big(\mathrm{Var}(M)_\infty\big) < \infty$.

Theorem 4. *Let $M = (M_t)_{t\in\mathbb{R}_+}$ be a local martingale with the representation*

$$M_t = F(t)\mathbb{1}_{\{t<\gamma\}} + L\mathbb{1}_{\{t\geqslant\gamma\}}, \quad t \in \mathbb{R}_+, \tag{23}$$

where $L = H(\gamma) + L'$, a pair (F, H) satisfies Condition M and a random variable L' satisfies (19). Then in Case B the local martingale M has type 4. In Case A all types are possible. Namely,

(i) *M has type 1 if and only if $\mathsf{E}\big(|L'|\mathbb{1}_{\{\gamma<\infty\}}\big) = \infty$ or $\int_{[0,t_G)} |H(s)|\,dG(s) = \infty$.*

(ii) *If $\mathsf{P}(\gamma = \infty) > 0$, $\mathsf{E}\big(|L'|\mathbb{1}_{\{\gamma<\infty\}}\big) < \infty$, and $\int_{\mathbb{R}_+} |H(s)|\,dG(s) < \infty$ then M has type 4.*

(iii) *If $\mathsf{P}(\gamma = \infty) = 0$, $\mathsf{E}|L'| < \infty$, and $\int_{[0,t_G)} |H(s)|\,dG(s) < \infty$ then*

(iii.i) *M has type 2a (resp., 2b) if and only if $\lim_{t\uparrow\uparrow t_G} F(t)\overline{G}(t) > 0$ (resp., $\lim_{t\uparrow\uparrow t_G} F(t)\overline{G}(t) < 0$);*

(iii.ii) *M has type 3 if and only if*

$$\lim_{t\uparrow\uparrow t_G} F(t)\overline{G}(t) = 0 \qquad \text{and} \qquad \int_{[0,t_G)} \overline{G}(s)\Big|\frac{dF}{dG}(s)\Big|\,dG(s) = \infty;$$

(iii.iii) *M has type 4 if and only if*

$$\int\limits_{[0,t_G)} \overline{G}(s)\Big|\frac{dF}{dG}(s)\Big|\,dG(s) < \infty. \tag{24}$$

Remark 5. It follows from (13) that the limit $\lim_{t\to t_G} F(t)\overline{G}(t)$ in (iii.i) and (iii.ii) exists. Also, $\int_{[0,t_G)} |H(s)|\,dG(s)$ in (i)–(iii) is finite if only if $F(t)\overline{G}(t)$ has a finite variation over $[0, t_G)$.

Remark 6. It follows from Theorem 4 that, in our model, every martingale M with $\mathsf{E}(\sup_t |M_t|) < \infty$ has an integrable total variation. Of course, on general spaces, there exist martingales M having finite variation on compacts and such that $\mathsf{E}(\sup_t |M_t|) < \infty$ and their total variation is not integrable, see, e.g., [9, Example 2.7, p. 103].

Example 2. Assume that $H\colon (0, 1) \to \mathbb{R}$ is a *monotone nondecreasing* function and, for definiteness, that it is right-continuous. Then it is the upper quantile function of $H(\gamma)$, where γ is uniformly distributed on $(0, 1)$. Assume also that H is integrable on $(0, 1)$ and $\int_0^1 H(s)\,ds = 0$, that is to say, that $H(\gamma)$ has zero mean. Put

$$F(t) = -(1-t)^{-1}\int\limits_0^t H(s)\,ds = (1-t)^{-1}\int\limits_t^1 H(s)\,ds.$$

We see that F satisfying (13) with $F(0) = 0$ is the Hardy–Littlewood maximal function corresponding to H. If we define M by (23) with $L = H(\gamma)$, then, by Theorem 4,

M is a uniformly integrable martingale with $M_\infty = H(\gamma)$ and $\sup_t M_t = F(\gamma)$. This example is essentially the example of Dubins and Gilat [7] of a uniformly integrable martingale with a given distribution of its terminal value, having maximal (with respect to the stochastic partial order) maximum (in time).

Example 3 ([13, Example 3.14])**.** Let $\Omega = (0, 1]$ be equipped with the Borel σ-field $\mathscr{F}$, and let P be the Lebesgue measure, $\gamma(\omega) = \omega$. Put $H(t) \equiv 0$. Then $F(t) = (1-t)^{-1}$ satisfies (13) with $F(0) = 1$. By Theorem 4, M defined by (23) is a supermartingale and local martingale but not a martingale. This seems to be the simplest example of a local martingale with continuous time, which is not a martingale. Note that, for $\omega = 1$, the trajectory $M_t(\omega) = (1-t)^{-1}\mathbb{1}_{\{t<1\}}$ has not a finite left-hand limit at 1. Moreover, if N is a modification of M, for $t < 1$, the values of $M_t(\omega)$ and $N_t(\omega)$ must coincide on the atom $\{t < \gamma\} = (t, 1]$ of $\mathscr{F}_t$, having the positive measure. Hence, $N_t(\omega) = M_t(\omega)$ for $\omega = 1$ for all $t < 1$. This is an example of a right-continuous supermartingale which has not a modification with *all* paths càdlàg. Of course, the usual assumptions are not satisfied in this example.

3 Proofs

Proof of Proposition 1. (i) and (iii) are evident from the definition of $\mathscr{F}_t$, and (ii) follows easily from (i).

Let us prove (iv). To prove that T is a stopping time, we must check that $\{T \leqslant t < \gamma\}$ is either $\varnothing$ or $\{t < \gamma\}$ for all $t \in \mathbb{R}_+$. This is trivial if $\{T < \gamma\} = \varnothing$. If there is a number r such that (4) holds, then $\{T \leqslant t < \gamma\}$ is either $\varnothing$ if $r > t$ or $\{t < \gamma\}$ if $r \leqslant t$.

