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Irregular barrier reflected BDSDEs with general jumps under stochastic Lipschitz and linear growth conditions

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Abstract In this paper, a solution is given to reflected backward doubly stochastic differential equations when the barrier is not necessarily right-continuous, and the noise is driven by two independent Brownian motions and an independent Poisson random measure. The existence and uniqueness of the solution is shown, firstly when the coefficients are stochastic Lipschitz, and secondly by weakening the conditions on the stochastic growth coefficient.

Keywords Reflected backward doubly stochastic differential equations, irregular barrier, Mertens decomposition, stochastic Lipschitz condition, stochastic linear growth condition

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1 Introduction

Nonlinear backward stochastic differential equations (BSDEs in short) were first introduced by Pardoux and Peng [27] with the uniform Lipschitz condition under which they proved the celebrated existence and uniqueness result. Since then, the theory of BSDEs has been intensively developed in the last years. The great interest in this the-

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ory comes from its connections with many other fields of research, such as mathematical finance [12, 11], stochastic control and stochastic games [10] and partial differential equations [28]. After Pardoux and Peng introduced the theory of BSDEs, they considered [29] a new kind of BSDEs, that is a class of backward doubly stochastic differential equations (BDSDEs in short) with two different directions of stochastic integrals with respect to two independent Brownian motions. They proved the existence and uniqueness of solutions to BDSDEs under uniform Lipschitz conditions on the coefficients.

In the setting of reflected BSDEs (resp. BDSDEs), an additional nondecreasing process is added in order to keep the solution above a certain lower-boundary process, called barrier (or obstacle), and to do this in a minimal fashion. The reflected BSDEs (RBSDEs in short) were introduced by El Karoui et al. [13], again under the uniform Lipschitz condition on the coefficients. The authors of [13] proved the existence and uniqueness results in the case of a Brownian filtration and a continuous barrier. The reflected BDSDEs (RBDSDEs in short) were introduced by Bahlali et al. [6] where the authors studied the case of RBDSDEs with continuous coefficients, and proved the existence and uniqueness of the solution.

To the best of our knowledge, the paper by Grigorova et al. [14] is the first one which studied RBSDEs in the case where the barrier is not necessarily right-continuous (just right upper semi-continuous). The authors of [14] studied the existence and uniqueness result under the Lipschitz assumption on the coefficients in a filtration that supports a Brownian motion and an independent Poisson random measure. Later, several authors have studied the RBSDEs following Grigorova et al. [14] (see e.g. [1–3, 17, 20, 23]). Recently, Berrhazi et al. [7] discussed the case of RBDSDE with a right upper semi-continuous barrier under Lipschitz coefficients.

Our aim in this paper is to extend the work on RBDSDEs with jumps (RBDSDEJs in short) to the case of an irregular barrier (which is assumed to be not necessarily right-continuous). The specificity of such equations lies in the fact that the two independent Brownian motions are coupled with an independent Poisson random measure. We'll prove the existence and uniqueness of the solution to such equations under the so-called stochastic Lipschitz coefficients. The interest in this last condition is based on the fact that, unfortunately, in many applications, the usual Lipschitz conditions cannot be satisfied. For example, the pricing of the American claim is equivalent to solving the linear RBDSE

$$\begin{cases} -dV_t = (r_t V_t + \theta_t Z_t)dt - Z_t dW_t + dK_t, \quad V_T = \xi_T; \\ V_t \geq \xi_t, \quad (V_t - \xi_t)dK_t = 0 \quad \text{a.s.} \end{cases} \tag{1}$$

where ξ_t is the amount received from the seller at time t, r_t is the interest rate process and θ_t is the risk premium process. The additional process K is needed for this problem because there exists no replicating strategy for the option. We have to use a super-replicating strategy with a consumption process K. The minimality condition on K just states that we only invest money in the portfolio when $V_t > \xi_t$. Here both r_t and θ_t are not bounded in general. So, it is not possible to solve the RBSDE (1) by the result of El Karoui et al. [13]. Thus, in order to study more general RBSDEs (resp. RBDSDEs), one needs to relax the uniform Lipschitz conditions on the coefficients. To this direction, several attempts have been done. Among others, we refer to

[4, 5, 9, 15, 21–24] for the case of BSDEs, and [16, 25, 26, 30] for BDSDEs.

In our paper, we use a generalization of the Doob–Meyer decomposition called the Mertens decomposition. This decomposition is used for strong optional supermartingales which are not necessarily right-continuous. We also use some tools from the optimal stopping theory, as well as a generalization of the Itô formula to the case of a strong optional supermartingale called the Gal'chouk–Lenglart formula due to Lenglart [19].

The paper is organized as follows. In Section 2, we give some notations, assumptions and main contributions needed in this paper. In Section 3, we prove the existence and uniqueness of the solution to RBDSDEJs with a stochastic Lipschitz coefficients (f, g) and an irregular barrier ξ, and we also give a comparison theorem for solutions. Section 4 is devoted to prove the existence of a minimal solution to RBDSDEJs under a stochastic growth coefficient f.

2 Definitions and preliminary results

Let $0 < T < +\infty$ be a non-random horizon time, Ω be a non-empty set, $\mathscr{F}$ be a σ-algebra of sets of Ω and $\mathbf{P}$ be a probability measure defined on $\mathscr{F}$. The triple $(\Omega, \mathscr{F}, \mathbf{P})$ defines a probability space which is assumed to be complete. We assume there are three mutually independent processes:

- a d-dimensional Brownian motion $(W_t)_{t\leq T}$,
- a ℓ-dimensional Brownian motion $(B_t)_{t\leq T}$,
- a random Poisson measure μ on $E \times \mathbf{R}_+$ with compensator $\nu(dt, de) = \lambda(de)dt$, where the space $E = \mathbf{R}^\ell - \{0\}$ is equipped with its Borel field $\mathcal{E}$ such that $\{\widetilde{\mu}([0,t] \times \mathcal{B}) = (\mu - \nu)[0,t] \times \mathcal{B}\}$ is a martingale for any $\mathcal{B} \in \mathcal{E}$ satisfying $\lambda(\mathcal{B}) < \infty$. λ is a σ-finite measure on $\mathcal{E}$ and satisfies

$$\int_E (1 \wedge |e|^2)\lambda(de) < \infty.$$

We consider the family $(\mathscr{F}_t)_{t\leq T}$ given by

$$\mathscr{F}_t = \mathscr{F}_t^W \vee \mathscr{F}_{t,T}^B \vee \mathscr{F}_t^\mu, \quad 0 \leq t \leq T,$$

where for any process $(\eta_t)_{t\leq T}$, $\mathscr{F}_{s,t}^\eta = \sigma\{\eta_r - \eta_s,\ s \leq r \leq t\} \vee \mathcal{N}$, $\mathscr{F}_t^\eta = \mathscr{F}_{0,t}^\eta$. Here $\mathcal{N}$ denotes the class of $\mathbf{P}$-null sets of $\mathscr{F}$. Note that the family $(\mathscr{F}_t)_{t\leq T}$ does not constitute a classical filtration.

For an integer $k \geq 1$, $|\ .\ |$ and $\langle .,. \rangle$ stand for the Euclidian norm and the inner product in $\mathbf{R}^k$, $\mathcal{T}_{[t,T]}$ denotes the set of stopping times τ such that $\tau \in [t, T]$ and $\mathscr{P}$ denotes the σ-algebra of $\mathscr{F}_t$-predictable sets of $\Omega \times [0, T]$.

For every $\mathscr{F}_t$-measurable process $(a_t)_{t\leq T}$, we define an increasing process $(A_t)_{t\leq T}$ by setting $A_t = \int_0^t a_s^2 ds$.

For every $\beta > 0$, we consider the following sets (where $\mathbf{E}$ denotes the mathematical expectation with respect to the probability measure $\mathbf{P}$):

- $\mathscr{S}^2(\mathbf{R}^k)$ and $\mathscr{S}^2_\beta(\mathbf{R}^k)$ are the spaces of $\mathscr{F}_t$-adapted optional processes $\Psi : \Omega \times [0, T] \longrightarrow \mathbf{R}^k$ which satisfy, respectively,

$$\|\Psi\|^2_{\mathscr{S}^2(\mathbf{R}^k)} = \mathbf{E}\left(\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[0,T]}} |\Psi_\tau|^2\right) < +\infty$$

and

$$\|\Psi\|^2_{\mathscr{S}^2_\beta(\mathbf{R}^k)} = \mathbf{E}\left(\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[0,T]}} e^{\beta A_\tau}|\Psi_\tau|^2\right) < +\infty.$$

- $\mathscr{M}^2(\mathbf{R}^{k\times d})$, $\mathscr{M}^2_\beta(\mathbf{R}^{k\times d})$ and $\mathscr{M}^{2,a}_\beta(\mathbf{R}^k)$ are the spaces of $\mathscr{F}_t$-progressively measurable processes $\Psi : \Omega \times [0, T] \longrightarrow \mathbf{R}^{k\times d}$ (resp. $\mathbf{R}^k$) which satisfy, respectively,

$$\|\Psi\|^2_{\mathscr{M}^2(\mathbf{R}^{k\times d})} = \mathbf{E}\left(\int_0^T |\Psi_t|^2\, dt\right) < +\infty,$$
$$\|\Psi\|^2_{\mathscr{M}^2_\beta(\mathbf{R}^{k\times d})} = \mathbf{E}\left(\int_0^T e^{\beta A_t}|\Psi_t|^2\, dt\right) < +\infty$$

and

$$\|\Psi\|^2_{\mathscr{M}^{2,a}_\beta(\mathbf{R}^k)} = \mathbf{E}\left(\int_0^T e^{\beta A_t} a_t^2 |\Psi_t|^2\, dt\right) < +\infty.$$

- $\mathscr{L}_\lambda$ is the space of $\mathscr{P}\otimes\mathcal{E}$-measurable mappings $U : E \longrightarrow \mathbf{R}^k$ such that

$$\|U\|^2_\lambda = \int_E |U(e)|^2 \lambda(de) < +\infty.$$

- $\mathscr{L}^2_\beta(\mathbf{R}^k)$ is the space of $\mathscr{P}\otimes\mathcal{E}$-measurable processes $U : \Omega \times [0, T] \times E \longrightarrow \mathbf{R}^k$ such that

$$\|U\|^2_{\mathscr{L}^2_\beta(\mathbf{R}^k)} = \mathbf{E}\left(\int_0^T e^{\beta A_t}\, \|U_t\|^2_\lambda\, dt\right) < +\infty.$$

Notice that the space

$$\mathcal{A}^2_\beta(\mathbf{R}^k) = \mathscr{M}^{2,a}_\beta(\mathbf{R}^k) \times \mathscr{M}^2_\beta(\mathbf{R}^{k\times d}) \times \mathscr{L}^2_\beta(\mathbf{R}^k)$$

endowed with the norm

$$\|(Y, Z, U)\|^2_{\mathcal{A}^2_\beta(\mathbf{R}^k)} = \|Y\|^2_{\mathscr{M}^{2,a}_\beta(\mathbf{R}^k)} + \|Z\|^2_{\mathscr{M}^2_\beta(\mathbf{R}^{k\times d})} + \|U\|^2_{\mathscr{L}^2_\beta(\mathbf{R}^k)}$$

is a Banach space as is the space

$$\mathscr{B}^2_\beta(\mathbf{R}^k) = (\mathscr{M}^{2,a}_\beta(\mathbf{R}^k) \cap \mathscr{S}^2_\beta(\mathbf{R}^k)) \times \mathscr{M}^2_\beta(\mathbf{R}^{k\times d}) \times \mathscr{L}^2_\beta(\mathbf{R}^k)$$

with the norm

$$\|(Y, Z, U)\|^2_{\mathscr{B}^2_\beta(\mathbf{R}^k)} = \|Y\|^2_{\mathscr{S}^2_\beta(\mathbf{R}^k)} + \|Y\|^2_{\mathscr{M}^{2,a}_\beta(\mathbf{R}^k)} + \|Z\|^2_{\mathscr{M}^2_\beta(\mathbf{R}^{k\times d})} + \|U\|^2_{\mathscr{L}^2_\beta(\mathbf{R}^k)}.$$

For a làdlàg (limited from right and left) process $(Y_t)_{t\leq T}$, we denote by:

- $Y_{t-} = \lim_{s\nearrow t} Y_s$ the left-hand limit of Y at $t \in [0,T]$, $(Y_{0-} = Y_0)$, $Y_- := (Y_{t-})_{t\leq T}$ and $\Delta Y_t := Y_t - Y_{t-}$ the size of the left jump of Y at t.
- $Y_{t+} = \lim_{s\searrow t} Y_s$ the right-hand limit of Y at $t \in [0,T]$, $(Y_{T+} = Y_T)$, $Y_+ := (Y_{t+})_{t\leq T}$ and $\Delta_+ Y_t := Y_{t+} - Y_t$ the size of the right jump of Y at t.

Let $f : \Omega \times [0,T] \times \mathbf{R}^k \times \mathbf{R}^{k\times d} \times \mathbf{R}^k \to \mathbf{R}^k$, $g : \Omega \times [0,T] \times \mathbf{R}^k \times \mathbf{R}^{k\times d} \times \mathbf{R}^k \to \mathbf{R}^{k\times \ell}$, and $(\xi_t)_{t\leq T}$ be an optional process which is assumed to be right upper semi-continuous and limited from left. The process $(\xi_t)_{t\leq T}$ will be called **irregular barrier**. We are interested in the following RBDSDEJs associated with parameters (f, g, ξ):

$$\begin{cases} Y_\tau = \xi_T + \int_\tau^T f(s,\Theta_s)ds + \int_\tau^T g(s,\Theta_s)dB_s - \int_\tau^T Z_s dW_s \\ \qquad - \int_\tau^T \int_E U_s(e)\widetilde{\mu}(ds,de) + K_T - K_\tau + C_{T-} - C_{\tau-} \quad \tau \in \mathcal{T}_{[0,T]}, \\ Y_\tau \geq \xi_\tau \quad \forall \tau \in \mathcal{T}_{[0,T]}, \\ K = K^c + K^d \text{ (continuous + purely discontinuous part) is a} \\ \text{nondecreasing right-continuous predictable process with} \\ K_0 = 0 \text{ such that} \\ \int_0^T \mathbb{1}_{\{Y_t > \xi_t\}} dK_t^c = 0 \text{ a.s. and } (Y_{\tau-} - \xi_{\tau-})\Delta K_\tau^d = 0 \text{ a.s.} \quad \forall \tau \in \mathcal{T}^p_{[0,T]}, \\ C \text{ is a nondecreasing right-continuous predictable purely} \\ \text{discontinuous process with } C_{0-} = 0 \text{ such that} \\ (Y_\tau - \xi_\tau)\Delta C_\tau = 0 \text{ a.s.} \quad \forall \tau \in \mathcal{T}_{[0,T]}. \end{cases} \tag{2}$$

Here Θ_s stands for the triple (Y_s, Z_s, U_s).

Let us consider the filtration $(\mathcal{G}_t)_{t\leq T}$ given by $\mathcal{G}_t = \mathscr{F}_t^W \vee \mathscr{F}_T^B \vee \mathscr{F}_t^\mu$, $0 \leq t \leq T$ which is assumed to be right-continuous and quasi-left-continuous, and make precise the notion of solution to RBDSDEJ (2).

Definition 1. Let ξ be an irregular barrier. A process (Y, Z, U, K, C) is called a solution to RBDSDEJ associated with parameters (f, g, ξ), if it satisfies the system (2) and

- $(Y, Z, U) \in \mathscr{B}^2_\beta(\mathbf{R}^k)$,
- $(K, C) \in \mathscr{S}^2(\mathbf{R}^k) \times \mathscr{S}^2(\mathbf{R}^k)$.

