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Distance from fractional Brownian motion with associated Hurst index $0 < H < 1/2$ to the subspaces of Gaussian martingales involving power integrands with an arbitrary positive exponent

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Abstract We find the best approximation of the fractional Brownian motion with the Hurst index $H \in (0, 1/2)$ by Gaussian martingales of the form $\int_0^t s^\gamma dW_s$, where W is a Wiener process, $\gamma > 0$.

Keywords Fractional Brownian motion, martingale, approximation
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1 Introduction

The subject of the present paper is a fractional Brownian motion (fBm) $B^H = \{B_t^H, t \geq 0\}$ with the Hurst index $H \in \left(0, \frac{1}{2}\right)$. Generally speaking, a fBm with the Hurst index $H \in (0, 1)$ is a Gaussian process with zero mean and the covariance

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function of the form

$$\mathrm{E}B_t^H B_s^H = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

Its properties are rather different for $H \in \left(0, \frac{1}{2}\right)$ and $H \in \left(\frac{1}{2}, 1\right)$. In particular, $H \in \left(0, \frac{1}{2}\right)$ implies short-term dependence. In contrast, $H \in \left(\frac{1}{2}, 1\right)$ implies long-term dependence. Moreover, technically it is easier to deal with fBms having $H \in \left(\frac{1}{2}, 1\right)$. Due to this and many other reasons, fBm with $H \in \left(\frac{1}{2}, 1\right)$ has been much more intensively studied in the recent years. However, the financial markets in which trading takes place quite often, demonstrate the presence of a short memory, and therefore the volatility in such markets (so called rough volatility) is well modeled by fBm with $H \in \left(0, \frac{1}{2}\right)$, see e.g. [7]. Thus interest to fBm with small Hurst indices has substantially increased recently. Furthermore, it is well known that a fractional Brownian motion is neither a Markov process nor semimartingale, and especially it is neither martingale nor a process with independent increments unless $H = \frac{1}{2}$. That is why it is naturally to search the possibility of the approximation of fBm in a certain metric by simpler processes, such as Markov processes, martingales, semimartingales or a processes of bounded variation. As for the processes of bounded variation and semimartingales, corresponding results are presented in [1, 2] and [12]. In the papers [4, 5, 8, 10] approximation of a fractional Brownian motion with Gaussian martingales was studied and summarized in the monograph [3], but most of problems were considered only for $H \in \left(\frac{1}{2}, 1\right)$, for the reasons stated above.

In the present paper we continue to consider the approximation of a fractional Brownian motion by Gaussian martingales but concentrate on the case $H \in \left(0, \frac{1}{2}\right)$.

Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a complete probability space with a filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\ge 0}$ satisfying the standard assumptions. We start with the Molchan representation of fBm via the Wiener process on a finite interval. Namely, it was established in [9] that the fBm $\{B_t^H, \mathcal{F}_t, t \ge 0\}$ can be represented as

$$B_t^H = \int_0^t z(t,s)dW_s, \tag{1}$$

where $\{W_t, \ t \in [0, \ T]\}$ is a Wiener process,

$$z(t,s) = c_H\bigg(t^{H-1/2}s^{1/2-H}(t-s)^{H-1/2} \\ -(H-1/2)s^{1/2-H}\int_s^t u^{H-3/2}(u-s)^{H-1/2}du\bigg),$$

is the Molchan kernel,

$$c_H = \left(\frac{2H\cdot\Gamma(\frac{3}{2}-H)}{\Gamma(H+\frac{1}{2})\cdot\Gamma(2-2H)}\right)^{1/2}, \tag{2}$$

and $\Gamma(x)$, $x > 0$, is the Gamma function.

Let us consider a problem of the distance between a fractional Brownian motion and the space of square integrable martingales (initially not obligatory Gaussian), adapted to the same filtration. So, we are looking for a square integrable $\mathbb{F}$-martingale M with the bracket that is absolutely continuous w.r.t. (with respect to) the Lebesgue measure such that it minimizes the value

$$\rho_H(M)^2 := \sup_{t\in[0,T]} \mathrm{E}(B_t^H - M_t)^2.$$

We observe first that B^H and W generate the same filtration, so any square integrable $\mathbb{F}$-martingale M with the bracket that is absolutely continuous w.r.t. the Lebesgue measure, admits a representation

$$M_t = \int_0^t a(s)dW_s, \tag{3}$$

where a is an $\mathbb{F}$-adapted square integrable process such that $\langle M\rangle_t = \int_0^t a^2(s)ds$. Hence we can write, see [10],

$$\begin{aligned}\mathrm{E}(B_t^H - M_t)^2 &= \mathrm{E}\left(\int_0^t (z(t,s)-a(s))dW_s\right)^2 = \int_0^t \mathrm{E}(z(t,s)-a(s))^2 ds\\ &= \int_0^t (z(t,s)-\mathrm{E}a(s))^2 ds + \int_0^t \mathrm{Var}\, a(s)ds.\end{aligned}$$

Consequently, it is enough to minimize $\rho_H(M)$ over continuous *Gaussian* martingales. Such martingales have orthogonal and therefore independent increments. Then the fact that they have representation (3) with a non-random a follows, e.g., from [11].