Conversely, let T be a stopping time. If $T \geqslant \gamma$ for all ω, then there is nothing to prove. Assume that the set $\{T < \gamma\} \neq \varnothing$. Then there are real numbers q such that $\{T \leqslant q < \gamma\} \neq \varnothing$. For such q, by the definition of $\mathscr{F}_q$, $\{T \leqslant q < \gamma\} = \{q < \gamma\}$, or, equivalently, $\{T \leqslant q\} \supseteq \{q < \gamma\}$. Let r be the greatest lower bound of such q. The sets $\{q < \gamma\} \uparrow \{r < \gamma\}$ and $\{T \leqslant q\} \downarrow \{T \leqslant r\}$ as $q \downarrow r$. Thus,

$$\{T < \gamma\} = \bigcup_{q\colon \{T \leqslant q < \gamma\} \neq \varnothing} \{q < \gamma\} = \{r < \gamma\} \subseteq \{T \leqslant r\}.$$

Since $\{T \leqslant t < \gamma\} = \varnothing$ for any $t < r$, we have (4). □

Proof of Proposition 2. The first statement in (i) follows from Proposition 1 (i). Since $\mathsf{P}(t < \gamma \wedge t_G) > 0$ for every $t < t_G$, we obtain that X_t and Y_t take the same constant value on $\{t < \gamma \wedge t_G\}$.

Since a random variable Y_t is $\mathscr{F}_{t-}$-measurable for a predictable process Y, Y_t is constant on $\{t \leqslant \gamma\}$. Denote by $C(t)$, $t \in \mathcal{T}$, the value of Y_t on $\{t \leqslant \gamma\}$. Since $\mathsf{P}(\gamma \geqslant t) > 0$ for $t \in \mathcal{T}$, there is an ω such that $C(s) \equiv Y_s(\omega)$, $s \leqslant t$, and the measurability of C follows.

Let us prove (iii) in Case B. Then we obtain that $X_t = F(t)$ for all $t < t_G$ on the set $\{\gamma = t_G\}$, which has a positive probability. However, almost all paths of X_t have a finite variation over $[0, t_G)$, and the claim follows. The proof in Case A is similar.

Now let us prove (5) in the case where $M = (M_t)_{t\in\mathbb{R}_+}$ is a uniformly integrable (a.s. càdlàg) martingale. We can find a random variable M_∞ that is $\mathscr{F}_\infty$-measurable and such that $\lim_{n\to\infty} M_n = M_\infty$ P-a.s. Since $\{t < \gamma\}$ is an atom of $\mathscr{F}_t$ and has a positive probability for $t < t_G$, we obtain from the martingale property that $M_t(\omega) = F(t)$ for all $\omega \in \{t < \gamma\}$, where

$$F(t) = \frac{\mathsf{E}\big(M_\infty \mathbb{1}_{\{t<\gamma\}}\big)}{\overline{G}(t)}, \quad t < t_G.$$

It is clear that the nominator and the denominator are right-continuous functions of bounded variation on $[0, t_G]$, hence $F(t)$, $0 \leqslant t < t_G$ is a càdlàg function on $\mathcal{T}$ and has a finite variation on $[0, t_G)$ in Case B and on every $[0, t]$, $0 \leqslant t < t_G$, in Case A.

Now set $L = M_\infty \mathbb{1}_{\{\gamma<\infty\}}$. Then $L\mathbb{1}_{\{\gamma\leqslant t\}} = M_\infty \mathbb{1}_{\{\gamma\leqslant t\}}$ is $\mathscr{F}_t$-measurable, and hence P-a.s.

$$M_t \mathbb{1}_{\{\gamma\leqslant t\}} = \mathsf{E}(M_\infty \mathbb{1}_{\{\gamma\leqslant t\}} | \mathscr{F}_t) = L\mathbb{1}_{\{\gamma\leqslant t\}}.$$

Thus we have obtained, that, for a given $t \in \mathbb{R}_+$, M_t is equal P-a.s. to the right-hand side of (5) with L and $F(t)$ as above. Since both the left-hand side and the right-hand side of (5) are almost surely right-continuous, they are indistinguishable. Moreover, if we change $F(t)$ for $t \geqslant t_G$, the right-hand side of (5) will change on an evanescent set. Thus we can put, say, $F(t) = 0$ for $t \geqslant t_G$, and then the right-hand side of (5) is a regular right-continuous process with finite variation, and indistinguishable from M.

Now let M be a local martingale and $\{T_n\}$ be a localizing sequence of stopping times, i.e. $T_n \uparrow \infty$ a.s. and M^{T_n} is a uniformly integrable martingale for each n. We have proved that almost all paths of M^{T_n} have finite variation. It follows that almost all paths of M have finite variation. This proves (iv).

Next, let M be a σ-martingale, i.e. M is a semimartingale and there is an increasing sequence of predictable sets Σ_n such that $\cup_n \Sigma_n = \Omega \times \mathbb{R}_+$ and the integral process $\mathbb{1}_{\Sigma_n} \cdot M$ is a uniformly integrable martingale for every n. It does not matter if we integrate over $[0, t]$ or $(0, t]$, so let us agree that the domain of integration does not include 0. Since the integrand is bounded and every semimartingale is a process with finite variation in our model, the integral can be considered in the Lebesgue–Stieltjes sense, as well as other integrals appearing in the proof. Since $\mathbb{1}_{\Sigma_n} \cdot M$ is stopped at γ for every n with probability one, we have

$$\int \mathbb{1}_{]\!]\gamma,\infty[\![\cap\Sigma_n}(t)\, d\,\mathrm{Var}(M)_t = \int \mathbb{1}_{]\!]\gamma,\infty[\![}(t)\, d\,\mathrm{Var}(\mathbb{1}_{\Sigma_n} \cdot M)_t = 0 \quad \text{P-a.s.}$$

for every n, therefore,

$$\int \mathbb{1}_{]\!]\gamma,\infty[\![}(t)\, d\,\mathrm{Var}(M)_t = 0 \quad \text{P-a.s.}$$

Combining with (i), we prove representation (5). □

Remark 7. As it was already explained in the introduction, we can prove directly, without assuming that paths are a.s. càdlàg, that any uniformly integrable martingale has a regular modification. The proof is essentially the same as above where we proved that a.s. càdlàg uniformly integrable martingale has representation (5).