Remark 2.1. We note that a process $(Y, Z, U, K, C) \in \mathscr{B}^2_\beta(\mathbf{R}^k) \times \mathscr{S}^2(\mathbf{R}^k) \times \mathscr{S}^2(\mathbf{R}^k)$ satisfies the equation (2) if and only if

$$\begin{aligned} Y_t &= \xi_T + \int_t^T f(s,\Theta_s)ds + \int_t^T g(s,\Theta_s)dB_s - \int_t^T Z_s dW_s \\ &\quad - \int_t^T \int_E U_s(e)\widetilde{\mu}(ds,de) + K_T - K_t + C_{T-} - C_{t-}. \end{aligned}$$

Remark 2.2. If (Y, Z, U, K, C) is a solution to RBDSDEJ (2), then $\Delta C_t = Y_t - Y_{t+}$ for all $t \leq T$ outside an evanescent set. It follows that $Y_t \geq Y_{t+}$ for all $t \leq T$, which implies that Y is necessarily right upper semi-continuous. Moreover, the process $\left(Y_t + \int_0^t f(s, \Theta_s)ds\right)_{t\leq T}$ is a strong supermartingale. Actually, by using Hölder's inequality and the stochastic Lipschitz condition on f (below), we obtain, for each $\tau \in \mathcal{T}_{[0,T]}$,

$$\begin{aligned}
&\mathbb{E}\left|Y_\tau + \int_0^\tau f(s, \Theta_s)ds\right|^2 \\
\leq\ & 2\left(\mathbb{E}\,|Y_\tau|^2 + \frac{1}{\beta}\mathbb{E}\int_0^T e^{\beta A_s}\left|\frac{f(s, \Theta_s)}{a_s}\right|^2 ds\right) \\
\leq\ & 2\left(\|Y\|^2_{\mathcal{S}^2_\beta(\mathbf{R}^k)} + \frac{4}{\beta}\|Y\|^2_{\mathcal{M}^{2,a}_\beta(\mathbf{R}^k)} + \frac{4}{\beta}\|Z\|^2_{\mathcal{M}^2_\beta(\mathbf{R}^{k\times d})} + \frac{4}{\beta}\|U\|^2_{\mathcal{L}^2_\beta(\mathbf{R}^k)}\right. \\
&\left. + \frac{4}{\beta}\left\|\frac{f(.,0)}{a}\right\|^2_{\mathcal{M}^2_\beta(\mathbf{R}^k)}\right) < +\infty.
\end{aligned}$$

Moreover, for all $\tau, \nu \in \mathcal{T}_{[0,T]}$ with $\nu \leq \tau$ we have

$$\mathbb{E}\left[Y_\tau - Y_\nu - \int_\nu^\tau f(s, \Theta_s)ds|\mathcal{G}_\nu\right] = \mathbb{E}\left[K_\nu - K_\tau + C_{\nu^-} - C_{\tau^-}|\mathcal{G}_\nu\right] \ \ a.s.$$

Since K and C are nondecreasing processes, and $\left(Y_t + \int_0^t f(s, \Theta_s)ds\right)_{t\leq T}$ is a $\mathcal{G}_t$-adapted process then the observation follows.

Remark 2.3. In our framework the filtration is quasi-left-continuous, martingales have only totally inaccessible jumps and Y has two type of left-jumps: totally inaccessible jumps which stem from stochastic integral with respect to $\widetilde{\mu}$, and predictable jumps which come from the predictable jumps of the irregular barrier ξ. The latter are the source of the predictability of K. Moreover, the processes K and μ do not have jumps in common.

Remark 2.4 (The particular case of a right-continuous barrier)**.** If the barrier ξ is right-continuous, we have $Y_t \geq Y_{t+} \geq \xi_{t+} = \xi_t$. Hence, if t is such that $Y_t = \xi_t$, then $Y_t = Y_{t+} = \xi_t$. If t is such that $Y_t > \xi_t$, then by the minimality condition on C, $Y_t - Y_{t+} = C_t - C_{t-} = 0$. Thus, in both cases, $Y_t = Y_{t+}$, so Y is right-continuous. Moreover, the right-continuity of Y combined with the fact that $\Delta C_t = Y_t - Y_{t+}$ give $C_t = C_{t-}$ for all $t \leq T$. As C is right-continuous, purely discontinuous and such that $C_{0-} = 0$, we deduce $C = 0$. Thus, we recover the usual formulation of RBDSDEJs with a right-continuous barrier.

Proposition 2.5. *Let $(Y, Z, U) \in \mathcal{B}^2_\beta(\mathbf{R}^k)$ with Y being a làdlàg process, and let a coefficient $g(.) \in \mathcal{M}^2_\beta(\mathbf{R}^{k\times\ell})$. Then*

$$\left(\int_0^t e^{\beta A_s}\left(\langle Y_{s-}, Z_s dW_s\rangle + \langle Y_{s-}, g(s)dB_s\rangle + \int_E \langle Y_{s-}, U_s(e)\widetilde{\mu}(ds, de)\rangle\right)\right)_{t\leq T}$$

is a martingale.

Proof. Using the left-continuity of trajectories of the process Y_{s-}, we have

$$|Y_{s-}(\omega)|^2 \leq \sup_{t\in[0,T]\cap\mathbb{Q}} |Y_{t-}(\omega)|^2 \quad \forall(s,\omega)\in[0,T]\times\Omega.$$

On the other hand, we have $|Y_{t-}|^2 \leq \operatorname{ess\,sup}_{\tau\in\mathcal{T}_{[0,T]}} |Y_\tau|^2$ which implies

$$\sup_{t\in[0,T]\cap\mathbb{Q}} |Y_{t-}|^2 \leq \operatorname{ess\,sup}_{\tau\in\mathcal{T}_{[0,T]}} |Y_\tau|^2.$$

Then for all $\tau \leq T$

$$\begin{aligned}\int_0^\tau e^{2\beta A_s}|Y_{s-}|^2|Z_s|^2 ds &\leq \int_0^\tau e^{2\beta A_s} \sup_{t\in[0,T]\cap\mathbb{Q}} |Y_{t-}|^2|Z_s|^2 ds \\ &\leq \int_0^\tau e^{2\beta A_s} \operatorname{ess\,sup}_{\tau\in\mathcal{T}_{[0,T]}} |Y_\tau|^2|Z_s|^2 ds.\end{aligned}$$

Further, we have

$$\int_0^\tau e^{2\beta A_s} \operatorname{ess\,sup}_{\tau\in\mathcal{T}_{[0,T]}} |Y_\tau|^2|Z_s|^2 ds \leq \operatorname{ess\,sup}_{\tau\in\mathcal{T}_{[0,T]}} e^{\beta A_\tau}|Y_\tau|^2 \int_0^\tau e^{\beta A_s}|Z_s|^2 ds.$$

Hence

$$\begin{aligned}\mathbf{E}\sqrt{\int_0^\tau e^{2\beta A_s}|Y_{s-}|^2|Z_s|^2 ds} &\leq \mathbf{E}\sqrt{\operatorname{ess\,sup}_{\tau\in\mathcal{T}_{[0,T]}} e^{\beta A_\tau}|Y_\tau|^2 \int_0^T e^{\beta A_s}|Z_s|^2 ds} \\ &\leq \frac{1}{2}\left(\|Y\|^2_{\mathscr{S}^2_\beta(\mathbf{R}^k)} + \|Z\|^2_{\mathscr{M}^2_\beta(\mathbf{R}^{k\times d})}\right).\end{aligned}$$

Since $(Y,Z)\in\mathscr{S}^2_\beta(\mathbf{R}^k)\times\mathscr{M}^2_\beta(\mathbf{R}^{k\times d})$, we get the finite expectation. Since the process $\left(\int_0^t e^{\beta A_s}\langle Y_s, Z_s dW_s\rangle\right)_{t\leq T}$ is adapted, it is a martingale.

By the same arguments,

$$\left(\int_0^t\int_E e^{\beta A_s}\langle Y_{s-}, U_s(e)\widetilde{\mu}(ds,de)\rangle\right)_{t\leq T} \quad \text{and} \quad \left(\int_0^t e^{\beta A_s}\langle Y_{s-}, g(s)dB_s\rangle\right)_{t\leq T}$$

are martingales since

$$\begin{aligned}&\mathbf{E}\sqrt{\int_0^\tau\int_E e^{2\beta A_s}|Y_{s-}|^2|U_s(e)|^2\lambda(de)ds} \\ &\qquad\leq \mathbf{E}\sqrt{\operatorname{ess\,sup}_{\tau\in\mathcal{T}_{[0,T]}} e^{\beta A_\tau}|Y_\tau|^2 \int_0^T e^{\beta A_s}\|U_s\|^2_\lambda ds} \\ &\qquad\leq \frac{1}{2}\left(\|Y\|^2_{\mathscr{S}^2_\beta(\mathbf{R}^k)} + \|U\|^2_{\mathscr{L}^2_\beta(\mathbf{R}^k)}\right)\end{aligned}$$

and

$$\mathbf{E}\sqrt{\int_0^\tau e^{2\beta A_s}|Y_{s-}|^2|g(s)|^2 ds} \leq \mathbf{E}\sqrt{\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[0,T]}} e^{\beta A_\tau}|Y_\tau|^2 \int_0^T e^{\beta A_s}|g(s)|^2 ds}$$
$$\leq \frac{1}{2}\left(\|Y\|^2_{\mathcal{S}^2_\beta(\mathbf{R}^k)} + \|g\|^2_{\mathcal{M}^2_\beta(\mathbf{R}^{k\times\ell})}\right). \qquad \square$$

Let us recall some results from the general theory of optional processes, which will be useful in the sequel.

Theorem 2.6 (Mertens decomposition)**.** *Let $\tilde{Y}$ be a strong optional supermartingale of class (D). There exists a unique uniformly integrable martingale (càdlàg) N, a unique nondecreasing right-continuous predictable process K with $K_0 = 0$ and $\mathbf{E}|K_T|^2 < +\infty$, and a unique nondecreasing right-continuous adapted purely discontinuous process C with $C_{0-} = 0$ and $\mathbf{E}|C_T|^2 < +\infty$, such that*

$$\tilde{Y}_t = N_t - K_t - C_{t-} \quad \forall t \leq T \ a.s.$$

Theorem 2.7 (Dellacherie–Meyer)**.** *Let K be a nondecreasing predictable process. Let U be the potential of the process K, i.e. $U := \mathbf{E}[K_T|\mathcal{G}_t] - K_t$ for all $t \leq T$. We assume that there exists a positive $\mathcal{G}_T$-measurable random variable X such that $|U_\nu| \leq \mathbf{E}[X|\mathcal{G}_\nu]$ a.s. for all $\nu \in \mathcal{T}_{[0,T]}$. Then $\mathbf{E}|K_T|^2 \leq c\mathbf{E}|X|^2$, where c is a positive constant.*

The proof is established in chapter VI, Theorem 99, [8] for the case of a nondecreasing process which is not necessarily right-continuous nor left-continuous.

Corollary 2.8. *Let Y be a strong optional supermartingale of class (D) such that, for all $\nu \in \mathcal{T}_{[0,T]}$, $|Y_\nu| \leq \mathbf{E}[X|\mathcal{G}_\nu]$ a.s., where X is a nonnegative $\mathcal{G}_T$-measurable random variable. Let $\tilde{K}$ be the Mertens process associated with Y. Then there exists a positive constant c such that $\mathbf{E}|\tilde{K}_T|^2 \leq c\mathbf{E}|X|^2$.*

The proof is established in [23].

Theorem 2.9 (Gal'chouk–Lenglart formula)**.** *Let $n \in \mathbb{N}$. Let Y be an n-dimensional optional semimartingale with the decomposition $Y^k = Y_0^k + M^k + R^k + O^k$, for all $k = 1, \ldots, n$, where M^k is a (càdlàg) local martingale, R^k is a right-continuous process of finite variation such that $R_0^k = 0$ and O^k is a left-continuous process of finite variation which is purely discontinuous and such that $O_0^k = 0$. Let F be a twice continuously differentiable function on $\mathbb{R}^n$. Then, almost surely, for all $t \geq 0$,*

$$\begin{aligned} F(Y_t) &= F(Y_0) + \sum_{k=1}^n \int_0^t D^k F(Y_{s-}) d(M^k + R^k)_s + \sum_{k=1}^n \int_0^t D^k F(Y_s) dO^k_{s+} \\ &+ \frac{1}{2}\sum_{k,l=1}^n \int_0^t D^k D^l F(Y_{s-}) d[M^{k,c}, M^{l,c}]_s \\ &+ \sum_{0<s\leq t}\left[F(Y_s) - F(Y_{s-}) - \sum_{k=1}^n D^k F(Y_{s-})\Delta Y_s^k\right] \end{aligned}$$

$$+\sum_{0\le s<t}\left[F(Y_{s+})-F(Y_s)-\sum_{k=1}^{n}D^kF(Y_s)\Delta_+Y_s^k\right],$$

where D^k denotes the differentiation operator with respect to the k-th coordinate, and $M^{k,c}$ denotes the continuous part of M^k.

Corollary 2.10. *Let Y be an optional semimartingale with the decomposition $Y = Y_0 + M + R + O$ where M, R and O are as in Theorem 2.9. Then, almost surely, for all $t \ge 0$,*

$$\begin{aligned}
&e^{\beta A_t}|Y_t|^2+\beta\int_t^T e^{\beta A_s}a_s^2|Y_s|^2ds\\
=\;&e^{\beta A_T}|Y_T|^2+2\int_t^T e^{\beta A_s}Y_{s-}d(M+R)_s+\int_t^T e^{\beta A_s}Y_sdO_{s+}\\
&+\int_t^T e^{\beta A_s}d[M^c,M^c]_s-\sum_{t<s\le T}e^{\beta A_s}(\Delta Y_s)^2-\sum_{t\le s<T}e^{\beta A_s}(\Delta_+Y_s)^2.
\end{aligned}$$

Proof. To prove the corollary, it suffices to apply the change of variables formula from Theorem 2.9 with $F(X,Y)=XY^2$ for $X_t=e^{\beta A_t}$. □

Lemma 2.11. *Let $Y \in \mathcal{S}_\beta^2(\mathbf{R}^k)$, $\vartheta \in \mathcal{M}_\beta^2(\mathbf{R}^k)$, $\zeta \in \mathcal{M}_\beta^2(\mathbf{R}^{k\times\ell})$, $\pi \in \mathcal{M}_\beta^2(\mathbf{R}^{k\times d})$ and $\phi \in \mathcal{L}_\beta^2(\mathbf{R}^k)$ be such that*

$$Y_t=Y_0-\int_0^t\vartheta_sds-\int_0^t\zeta_sdB_s+\int_0^t\pi_sdW_s+\int_0^t\int_E\phi_s(e)\widetilde{\mu}(ds,de)-K_t-C_{t-},$$

where $\mathbf{E}|K_T|^2+\mathbf{E}|C_T|^2<+\infty$. Then Y is an optional semimartingale with the decomposition $Y = Y_0 + M + R + O$ where $M_t = -\int_0^t\zeta_sdB_s+\int_0^t\pi_sdW_s+\int_0^t\int_E\phi_s(e)\widetilde{\mu}(ds,de)$, $R_t=-\int_0^t\vartheta_sds-K_t$ and $O_t=-C_{t-}$, and we have, for any $\beta>0$ and $t\le T$,

$$\begin{aligned}
&e^{\beta A_t}|Y_t|^2+\beta\int_t^T e^{\beta A_s}a_s^2|Y_s|^2ds+\int_t^T e^{\beta A_s}|\pi_s|^2ds\\
=\;&e^{\beta A_T}|Y_T|^2+2\int_t^T e^{\beta A_s}\langle Y_{s-},\vartheta_s\rangle ds+2\int_t^T e^{\beta A_s}\langle Y_{s-},\zeta_sdB_s\rangle\\
&-2\int_t^T e^{\beta A_s}\langle Y_{s-},\pi_sdW_s\rangle-2\int_t^T\int_E e^{\beta A_s}\langle Y_{s-},\phi_s(e)\widetilde{\mu}(de,ds)\rangle\\
&+\int_t^T e^{\beta A_s}|\zeta_s|^2ds+2\int_t^T e^{\beta A_s}\langle Y_{s-},dK_s\rangle+2\int_t^T e^{\beta A_s}\langle Y_s,dC_s\rangle\\
&-\sum_{t<s\le T}e^{\beta A_s}(\Delta Y_s)^2-\sum_{t\le s<T}e^{\beta A_s}(\Delta_+Y_s)^2.
\end{aligned}$$

3 Reflected BDSDEJs with stochastic Lipschitz coefficients

3.1 Assumptions

We assume that the parameters (f, g, ξ) satisfy the following assumptions **(A1)**, for some $\beta > 0$ (where we define for all $t \le T$, $h(t,0)=h(t,0,0,0)$, for $h \in \{f, g\}$

to ease the reading).

(A1.1): f and g are jointly measurable, and there exists a constant $\alpha \in]0,1[$ and four non-negative, $\mathscr{F}_t^W$-measurable processes $(\gamma_t)_{t\leq T}$, $(\kappa_t)_{t\leq T}$, $(\sigma_t)_{t\leq T}$ and $(\varrho_t)_{t\leq T}$ such that for all $(y,y')\in(\mathbf{R}^k)^2$, $(z,z')\in(\mathbf{R}^{k\times d})^2$ and $(u,u')\in(\mathscr{L}_\lambda)^2$,

$$|f(t,y,z,u)-f(t,y',z',u')|\leq \gamma_t|y-y'|+\kappa_t|z-z'|+\sigma_t\left\|u-u'\right\|_\lambda,$$
$$|g(t,y,z,u)-g(t,y',z',u')|^2\leq \varrho_t|y-y'|^2+\alpha\left(|z-z'|^2+\left\|u-u'\right\|_\lambda^2\right).$$

(A1.2): For all $0\leq t\leq T,\ a_t^2=\gamma_t+\kappa_t^2+\sigma_t^2+\varrho_t>0$.