Now let $a : [0,T] \to \mathbb{R}$ be a nonrandom measurable function of the class $L_2[0,T]$; that is, a is such that the stochastic integral $\int\limits_0^t a(s)dW_s$, $t \in [0,\ T]$, is well defined w.r.t. the Wiener process $\{W_t,\quad t \in [0,\ T]\}$ (this integral is usually called the Wiener integral if the integrand is a nonrandom function). So, the problem is to find

$$\inf_{a\in L_2[0,T]} \sup_{0\le t\le T} \mathrm{E}\left(B_t^H - \int\limits_0^t a(s)dW_s\right)^2 = \inf_{a\in L_2[0,T]} \sup_{0\le t\le T} \int\limits_0^t (z(t,s)-a(s))^2 ds.$$

Note that the expression to be minimized does not involve neither the fractional Brownian motion nor the Wiener process, so the problem becomes purely analytic. Moreover, since the problem posed in a general form is not observable and accessible for solution, we restrict ourselves to one particular subclass of functions. We study the class

$$\{a(s) = s^\gamma, \gamma > 0\}.$$

Our main result is Theorem 1, which shows where $\max\limits_{t\in[0,1]} \mathrm{E}\left(B_t^H - \int_0^t a(s)dW_s\right)^2$ could be reached, depending on $\gamma > 0$. We also provide remarks after the theorem.

2 Distance from fBm with $H \in (0, 1/2)$ to the subspaces of Gaussian martingales involving power integrands

Consider a class of power functions with an arbitrary positive exponent. Thus, we now introduce the class

$$\{a(s) = s^{\gamma}, \gamma > 0\}.$$

For the sake of simplicity, let $T = 1$.

Theorem 1. *Let $a = a(s)$ be a function of the form $a(s) = s^{\gamma}$, $\gamma > 0$, $H \in (0, 1/2)$. Then:*

(i) *For all $\gamma > 0$ the maximum $\max_{t \in [0,1]} \mathrm{E}\left(B_t^H - \int_0^t s^{\gamma} dW_s\right)^2$ is reached at one of the following points: $t = 1$ or $t = t_1$, where*

$$t_1 = \Bigg(c_H B\left(\gamma - H + \frac{3}{2}, H + \frac{1}{2}\right)(\gamma + 1) - \sqrt{c_H^2 \left(B\left(\gamma - H + \frac{3}{2}, H + \frac{1}{2}\right)(\gamma + 1)\right)^2 - 2H} \Bigg)^{\frac{1}{\gamma - H + \frac{1}{2}}}.$$

(ii) *For any $H \in (0, 1/2)$ there exists $\gamma_0 = \gamma_0(H) > 0$ such that for $\gamma > \gamma_0$ the maximum*

$$\max_{t \in [0,1]} \mathrm{E}\left(B_t^H - \int_0^t s^{\gamma} dW_s\right)^2 = t_1^{2H} - 2t_1^{\gamma + \frac{1}{2} + H} c_H B\left(\gamma - H + \frac{3}{2}, H + \frac{1}{2}\right)\frac{\gamma + 1}{\gamma + \frac{1}{2} + H} + \frac{1}{2\gamma + 1} t_1^{2\gamma + 1}$$

and is reached at the point t_1. Here $B(x, y)$, $x, y > 0$, is a beta function.

Proof. According to Lemma 2.20 [3], the distance between the fractional Brownian motion and the integral $\int_0^t s^{\gamma} dW_s$ w.r.t. Wiener process $t \in [0, 1]$ equals

$$\begin{aligned} E\left(B_t^H - \int_0^t s^{\gamma} dW_s\right)^2 &= E\left(B_t^H\right)^2 - 2E\left(\int_0^t z(t,s) dW_s \int_0^t s^{\gamma} dW_s\right) \\ &+ E\left(\int_0^t s^{\gamma} dW_s\right)^2 = t^{2H} - 2\int_0^t z(t,s) s^{\gamma} ds + \int_0^t s^{2\gamma} ds \\ &= t^{2H} - 2t^{\gamma + H + \frac{1}{2}} c_H B\left(\gamma - H + \frac{3}{2}, H + \frac{1}{2}\right)\frac{\gamma + 1}{\gamma + H + \frac{1}{2}} \\ &+ \frac{t^{2\gamma + 1}}{2\gamma + 1} := h(t, \gamma), \end{aligned} \tag{4}$$

where c_H is taken from (2).