Proof of Theorem 1. First, we prove that statements (ii) and (iii) are equivalent. The implication (ii)⇒(iii) follows trivially from the definition of a martingale. Conversely, let (iii) hold. The process $(M_t)_{t\in\mathcal{T}}$ is right-continuous, adapted by Proposition 1 (i), and integrable, see (7). Moreover, due to (6),

$$M_t - M_s = 0 \quad \text{on } \{s \geqslant \gamma\},$$

where $0 \leqslant s < t \in \mathcal{T}$. Hence,

$$\mathsf{E}[M_t - M_s | \mathscr{F}_s] = 0 \quad \text{on } \{s \geqslant \gamma\}.$$

But $\mathsf{E}[M_t - M_s|\mathscr{F}_s]$ is $\mathscr{F}_s$-measurable and, thus, equals a constant on $\{s < \gamma\}$. And this constant must be zero since $\mathsf{E}(M_t - M_s) = 0$ by (8).

The implication (ii)⇒(i) is trivial if $t_G = \infty$ or $t_G \in \mathcal{T}$. So we assume that $t_G < \infty$ and $\overline{G}(t) \downarrow 0$ as $t \upuparrows t_G$. Let $t_1 < \cdots < t_n < \cdots < t_G$, $t_n \to t_G$, be an increasing sequence, then $\overline{G}(t_n) \to 0$. Put

$$T_n = \begin{cases} t_n, & \text{if } \gamma > t_n; \\ +\infty, & \text{otherwise.} \end{cases}$$

Then T_n is a stopping time by Proposition 1 (iv), $T_n \uparrow \infty$ a.s., and $M_{t\wedge T_n} = M_{t\wedge t_n}$. Hence, M^{T_n} is a martingale and M is a local martingale.

It remains to prove the implication (i)⇒(ii). Let $M = (M_t)_{t\in\mathbb{R}_+}$ be a local martingale with a localizing sequence $\{T_n\}$, i.e. $T_n \uparrow \infty$ a.s. and M^{T_n} is a uniformly integrable martingale for each n. If $\mathsf{P}(T_n \geqslant \gamma) = 1$ for some n, then $M = M^{T_n}$ is a uniformly integrable martingale, and there is nothing to prove. So assume that $\mathsf{P}(T_n < \gamma) > 0$ for all n. By Proposition 1 (iv), there is a number r_n such that $\{T_n < \gamma\} = \{T_n = r_n < \gamma\} = \{r_n < \gamma\}$. It follows from $\mathsf{P}(r_n < \gamma) > 0$ that $r_n < t_G$. In Case B we get $\mathsf{P}(T_n < \gamma) = \mathsf{P}(r_n < \gamma) \geqslant \mathsf{P}(\gamma = t_G) > 0$ for every n, a contradiction with $T_n \to \infty$ a.s. In Case A, if $\mathsf{P}(\gamma = \infty) > 0$, then it follows from $T_n \to \infty$ a.s. that $r_n \to \infty$. In remaining cases where $\mathsf{P}(\gamma = t_G) = 0$, we obtain from $\mathsf{P}(r_n < \gamma) \to 0$ that $r_n \to t_G$, $n \to \infty$. The claim follows since $M_{t\wedge T_n} = M_{t\wedge r_n}$, and hence $(M_t)_{t\leqslant r_n}$ is a martingale. □

Proof of Proposition 3. (a) It is obvious that (13) is equivalent to (15). It also follows from (13) that in Case B (14) is equivalent to (16). Thus it remains to prove that F defined in (a) satisfies $F \overset{\text{loc}}{\ll} G$. Since $\overline{G}(s) \geqslant \overline{G}(t) > 0$ for any $s < t < t_G$, we have

$$\frac{1}{\overline{G}(t)} = \frac{1}{\overline{G}(0)} + \int_{(0,t]} \frac{1}{\overline{G}(s)\overline{G}(s-)}\, dG(s), \quad t < t_G.$$

On the other hand, from (15)

$$F(t)\overline{G}(t) = F(0)\overline{G}(0) - \int_{(0,t]} H(s)\, dG(s), \quad t < t_G.$$

Combining, we obtain from integration by parts that

$$F(t) = F(t)\overline{G}(t)\frac{1}{\overline{G}(t)} = F(0) - \int_{(0,t]} \frac{H(s)}{\overline{G}(s-)}\, dG(s) + \int_{(0,t]} \frac{F(s)}{\overline{G}(s-)}\, dG(s), \quad t < t_G.$$

This shows that $F \overset{\text{loc}}{\ll} G$ in Case A. In Case B we must show additionally that the function $\frac{|F(s)|+|H(s)|}{\overline{G}(s-)}$ is dG-integrable over $(0, t_G)$. But $1/\overline{G}(s-) \leqslant 1/\mathsf{P}(\gamma = t_G)$, $s \leqslant t_G$, and $F(s)$ is bounded on $[0, t_G)$ in view of (15). The claim follows.