(A1.3): For any $(t,y,z,u)\in[0,T]\times\mathbf{R}^k\times\mathbf{R}^{k\times d}\times\mathscr{L}_\lambda$, $f(t,y,z,u)$ and $g(t,y,z,u)$ are $\mathscr{F}_t$-measurable with $\frac{f(.,0)}{a}\in\mathscr{M}_\beta^2(\mathbf{R}^k)$ and $g(.,0)\in\mathscr{M}_\beta^2(\mathbf{R}^{k\times\ell})$.

(A1.4): The irregular barrier $(\xi_t)_{t\leq T}$ is in $\mathscr{S}_{2\beta}^2(\mathbf{R}^k)$.

3.2 Existence and uniqueness of solution

Before proving the existence and uniqueness, let us establish the corresponding result in the case where the coefficients f and g do not depend on the variables Y, Z and U. So we consider the RBDSDEJ, $\forall\tau\in\mathcal{T}_{[0,T]}$,

$$\begin{cases} Y_\tau=\xi_T+\displaystyle\int_\tau^T f(s)ds+\int_\tau^T g(s)dB_s-\int_\tau^T Z_sdW_s-\int_\tau^T\int_E U_s(e)\widetilde{\mu}(ds,de)\\ \qquad +K_T-K_\tau+C_{T-}-C_{\tau-},\\ Y_\tau\geq\xi_\tau,\\ \displaystyle\int_0^T \mathbb{1}_{\{Y_t>\xi_t\}}dK_t^c=0,\quad (Y_{\tau-}-\xi_{\tau-})\Delta K_\tau^d=0\quad\text{and}\quad (Y_\tau-\xi_\tau)\Delta C_\tau=0\text{ a.s.}\end{cases} \tag{3}$$

where $K=K^c+K^d$ (continuous + purely discontinuous part) is a nondecreasing right-continuous predictable process with $K_0=0$ and C is a nondecreasing right-continuous predictable purely discontinuous process with $C_{0-}=0$. Moreover, the irregular barrier ξ satisfies **(A1.4)** and the coefficients (f,g) satisfy the following condition:

(A1.5): For any $t\leq T$, $f(t)$ and $g(t)$ are $\mathscr{F}_t$-measurable with $\frac{f(.)}{a}\in\mathscr{M}_\beta^2(\mathbf{R}^k)$ and $g(.)\in\mathscr{M}_\beta^2(\mathbf{R}^{k\times\ell})$.

Let us prove an a priori estimate of the solution in the following lemma.

Lemma 3.1. *Let (Y^1,Z^1,U^1,K^1,C^1) and (Y^2,Z^2,U^2,K^2,C^2) be two solutions to RBDSDEJs with parameters $(f^1(.),g^1(.),\xi^1)$ and $(f^2(.),g^2(.),\xi^2)$, respectively. We denote $\overline{\Re}:=\Re^1-\Re^2$ for $\Re\in\{Y,Z,U,K,C,f,g,\xi\}$. Then there exists a constant $\kappa(\beta)$ depending on β such that for all $\beta>1$*

$$\begin{aligned}&\|\overline{Y}\|^2_{\mathscr{B}_\beta^2(\mathbf{R}^k)}+\|\overline{Z}\|^2_{\mathscr{M}_\beta^2(\mathbf{R}^{k\times d})}+\|\overline{U}\|^2_{\mathscr{L}_\beta^2(\mathbf{R}^k)}\\ \leq\ &\kappa(\beta)\left(\|\overline{\xi}\|^2_{\mathscr{S}_{2\beta}^2(\mathbf{R}^k)}+\left\|\frac{\overline{f}}{a}\right\|^2_{\mathscr{M}_\beta^2(\mathbf{R}^k)}+\|\overline{g}\|^2_{\mathscr{M}_\beta^2(\mathbf{R}^{k\times\ell})}\right).\end{aligned}$$

Proof. Let $\tau \in \mathcal{T}_{[0,T]}$. It is obvious that the process $\overline{Y}$ is an optional semimartingale with the decomposition $\overline{Y}_\tau = \overline{Y}_0 + M_\tau + R_\tau + O_\tau$ where $M_\tau = -\int_0^\tau \overline{g}(s)dB_s + \int_0^\tau \overline{Z}_s dW_s + \int_0^\tau \int_E \overline{U}_s(e)\widetilde{\mu}(ds,de)$, $R_\tau = -\int_0^\tau \overline{f}(s)ds - \overline{K}_\tau$ and $O_\tau = -\overline{C}_{\tau-}$. Then, from Lemma 2.11, we have

$$\begin{aligned}
& e^{\beta A_t}|\overline{Y}_t|^2 + \beta \int_t^T e^{\beta A_s} a_s^2 |\overline{Y}_s|^2 ds + \int_t^T e^{\beta A_s}|\overline{Z}_s|^2 ds \\
= \; & e^{\beta A_T}|\overline{\xi}_T|^2 + 2\int_t^T e^{\beta A_s}\langle \overline{Y}_{s-}, \overline{f}(s)\rangle ds + 2\int_t^T e^{\beta A_s}\langle \overline{Y}_{s-}, d\overline{K}_s\rangle \\
& -2\int_t^T e^{\beta A_s}\langle \overline{Y}_{s-}, \overline{Z}_s dW_s\rangle - 2\int_t^T\int_E e^{\beta A_s}\langle \overline{Y}_{s-}, \overline{U}_s(e)\widetilde{\mu}(ds,de)\rangle \\
& +2\int_t^T e^{\beta A_s}\langle \overline{Y}_{s-}, \overline{g}(s)dB_s\rangle + \int_t^T e^{\beta A_s}|\overline{g}(s)|^2 ds + 2\int_t^T e^{\beta A_s}\langle \overline{Y}_s, d\overline{C}_s\rangle \\
& - \sum_{t<s\le T} e^{\beta A_s}(\Delta \overline{Y}_s)^2 - \sum_{t\le s<T} e^{\beta A_s}(\Delta_+ \overline{Y}_s)^2. \qquad (4)
\end{aligned}$$

From Remark 2.3, the processes $\overline{K}$ and μ do not have jumps in common, but $\overline{K}$ jumps at predictable stopping times and μ jumps only at totally inaccessible stopping times. Then we can note that

$$\sum_{t<s\le T} e^{\beta A_s}(\Delta \overline{Y}_s)^2 = \int_t^T\int_E e^{\beta A_s}|\overline{U}_s(e)|^2 \mu(ds,de) + \sum_{t<s\le T} e^{\beta A_s}(\Delta \overline{K}_s)^2.$$

Hence

$$\begin{aligned}
& \int_t^T e^{\beta A_s}\|\overline{U}_s\|_\lambda^2 ds - \sum_{t<s\le T} e^{\beta A_s}(\Delta \overline{Y}_s)^2 \\
= \; & \int_t^T e^{\beta A_s}\|\overline{U}_s\|_\lambda^2 ds - \int_t^T\int_E e^{\beta A_s}|\overline{U}_s(e)|^2\mu(ds,de) - \sum_{t<s\le T} e^{\beta A_s}(\Delta \overline{K}_s)^2 \\
\le \; & -\int_t^T\int_E e^{\beta A_s}|\overline{U}_s(e)|^2 \widetilde{\mu}(ds,de).
\end{aligned}$$

On the other hand, by using the Skorokhod and minimality conditions on $\overline{K}$ and $\overline{C}$ we can show that $\langle \overline{Y}_{s-}, d\overline{K}_s\rangle \le 0$ and $\langle \overline{Y}_s, d\overline{C}_s\rangle \le 0$. Indeed, for all $s \le T$

$$\begin{aligned}
\langle \overline{Y}_{s-}, d\overline{K}_s\rangle &= \langle Y^1_{s-} - \xi_{s-}, dK_s^{1,c} + \Delta K_s^{1,d}\rangle - \langle Y^2_{s-} - \xi_{s-}, dK_s^{1,c} + \Delta K_s^{1,d}\rangle \\
& \quad -\langle Y^1_{s-} - \xi_{s-}, dK_s^{2,c} + \Delta K_s^{2,d}\rangle + \langle Y^2_{s-} - \xi_{s-}, dK_s^{2,c} + \Delta K_s^{2,d}\rangle \\
&= -\langle Y^2_{s-} - \xi_{s-}, dK_s^{1,c} + \Delta K_s^{1,d}\rangle - \langle Y^1_{s-} - \xi_{s-}, dK_s^{2,c} + \Delta K_s^{2,d}\rangle \\
&\le 0, \quad \text{since } Y^i \ge \xi \text{ for } i = 1, 2.
\end{aligned}$$

Furthermore we have $\langle \overline{Y}_s, d\overline{C}_s\rangle = \langle \overline{Y}_s, \Delta \overline{C}_s\rangle$, and by the same arguments, we have, for all $s \le T$,

$$\langle \overline{Y}_s, \Delta \overline{C}_s\rangle = \langle Y^1_s - \xi_s, \Delta C^1_s\rangle - \langle Y^2_s - \xi_s, \Delta C^1_s\rangle - \langle Y^1_s - \xi_s, \Delta C^2_s\rangle$$

$$\begin{aligned} & -\langle \xi_s - Y_s^2, \Delta C_s^2 \rangle \\ = \ & 0 - \langle Y_s^2 - \xi_s, \Delta C_s^1 \rangle - \langle Y_s^1 - \xi_s, \Delta C_s^2 \rangle - 0 \\ \leq \ & 0, \quad \text{since } Y^i \geq \xi \text{ for } i = 1, 2. \end{aligned}$$

Moreover, by using the fact that

$$2\langle \overline{Y}_s, \overline{f}(s) \rangle \leq (\beta - 1) a_s^2 |\overline{Y}_s|^2 + \frac{1}{\beta - 1} \frac{|\overline{f}(s)|^2}{a_s^2} \quad \forall \beta > 1,$$

the inequality (4) becomes

$$\begin{aligned} & e^{\beta A_t} |\overline{Y}_t|^2 + \int_t^T e^{\beta A_s} a_s^2 |\overline{Y}_s|^2 ds + \int_t^T e^{\beta A_s} |\overline{Z}_s|^2 ds + \int_t^T e^{\beta A_s} \|\overline{U}_s\|_\lambda^2 ds \\ \leq \ & \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{[0,T]}} e^{2\beta A_\tau} |\overline{\xi}_\tau|^2 + \frac{1}{\beta - 1} \int_t^T e^{\beta A_s} \left| \frac{\overline{f}(s)}{a_s} \right|^2 ds - 2 \int_t^T e^{\beta A_s} \langle \overline{Y}_{s-}, \overline{Z}_s dW_s \rangle \\ & -2 \int_t^T \int_E e^{\beta A_s} \langle \overline{Y}_{s-}, \overline{U}_s(e) \widetilde{\mu}(ds, de) \rangle + 2 \int_t^T e^{\beta A_s} \langle \overline{Y}_{s-}, \overline{g}(s) dB_s \rangle \\ & + \int_t^T e^{\beta A_s} |\overline{g}(s)|^2 ds. \end{aligned} \tag{5}$$

Taking the expectation on the both sides of the inequality (5) and using Proposition 2.5, we get, for all $\beta > 1$,

$$\begin{aligned} & \|\overline{Y}\|^2_{\mathscr{M}_\beta^{2,a}(\mathbf{R}^k)} + \|\overline{Z}\|^2_{\mathscr{M}_\beta^2(\mathbf{R}^{k \times d})} + \|\overline{U}\|^2_{\mathscr{L}_\beta^2(\mathbf{R}^k)} \\ \leq \ & \|\overline{\xi}\|^2_{\mathscr{S}_{2\beta}^2(\mathbf{R}^k)} + \frac{1}{\beta - 1} \left\| \frac{\overline{f}}{a} \right\|^2_{\mathscr{M}_\beta^2(\mathbf{R}^k)} + \|\overline{g}\|^2_{\mathscr{M}_\beta^2(\mathbf{R}^{k \times \ell})}. \end{aligned} \tag{6}$$

On the other hand, by taking the essential supremum over $\tau \in \mathcal{T}_{[0,T]}$ and then the expectation on both sides of inequality (5) we obtain

$$\begin{aligned} & \mathbf{E} \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{[0,T]}} e^{\beta A_\tau} |\overline{Y}_\tau|^2 \\ \leq \ & \mathbf{E} \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{[0,T]}} e^{2\beta A_\tau} |\overline{\xi}_\tau|^2 + \frac{1}{\beta - 1} \mathbf{E} \int_0^T e^{\beta A_s} \left| \frac{\overline{f}(s)}{a_s} \right|^2 ds \\ & + 2\mathbf{E} \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{[0,T]}} \left| \int_0^\tau e^{\beta A_s} \langle \overline{Y}_{s-}, \overline{Z}_s dW_s \rangle \right| \\ & + 2\mathbf{E} \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{[0,T]}} \left| \int_0^\tau \int_E e^{\beta A_s} \langle \overline{Y}_{s-}, \overline{U}_s(e) \widetilde{\mu}(ds, de) \rangle \right| \\ & + 2\mathbf{E} \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{[0,T]}} \left| \int_0^\tau e^{\beta A_s} \langle \overline{Y}_{s-}, \overline{g}(s) dB_s \rangle \right| + \int_0^T e^{\beta A_s} |\overline{g}(s)|^2 ds. \end{aligned}$$

From the Burkhölder–Davis–Gundy inequality, there exists a universal constant c such that

$$\begin{aligned} 2\mathbf{E}\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[0,T]}}\left|\int_0^\tau e^{\beta A_s}\langle\overline{Y}_{s-},\overline{Z}_s dW_s\rangle\right| &\leq 2c\mathbf{E}\sqrt{\int_0^T e^{2\beta A_s}|\overline{Y}_{s-}|^2|\overline{Z}_s|^2 ds} \\ &\leq \frac{1}{4}\|\overline{Y}\|^2_{\mathscr{S}^2_\beta(\mathbf{R}^k)} + 4c^2\|\overline{Z}\|^2_{\mathscr{M}^2_\beta(\mathbf{R}^{k\times d})}, \end{aligned}$$

$$\begin{aligned} & 2\mathbf{E}\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[0,T]}}\left|\int_0^\tau\int_E e^{\beta A_s}\langle\overline{Y}_{s-},\overline{U}_s(e)\widetilde{\mu}(ds,de)\rangle\right| \\ \leq\ & 2c\mathbf{E}\sqrt{\int_0^T e^{2\beta A_s}|\overline{Y}_{s-}|^2\|\overline{U}_s\|^2_\lambda ds} \leq \frac{1}{4}\|\overline{Y}\|^2_{\mathscr{S}^2_\beta(\mathbf{R}^k)} + 4c^2\|\overline{U}\|^2_{\mathscr{L}^2_\beta(\mathbf{R}^k)} \end{aligned}$$

and

$$\begin{aligned} 2\mathbf{E}\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[0,T]}}\left|\int_0^\tau e^{\beta A_s}\langle\overline{Y}_{s-},\overline{g}(s)dB_s\rangle\right| &\leq 2c\mathbf{E}\sqrt{\int_0^T e^{2\beta A_s}|\overline{Y}_{s-}|^2|\overline{g}(s)|^2 ds} \\ &\leq \frac{1}{4}\|\overline{Y}\|^2_{\mathscr{S}^2_\beta(\mathbf{R}^k)} + 4c^2\|\overline{g}\|^2_{\mathscr{M}^2_\beta(\mathbf{R}^{k\times\ell})}. \end{aligned}$$

Consequently,

$$\begin{aligned} \|\overline{Y}\|^2_{\mathscr{S}^2_\beta(\mathbf{R}^k)} \leq\ & 4\left(\|\overline{\xi}\|^2_{\mathscr{S}^2_{2\beta}(\mathbf{R}^k)} + \frac{1}{\beta-1}\left\|\frac{\overline{f}}{a}\right\|^2_{\mathscr{M}^2_\beta(\mathbf{R}^k)} + (4c^2+1)\|\overline{g}\|^2_{\mathscr{M}^2_\beta(\mathbf{R}^{k\times\ell})}\right. \\ & \left. +4c^2\|\overline{Z}\|^2_{\mathscr{M}^2_\beta(\mathbf{R}^{k\times d})} + 4c^2\|\overline{U}\|^2_{\mathscr{L}^2_\beta(\mathbf{R}^k)}\right). \qquad (7) \end{aligned}$$

The desired result is obtained by combining the estimates (6) and (7) for $\beta > 1$. □

In the following, we state the existence and uniqueness result for the solution to RBDSDEJ (3).

Proposition 3.2. *Under the assumptions* **(A1.4)** *and* **(A1.5)***, the RBDSDEJ* (3) *admits a unique solution* $(Y, Z, U, K, C) \in \mathscr{B}^2_\beta(\mathbf{R}^k) \times \mathscr{S}^2(\mathbf{R}^k) \times \mathscr{S}^2(\mathbf{R}^k)$ *for all* $\beta > 1$*, and for each* $\nu \in \mathcal{T}_{[0,T]}$ *we have*

$$Y_\nu = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[\nu,T]}}\mathbf{E}\left[\xi_\tau + \int_\nu^\tau f(t)dt + \int_\nu^\tau g(t)dB_t|\mathcal{G}_\nu\right] \quad a.s.$$

Proof. Let $\nu \in \mathcal{T}_{[0,T]}$. We define the value function $\overline{\overline{Y}}(\nu)$ by

$$\overline{\overline{Y}}(\nu) = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[\nu,T]}}\mathbf{E}\left[\xi_\tau + \int_\nu^\tau f(t)dt + \int_\nu^\tau g(t)dB_t|\mathcal{G}_\nu\right],$$

and $\widetilde{Y}(\nu)$ by

$$\begin{aligned}\widetilde{Y}(\nu) &= \overline{\overline{Y}}(\nu) + \int_0^{\nu} f(t)dt + \int_0^{\nu} g(t)dB_t \\ &= \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[\nu,T]}} \mathbf{E}\left[\xi_\tau + \int_0^{\tau} f(t)dt + \int_0^{\tau} g(t)dB_t|\mathcal{G}_\nu\right].\end{aligned}$$

The process $\left(\xi_t + \int_0^t f(s)ds + \int_0^t g(s)dB_s\right)_{t\le T}$ is progressively measurable. Therefore, the family $(\widetilde{Y}(\nu))_{\nu\in\mathcal{T}_{[0,T]}}$ is a supermartingale family. This observation with the Remark **b.** page 435 in [8] ensures the existence of a strong optional supermartingale $\widetilde{Y}$ such that $\widetilde{Y}_\nu = \widetilde{Y}(\nu)$ for all $\nu \in \mathcal{T}_{[0,T]}$. Thus, we have $\overline{\overline{Y}}(\nu) = \widetilde{Y}_\nu - \int_0^{\nu} f(t)dt - \int_0^{\nu} g(t)dB_t$. On the other hand, almost all trajectories of the strong optional supermartingale are làdlàg, then the làdlàg optional process $(\overline{\overline{Y}}_t)_{t\le T} := \left(\widetilde{Y}_t - \int_0^t f(s)ds - \int_0^t g(s)dB_s\right)_{t\le T}$ aggregates the family $(\overline{\overline{Y}}(\nu))_{\nu\in\mathcal{T}_{[0,T]}}$.

Now, it remains to show that the candidate $\overline{\overline{Y}} \in \mathscr{S}^2_\beta(\mathbf{R}^k)$. Using the Jensen's, Young's and Hölder's inequalities respectively, we obtain

$$\begin{aligned}
& e^{\frac{\beta}{2}A_\nu}|\overline{\overline{Y}}_\nu| \\
= & \left|\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[\nu,T]}} \mathbf{E}\left[e^{\frac{\beta}{2}A_\nu}\xi_\tau + e^{\frac{\beta}{2}A_\nu}\int_\nu^{\tau} f(t)dt + e^{\frac{\beta}{2}A_\nu}\int_\nu^{\tau} g(t)dB_t|\mathcal{G}_\nu\right]\right| \\
\le & \mathbf{E}\Bigg[\Bigg\{\Bigg|\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[\nu,T]}} e^{\frac{\beta}{2}A_\nu}\xi_\tau + e^{\frac{\beta}{2}A_\nu}\int_\nu^{T} f(t)dt \\
& \qquad + \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[\nu,T]}} e^{\frac{\beta}{2}A_\nu}\int_\nu^{\tau} g(t)dB_t\Bigg|^2\Bigg\}^{\frac{1}{2}}|\mathcal{G}_\nu\Bigg] \\
\le & \sqrt{3}\mathbf{E}\Bigg[\Bigg\{\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[\nu,T]}} e^{\beta A_\tau}|\xi_\tau|^2 + e^{\beta A_\nu}\left|\int_\nu^{T} f(t)dt\right|^2 \\
& \qquad + \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[\nu,T]}} e^{\beta A_\nu}\left|\int_\nu^{\tau} g(t)dB_t\right|^2\Bigg\}^{\frac{1}{2}}|\mathcal{G}_\nu\Bigg] \\
\le & \sqrt{3}\mathbf{E}\Bigg[\Bigg\{\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[\nu,T]}} e^{2\beta A_\tau}|\xi_\tau|^2 + e^{\beta A_\nu}\left(\int_\nu^{T} e^{-\beta A_t}a_t^2 dt\right) \times \\
& \qquad \left(\int_\nu^{T} e^{\beta A_t}\left|\frac{f(t)}{a_t}\right|^2 dt\right) + c\int_0^{T} e^{\beta A_t}|g(t)|^2 dt\Bigg\}^{\frac{1}{2}}|\mathcal{G}_\nu\Bigg] \\
\le & \sqrt{3}\mathbf{E}\Bigg[\Bigg\{\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[0,T]}} e^{2\beta A_\tau}|\xi_\tau|^2 + \frac{1}{\beta}\int_0^{T} e^{\beta A_t}\left|\frac{f(t)}{a_t}\right|^2 dt \\
& \qquad + c\int_0^{T} e^{\beta A_t}|g(t)|^2 dt\Bigg\}^{\frac{1}{2}}|\mathcal{G}_\nu\Bigg].
\end{aligned}$$

Taking the essential supremum over $\nu \in \mathcal{T}_{[0,T]}$ on the above sides and using the Doob's martingale inequality, we conclude that

$$\begin{aligned}
&\mathbf{E} \operatorname*{ess\,sup}_{\nu\in\mathcal{T}_{[0,T]}} e^{\beta A_\nu}|\overline{\overline{Y}}_\nu|^2 \\
\leq\;& \kappa'(\beta)\mathbf{E}\left(\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[\nu,T]}} e^{2\beta A_\tau}|\xi_\tau|^2 + \int_0^T e^{\beta A_t}\left|\frac{f(t)}{a_t}\right|^2 dt + \int_0^T e^{\beta A_t}|g(t)|^2 dt\right)
\end{aligned}$$

where $\kappa'(\beta)$ is a positive constant depending on β. It follows that $\overline{\overline{Y}} \in \mathcal{S}^2_\beta(\mathbf{R}^k)$.

Note that the strong optional supermartingale $\widetilde{Y}$ is of class (D) (i.e. the set of all random variables $\widetilde{Y}_\nu$, for each finite stopping time ν, is uniformly integrable). Then by the Mertens decomposition (see Theorem 2.6), there exists a uniformly integrable martingale (càdlàg) N, a nondecreasing right-continuous predictable process K (with $K_0 = 0$) such that $\mathbf{E}|K_T|^2 < +\infty$ and a nondecreasing right-continuous adapted purely discontinuous process C (with $C_{0-} = 0$) such that $\mathbf{E}|C_T|^2 < +\infty$, with the following equality:

$$\widetilde{Y}_\tau = N_\tau - K_\tau - C_{\tau-} \quad \forall \tau \in \mathcal{T}_{[0,T]}.$$

By an extension of Itô's martingale representation Theorem, there exists a unique pair of predictable processes $(Z, U) \in \mathcal{M}^2(\mathbf{R}^{k\times d}) \times \mathcal{L}^2(\mathbf{R}^k)$ such that

$$N_\tau = N_0 + \int_0^\tau Z_s dW_s + \int_0^\tau \int_E U_s(e)\widetilde{\mu}(ds, de).$$

Hence for each $\tau \in \mathcal{T}_{[0,T]}$

$$\begin{aligned}
\overline{\overline{Y}}_\tau \;=\;& -\int_0^\tau f(s)ds - \int_0^\tau g(s)dB_s + N_0 + \int_0^\tau Z_s dW_s \\
& -\int_0^\tau \int_E U_s(e)\widetilde{\mu}(ds, de) - K_\tau - C_{\tau-}
\end{aligned} \tag{8}$$

with $\overline{\overline{Y}}_T = \overline{\overline{Y}}(T) = \xi_T$ and $\overline{\overline{Y}}_\tau = \overline{\overline{Y}}(\tau) \geq \xi_\tau$ a.s for all $\tau \in \mathcal{T}_{[0,T]}$. Next, let us focus on the Skorokhod and minimality conditions. Since $\Delta_+\overline{\overline{Y}}_\tau = \mathbb{1}_{\{\overline{\overline{Y}}_\tau=\xi_\tau\}}\Delta_+\overline{\overline{Y}}_\tau$ a.s.(see Remark A.4 in [14]), from (8) we have $\Delta C_\tau = -\Delta_+\overline{\overline{Y}}_\tau$ a.s., then $\Delta C_\tau = \mathbb{1}_{\{\overline{\overline{Y}}_\tau=\xi_\tau\}}\Delta C_\tau$ a.s. It follows that the minimality condition on C is satisfied. Further, due to a result from the optimal stopping theory (see Proposition B.11 in [18]), for each predictable stopping time τ, we have $\int_0^T \mathbb{1}_{\{\overline{\overline{Y}}_t>\xi_t\}}dK^c_t = 0$ a.s. and $\Delta K^d_\tau = \mathbb{1}_{\{\overline{\overline{Y}}_{\tau-}=\xi_{\tau-}\}}\Delta K^d_\tau$ a.s. Then the process K satisfies the Skorokhod condition. Thus, we found a process $(\overline{\overline{Y}}, Z, U, K, C)$ which satisfies the RBDSDEJ (3).

Now, it remains to show that $(\overline{\overline{Y}}, Z, U, K, C) \in \mathcal{B}^2_\beta(\mathbf{R}^k) \times \mathcal{S}^2(\mathbf{R}^k) \times \mathcal{S}^2(\mathbf{R}^k)$. Indeed, let $\widetilde{K}_t := K_t + C_{t-}$ be the Mertens process associated with $\widetilde{Y}$. By the definition of $\widetilde{Y}_\nu$, we see that

$$|\widetilde{Y}_\nu| \;=\; \left|\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[\nu,T]}} \mathbf{E}\left[\xi_\tau + \int_0^\tau f(t)dt + \int_0^\tau g(t)dB_t|\mathcal{G}_\nu\right]\right|$$

$$\leq \quad \mathbf{E}\left[\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[\nu,T]}}|\xi_\tau| + \int_0^T |f(t)|dt + \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[\nu,T]}}\left|\int_0^\tau g(t)dB_t\right| |\mathcal{G}_\nu\right].$$

From Corollary 2.8, there exists a positive constant c such that

$$\begin{aligned}
\mathbf{E}|\widetilde{K}_T|^2 &\leq c\mathbf{E}\left|\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[\nu,T]}}|\xi_\tau| + \int_0^T |f(t)|dt + \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[\nu,T]}}\left|\int_0^\tau g(t)dB_t\right|\right|^2 \\
&\leq c(\beta)\left(\|\xi\|^2_{\mathcal{S}^2_{2\beta}(\mathbf{R}^k)} + \left\|\frac{f}{a}\right\|^2_{\mathcal{M}^2_\beta(\mathbf{R}^k)} + \|g\|^2_{\mathcal{M}^2_\beta(\mathbf{R}^{k\times\ell})}\right)
\end{aligned}$$

where $c(\beta)$ is a positive constant depending on β. $\widetilde{K}$ is nondecreasing, and it implies that

$$\mathbf{E}\operatorname*{ess\,sup}_{\nu\in\mathcal{T}_{[0,T]}}|\widetilde{K}_\tau|^2 \leq \mathbf{E}|\widetilde{K}_T|^2 < +\infty.$$

It follows that $\widetilde{K} \in \mathcal{S}^2(\mathbf{R}^k)$, then $(K, C) \in \mathcal{S}^2(\mathbf{R}^k) \times \mathcal{S}^2(\mathbf{R}^k)$. On the other hand, from Lemma 2.11 we have

$$\begin{aligned}
& e^{\beta A_t}|\overline{\overline{Y}}_t|^2 + \beta\int_t^T e^{\beta A_s}a_s^2|\overline{\overline{Y}}_s|^2ds + \int_t^T e^{\beta A_s}|Z_s|^2ds \\
= \;& e^{\beta A_T}|\xi_T|^2 + 2\int_t^T e^{\beta A_s}\langle\overline{\overline{Y}}_{s-}, f(s)\rangle ds + 2\int_t^T e^{\beta A_s}\langle\overline{\overline{Y}}_{s-}, dK_s\rangle \\
& -2\int_t^T e^{\beta A_s}\langle\overline{\overline{Y}}_{s-}, Z_sdW_s\rangle - 2\int_t^T\int_E e^{\beta A_s}\langle\overline{\overline{Y}}_{s-}, U_s(e)\widetilde{\mu}(ds,de)\rangle \\
& +2\int_t^T e^{\beta A_s}\langle\overline{\overline{Y}}_{s-}, g(s)dB_s\rangle + \int_t^T e^{\beta A_s}|g(s)|^2ds + 2\int_t^T e^{\beta A_s}\langle\overline{\overline{Y}}_s, dC_s\rangle \\
& -\sum_{t<s\leq T} e^{\beta A_s}(\Delta\overline{\overline{Y}}_s)^2 - \sum_{t\leq s<T} e^{\beta A_s}(\Delta_+\overline{\overline{Y}}_s)^2.
\end{aligned}$$

From Remark 2.3, the processes K and μ do not have jumps in common, but K jumps at predictable stopping times and μ jumps only at totally inaccessible stopping times, then we can write

$$\sum_{t<s\leq T} e^{\beta A_s}(\Delta\overline{\overline{Y}}_s)^2 = \int_t^T\int_E e^{\beta A_s}|U_s(e)|^2\mu(ds,de) + \sum_{t<s\leq T} e^{\beta A_s}(\Delta K_s)^2.$$

Hence

$$\begin{aligned}
& \int_t^T e^{\beta A_s}\|U_s\|^2_\lambda ds - \sum_{t<s\leq T} e^{\beta A_s}(\Delta\overline{\overline{Y}}_s)^2 \\
= \;& \int_t^T e^{\beta A_s}\|U_s\|^2_\lambda ds - \int_t^T\int_E e^{\beta A_s}|U_s(e)|^2\mu(ds,de) - \sum_{t<s\leq T} e^{\beta A_s}(\Delta K_s)^2 \\
\leq \;& -\int_t^T\int_E e^{\beta A_s}|U_s(e)|^2\widetilde{\mu}(ds,de).
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\int_t^T e^{\beta A_s} a_s^2 |\overline{\overline{Y}}_s|^2 ds + \int_t^T e^{\beta A_s} |Z_s|^2 ds + \int_t^T e^{\beta A_s} \|U_s\|_\lambda^2 ds \\
\leq\ & e^{\beta A_T} |\xi_T|^2 + \frac{1}{\beta - 1} \int_t^T e^{\beta A_s} \left| \frac{f(s)}{a_s} \right|^2 ds - 2 \int_t^T e^{\beta A_s} \langle \overline{\overline{Y}}_{s-}, Z_s dW_s \rangle \\
& -2 \int_t^T \int_E e^{\beta A_s} \langle \overline{\overline{Y}}_{s-}, U_s(e) \widetilde{\mu}(ds, de) \rangle + 2 \int_t^T e^{\beta A_s} \langle \overline{\overline{Y}}_{s-}, g(s) dB_s \rangle \\
& + \int_t^T e^{\beta A_s} |g(s)|^2 ds + 2 \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{[0,T]}} e^{2\beta A_\tau} |\xi_\tau|^2 + K_T^2 + C_T^2.
\end{aligned}$$

Here we have used also the Skorokhod and minimality conditions on K and C. Next, by taking the expectation on both sides of above inequality, we get

$$\begin{aligned}
& \|\overline{\overline{Y}}\|^2_{\mathscr{M}_\beta^{2,a}(\mathbf{R}^k)} + \|Z\|^2_{\mathscr{M}_\beta^2(\mathbf{R}^{k\times d})} + \|U\|^2_{\mathscr{L}_\beta^2(\mathbf{R}^k)} \\
\leq\ & 3\|\xi\|^2_{\mathscr{S}_{2\beta}^2(\mathbf{R}^k)} + \frac{1}{\beta - 1} \left\| \frac{f}{a} \right\|^2_{\mathscr{M}_\beta^2(\mathbf{R}^k)} + \|g\|^2_{\mathscr{M}_\beta^2(\mathbf{R}^{k\times\ell})} + \mathbf{E}|K_T|^2 + \mathbf{E}|C_T|^2.
\end{aligned}$$

Then $(\overline{\overline{Y}}, Z, U) \in \mathscr{M}_\beta^{2,a}(\mathbf{R}^k) \times \mathscr{M}_\beta^2(\mathbf{R}^{k\times d}) \times \mathscr{L}_\beta^2(\mathbf{R}^k)$.