(b) It is clear that the function $H(t)$, $t \in \mathcal{T}$, defined as in the statement, belongs to $L^1_{\text{loc}}(dG)$. Integrating by parts, we get, for $t \in [0, t_G)$,

$$\begin{aligned} F(t)\overline{G}(t) &= F(0)\overline{G}(0) - \int_{(0,t]} F(s)\,dG(s) + \int_{(0,t]} \overline{G}(s-)\,dF(s) \\ &= F(0)\overline{G}(0) - \int_{(0,t]} F(s)\,dG(s) + \int_{(0,t]} \overline{G}(s-)\frac{dF}{dG}(s)\,dG(s) \\ &= F(0)\overline{G}(0) - \int_{(0,t]} H(s)\,dG(s), \end{aligned}$$

i.e. (13) holds. Therefore, Condition M is satisfied. Conversely, let (13) hold. In the proof of part (a) we deduced from (15) (and, hence, from (13)) that

$$\frac{dF}{dG}(t) = -\frac{H(t)}{\overline{G}(t-)} + \frac{F(t)}{\overline{G}(t-)}, \quad dG\text{-a.s.},$$

and (17) follows. □

Proof of Theorem 2. Let $M = (M_t)_{t\in\mathbb{R}_+}$ be a local martingale. By Proposition 2 (v) and Theorem 1, M has representation (6) and, moreover, (7) and (8) hold. Define the function $H(t)$, $t \in \mathcal{T}$, by (10). Then, see (9),

$$\mathsf{E}\big(|H(\gamma)|\mathbb{1}_{\{\gamma\leqslant t\}}\big) \leqslant \mathsf{E}\big(|L|\mathbb{1}_{\{\gamma\leqslant t\}}\big) < \infty,$$

which implies $H \in L^1_{\text{loc}}(dG)$. Putting $L' = \big(L - H(\gamma)\big)\mathbb{1}_{\{\gamma<\infty\}}$, we obtain (19) as well. Now it follows from (20) and the second relation in (19) that

$$\mathsf{E}(M_t) = F(t)\overline{G}(t) + \int_{[0,t]} H(s)\,dG(s), \quad t \in \mathcal{T},$$

so (13) and (14) follow from (8). Finally, (11) follows from Proposition 3 (a).

Conversely, let (20) hold true with a pair (F, H) satisfying Condition M and a random variable L' satisfying (19). Then, putting $L = H(\gamma) + L'$, we obtain (6) and (7). It remains to note that (13) and (14) (in case B) imply (8). □

Proof of Corollary 1. The required decomposition follows from (20) if we put

$$M'_t = F(t)\mathbb{1}_{\{t<\gamma\}} + H(\gamma)\mathbb{1}_{\{t\geqslant\gamma\}}, \qquad M''_t = L'\mathbb{1}_{\{t\geqslant\gamma\}}.$$

Let a local martingale M with a representation (20) vanish on $\{t < \gamma\}$ and $\mathsf{E}M_0 = 0$. Then $F(t) \equiv 0$ for $t < t_G$ and $0 = \mathsf{E}M_0 = H(0)\overline{G}(0) + \mathsf{E}\big(L'\mathbb{1}_{\{\gamma=0\}}\big) = H(0)\overline{G}(0)$ in view of the second relation in (19). By Theorem 2, it follows from (13) and (14) that $H(t) = 0$ dG-a.s. Now, if M is also adapted with respect to the smallest filtration making γ a stopping time, then $M_\gamma = \big(H(\gamma) + L'\big)\mathbb{1}_{\{\gamma<\infty\}} = L'\mathbb{1}_{\{\gamma<\infty\}}$ is $\sigma\{\gamma\}$-measurable. Using again the second relation in (19), we conclude that $L'\mathbb{1}_{\{\gamma<\infty\}} = 0$ a.s. This proves the unicity. □

Proof of Theorem 3. To prove sufficiency it is enough to consider the case, where $H \equiv 0$ and $F \equiv 0$. In view of the first condition in (22), there exists a Borel function $J\colon (0,\infty] \to \mathbb{R}_+$ such that $\mathsf{E}\big[|L'|\big|\gamma = t\big] = J(t)$. Put $\Sigma_n = \Omega \times \{t \in (0,+\infty)\colon J(t) \leqslant n\}$ and consider the Lebesgue–Stieltjes integral process $\mathbb{1}_{\Sigma_n}\cdot M_t = L'\mathbb{1}_{\big\{\mathsf{E}\big[|L'|\big|\gamma\big]\leqslant n\big\}}\mathbb{1}_{\{t\geqslant\gamma>0\}}$. By Theorem 2, cf. condition (19), it is a local martingale. Since Σ_n are predictable and $\cup_n \Sigma_n = \Omega\times\mathbb{R}_+$, M is a σ-martingale.

Conversely, let M be a σ-martingale. It is easy to check that to prove necessity it is enough to consider the case $M_0 = 0$. According to Proposition 2 (v) and Remark 1

$$M_t = F(t)\mathbb{1}_{\{t<\gamma\}} + L\mathbb{1}_{\{t\geqslant\gamma\}}, \tag{25}$$

where L is a random variable, $F(t)$, $0 \leqslant t < t_G$, is a deterministic function with finite variation over $[0,t]$ for every $t < t_G$ in case A and over $[0,t_G)$ in Case B. By the definition of σ-martingales, there is an increasing sequence of predictable sets Σ_n such that $\cup_n \Sigma_n = \Omega\times\mathbb{R}_+$ and the integral process $\mathbb{1}_{\Sigma_n}\cdot M$ is a local martingale for every n. It was mentioned in the proof of Proposition 2 that the integral is understood as the Lebesgue–Stieltjes integral. By Proposition 2 (ii), there are Borel subsets D_n of $\mathbb{R}_+$ such that $\mathbb{1}_{\Sigma_n}(\omega,t)\mathbb{1}_{\{\gamma(\omega)\geqslant t\}} = \mathbb{1}_{D_n}(t)\mathbb{1}_{\{\gamma(\omega)\geqslant t\}}$, in particular,

$$\cup_n D_n \supseteq \mathcal{T}. \tag{26}$$

According to Theorem 2 and Proposition 3, $\mathbb{1}_{\Sigma_n}\cdot M$ has a representation

$$\mathbb{1}_{\Sigma_n}\cdot M_t = F^n(t)\mathbb{1}_{\{t<\gamma\}} + \big(H^n(\gamma) + L^n\big)\mathbb{1}_{\{t\geqslant\gamma\}}$$

where $\mathsf{E}\big(|L^n|\mathbb{1}_{\{\gamma\leqslant t\}}\big) < \infty$, $t\in\mathcal{T}$, $\mathsf{E}[L^n|\gamma] = 0$, $H^n \in L^1_{\mathrm{loc}}(dG)$, $F^n \overset{\mathrm{loc}}{\ll} G$,

$$H^n(t) = F^n(t) - \overline{G}(t-)\frac{dF^n}{dG}(t) = F^n(t-) - \overline{G}(t)\frac{dF^n}{dG}(t), \tag{27}$$

$0 < t < t_G$, and in Case B $H^n(t_G) := \lim_{t\uparrow\uparrow t_G} F^n(t)$.

Combining with (25), we get

$$F^n(t) = \int_{(0,t]} \mathbb{1}_{D_n}(s)\,dF(s), \quad 0 < t < t_G, \tag{28}$$

and

$$H^n(\gamma) + L^n = \int_{(0,\gamma)} \mathbb{1}_{D_n}(s)\,dF(s) + \mathbb{1}_{D_n}(\gamma)\big(L - F(\gamma-)\big) \quad \text{a.s.} \tag{29}$$

Since $F^n \overset{\mathrm{loc}}{\ll} G$, it follows from (28) and (26) that $F \overset{\mathrm{loc}}{\ll} G$. Substituting (28) in (29) and taking conditional expectation given γ, we get

$$H^n(\gamma) - F^n(\gamma-) = \mathbb{1}_{D_n}(\gamma)\big(H(\gamma) - F(\gamma-)\big) \quad \text{a.s.,}$$

i.e.

$$H^n(t) - F^n(t-) = \mathbb{1}_{D_n}(t)\big(H(t) - F(t-)\big) \quad dG(t)\text{-a.s.} \tag{30}$$

It follows from (28) and (27) that

$$-\overline{G}(t)\mathbb{1}_{D_n}(t)\frac{dF}{dG}(t) = -\overline{G}(t)\frac{dF^n}{dG}(t) = \mathbb{1}_{D_n}(t)\big(H(t) - F(t-)\big) \quad dG(t)\text{-a.s.},$$

so, taking (26) into account, we obtain

$$H(t) = F(t) - \overline{G}(t-)\frac{dF}{dG}(t) \quad dG(t)\text{-a.s.}$$

Additionally, in Case B, the left-hand side of (30) at $t = t_G$ vanishes, hence, $H(t_G) := \lim_{t\uparrow\uparrow t_G} F(t)$. It remains to put $L' = L - H(\gamma)$. □

Proof of Theorem 4. In Case B

$$\mathrm{Var}(M)_\infty \leqslant 2\,\mathrm{Var}(F)_{t_G-} + |L|,$$

and the first term is finite by Remark 1, while $\mathsf{E}|L| < \infty$ due to (7). Thus, we proceed to Case A.

(i) Note that

$$\int\limits_{[0,t_G)} |H(s)|\,dG(s) = \mathsf{E}\big(|H(\gamma)|\mathbb{1}_{\{\gamma<t_G\}}\big) = \mathsf{E}\big(|H(\gamma)|\mathbb{1}_{\{\gamma<\infty\}}\big)$$

and $\mathsf{E}\big(|L|\mathbb{1}_{\{\gamma<\infty\}}\big) < \infty$ if and only if

$$\mathsf{E}\big(|H(\gamma)|\mathbb{1}_{\{\gamma<\infty\}}\big) < \infty \qquad \text{and} \qquad \mathsf{E}\big(|L'|\mathbb{1}_{\{\gamma<\infty\}}\big) < \infty.$$

Next, if M_∞ is well defined, then

$$M_\infty = L\mathbb{1}_{\{\gamma<\infty\}} + \lim_{t\to\infty} F(t)\mathbb{1}_{\{\gamma=\infty\}}.$$

Finally, if $\mathsf{P}(\gamma = \infty) > 0$, then it follows from (13) that $\lim_{t\to\infty} F(t)$ exists and is finite if $\int\limits_{[0,t_G)} |H(s)|\,dG(s) < \infty$. Now, combining all above, we arrive at (i).

(ii) If $\mathsf{P}(\gamma = \infty) > 0$, then

$$\mathrm{Var}(M)_\infty \leqslant 2\,\mathrm{Var}(F)_\infty + |L|\mathbb{1}_{\{\gamma<\infty\}},$$

and the last term on the right has finite expectation by assumptions. Since $\overline{G}(t) \geqslant \mathsf{P}(\gamma = \infty) > 0$ in the case under consideration, it follows from assumptions and (13) that F has a finite variation over $\mathbb{R}_+$.

From now on we assume that $\mathsf{E}\big(|L'|\mathbb{1}_{\{\gamma<\infty\}}\big) < \infty$, $\int_{[0,t_G)} |H(s)|\,dG(s) < \infty$, and $\mathsf{P}(\gamma = t_G) = 0$. Then M is a martingale on $[0, t_G)$ by Theorem 1 and it coincides with $M_\infty = L$ for $t \geqslant t_G$. Hence, it is a (necessarily closed) submartingale (resp. supermartingale) if and only if $\mathsf{E}[L - M_t|\mathscr{F}_t] \geqslant 0$ (resp. $\leqslant 0$), for $t < t_G$. As in the proof of Theorem 1,

$$L - M_t = 0 \quad \text{on } \{t \geqslant \gamma\},$$

hence,

$$\mathsf{E}[L - M_t | \mathscr{F}_t] = \mathrm{const}\mathbb{1}_{\{t<\gamma\}}.$$

Taking expectations, we see that this constant has the same sign as $\mathsf{E}(L - M_t) = \mathsf{E}(L - M_0)$. However,

$$\mathsf{E}(L - M_0) = \mathsf{E}\big(H(\gamma) - M_0\big) = \int\limits_{(0,t_G)} H(s)\, dG(s) - F(0)\overline{G}(0) = -\lim_{t \upuparrows t_G} F(t)\overline{G}(t),$$

and (iii.i) follows.