Finally, it is remarkabe that the uniqueness of the solution comes from the uniqueness of the Mertens decomposition and the Itô's martingale representation Theorem, and if $\overline{\overline{Y}}$ and Y are two first-components of the solution, then by Lemma 3.1 we have immediately $\overline{\overline{Y}} = Y$. □

Proposition 3.3. *Assume that the assumptions* **(A1.1)–(A1.4)** *are true. Then, if* $(y, z, u) \in \mathscr{B}_\beta^2(\mathbf{R}^k)$ *for* $\beta > 1$*, there exists a unique process* $(Y, Z, U, K, C) \in \mathscr{B}_\beta^2(\mathbf{R}^k) \times \mathscr{S}^2(\mathbf{R}^k) \times \mathscr{S}^2(\mathbf{R}^k)$ *being a solution to the following RBDSDEJ, for all* $\tau \in \mathcal{T}_{[0,T]}$,

$$\begin{cases}
Y_\tau = \xi_T + \displaystyle\int_\tau^T f(s, y_s, z_s, u_s) ds + \int_\tau^T g(s, y_s, z_s, u_s) dB_s - \int_\tau^T Z_s dW_s \\
\qquad - \displaystyle\int_\tau^T \int_E U_s(e) \widetilde{\mu}(ds, de) + K_T - K_\tau + C_{T-} - C_{\tau-}, \\
Y_\tau \geq \xi_\tau, \\
\displaystyle\int_0^T \mathbb{1}_{\{Y_t > \xi_t\}} dK_t^c = 0, \quad (Y_{\tau-} - \xi_{\tau-}) \Delta K_\tau^d = 0 \quad \text{and} \quad (Y_\tau - \xi_\tau) \Delta C_\tau = 0 \ a.s.
\end{cases}$$

Proof. Given $(y, z, u) \in \mathscr{B}_\beta^2(\mathbf{R}^k)$, we define $\widehat{f}(t) = f(t, y_t, z_t, u_t)$ and $\widehat{g}(t) = g(t, y_t, z_t, u_t)$. Let us show that $\widehat{f}$ and $\widehat{g}$ satisfy **(A1.5)**. From the assumptions **(A1.1)** and **(A1.2)**, we have

$$|\widehat{f}(s)|^2 \leq 4 \left(a_s^4 |y_s|^2 + a_s^2 |z_s|^2 + a_s^2 \|u_s\|_\lambda^2 + |f(s, 0)|^2 \right)$$

and

$$|\widehat{g}(s)|^2 \leq 2 \left(a_s^2 |y_s|^2 + \alpha (|z_s|^2 + \|u_s\|_\lambda^2) + |g(s, 0)|^2 \right).$$

Thus gathering these inequalities, we deduce that

$$\begin{aligned}
&\mathbf{E}\left(\int_0^T e^{\beta A_s}\left|\frac{\widehat{f}(s)}{a_s}\right|^2 ds+\int_0^T e^{\beta A_s}|\widehat{g}(s)|^2 ds\right)\\
\leq\ &\mathbf{E}\left(6\int_0^T e^{\beta A_s}a_s^2|y_s|^2 ds+(4+2\alpha)\int_0^T e^{\beta A_s}(|z_s|^2+\|u_s\|_\lambda^2)ds\right)\\
&+\mathbf{E}\left(4\int_0^T e^{\beta A_s}\left|\frac{f(s,0)}{a_s}\right|^2 ds+2\int_0^T e^{\beta A_s}|g(s,0)|^2 ds\right).
\end{aligned}$$

This implies that $\widehat{f}$ and $\widehat{g}$ satisfy **(A1.5)** since $(y,z,u)\in\mathscr{B}_\beta^2(\mathbf{R}^k)$ and in view of the assumption **(A1.3)**. Hence the result follows from Proposition 3.2. □

We are now in position to study the solvability of our RBDSDEJ (2) associated with parameters $(f(.,\Theta), g(.,\Theta), \xi)$.

Theorem 3.4. *Under the assumptions* **(A1.1)–(A1.4)**, *there exists* $\beta_0>0$ *such that for all* $\beta\geq\beta_0$ *the RBDSDEJ* (2) *admits a unique solution* $(Y,Z,U,K,C)\in\mathscr{B}_\beta^2(\mathbf{R}^k)\times\mathscr{S}^2(\mathbf{R}^k)\times\mathscr{S}^2(\mathbf{R}^k)$.

Proof. (i) Existence. Our strategy in the proof of existence is to use the Picard approximate sequence. To this end, we consider the sequence $(\Theta^n)_{n\geq 0}:=(Y^n,Z^n,U^n)_{n\geq 0}\in\mathscr{B}_\beta^2(\mathbf{R}^k)$ defined recursively by $Y^0=Z^0=U^0=0$ and for any $n\geq 1$, $\tau\in\mathcal{T}_{[0,T]}$,

$$\begin{cases}
Y_\tau^{n+1}=\xi_T+\int_\tau^T f\left(s,\Theta_s^n\right)ds+\int_\tau^T g\left(s,\Theta_s^n\right)dB_s-\int_\tau^T Z_s^{n+1}dW_s\\
\qquad\qquad -\int_\tau^T\int_E U_s^{n+1}(e)\widetilde{\mu}(ds,de)+K_T^{n+1}-K_\tau^{n+1}+C_{T-}^{n+1}-C_{\tau-}^{n+1},\\
Y_\tau^{n+1}\geq\xi_\tau \text{ a.s.},\\
\int_0^T \mathbb{1}_{\{Y_t^{n+1}>\xi_t\}}dK_t^{c,n+1}=0 \text{ a.s.}\quad\text{and}\quad (Y_{\tau-}^{n+1}-\xi_{\tau-})\Delta K_\tau^{d,n+1}=0\text{ a.s.},\\
(Y_\tau^{n+1}-\xi_\tau)\Delta C_\tau^{n+1}=0\text{ a.s.}
\end{cases}\tag{9}$$

Since for $n\geq 0$, $(Y^n,Z^n,U^n)\in\mathscr{B}_\beta^2(\mathbf{R}^k)$, by virtue of Proposition 3.3, RBDSDEJ (9) has a unique solution $(Y^{n+1},Z^{n+1},U^{n+1},K^{n+1},C^{n+1})\in\mathscr{B}_\beta^2(\mathbf{R}^k)\times\mathscr{S}^2(\mathbf{R}^k)\times\mathscr{S}^2(\mathbf{R}^k)$.

In the sequel, we shall show that $(Y^n,Z^n,U^n)_{n\geq 0}$ is a Cauchy sequence in the Banach space $\mathscr{B}_\beta^2(\mathbf{R}^k)$. We define $\overline{\Re}^{n+1}=\Re^{n+1}-\Re^n$ for $\Re\in\{Y,Z,U,K,C\}$, and

$$\forall h\in\{f,g\},\quad \overline{h}_\Theta^n(t)=h(t,\Theta_t^n)-h(t,\Theta_t^{n-1}),\quad t\leq T.$$

We derive that for any $n\geq 1$ the process $(\overline{Y}^{n+1},\overline{Z}^{n+1},\overline{U}^{n+1},\overline{K}^{n+1},\overline{C}^{n+1})$ satisfies the following equation

$$\overline{Y}_t^{n+1}\ =\ \int_t^T\overline{f}_\Theta^n(s)ds+\int_t^T\overline{g}_\Theta^n(s)dB_s-\int_t^T\overline{Z}_s^{n+1}dW_s$$

$$-\int_t^T\int_E \overline{U}_s^{n+1}(e)\widetilde{\mu}(ds,de)+\overline{K}_T^{n+1}-\overline{K}_t^{n+1}+\overline{C}_{T-}^{n+1}-\overline{C}_{t-}^{n+1}.$$

Applying the Lemma 2.11, we have

$$\begin{aligned}
& e^{\beta A_t}|\overline{Y}_t^{n+1}|^2+\beta\int_t^T e^{\beta A_s}a_s^2|\overline{Y}_s^{n+1}|^2ds+\int_t^T e^{\beta A_s}|\overline{Z}_s^{n+1}|^2ds\\
= \;& 2\int_t^T e^{\beta A_s}\langle\overline{Y}_{s-}^{n+1},\overline{f}_\Theta^n(s)\rangle ds+2\int_t^T e^{\beta A_s}\langle\overline{Y}_{s-}^{n+1},d\overline{K}_s^{n+1}\rangle\\
& -2\int_t^T e^{\beta A_s}\langle\overline{Y}_{s-}^{n+1},\overline{Z}_s^{n+1}dW_s\rangle-2\int_t^T\int_E e^{\beta A_s}\langle\overline{Y}_{s-}^{n+1},\overline{U}_s^{n+1}(e)\widetilde{\mu}(ds,de)\rangle\\
& +2\int_t^T e^{\beta A_s}\langle\overline{Y}_{s-}^{n+1},\overline{g}_\Theta^n(s)dB_s\rangle+\int_t^T e^{\beta A_s}|\overline{g}_\Theta^n(s)|^2ds+2\int_t^T e^{\beta A_s}\langle\overline{Y}_s^{n+1},d\overline{C}_s^{n+1}\rangle\\
& -\sum_{t<s\le T}e^{\beta A_s}(\Delta\overline{Y}_s^{n+1})^2-\sum_{t\le s<T}e^{\beta A_s}(\Delta_+\overline{Y}_s^{n+1})^2.
\end{aligned}\tag{10}$$

From Remark 2.3, the processes $\overline{K}^{n+1}$ and μ do not have jumps in common, but $\overline{K}^{n+1}$ jumps at predictable stopping times and μ jumps only at totally inaccessible stopping times, then

$$\int_t^T e^{\beta A_s}\|\overline{U}_s^{n+1}\|_\lambda^2ds-\sum_{t<s\le T}e^{\beta A_s}(\Delta\overline{Y}_s^{n+1})^2\le-\int_t^T\int_E e^{\beta A_s}|\overline{U}_s^{n+1}(e)|^2\widetilde{\mu}(ds,de).$$

On the other hand, by using the Skorokhod and minimality conditions on $\overline{K}^{n+1}$ and $\overline{C}^{n+1}$ we can show that $\langle\overline{Y}_{s-}^{n+1},d\overline{K}_s^{n+1}\rangle\le 0$ and $\langle\overline{Y}_s^{n+1},d\overline{C}_s^{n+1}\rangle\le 0$. Moreover, from the assumptions **(A1.1)**–**(A1.2)**, we deduce that for any $\varepsilon>0$,

$$\begin{aligned}
2\langle\overline{Y}_s^{n+1},\overline{f}_\Theta^n(s)\rangle &\le 2|\overline{Y}_s^{n+1}|\left(\gamma_s|\overline{Y}_s^n|+\kappa_s|\overline{Z}_s^n|+\sigma_s\|\overline{U}_s^n\|_\lambda\right)\\
&\le \left(\gamma_s+\frac{1}{\varepsilon}[\kappa_s^2+\sigma_s^2]\right)|\overline{Y}_s^{n+1}|^2+\gamma_s|\overline{Y}_s^n|^2+\varepsilon\left(|\overline{Z}_s^n|^2+\|\overline{U}_s^n\|_\lambda^2\right)\\
&\le \left(1+\frac{1}{\varepsilon}\right)a_s^2|\overline{Y}_s^{n+1}|^2+a_s^2|\overline{Y}_s^n|^2+\varepsilon\left(|\overline{Z}_s^n|^2+\|\overline{U}_s^n\|_\lambda^2\right)
\end{aligned}$$

and

$$|\overline{g}_\Theta^n(s)|^2\le a_s^2|\overline{Y}_s^n|^2+\alpha\left(|\overline{Z}_s^n|^2+\|\overline{U}_s^n\|_\lambda^2\right).$$

Plugging these inequalities in (10), and taking the expectation in both side, we deduce that, for any $\beta>0$ and $\varepsilon>0$,

$$\begin{aligned}
&\left(\beta-1-\frac{1}{\varepsilon}\right)\mathbf{E}\int_t^T e^{\beta A_s}a_s^2|\overline{Y}_s^{n+1}|^2ds+\mathbf{E}\int_t^T e^{\beta A_s}|\overline{Z}_s^{n+1}|^2ds\\
&+\mathbf{E}\int_t^T e^{\beta A_s}\|\overline{U}_s^{n+1}\|_\lambda^2ds
\end{aligned}$$

$$
\begin{aligned}
\leq\ & 2\mathbf{E}\int_t^T e^{\beta A_s} a_s^2 |\overline{Y}_s^n|^2 ds \\
& +(\varepsilon+\alpha)\left(\mathbf{E}\int_t^T e^{\beta A_s}|\overline{Z}_s^n|^2 ds + \mathbf{E}\int_t^T e^{\beta A_s}\|\overline{U}_s^n\|_\lambda^2 ds\right).
\end{aligned}
$$

Fix $\varepsilon > 0$ and define $\overline{c} = 2/(\varepsilon+\alpha)$ and $\beta_0 = 1+\overline{c}+1/\varepsilon$. Choosing $\beta \geq \beta_0$, we obtain

$$
\begin{aligned}
& \mathbf{E}\left[\overline{c}\int_t^T e^{\beta A_s} a_s^2 |\overline{Y}_s^{n+1}|^2 ds + \int_t^T e^{\beta A_s}|\overline{Z}_s^{n+1}|^2 ds + \int_t^T e^{\beta A_s}\|\overline{U}_s^{n+1}\|_\lambda^2 ds\right] \\
\leq\ & (\varepsilon+\alpha)\mathbf{E}\left[\overline{c}\int_t^T e^{\beta A_s} a_s^2 |\overline{Y}_s^{n}|^2 ds + \int_t^T e^{\beta A_s}|\overline{Z}_s^{n}|^2 ds + \int_t^T e^{\beta A_s}\|\overline{U}_s^{n}\|_\lambda^2 ds\right]
\end{aligned}
$$

and by iterations we deduce that

$$
\begin{aligned}
& \overline{c}\left\|\overline{Y}^{n+1}\right\|^2_{\mathscr{M}_\beta^{2,a}(\mathbf{R}^k)} + \left\|\overline{Z}^{n+1}\right\|^2_{\mathscr{M}_\beta^{2}(\mathbf{R}^k)} + \left\|\overline{U}^{n+1}\right\|^2_{\mathscr{L}_\beta^{2}(\mathbf{R}^k)} \\
\leq\ & (\varepsilon+\alpha)^n\left(\overline{c}\left\|\overline{Y}^{1}\right\|^2_{\mathscr{M}_\beta^{2,a}(\mathbf{R}^k)} + \left\|\overline{Z}^{1}\right\|^2_{\mathscr{M}_\beta^{2}(\mathbf{R}^k)} + \left\|\overline{U}^{1}\right\|^2_{\mathscr{L}_\beta^{2}(\mathbf{R}^k)}\right).
\end{aligned}
$$

Hence, choosing $\varepsilon > 0$ such that $\varepsilon+\alpha < 1$, we deduce that $(Y^n, Z^n, U^n)_{n\geq 1}$ is a Cauchy sequence in the Banach space $\mathcal{A}_\beta^2(\mathbf{R}^k)$. It remains to show that $(Y^n)_{n\geq 1}$ is a Cauchy sequence in $\mathscr{S}_\beta^2(\mathbf{R}^k)$. To this end, we define for any integers $n, m \geq 1$ $\mathfrak{R}^{n,m} = \mathfrak{R}^n - \mathfrak{R}^m$ for $\mathfrak{R} \in \{Y, Z, U, K, C\}$, and

$$
\forall h \in \{f,\ g\}, \quad h_\Theta^{n,m}(t) = h(t,\Theta_t^n) - h(t,\Theta_t^m), \quad t \leq T.
$$

Then it is readily seen that

$$
\begin{aligned}
Y_t^{n+1,m+1} =\ & \int_t^T f_\Theta^{n,m}(s)ds + \int_t^T g_\Theta^{n,m}(s)dB_s - \int_t^T Z_s^{n+1,m+1}dW_s \\
& -\int_t^T\int_E U_s^{n+1,m+1}(e)\widetilde{\mu}(ds,de) + K_T^{n+1,m+1} - K_t^{n+1,m+1} \\
& +C_{T-}^{n+1,m+1} - C_{t-}^{n+1,m+1}. \qquad (11)
\end{aligned}
$$