The same proof shows that if $\int_{(0,t_G)} H(s)\, dG(s) = F(0)\overline{G}(0)$, then M is a uniformly integrable martingale. Therefore, to prove (iii.ii) and (iii.iii) it is enough to show that $\mathsf{E}(\sup_t |M_t|) < \infty$ implies (24), and that (24) implies $\mathsf{E}\big(\mathrm{Var}(M)_\infty\big) < \infty$.

If M is a local martingale with $\mathsf{E}(\sup_t |M_t|) < \infty$, then $\mathsf{E}\big(|\Delta M_\gamma|\mathbb{1}_{\{\gamma<\infty\}}\big) < \infty$. But $|\Delta M_\gamma|\mathbb{1}_{\{\gamma<\infty\}} = |L - F(\gamma-)|\mathbb{1}_{\{\gamma<\infty\}}$, hence, taking conditional expectation given γ, we get

$$\int\limits_{[0,t_G)} |H(s) - F(s-)|\, dG(s) < \infty.$$

In view of (21) which is equivalent to (17), we obtain (24).

Conversely, let (24) hold. Then

$$\begin{aligned}
&\mathrm{Var}(M)_\infty \\
&\quad = |L|\mathbb{1}_{\{\gamma=0\}} + |F(0)|\mathbb{1}_{\{\gamma>0\}} + \int\limits_{(0,\gamma)} \Big|\frac{dF}{dG}(s)\Big|\, dG(s) + |L - F(\gamma-)|\mathbb{1}_{\{0<\gamma<\infty\}} \\
&\quad \leqslant 2|L|\mathbb{1}_{\{\gamma<\infty\}} + 2|F(0)|\mathbb{1}_{\{\gamma>0\}} + 2\int\limits_{(0,\gamma)} \Big|\frac{dF}{dG}(s)\Big|\, dG(s)
\end{aligned}$$

and

$$\begin{aligned}
\mathsf{E}\Big(\int_{(0,\gamma)} \Big|\frac{dF}{dG}(s)\Big|\, dG(s)\Big) &= \int\limits_{[0,t_G)} \int\limits_{(0,u)} \Big|\frac{dF}{dG}(s)\Big|\, dG(s)\, dG(u) \\
&= \int\limits_{[0,t_G)} \Big|\frac{dF}{dG}(s)\Big| \int\limits_{(s,t_G)} dG(u)\, dG(s) \\
&= \int\limits_{[0,t_G)} \overline{G}(s)\Big|\frac{dF}{dG}(s)\Big|\, dG(s) < \infty. \qquad \square
\end{aligned}$$

Remark 8. It follows from the last equalities in the proof that, due to (11) and (21) respectively, (24) implies that the following integrals are also finite:

$$\int\limits_{[0,t_G)} |F(s-)|\, dG(s) < \infty \qquad \text{and} \qquad \int\limits_{[0,t_G)} |H(s)|\, dG(s) < \infty.$$

However, it may happen that

$$\int_{[0,t_G)} |F(s)|\,dG(s) = \infty,$$

see an example in [13, Remark 3.11].

4 Complements

4.1 Single jump processes and their compensators

Let us consider the same setting as in Section 2 and let V be a finite random variable. For simplicity, we assume that $\{\gamma = 0\} \subseteq \{V = 0\}$. Then

$$X_t = V\mathbb{1}_{\{t \geqslant \gamma\}}$$

is an adapted process of finite variation on compact intervals.

Lemma 1. *The process $X = (X_t)_{\mathbb{R}_+}$ is of locally integrable variation if and only if*

$$\mathsf{E}\big(|V|\mathbb{1}_{\{\gamma \leqslant t\}}\big) < \infty, \quad t \in \mathcal{T}. \tag{31}$$

Proof. Let (31) hold. If $t_G \in \mathcal{T}$, then $\mathsf{E}\big(|V|\mathbb{1}_{\{\gamma \leqslant t_G\}}\big) < \infty$ means that the process X itself has integrable variation. In Case A, put

$$T_n = \begin{cases} t_n, & \text{if } \gamma > t_n; \\ +\infty, & \text{otherwise.} \end{cases}$$

where $t_n \upuparrows t_G$. Then $T_n \uparrow \infty$ a.s. and $\mathrm{Var}(X^{T_n})_\infty = |V|\mathbb{1}_{\{\gamma \leqslant T_n\}} = |V|\mathbb{1}_{\{\gamma \leqslant t_n\}}$.

Conversely, let $\{T_n\}$ be a localizing sequence of stopping times such that $\mathsf{E}\big(|V|\mathbb{1}_{\{\gamma \leqslant T_n\}}\big) < \infty$. If $\mathsf{P}(\gamma \leqslant T_n) = 1$ for n large enough, then V is integrable. So assume that $\mathsf{P}(\gamma > T_n) > 0$ for every n. By Proposition 1 (iv), there are numbers r_n such that $\{T_n < \gamma\} = \{T_n = r_n < \gamma\} = \{r_n < \gamma\}$. Thus, we have a sequence r_n such that $\mathsf{E}\big(|V|\mathbb{1}_{\{\gamma \leqslant r_n\}}\big) < \infty$. Since $T_n \to \infty$ a.s. and the sequence $\{T_n\}$ is increasing, in Case A it follows that $r_n \uparrow t_G$, and in Case B we come to a contradiction by repeating the arguments in the concluding part of the proof of Theorem 1. □

From now on we will assume that X is a process of locally integrable variation, i.e. (31) holds. Our aim is to find its compensator. We can introduce a function K similarly as the function H is introduced in (10):

$$K(t) = \mathsf{E}[V|\gamma = t], \quad t \in \mathcal{T}. \tag{32}$$

It is clear that $K \in L^1_{\mathrm{loc}}(dG)$ and $K(0) = 0$ if $\mathsf{P}(\gamma = 0) > 0$. Now define

$$F(t) = \int_{(0,t]} \overline{G}(s-)^{-1} K(s)\,dG(s), \quad 0 \leqslant t < t_G, \tag{33}$$

in particular, $F(0) = 0$. It follows that, in Case B, the function F has a bounded variation on $[0, t_G)$ and has a finite limit as $t \upuparrows t_G$, so we put

$$F(t_G) = \lim_{t \upuparrows t_G} F(t). \tag{34}$$

The next theorem takes its origin in [4], where the case when $V = 1$, γ is finite and $t_G = +\infty$ is considered.