Applying Lemma 2.11 to (11), and taking the essential supremum over $\tau \in \mathcal{T}_{[0,T]}$ and then the expectation on both sides we get

$$
\begin{aligned}
& \mathbf{E}\left(\operatorname*{ess\,sup}_{\nu\in\mathcal{T}_{[0,T]}} e^{\beta A_\tau}|Y_\tau^{n+1,m+1}|^2\right) + \beta\mathbf{E}\int_t^T e^{\beta A_s} a_s^2 |Y_s^{n+1,m+1}|^2 ds \\
& +\mathbf{E}\int_t^T e^{\beta A_s}|Z_s^{n+1,m+1}|^2 ds + \mathbf{E}\int_t^T e^{\beta A_s}\|U_s^{n+1,m+1}\|_\lambda^2 ds \\
\leq\ & 2\mathbf{E}\int_t^T e^{\beta A_s}\langle Y_{s-}^{n+1,m+1}, f_\Theta^{n,m}(s)\rangle ds + \mathbf{E}\int_t^T e^{\beta A_s}|g_\Theta^{n,m}(s)|^2 ds \\
& +2\mathbf{E}\operatorname*{ess\,sup}_{\nu\in\mathcal{T}_{[0,T]}}\left|\int_0^\tau e^{\beta A_s}\langle Y_{s-}^{n+1,m+1}, Z_s^{n+1,m+1}dW_s\rangle\right|
\end{aligned}
$$

$$+2\mathbf{E}\operatorname*{ess\,sup}_{\nu\in\mathcal{T}_{[0,T]}}\left|\int_0^\tau e^{\beta A_s}\langle Y_{s-}^{n+1,m+1}, g_\Theta^{n,m}(s)dB_s\rangle\right|$$
$$+2\mathbf{E}\operatorname*{ess\,sup}_{\nu\in\mathcal{T}_{[0,T]}}\left|\int_0^\tau\int_E e^{\beta A_s}\langle Y_{s-}^{n+1,m+1}, U_s^{n+1,m+1}(e)\widetilde{\mu}(ds,de)\rangle\right|.$$

But, for any $\varepsilon > 0$,

$$2\langle Y_s^{n+1,m+1}, f_\Theta^{n,m}(s)\rangle \le \frac{1}{\varepsilon}a_s^2|Y_s^{n+1,m+1}|^2 + \varepsilon\left|\frac{f_\Theta^{n,m}(s)}{a_s}\right|^2.$$

Moreover, by the Burkhölder–Davis–Gundy inequality, there exists a universal constant c such that

$$2\mathbf{E}\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[0,T]}}\left|\int_0^\tau e^{\beta A_s}\langle Y_{s-}^{n+1,m+1}, Z_s^{n+1,m+1}dW_s\rangle\right|$$
$$\le \frac{1}{4}\|Y^{n+1,m+1}\|^2_{\mathscr{S}_\beta^2(\mathbf{R}^k)} + 4c^2\|Z^{n+1,m+1}\|^2_{\mathscr{M}_\beta^2(\mathbf{R}^{k\times d})},$$

$$2\mathbf{E}\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[0,T]}}\left|\int_0^\tau\int_E e^{\beta A_s}\langle Y_{s-}^{n+1,m+1}, U_s^{n+1,m+1}(e)\widetilde{\mu}(ds,de)\rangle\right|$$
$$\le \frac{1}{4}\|Y^{n+1,m+1}\|^2_{\mathscr{S}_\beta^2(\mathbf{R}^k)} + 4c^2\|U^{n+1,m+1}\|^2_{\mathscr{L}_\beta^2(\mathbf{R}^k)}$$

and

$$2\mathbf{E}\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[0,T]}}\left|\int_0^\tau e^{\beta A_s}\langle Y_{s-}^{n+1,m+1}, g_\Theta^{n,m}(s)dB_s\rangle\right|$$
$$\le \frac{1}{4}\|Y^{n+1,m+1}\|^2_{\mathscr{S}_\beta^2(\mathbf{R}^k)} + 4c^2\|g_\Theta^{n,m}\|^2_{\mathscr{M}_\beta^2(\mathbf{R}^{k\times\ell})}.$$

Hence, there exists $\mathcal{C} > 0$ such that

$$\mathbf{E}\left(\operatorname*{ess\,sup}_{\nu\in\mathcal{T}_{[0,T]}} e^{\beta A_\tau}|Y_\tau^{n+1,m+1}|^2\right)$$
$$\le \mathcal{C}\left(\mathbf{E}\int_0^T e^{\beta A_s}\left|\frac{f_\Theta^{n,m}(s)}{a_s}\right|^2 ds + \mathbf{E}\int_0^T e^{\beta A_s}|g_\Theta^{n,m}(s)|^2 ds\right)$$
$$\le \mathcal{C}\Bigg(4\mathbf{E}\int_0^T e^{\beta A_s}a_s^2|Y_s^{n,m}|^2 ds$$
$$+(3+\alpha)\left(\mathbf{E}\int_0^T e^{\beta A_s}|Z_s^{n,m}|^2 ds + \mathbf{E}\int_0^T e^{\beta A_s}\|U_s^{n,m}\|_\lambda^2 ds\right)\Bigg).$$

Since $(Y^n, Z^n, U^n)_{n\ge1}$ is a Cauchy sequence in $\mathcal{A}_\beta^2(\mathbf{R}^k)$, we deduce that $(Y^n)_{n\ge1}$ is a Cauchy sequence in $\mathscr{S}_\beta^2(\mathbf{R}^k)$. Hence, $(Y^n, Z^n, U^n)_{n\ge1}$ is a Cauchy sequence in the Banach space $\mathscr{B}_\beta^2(\mathbf{R}^k)$, so it converges in $\mathscr{B}_\beta^2(\mathbf{R}^k)$ to a limit $\Theta = (Y, Z, U)$. Now let us show that (Y, Z, U), with the additional Mertens process (K, C), is a solution to RBDSDEJ (2).

Since $(Y^n, Z^n, U^n)_{n\geq 1}$ converges in $\mathscr{B}^2_\beta(\mathbf{R}^k)$ to a limit (Y, Z, U), we have

$$\lim_{n\to+\infty} \left\|(Y^n - Y, Z^n - Z, U^n - U)\right\|^2_{\mathscr{B}^2_\beta(\mathbf{R}^k)} = 0. \tag{12}$$

Using the Cauchy–Schwarz inequality and (12), we deduce from **(A1.1)** and **(A1.2)**

$$\begin{aligned}
&\mathbf{E}\bigg(\bigg|\int_t^T (f(s,\Theta^n_s) - f(s,\Theta_s))ds\bigg|^2\bigg)\\
&\qquad \leq \mathbf{E}\bigg(\frac{1}{\beta}\int_t^T e^{\beta A_s}\frac{|f(s,Y^n_s,Z^n_s,U^n_s) - f(s,Y_s,Z_s,U_s)|^2}{a_s^2}ds\bigg)\\
&\qquad \leq \frac{3}{\beta}\mathbf{E}\bigg(\int_t^T e^{\beta A_s}a_s^2|Y^n_s - Y_s|^2 ds + \int_t^T e^{\beta A_s}|Z^n_s - Z_s|^2 ds\\
&\qquad\quad + \int_t^T e^{\beta A_s}\|U^n_s - U_s\|^2_\lambda ds\bigg) \xrightarrow[n\to+\infty]{} 0.
\end{aligned}$$

Similarly, by the Burkhölder–Davis–Gundy inequality and (12), we have

$$\begin{aligned}
&\mathbf{E}\bigg(\sup_{0\leq t\leq T}\bigg|\int_t^T g(s,\Theta^n_s)dB_s - \int_t^T g(s,\Theta_s)dB_s\bigg|^2\bigg)\\
&\qquad \leq \mathbf{E}\left(\int_t^T |g(s,Y^n_s,Z^n_s,U^n_s) - g(s,Y_s,Z_s,U_s)|^2 ds\right)\\
&\qquad \leq \mathbf{E}\bigg(\int_t^T e^{\beta A_s}a_s^2|Y^n_s - Y_s|^2 ds + \alpha\int_t^T e^{\beta A_s}|Z^n_s - Z_s|^2 ds\\
&\qquad\quad + \alpha\int_t^T e^{\beta A_s}\|U^n_s - U_s\|^2_\lambda ds\bigg) \xrightarrow[n\to+\infty]{} 0.
\end{aligned}$$

Moreover, since $A_s \geq 0$ for all $s \leq T$, we have

$$\mathbf{E}\left(\sup_{0\leq t\leq T}\left|\int_t^T Z^n_s dW_s - \int_t^T Z_s dW_s\right|^2\right) \leq \mathbf{E}\int_t^T e^{\beta A_s}|Z^n_s - Z_s|^2 ds \xrightarrow[n\to+\infty]{} 0$$

and

$$\begin{aligned}
&\mathbf{E}\bigg(\sup_{0\leq t\leq T}\bigg|\int_t^T\int_E U^n_s(e)\widetilde{\mu}(de,ds) - \int_t^T\int_E U_s(e)\widetilde{\mu}(de,ds)\bigg|^2\bigg)\\
&\qquad\qquad \leq \mathbf{E}\int_t^T e^{\beta A_s}\|U^n_s - U_s\|^2_\lambda ds \xrightarrow[n\to+\infty]{} 0.
\end{aligned}$$

For each $\tau \in \mathcal{T}_{[0,T]}$, let

$$\begin{aligned}
\widetilde{K}_\tau = K_\tau - C_{\tau-} &= Y_0 - Y_\tau - \int_0^\tau f(s,\Theta_s)\,ds - \int_0^\tau g(s,\Theta_s)\,dB_s\\
&\quad + \int_0^\tau Z_s dW_s + \int_0^\tau\int_E U_s(e)\widetilde{\mu}(ds,de).
\end{aligned}$$

Then, we can easily show that $\|\widetilde{K}^n - \widetilde{K}\|^2_{\mathscr{S}^2} \longrightarrow 0$, as $n \longrightarrow +\infty$. So, letting $n \longrightarrow +\infty$ in (9), we deduce that (Y, Z, U, K, C) is a solution to RBDSDEJ (2).

(ii) Uniqueness. Let $(Y^1, Z^1, U^1, K^1, C^1)$ and $(Y^2, Z^2, U^2, K^2, C^2)$ be two solutions to RBDSDEJ (2). We define $\overline{\Re} = \Re^1 - \Re^2$ for $\Re \in \{Y, Z, U, K, C\}$ and

$$\forall h \in \{f, g\}, \quad \overline{h}_\Theta(t) = h(t, \Theta^1_t) - h(t, \Theta^2_t), \quad t \leq T.$$

Thus the process $(\overline{Y}, \overline{Z}, \overline{U}, \overline{K}, \overline{C})$ satisfies the following equation

$$\begin{aligned} \overline{Y}_t &= \int_t^T \overline{f}_\Theta(s)ds + \int_t^T \overline{g}_\Theta(s)dB_s - \int_t^T \overline{Z}_s dW_s + \int_t^T \int_E \overline{U}_s(e)\widetilde{\mu}(ds, de) \\ &\quad + \overline{K}_T - \overline{K}_t + \overline{C}_{T-} - \overline{C}_{t-}. \end{aligned} \tag{13}$$

Applying Lemma 2.11 to (13) and taking into consideration Remark 2.3, we have

$$\begin{aligned} &\mathbf{E}\left[e^{\beta A_t}|\overline{Y}_t|^2\right] + \beta \mathbf{E}\int_t^T e^{\beta A_s} a_s^2 |\overline{Y}_s|^2 ds + \mathbf{E}\int_t^T e^{\beta A_s}(|\overline{Z}_s|^2 + \|\overline{U}_s\|^2_\lambda)ds \\ &\qquad \leq 2\mathbf{E}\int_t^T e^{\beta A_s}\langle \overline{Y}_s, \overline{f}_\Theta(s)\rangle ds + \mathbf{E}\int_t^T e^{\beta A_s}|\overline{g}_\Theta(s)|^2 ds. \end{aligned}$$

By the same computations as before (by using the assumptions **(A1.1)–(A1.2)**), we have, for any $\varepsilon > 0$,

$$2\langle \overline{Y}_s, \overline{f}_\Theta(s)\rangle \leq \left(2 + \frac{2}{\varepsilon}\right) a_s^2 |\overline{Y}_s|^2 + \varepsilon(|\overline{Z}_s|^2 + |\overline{U}_s|^2),$$

and

$$|\overline{g}_\Theta(s)|^2 \leq a_s^2|\overline{Y}_s|^2 + \alpha(|\overline{Z}_s|^2 + |\overline{U}_s|^2).$$

Hence, choosing $\varepsilon > 0$, $\beta > 0$ such that $\varepsilon + \alpha < 1$ and $\beta > 3 + 2/\varepsilon$, we deduce that

$$\begin{aligned} &\mathbf{E}\left[e^{\beta A_t}|\overline{Y}_t|^2\right] + \left(\beta - 3 - \frac{2}{\varepsilon}\right)\mathbf{E}\int_t^T e^{\beta A_s} a_s^2 |\overline{Y}_s|^2 ds \\ &\qquad + (1 - \varepsilon - \alpha)\mathbf{E}\left[\int_t^T e^{\beta A_s}|\overline{Z}_s|^2 ds + \int_t^T e^{\beta A_s}\|\overline{U}_s\|^2_\lambda ds\right] \leq 0. \end{aligned}$$

It follows that $(\overline{Y}, \overline{Z}, \overline{U}) = (0, 0, 0)$, and thus $(\overline{K}, \overline{C}) = (0, 0)$. □

3.3 Comparison theorem

In all what follows, we are interested in one-dimensional RBDSDEJs (i.e. $k = 1$). We consider the RBDSDEJs associated with parameters $(f^i(., \Theta), g(., \Theta), \xi^i)$ for $i = 1, 2$ where Θ^i stands for the process (Y^i, Z^i, U^i). Let us state the following assumption

(A1.6): $\begin{cases} \xi^1_t \leq \xi^2_t \text{ a.s.} \quad \forall t \leq T \\ f^1(t, y, z, u) \leq f^2(t, y, z, u) \text{ a.s.} \quad \forall (t, y, z, u) \in [0, T] \times \mathbf{R} \times \mathbf{R}^d \times \mathscr{L}_\lambda. \end{cases}$

Then we have the following comparison result.

Theorem 3.5. *Let* $(Y^i, Z^i, U^i, K^i, C^i)$ *be a solution to RBDSDEJs associated with parameters* $(f^i(., \Theta), g(., \Theta), \xi^i)$ *for* $i = 1, 2$. *Under the assumptions* **(A1.1)–(A1.4)** *and* **(A1.6)** *we have*

$$\forall t \leq T, \quad Y_t^1 \leq Y_t^2, \quad \mathbf{P}\text{-}a.s.$$

Proof. Define $\widehat{\Re} = \Re^1 - \Re^2$ for $\Re \in \{Y, Z, U, K, C, \xi\}$. Then the process $(\widehat{Y}, \widehat{Z}, \widehat{U}, \widehat{K}, \widehat{C})$ satisfies the following equation

$$\begin{aligned} \widehat{Y}_t = & \ \widehat{\xi}_T + \int_t^T [f^1(s, \Theta_s^1) - f^2(s, \Theta_s^2)]ds + \int_t^T [g(s, \Theta_s^1) - g(s, \Theta_s^2)]dB_s \\ & - \int_t^T \widehat{Z}_s dW_s - \int_t^T \int_E \widehat{U}_s(e)\widetilde{\mu}(ds, de) + \widehat{K}_T - \widehat{K}_t + \widehat{C}_{T-} - \widehat{C}_{t-}. \end{aligned}$$

Applying Lemma 2.11, taking into account Remark 2.3 and taking the expectation, we obtain, for all $t \leq T$,

$$\begin{aligned} & \mathbf{E}\left[e^{\beta A_t}|\widehat{Y}_t^+|^2\right] + \beta \mathbf{E} \int_t^T \mathbb{1}_{\{\widehat{Y}_s > 0\}} e^{\beta A_s} a_s^2 |\widehat{Y}_s|^2 ds \\ & + \mathbf{E} \int_t^T \mathbb{1}_{\{\widehat{Y}_s > 0\}} e^{\beta A_s} |\widehat{Z}_s|^2 ds + \mathbf{E} \int_t^T \mathbb{1}_{\{\widehat{Y}_s > 0\}} e^{\beta A_s} \|\widehat{U}_s\|_\lambda^2 ds \\ \leq & \ \mathbf{E}\left[e^{\beta A_T}|\widehat{\xi}_T^+|^2\right] + 2\mathbf{E} \int_t^T \mathbb{1}_{\{\widehat{Y}_s > 0\}} e^{\beta A_s} \widehat{Y}_s^+ [f^1(s, \Theta_s^1) - f^2(s, \Theta_s^2)]ds \\ & + \mathbf{E} \int_t^T \mathbb{1}_{\{\widehat{Y}_s > 0\}} e^{\beta A_s} |g(s, \Theta_s^1) - g(s, \Theta_s^2)|^2 ds. \end{aligned} \tag{14}$$

By assumption **(A1.6)**, we have $\mathbf{E}[e^{\beta A_T}|\widehat{\xi}_T^+|] = 0$ and $\widehat{Y}_s^+[f^1(s, \Theta_s^1) - f^2(s, \Theta_s^1)] \leq 0$, and due to the assumptions **(A1.1)–(A1.2)**, we get, for any $\varepsilon > 0$,

$$\begin{aligned} & 2\mathbf{E} \int_t^T \mathbb{1}_{\{\widehat{Y}_s > 0\}} e^{\beta A_s} \widehat{Y}_s^+ [f^1(s, \Theta_s^1) - f^2(s, \Theta_s^2)]ds \\ \leq & \ 2\mathbf{E} \int_t^T \mathbb{1}_{\{\widehat{Y}_s > 0\}} e^{\beta A_s} \widehat{Y}_s^+ [f^2(s, \Theta_s^1) - f^2(s, \Theta_s^2)]ds \\ \leq & \ \left(2 + \frac{2}{\varepsilon}\right) \mathbf{E} \int_t^T \mathbb{1}_{\{\widehat{Y}_s > 0\}} e^{\beta A_s} a_s^2 |\widehat{Y}_s^+|^2 ds + \varepsilon \mathbf{E} \int_t^T \mathbb{1}_{\{\widehat{Y}_s > 0\}} e^{\beta A_s} |\widehat{Z}_s|^2 ds \\ & + \varepsilon \mathbf{E} \int_t^T \mathbb{1}_{\{\widehat{Y}_s > 0\}} e^{\beta A_s} \|\widehat{U}_s\|_\lambda^2 ds \end{aligned}$$

and

$$\begin{aligned} & \mathbf{E} \int_t^T \mathbb{1}_{\{\widehat{Y}_s > 0\}} e^{\beta A_s} |g(s, \Theta_s^1) - g(s, \Theta_s^2)|^2 ds \\ \leq & \ \mathbf{E} \int_t^T \mathbb{1}_{\{\widehat{Y}_s > 0\}} e^{\beta A_s} a_s^2 |\widehat{Y}_s|^2 ds + \alpha \mathbf{E} \int_t^T \mathbb{1}_{\{\widehat{Y}_s > 0\}} e^{\beta A_s} (|\widehat{Z}_s|^2 + \|\widehat{U}_s\|_\lambda^2) ds. \end{aligned}$$

Plugging these two last inequalities in (14), we deduce that, for any $\beta > 0$ and $\varepsilon > 0$,

$$
\begin{aligned}
& \mathbf{E}\left[e^{\beta A_t}|\widehat{Y}_t^+|^2\right] + \beta \mathbf{E}\int_t^T \mathbb{1}_{\{\widehat{Y}_s>0\}} e^{\beta A_s} a_s^2 |\widehat{Y}_s|^2 ds \\
& +\mathbf{E}\int_t^T \mathbb{1}_{\{\widehat{Y}_s>0\}} e^{\beta A_s}|\widehat{Z}_s|^2 ds + \mathbf{E}\int_t^T \mathbb{1}_{\{\widehat{Y}_s>0\}} e^{\beta A_s}\|\widehat{U}_s\|_\lambda^2 ds \\
\leq\ & \left(3+\frac{2}{\varepsilon}\right)\mathbf{E}\int_t^T \mathbb{1}_{\{\widehat{Y}_s>0\}} e^{\beta A_s} a_s^2 |\widehat{Y}_s^+|^2 ds \\
& +(\varepsilon+\alpha)\left(\mathbf{E}\int_t^T \mathbb{1}_{\{\widehat{Y}_s>0\}} e^{\beta A_s}|\widehat{Z}_s|^2 ds + \mathbf{E}\int_t^T \mathbb{1}_{\{\widehat{Y}_s>0\}} e^{\beta A_s}\|\widehat{U}_s\|_\lambda^2 ds\right).
\end{aligned}
$$

Choosing $\varepsilon = (1-\alpha)/2$ and taking $\beta > 3 + 2/\varepsilon$, we derive that

$$
|\widehat{Y}_t^+|^2 = 0 \quad \text{a.s.} \quad \forall t \leq T, \quad \text{i.e.,} \quad Y_t^1 \leq Y_t^2 \quad \text{a.s.} \quad \forall t \leq T. \qquad \square
$$

4 Reflected BDSDEJs with stochastic growth condition

In this section we are interested in weakening the conditions on the coefficient f. We are also interested in one-dimensional RBDSDEJs (i.e. $k = 1$). Let us state the new working assumptions.

4.1 Assumptions

We assume that the data (f, g, ξ) satisfy the following assumptions **(A2)**:

(A2.1): There exist four non-negative $\mathscr{F}_t^W$-measurable processes $(\gamma_t)_{t\leq T}$, $(\kappa_t)_{t\leq T}$, $(\sigma_t)_{t\leq T}$ and $(\varrho_t)_{t\leq T}$ such that the condition **(A1.2)** holds, and there exists another $\mathcal{F}_t$-progressively measurable nonnegative process $(\zeta_t)_{t\leq T}$ such that $\dfrac{\zeta}{a} \in \mathscr{M}_\beta^2(\mathbf{R})$ and for all $(t, y, z, u) \in [0, T] \times \mathbf{R} \times \mathbf{R}^d \times \mathscr{L}_\lambda$,

$$
|f(t, y, z, u)| \leq \zeta_t + \gamma_t|y| + \kappa_t|z| + \sigma_t\|u\|_\lambda.
$$

(A2.2): $f(\omega, t, \cdot, \cdot, \cdot) : \mathbf{R} \times \mathbf{R}^d \times \mathscr{L}_\lambda \to \mathbf{R}$ is continuous.

(A2.3): The coefficient g satisfies **(A1.1)** for $\alpha \in]0, 1/2[$.

(A2.4): The irregular barrier $(\xi_t)_{t\leq T}$ satisfies **(A1.4)**.

4.2 Existence of a minimal solution

In this section, we will prove the existence of a minimal solution to RBDSDEJ (2) under the conditions **(A2)**. First let us define a minimal solution as follows.

Definition 2. A solution (Y, Z, U, K, C) to RBDSDEJ (2) is called a minimal solution if for any other solution $(Y^*, Z^*, U^*, K^*, C^*)$ to (2) we have, for each $t \leq T$, $Y_t \leq Y_t^*$.

For fixed (ω, t) in $\Omega \times [0, T]$, we define the sequence $f_n(t, y, z, u)$ associated to the coefficient f as follows: for all $(y, y') \in \mathbf{R}^2$, $(z, z') \in \mathbf{R}^d \times \mathbf{R}^d$ and $(u, u') \in \mathscr{L}_\lambda \times \mathscr{L}_\lambda$,

$$f_n(t, y, z, u) = \inf_{y', z', u'} [f(t, y', z', u') + n(\gamma_t |y - y'| + \kappa_t |z - z'| + \sigma_t \|u - u'\|_\lambda)].$$

From Proposition 4.2 in [30], the sequence f_n is well defined for each $n \geq 1$, and it satisfies:

- Linear growth condition: $\forall n \geq 1$,

$$\forall (y, z, u) \in \mathbf{R} \times \mathbf{R}^d \times \mathscr{L}_\lambda, \quad |f_n(t, y, z, u)| \leq \zeta_t + \gamma_t |y| + \kappa_t |z| + \sigma_t \|u\|_\lambda. \tag{15}$$

- Monotonicity: $\forall (y, z, u) \in \mathbf{R} \times \mathbf{R}^d \times \mathscr{L}_\lambda$, $f_n(t, y, z, u)$ increases in n.
- Convergence: If $(y_n, z_n, u_n) \to (y, z, u)$ in $\mathbf{R} \times \mathbf{R}^d \times \mathscr{L}_\lambda$ as $n \to +\infty$, then

$$f_n(t, y_n, z_n, u_n) \xrightarrow[n \to +\infty]{} f(t, y, z, u). \tag{16}$$

- Lipschitz condition: $\forall n \geq 1$, and for all $(y, y') \in \mathbf{R}^2$, $(z, z') \in \mathbf{R}^d \times \mathbf{R}^d$ and $(u, u') \in \mathscr{L}_\lambda \times \mathscr{L}_\lambda$, we have

$$|f_n(t, y, z, u) - f_n(t, y', z', u')| \leq n\gamma_t |y - y'| + n\kappa_t |z - z'| + n\sigma_t \|u - u'\|_\lambda.$$

We also define the function

$$F(t, y, z, u) = \zeta_t + \gamma_t |y| + \kappa_t |z| + \sigma_t \|u\|_\lambda.$$

Now, from Theorem 3.4, there exist two processes $\overline{\Theta} := (\overline{Y}, \overline{Z}, \overline{U})$ and $\Theta^n := (Y^n, Z^n, U^n)$ which are the solutions to RBDSDEJs associated with parameters $(F(., \overline{\Theta}), g(., \overline{\Theta}), \xi)$ and $(f_n(., \Theta^n), g(., \Theta^n), \xi)$, respectively.

From the definitions of f_n and F together with (15), we observe that $\forall n \geq 1$, $f_n \leq f_{n+1} \leq F$. Then, due to Theorem 3.5 we have

$$\forall t \leq T, \quad Y_t^1 \leq Y_t^n \leq Y_t^{n+1} \leq \overline{Y}_t. \tag{17}$$

The proof of the main result of this section is based on the two next lemmas.

Lemma 4.1. *Under the assumption* ***(A2)****, there exists a positive constant* Λ *depending on* β *such that*

$$\| (\overline{Y}, \overline{Z}, \overline{U}) \|^2_{\mathscr{B}^2_\beta(\mathbf{R})} \leq \Lambda \left(\|\xi\|^2_{\mathscr{S}^2_{2\beta}(\mathbf{R})} + \left\| \frac{\zeta}{a} \right\|^2_{\mathscr{M}^2_\beta(\mathbf{R})} + \|g(., 0)\|^2_{\mathscr{M}^2_\beta(\mathbf{R}^\ell)} \right)$$

and for each $n \geq 1$

$$\|(Y^n, Z^n, U^n)\|^2_{\mathscr{B}^2_\beta(\mathbf{R})} \leq \Lambda \left(\|\xi\|^2_{\mathscr{S}^2_{2\beta}(\mathbf{R})} + \left\| \frac{\zeta}{a} \right\|^2_{\mathscr{M}^2_\beta(\mathbf{R})} + \|g(., 0)\|^2_{\mathscr{M}^2_\beta(\mathbf{R}^\ell)} \right).$$

Proof. We know that

$$\begin{aligned}\overline{Y}_t &= \xi_T + \int_t^T F(s,\overline{\Theta}_s)ds + \int_t^T g(s,\overline{\Theta}_s)dB_s - \int_t^T \overline{Z}_s dW_s \\ &\quad - \int_t^T \int_E \overline{U}_s(e)\widetilde{\mu}(ds,de) + \overline{K}_T - \overline{K}_t + \overline{C}_{T-} - \overline{C}_{t-}, \qquad (18)\end{aligned}$$

where $(\overline{K},\overline{C})$ satisfies the Skorokhod and minimality conditions. Then, applying Lemma 2.11 together with Remark 2.3, we deduce

$$\begin{aligned}& e^{\beta A_t}|\overline{Y}_t|^2 + \beta\int_t^T e^{\beta A_s}a_s^2|\overline{Y}_s|^2ds + \int_t^T e^{\beta A_s}|\overline{Z}_s|^2ds + \int_t^T e^{\beta A_s}\|\overline{U}_s\|_\lambda^2 ds \\ \leq\ & e^{\beta A_T}|\xi|^2 + 2\int_t^T e^{\beta A_s}\overline{Y}_s F(s,\overline{\Theta}_s)ds + 2\int_t^T e^{\beta A_s}\overline{Y}_{s-}g(s,\overline{\Theta}_s)dB_s \\ & -2\int_t^T e^{\beta A_s}\overline{Y}_{s-}\overline{Z}_s dW_s - 2\int_t^T\int_E e^{\beta A_s}\overline{Y}_{s-}\overline{U}_s(e)\widetilde{\mu}(ds,de) \\ & +\int_t^T e^{\beta A_s}|g(s,\overline{\Theta}_s)|^2ds + 2\int_t^T e^{\beta A_s}\overline{Y}_{s-}d\overline{K}_s + 2\int_t^T e^{\beta A_s}\overline{Y}_s d\overline{C}_s. \qquad (19)\end{aligned}$$

But for any $\beta > 0$ and $\varepsilon > 0$,

$$2\overline{Y}_s F(s,\overline{\Theta}_s) \leq \left(\frac{\beta}{2} + 2 + \frac{2}{\varepsilon}\right)a_s^2|\overline{Y}_s|^2 + \frac{2}{\beta}\left|\frac{\zeta_s}{a_s}\right|^2 + \varepsilon(|\overline{Z}_s|^2 + \|\overline{U}_s\|_\lambda^2)$$

and

$$\begin{aligned}|g(s,\overline{\Theta}_s)|^2 &\leq 2\left(|g(s,\overline{\Theta}_s) - g(s,0)|^2 + |g(s,0)|^2\right) \\ &\leq 2\left(a_s^2|\overline{Y}_s|^2 + \alpha(|\overline{Z}_s|^2 + \|\overline{U}_s\|_\lambda^2) + |g(s,0)|^2\right).\end{aligned}$$

Plugging these inequalities in (19) and taking expectation, we obtain

$$\begin{aligned}& \left(\frac{\beta}{2} - 4 - \frac{2}{\varepsilon}\right)\mathbf{E}\int_t^T e^{\beta A_s}a_s^2|\overline{Y}_s|^2ds + (1-\varepsilon-2\alpha)\mathbf{E}\int_t^T e^{\beta A_s}|\overline{Z}_s|^2ds \\ & +(1-\varepsilon-2\alpha)\mathbf{E}\int_t^T e^{\beta A_s}\|\overline{U}_s\|_\lambda^2ds \\ \leq\ & \mathbf{E}\left(e^{\beta A_T}|\xi|^2 + \frac{2}{\beta}\int_t^T e^{\beta A_s}\left|\frac{\zeta_s}{a_s}\right|^2ds + 2\int_t^T e^{\beta A_s}|g(s,0)|^2ds \right. \\ & \left. +2\int_t^T e^{\beta A_s}\overline{Y}_{s-}d\overline{K}_s + 2\int_t^T e^{\beta A_s}\overline{Y}_s d\overline{C}_s\right). \qquad (20)\end{aligned}$$

Moreover,

$$2\mathbf{E}\int_t^T e^{\beta A_s}\overline{Y}_{s-}d\overline{K}_s + 2\mathbf{E}\int_t^T e^{\beta A_s}\overline{Y}_s d\overline{C}_s$$

$$
\begin{aligned}
&\leq 2\mathbf{E} \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[0,T]}} e^{\beta A_\tau}|\xi_\tau| \int_0^T d(\overline{K}_t + \overline{C}_t) \\
&\leq \varepsilon \mathbf{E} \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[0,T]}} e^{2\beta A_\tau}|\xi_\tau|^2 + \frac{1}{\varepsilon}\mathbf{E}(\overline{K}_T + \overline{C}_T)^2,
\end{aligned}
$$

and from (18) we have

$$
\begin{aligned}
&\mathbf{E}(\overline{K}_T + \overline{C}_T)^2 \\
\leq\ & 6\mathbf{E}\left(\overline{Y}_0^2 + \xi_T^2 + \left|\int_0^T F(s,\overline{\Theta}_s)ds\right|^2 + \left|\int_0^T g(s,\overline{\Theta}_s)dB_s\right|^2 + \left|\int_0^T \overline{Z}_s dW_s\right|^2 \right. \\
&\left. + \left|\int_0^T \int_E \overline{U}_s(e)\widetilde{\mu}(ds,de)\right|^2\right) \\
\leq\ & 6\mathbf{E}\left(\overline{Y}_0^2 + \xi_T^2 + \frac{1}{\beta}\int_0^T e^{\beta A_s}\left|\frac{F(s,\overline{\Theta}_s)}{a_s}\right|^2 ds \right. \\
&\left. + c\left(\int_0^T |g(s,\overline{\Theta}_s)|^2 ds + \int_0^T |\overline{Z}_s|^2 ds + \int_0^T \|\overline{U}_s\|_\lambda^2 ds\right)\right) \\
\leq\ & 6\mathbf{E}\left(2\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[0,T]}} e^{2\beta A_\tau}|\xi_\tau|^2 + \frac{4}{\beta}\int_0^T e^{\beta A_s}\left|\frac{\zeta_s}{a_s}\right|^2 ds + 2c\int_0^T e^{\beta A_s}|g(s,0)|^2 ds \right. \\
& + \left(\frac{4}{\beta} + 2c\right)\int_0^T e^{\beta A_s} a_s^2 |\overline{Y}_s|^2 ds \\
&\left. + \left(\frac{4}{\beta} + 2\alpha c + c\right)\left(\int_0^T e^{\beta A_s}|\overline{Z}_s|^2 ds + \int_0^T e^{\beta A_s}\|\overline{U}_s\|_\lambda^2 ds\right)\right).
\end{aligned}
$$