Theorem 5. *Let V be a random variable satisfying (31), K and F defined in (32)–(34). Then the compensator A_t of the process $X_t = V\mathbb{1}_{\{t \geqslant \gamma\}}$ is given by*

$$A_t = F(t \wedge \gamma) \quad \textit{in Case A}$$

and

$$A_t = F(t \wedge \gamma) + K(t_G)\mathbb{1}_{\{\gamma \geqslant t_G\}}\mathbb{1}_{\{t \geqslant t_G\}} \quad \textit{in Case B}.$$

Proof. The process $t \wedge \gamma$ is adapted and continuous, hence, it is predictable. It follows that $F(t \wedge \gamma)$ is predictable. Next, in Case B, the set $\{(\omega, t) \colon \gamma(\omega) \geqslant t_G,\ t \geqslant t_G\}$ coincides with the intersection of predictable sets

$$\{\gamma \geqslant t_G\} \times [t_G, \infty) = \bigcap_n \Big[\{\gamma > t_G - n^{-1}\} \times (t_G - n^{-1}, \infty)\Big],$$

therefore, A is predictable. Hence it is enough to show that $M = A - X$ is a local martingale.

We use Theorem 2 and Proposition 3 (b). M has the representation (6) with the same function F and $L = F(\gamma)\mathbb{1}_{\{\gamma<\infty\}} - V\mathbb{1}_{\{\gamma<\infty\}} + K(t_G)\mathbb{1}_{\{\gamma=t_G<\infty\}}$. Define the function H as in Proposition 3 (b). Then it follows from (33) that $H(t) = F(t) - K(t)$, $0 < t < t_G$, and, in Case B, $H(t_G) = F(t_G)$. On the other hand, we have $\mathsf{E}[L|\gamma = t] = F(t) - K(t) = H(t)$, $0 < t < t_G$, and, in Case B, $\mathsf{E}[L|\gamma = t_G] = F(t_G) - K(t_G) + K(t_G) = F(t_G) = H(t_G)$. The claim follows. □

4.2 *Example: submartingales of class* (Σ)

Recall, see [22], that a nonnegative submartingale $X = (X_t)_{t\in\mathbb{R}_+}$ is called a submartingale of class (Σ) if $X_0 = 0$ and it can be decomposed as $X_t = N_t + A_t$, where $N = (N_t)_{t\in\mathbb{R}_+}$, $N_0 = 0$, is a local martingale, $A = (A_t)_{t\in\mathbb{R}_+}$, $A_0 = 0$, is a *continuous* increasing process, and the measure (dA_t) is carried by the set $\{t \colon X_t = 0\}$. A typical example is a process $X_t = \overline{L}_t - L_t$ which is the difference between the running maximum $\overline{L}_t$ of a continuous local martingale (L_t) and L_t itself.

Let $X = (X_t)_{t\in\mathbb{R}_+}$ be a nonnegative submartingale with the Doob–Meyer decomposition $X_t = N_t + A_t$, where $N_0 = A_0 = 0$, N is a local martingale, A is a predictable increasing process. Assume that $A_\infty < \infty$ a.s. and put $C_t = \inf\{s \geqslant 0 \colon A_s > t\}$. Then, see [10, Lemma 3.1], X is of class (Σ) if and only if a.s.

$$A_{C_t} = A_\infty \wedge t \qquad \text{and} \qquad X_{C_t} = X_\infty \mathbb{1}_{\{t \geqslant A_\infty\}},$$

where a finite limit $X_\infty := \lim_{t\to\infty} X_t$ exists a.s. by [10, Proposition 3.1]. Therefore, the process $M_t = -N_{C_t}$ has the representation

$$M_t = t \wedge \gamma - V\mathbb{1}_{\{t \geqslant \gamma\}}, \quad \text{where } \gamma = A_\infty \text{ and } V = X_\infty.$$

M may not be a local martingale. For example, take as L a Brownian motion stopped when it hits 1 and define $X = \overline{L} - L$, then $M_t = t \wedge 1$. However, if X is a submartingale of class (D) then N is a uniformly integrable martingale and M is also a uniformly integrable martingale (with respect to its own filtration and, by Theorem 1, with respect to the single jump filtration generated by γ on an original space). Now we can define a function K according to (32) and conclude that (33) is valid with $F(t) = t$. We may interpret (33) as the equation with known K and unknown G. This identity says that the Lebesgue measure on $[0, t_G)$ is absolutely continuous with respect to dG but not vice versa. However, if the function $K(t)$ does not vanish (dG-a.s.) then we obtain from (33) that dG is equivalent to the Lebesgue measure on $\mathcal{T}$, in particular, G is continuous, and

$$\mathsf{P}(\gamma > t) = \exp\Big(-\int_0^t \frac{dt}{K(t)}\Big), \quad t < t_G.$$

This statement coincides with Theorem 4.1 in [22]. If the function $K(t)$ may vanish, analysis of equation (33) with $F(t) = t$, known $K(t)$ and unknown $G(t)$ is done in [23].

A kind of a converse statement is proved in [11]. If, say, a martingale M satisfies

$$M_t = t \wedge \gamma - V \mathbb{1}_{\{t \geqslant \gamma\}},$$

where $\gamma < \infty$ and $V \geqslant 0$, then, using Monroe's theorem [20], we prove that there is a Brownian motion B and a finite stopping time T such that, for the stopped process $L = B^T$, the joint law of its terminal value L_∞ and its maximum $\overline{L}_\infty$ coincides with that of M, that is, with the law of $(\gamma - V, \gamma)$. In particular, this shows that a distribution function G is the law of the maximum of a uniformly integrable continuous martingale L with $L_0 = 0$ if and only if, with $F(t) = t$, $0 \leqslant t < t_G$, we have $F \overset{\text{loc}}{\ll} G$, $\int_{[0,t_G)} |H(s)|\, dG(s) < \infty$, where H is defined by (17), and $G(t) = o(t^{-1})$, see conditions for M to have type 3 or 4 in Theorem 4. This gives an alternative proof of the main result in [23].