Then (20) becomes

$$
\begin{aligned}
&\phi_1 \mathbf{E}\int_t^T e^{\beta A_s} a_s^2 |\overline{Y}_s|^2 ds + \phi_2 \mathbf{E}\int_t^T e^{\beta A_s}|\overline{Z}_s|^2 ds + \phi_2 \mathbf{E}\int_t^T e^{\beta A_s}\|\overline{U}_s\|_\lambda^2 ds \\
\leq\ & \Lambda_1 \mathbf{E}\left(\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[0,T]}} e^{2\beta A_\tau}|\xi_\tau|^2 + \int_0^T e^{\beta A_s}\left|\frac{\zeta_s}{a_s}\right|^2 ds + \int_0^T e^{\beta A_s}|g(s,0)|^2 ds\right).
\end{aligned}
$$

where $\phi_1 = \frac{\beta}{2} - 4 - \frac{2}{\varepsilon} - \frac{6}{\varepsilon}\left(\frac{4}{\beta} + 2c\right)$, $\phi_2 = 1 - \varepsilon - 2\alpha - \frac{6}{\varepsilon}\left(\frac{4}{\beta} + 2\alpha c + c\right)$ and Λ_1 is a nonnegative constant depending on β, c and ε. Now, choose $\varepsilon \leq 1 - 2\alpha$ with $0 < \alpha < 1/2$ and $\beta > 0$ such that $\varepsilon\beta(\beta - 12 - 24c) > 48$ (these choices are suitable to obtain a nonnegative ϕ_1 and ϕ_2). Hence

$$
\|\overline{Y}, \overline{Z}, \overline{U}\|^2_{\mathcal{A}_\beta^2(\mathbf{R})} \leq \Lambda_1\left(\|\xi\|^2_{\mathcal{S}^2_{2\beta}(\mathbf{R})} + \left\|\frac{\zeta}{a}\right\|^2_{\mathcal{M}_\beta^2(\mathbf{R})} + \|g(.,0)\|^2_{\mathcal{M}_\beta^2(\mathbf{R}^\ell)}\right). \tag{21}
$$

To conclude, we need an estimate of $\|\overline{Y}\|^2_{\mathscr{S}^2_\beta(\mathbf{R})}$. For this, using (19) once again and (21), we have

$$\begin{aligned}
&\mathbf{E}\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[0,T]}} e^{\beta A_\tau}|\overline{Y}_\tau|^2\\
\leq\ &\Lambda_1\left(\|\xi\|^2_{\mathscr{S}^2_{2\beta}(\mathbf{R})}+\left\|\frac{\zeta}{a}\right\|^2_{\mathscr{M}^2_\beta(\mathbf{R})}+\|g(.,0)\|^2_{\mathscr{M}^2_\beta(\mathbf{R}^\ell)}\right)\\
&+2\mathbf{E}\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[0,T]}}\left|\int_0^\tau e^{\beta A_s}\overline{Y}_{s-}g(s,\overline{\Theta}_s)dB_s\right|+2\mathbf{E}\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[0,T]}}\left|\int_0^\tau e^{\beta A_s}\overline{Y}_{s-}\overline{Z}_s dW_s\right|\\
&+2\mathbf{E}\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[0,T]}}\left|\int_0^\tau\int_E e^{\beta A_s}\overline{Y}_{s-}\overline{U}_s(e)\widetilde{\mu}(ds,de)\right|.
\end{aligned}$$

By the Burkhölder–Davis–Gundy inequality, there exists $c>0$ such that

$$2\mathbf{E}\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[0,T]}}\left|\int_0^\tau e^{\beta A_s}\overline{Y}_{s-}\overline{Z}_s dW_s\right|\leq\frac{1}{6}\|\overline{Y}\|^2_{\mathscr{S}^2_\beta(\mathbf{R})}+6c^2\|\overline{Z}\|^2_{\mathscr{M}^2_\beta(\mathbf{R}^d)},$$

$$\begin{aligned}
2\mathbf{E}\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[0,T]}}\left|\int_0^\tau e^{\beta A_s}\overline{Y}_{s-}g(s,\overline{\Theta}_s)dB_s\right| &\leq \frac{1}{6}\|\overline{Y}\|^2_{\mathscr{S}^2_\beta(\mathbf{R})}+6c^2\|g(.,\overline{\Theta})\|^2_{\mathscr{M}^2_\beta(\mathbf{R}^\ell)}\\
&\leq \frac{1}{6}\|\overline{Y}\|^2_{\mathscr{S}^2_\beta(\mathbf{R})}+12c^2\left(\|\overline{Y}\|^2_{\mathscr{M}^{2,a}_\beta(\mathbf{R})}+\alpha\|\overline{Z}\|^2_{\mathscr{M}^2_\beta(\mathbf{R}^d)}\right.\\
&\qquad\left.+\alpha\|\overline{U}\|^2_{\mathscr{L}^2_\beta(\mathbf{R})}+\|g(.,0)\|^2_{\mathscr{M}^2_\beta(\mathbf{R}^\ell)}\right)
\end{aligned}$$

and

$$2\mathbf{E}\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{[0,T]}}\left|\int_0^\tau\int_E e^{\beta A_s}\overline{Y}_{s-}\overline{U}_s(e)\widetilde{\mu}(ds,de)\right|\leq\frac{1}{6}\|\overline{Y}\|^2_{\mathscr{S}^2_\beta(\mathbf{R})}+6c^2\|\overline{U}\|^2_{\mathscr{L}^2_\beta(\mathbf{R})}.$$

Then, we derive that

$$\|\overline{Y}\|^2_{\mathscr{S}^2_\beta(\mathbf{R})}\leq\Lambda_2\left(\|\xi\|^2_{\mathscr{S}^2_{2\beta}(\mathbf{R})}+\left\|\frac{\zeta}{a}\right\|^2_{\mathscr{M}^2_\beta(\mathbf{R})}+\|g(.,0)\|^2_{\mathscr{M}^2_\beta(\mathbf{R}^\ell)}\right)\tag{22}$$

where Λ_2 is a nonnegative constant depending on β, c and ε. The desired result is obtained by combining the estimates (21) and (22) with $\Lambda=\Lambda_1\vee\Lambda_2$. As a consequence, from (17) we deduce that

$$\|Y^n\|^2_{\mathscr{S}^2_\beta(\mathbf{R})}\leq\Lambda_2\left(\|\xi\|^2_{\mathscr{S}^2_{2\beta}(\mathbf{R})}+\left\|\frac{\zeta}{a}\right\|^2_{\mathscr{M}^2_\beta(\mathbf{R})}+\|g(.,0)\|^2_{\mathscr{M}^2_\beta(\mathbf{R}^\ell)}\right).$$

Using the same computations as before, we can prove that

$$\|Y^n,Z^n,U^n\|^2_{\mathcal{A}^2_\beta(\mathbf{R})}\leq\Lambda_1\left(\|\xi\|^2_{\mathscr{S}^2_{2\beta}(\mathbf{R})}+\left\|\frac{\zeta}{a}\right\|^2_{\mathscr{M}^2_\beta(\mathbf{R})}+\|g(.,0)\|^2_{\mathscr{M}^2_\beta(\mathbf{R}^\ell)}\right).\quad\square$$

Lemma 4.2. *Under the assumption* **(A2)** *the sequence of processes* $(Y^n, Z^n, U^n)_{n\geq 1}$ *converges almost surely in* $\mathscr{B}^2_\beta(\mathbf{R})$ *for each* $\beta > 2$.

Proof. We know that

$$\begin{aligned} Y^n_t &= \xi_T + \int_t^T f_n(s,\Theta^n_s)ds + \int_t^T g(s,\Theta^n_s)dB_s - \int_t^T Z^n_s dW_s \\ &\quad - \int_t^T\int_E U^n_s(e)\widetilde{\mu}(ds,de) + K^n_T - K^n_t + C^n_{T-} - C^n_{t-}, \end{aligned} \tag{23}$$

where (K^n, C^n) satisfy the Skorokhod and minimality conditions. We define, for any integers $n, m \geq 1$, $\Re^{n,m} = \Re^n - \Re^m$ for $\Re \in \{Y, Z, U, K, C\}$,

$$\Delta f^{n,m}(t) = f_n(t,\Theta^n_t) - f_m(t,\Theta^m_t) \ \text{ and } \ \Delta g^{n,m}(t) = g(t,\Theta^n_t) - g(t,\Theta^m_t), \quad t \leq T.$$

Then, applying Lemma 2.11 together with Remark 2.3, we get

$$\begin{aligned} &\beta\mathbf{E}\int_t^T e^{\beta A_s} a_s^2 |Y^{n,m}_s|^2 ds + \mathbf{E}\int_t^T e^{\beta A_s}|Z^{n,m}_s|^2 ds + \mathbf{E}\int_t^T e^{\beta A_s}\|U^{n,m}_s\|^2_\lambda ds \\ \leq\ &\mathbf{E}\int_t^T e^{\beta A_s}|\Delta g^{n,m}(s)|^2 ds + 2\mathbf{E}\int_t^T e^{\beta A_s} Y^{n,m}_s \Delta f^{n,m}(s) ds. \end{aligned}$$

Using the assumption **(A2.3)** and the basic inequality $2ab \leq \epsilon a^2 + \frac{b^2}{\epsilon}$, we get

$$\begin{aligned} &(\beta-1)\left\|Y^{n,m}\right\|^2_{\mathscr{M}^{2,a}_\beta(\mathbf{R})} + (1-\alpha)\left(\left\|Z^{n,m}\right\|^2_{\mathscr{M}^2_\beta(\mathbf{R}^d)} + \left\|U^{n,m}\right\|^2_{\mathscr{L}^2_\beta(\mathbf{R})}\right) \\ \leq\ &\mathbf{E}\int_0^T e^{\beta A_s} a_s^2 |Y^{n,m}_s|^2 ds + \mathbf{E}\int_0^T e^{\beta A_s}\left|\frac{\Delta f^{n,m}(s)}{a_s}\right|^2 ds. \end{aligned}$$

Next, from the linear growth condition on f_n and f_m, and by Lemma 4.1, we find

$$\begin{aligned} &(\beta-2)\left\|Y^{n,m}\right\|^2_{\mathscr{M}^{2,a}_\beta(\mathbf{R})} + (1-\alpha)\left(\left\|Z^{n,m}\right\|^2_{\mathscr{M}^2_\beta(\mathbf{R}^d)} + \left\|U^{n,m}\right\|^2_{\mathscr{L}^2_\beta(\mathbf{R})}\right) \\ \leq\ &8\Lambda\left(\|\xi\|^2_{\mathscr{S}^2_{2\beta}(\mathbf{R})} + \left\|\frac{\zeta}{a}\right\|^2_{\mathscr{M}^2_\beta(\mathbf{R})} + \|g(.,0)\|^2_{\mathscr{M}^2_\beta(\mathbf{R}^\ell)}\right). \end{aligned}$$

Hence for $\beta > 2$, we deduce that (Y^n, Z^n, U^n) is a Cauchy sequence in $\mathcal{A}^2_\beta(\mathbf{R})$, so it converges in $\mathcal{A}^2_\beta(\mathbf{R})$. On the other hand, from (17) we deduce that there exists a process $Y \in \mathscr{S}^2_\beta(\mathbf{R})$ such that $Y^n \to Y$ a.s. as $n \to \infty$. The result follows. □

The main result in this section is what follows.

Theorem 4.3. *Under the assumptions* **(A2)**, *the RBDSDEJ* (2) *associated with parameters* $(f(.,\Theta), g(.,\Theta), \xi)$ *has a minimal solution* $(Y, Z, U, K, C) \in \mathscr{B}^2_\beta(\mathbf{R}) \times \mathscr{S}^2(\mathbf{R}) \times \mathscr{S}^2(\mathbf{R})$.

Proof. From (17), it is readily seen that $(Y^n)_{n\geq 1}$ converges to Y a.s. in $\mathscr{S}^2_\beta(\mathbf{R})$. Otherwise, due to Lemma 4.2 there exist two subsequences still noted as the whole sequences $(Z^n)_{n\geq 1}$ and $(U^n)_{n\geq 1}$ such that $\Theta^n = (Y^n, Z^n, U^n)$ converges to $\Theta = (Y, Z, U) \in \mathcal{A}^2_\beta(\mathbf{R})$ as $n \to +\infty$. By (16), we have

$$f_n(t, \Theta^n_t) \xrightarrow[n\to+\infty]{} f(t, \Theta_t), \quad t \leq T.$$

Furthermore, using the linear growth condition of f_n, it follows that

$$\begin{aligned}
&\mathbf{E}\int_0^T e^{\beta A_s}\left|\frac{f_n(s,\Theta^n_s)}{a_s}\right|^2 ds \\
\leq\ & 4\mathbf{E}\bigg(\int_0^T e^{\beta A_s}\left|\frac{\zeta_s}{a_s}\right|^2 ds + \sup_n \int_0^T e^{\beta A_s} a_s^2 |Y^n_s|^2 ds + \sup_n \int_0^T e^{\beta A_s}|Z^n_s|^2 ds \\
&+ \sup_n \int_0^T e^{\beta A_s}\|U^n_s\|^2_\lambda ds\bigg),
\end{aligned}$$

and by Lemma 4.1 we deduce that

$$\begin{aligned}
&\mathbf{E}\int_0^T e^{\beta A_s}\left|\frac{f_n(s,\Theta^n_s)}{a_s}\right|^2 ds \\
\leq\ & 4\left(\Lambda\|\xi\|^2_{\mathcal{S}^2_{2\beta}(\mathbf{R})} + (1+\Lambda)\left\|\frac{\zeta}{a}\right\|^2_{\mathcal{M}^2_\beta(\mathbf{R})} + \Lambda\|g(.,0)\|^2_{\mathcal{M}^2_\beta(\mathbf{R}^\ell)}\right).
\end{aligned}$$

Since

$$\mathbf{E}\left|\int_0^T f_n(s,\Theta^n_s)ds\right|^2 \leq \frac{1}{\beta}\mathbf{E}\int_0^T e^{\beta A_s}\left|\frac{f_n(s,\Theta^n_s)}{a_s}\right|^2 ds,$$

by Lebesgue's dominated convergence theorem, we deduce that, for almost all $t \leq T$,

$$\mathbf{E}\left|\int_0^T (f_n(s,\Theta^n_s) - f(s,\Theta_s))ds\right|^2 \xrightarrow[n\to+\infty]{} 0.$$

We have also, for almost all $t \leq T$,

$$\mathbf{E}\left|\int_0^T (g(s,\Theta^n_s) - g(s,\Theta_s))dB_s\right|^2 \xrightarrow[n\to+\infty]{} 0.$$

Moreover, we have

$$\mathbf{E}\left(\sup_{0\leq t\leq T}\left|\int_t^T Z^n_s dW_s - \int_t^T Z_s dW_s\right|^2\right) \leq \mathbf{E}\int_t^T e^{\beta A_s}|Z^n_s - Z_s|^2 ds \xrightarrow[n\to+\infty]{} 0$$

and

$$\begin{aligned}
&\mathbf{E}\bigg(\sup_{0\leq t\leq T}\left|\int_t^T\int_E U^n_s(e)\widetilde{\mu}(de,ds) - \int_t^T\int_E U_s(e)\widetilde{\mu}(de,ds)\right|^2\bigg) \\
&\qquad\qquad\leq\ \mathbf{E}\int_t^T e^{\beta A_s}\|U^n_s - U_s\|^2_\lambda ds \xrightarrow[n\to+\infty]{} 0.
\end{aligned}$$

Next, for each $\tau \in \mathcal{T}_{[0,T]}$, let

$$\widetilde{K}_\tau = K_\tau - C_{\tau-} = Y_0 - Y_\tau - \int_0^\tau f(s,\Theta_s)\,ds - \int_0^\tau g(s,\Theta_s)\,dB_s + \int_0^\tau Z_s dW_s + \int_0^\tau \int_E U_s(e)\widetilde{\mu}(ds,de).$$

Then, we can easy show that $\|\widetilde{K}^n - \widetilde{K}\|^2_{\mathscr{S}^2} \longrightarrow 0$, as $n \longrightarrow +\infty$. So, letting $n \longrightarrow +\infty$ in (23), we deduce that (Y,Z,U,K,C) is a solution to RBDSDEJ (2).

Now, let $(Y^*, Z^*, U^*, K^*, C^*) \in \mathscr{B}^2_\beta(\mathbf{R}) \times \mathscr{S}^2(\mathbf{R}) \times \mathscr{S}^2(\mathbf{R})$ be another solution to RBDSDEJ (2). By virtue of Theorem 3.5, we deduce that

$$\forall n \geq 1, \quad Y^n \leq Y^*.$$

Therefore, by passing to the limit $Y \leq Y^*$ one proves that Y is the minimal solution to RBDSDEJ (2). □

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