Acknowledgments

We thank three anonymous referees for careful reading of the paper and constructive comments and suggestions for improving the presentation. A special thanks goes to the referee who suggested Theorem 3 on a characterisation of σ-martingales.

References

[1] Boel, R., Varaiya, P., Wong, E.: Martingales on jump processes. I. Representation results. SIAM J. Control **13**(5), 999–1021 (1975). MR0400379. https://doi.org/10.1137/0313063

[2] Chou, C.-S., Meyer, P.-A.: Sur la représentation des martingales comme intégrales stochastiques dans les processus ponctuels. In: Séminaire de Probabilités, IX. Lecture Notes in Math., vol. 465, pp. 226–236. Springer (1975). MR0436310. https://doi.org/10.1007/BFb0102993

[3] Davis, M.H.A.: The representation of martingales of jump processes. SIAM J. Control Optim. **14**(4), 623–638 (1976). MR0418221. https://doi.org/10.1137/0314041

[4] Dellacherie, C.: Un exemple de la théorie générale des processus. In: Séminaire de Probabilités, IV. Lecture Notes in Math., vol. 124, pp. 60–70. Springer (1970). MR0263157. https://doi.org/10.1007/BFb0059333

[5] Dellacherie, C., Meyer, P.-A.: Probabilities and Potential. North-Holland Mathematics Studies, vol. 29, p. 189. North-Holland Publishing Co., Amsterdam-New York (1978). MR521810

[6] Dellacherie, C., Meyer, P.-A.: Probabilities and Potential. B. North-Holland Mathematics Studies, vol. 72, p. 463. North-Holland Publishing Co., Amsterdam (1982). MR745449

[7] Dubins, L.E., Gilat, D.: On the distribution of maxima of martingales. Proc. Am. Math. Soc. **68**(3), 337–338 (1978). MR0494473. https://doi.org/10.1090/S0002-9939-1978-0494473-4

[8] Elliott, R.J.: Stochastic integrals for martingales of a jump process with partially accessible jump times. Z. Wahrscheinlichkeitstheor. Verw. Geb. **36**(3), 213–226 (1976). MR0420846. https://doi.org/10.1007/BF00532546

[9] Gushchin, A.A.: Stochastic Calculus for Quantitative Finance, p. 185. ISTE Press, London; Elsevier Ltd, Oxford (2015). MR3410512

[10] Gushchin, A.A.: The joint law of terminal values of a nonnegative submartingale and its compensator. Theory Probab. Appl. **62**(2), 216–235 (2018). MR3649035. https://doi.org/10.1137/S0040585X97T988575

[11] Gushchin, A.A.: The joint law of a max-continuous local submartingale and its maximum. Theory Probab. Appl. **65** (2020)

[12] He, S.W.: Some remarks on single jump processes. In: Seminar on Probability, XVII. Lecture Notes in Math., vol. 986, pp. 346–348. Springer (1983). MR0770423. https://doi.org/10.1007/BFb0068327

[13] Herdegen, M., Herrmann, S.: Single jump processes and strict local martingales. Stoch. Process. Appl. **126**(2), 337–359 (2016). MR3434986. https://doi.org/10.1016/j.spa.2015.09.003

[14] Jacod, J., Skorohod, A.V.: Jumping filtrations and martingales with finite variation. In: Séminaire de Probabilités, XXVIII. Lecture Notes in Math., vol. 1583, pp. 21–35. Springer (1994). MR1329098. https://doi.org/10.1007/BFb0073831

[15] Jacod, J.: Multivariate point processes: predictable projection, Radon-Nikodým derivatives, representation of martingales. Z. Wahrscheinlichkeitstheor. Verw. Geb. **31**, 235–253 (1975). MR0380978. https://doi.org/10.1007/BF00536010

[16] Jacod, J.: Un théorème de représentation pour les martingales discontinues. Z. Wahrscheinlichkeitstheor. Verw. Geb. **34**(3), 225–244 (1976). MR0418222. https://doi.org/10.1007/BF00532705

[17] Jacod, J.: Calcul Stochastique et Problèmes de Martingales. Lecture Notes in Mathematics, vol. 714, p. 539. Springer (1979). MR542115

[18] Jeanblanc, M., Rutkowski, M.: Modelling of default risk: an overview. In: Mathematical Finance: Theory and Practice, pp. 171–269. Higher Education Press, Beijing (2000)

[19] Jeanblanc, M., Yor, M., Chesney, M.: Mathematical Methods for Financial Markets. Springer Finance, p. 732. Springer (2009). MR2568861. https://doi.org/10.1007/978-1-84628-737-4

[20] Monroe, I.: On embedding right continuous martingales in Brownian motion. Ann. Math. Stat. **43**(4), 1293–1311 (1972). MR0343354. https://doi.org/10.1214/aoms/1177692480

[21] Neveu, J.: Processus ponctuels. In: École D'Été de Probabilités de Saint-Flour, VI—1976, pp. 249–445. Springer (1977). MR0474493. https://doi.org/10.1007/BFb0097494

[22] Nikeghbali, A.: A class of remarkable submartingales. Stoch. Process. Appl. **116**(6), 917–938 (2006). MR2254665. https://doi.org/10.1016/j.spa.2005.12.003

[23] Vallois, P.: Sur la loi du maximum et du temps local d'une martingale continue uniformement intégrable. Proc. Lond. Math. Soc. (3) **69**(2), 399–427 (1994). MR1281971. https://doi.org/10.1112/plms/s3-69.2.399