Let us calculate the partial derivative of $h(t,\gamma)$ in t:

$$\frac{\partial h(t,\gamma)}{\partial t} = t^{2H-1}\Big(2H - 2t^{\gamma-H+\frac{1}{2}}c_H \cdot B\left(\gamma - H + \frac{3}{2}, H + \frac{1}{2}\right)(\gamma+1) + t^{2(\gamma-H+\frac{1}{2})}\Big).$$

Let us verify whether there is $t \in [0,1]$ such that $\frac{\partial h(t,\gamma)}{\partial t} = 0$, i.e.

$$t^{2(\gamma-H+\frac{1}{2})} - 2t^{\gamma-H+\frac{1}{2}}c_H B\left(\gamma - H + \frac{3}{2}, H + \frac{1}{2}\right)(\gamma+1) + 2H = 0.$$

Changing the variable $t^{\gamma-H+\frac{1}{2}} =: x$, we obtain the following quadratic equation:

$$x^2 - 2xc_H B\left(\gamma - H + \frac{3}{2}, H + \frac{1}{2}\right)(\gamma+1) + 2H = 0. \tag{5}$$

The discriminant $D = D(\gamma)$ of the quadratic equation (5) equals

$$\begin{aligned} D(\gamma) &= 4c_H^2\left(B\left(\gamma - H + \frac{3}{2}, H + \frac{1}{2}\right)(\gamma+1)\right)^2 - 8H \\ &= 8H\left(\left(B\left(\gamma - H + \frac{3}{2}, H + \frac{1}{2}\right)(\gamma+1)\right)^2 \frac{\Gamma(\frac{3}{2}-H)}{\Gamma(H+\frac{1}{2})\Gamma(2-2H)} - 1\right) \\ &= 8H\left(\frac{\Gamma(H+\frac{1}{2})\Gamma(\frac{3}{2}-H)}{\Gamma(2-2H)}\left(\frac{\Gamma(\gamma-H+\frac{3}{2})}{\Gamma(\gamma+1)}\right)^2 - 1\right). \end{aligned} \tag{6}$$

Now we are going to show that $D(0) > 0$ and $D(\gamma)$ is increasing in $\gamma > 0$. For this we study separately the function $f(H) := \frac{\Gamma(H+\frac{1}{2})(\Gamma(\frac{3}{2}-H))^3}{\Gamma(2-2H)}$ for $H \in (0, \frac{1}{2})$.

Let us calculate the values of this function at the following points: $f(0) = \frac{\pi^2}{8} > 1$, $f(\frac{1}{2}) = 1$. To establish that $f(H)$ is decreasing in H, consider Lemma 3 and the following calculation:

$$\begin{aligned} (\ln f(H))'_H &= \left(\ln \frac{\Gamma(H+\frac{1}{2})(\Gamma(\frac{3}{2}-H))^3}{\Gamma(2-2H)}\right)'_H \\ &= \left(\ln\Gamma\left(H+\frac{1}{2}\right) + 3\ln\Gamma\left(\frac{3}{2}-H\right) - \ln\Gamma(2-2H)\right)'_H \\ &= \frac{\Gamma'(H+\frac{1}{2})}{\Gamma(H+\frac{1}{2})} - 3\frac{\Gamma'(\frac{3}{2}-H)}{\Gamma(\frac{3}{2}-H)} + 2\frac{\Gamma'(2-2H)}{\Gamma(2-2H)} \\ &= \int_0^1 \frac{1-t^{H-\frac{1}{2}}}{1-t}dt - C - 3\left(\int_0^1 \frac{1-t^{\frac{1}{2}-H}}{1-t}dt - C\right) \\ &\quad + 2\left(\int_0^1 \frac{1-t^{1-2H}}{1-t}dt - C\right) \end{aligned}$$

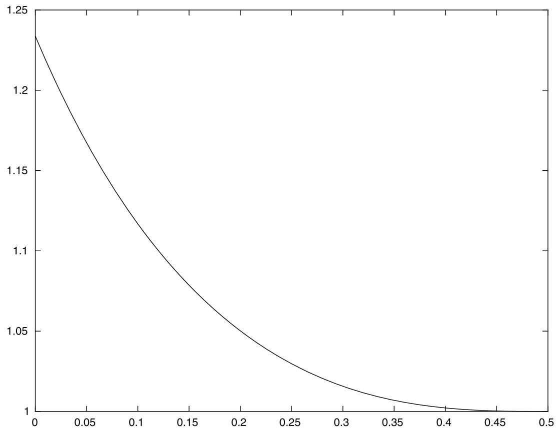

Fig. 1. The behavior of $f(H)$ for $H \in (0, \frac{1}{2})$

$$= \int_0^1 \frac{1 - t^{H-\frac{1}{2}} - 3 + 3t^{\frac{1}{2}-H} + 2 - 2t^{1-2H}}{1-t} dt$$
$$= \int_0^1 \frac{3t^{\frac{1}{2}-H} - 2t^{1-2H} - t^{H-\frac{1}{2}}}{1-t} dt$$
$$= \int_0^1 \frac{t^{H-\frac{1}{2}}(3t^{1-2H} - 2t^{\frac{3}{2}-3H} - 1)}{1-t} dt. \tag{7}$$

Let $t \in (0, 1)$. Obviously, in this case $t^{H-\frac{1}{2}} > 0$ and $1 - t > 0$. Changing the variables in (7) as $z := t^{\frac{1}{2}-H}$, we get

$$3z^2 - 2z^3 - 1 = -(1 - z)^2(2z + 1),$$

and this function is negative for all $z \in (0, 1)$. Hence, $(\ln f(H))' < 0$ and it means that $f(H)$ is decreasing. Furthermore, $f(H) > 1$ for every $H \in (0, \frac{1}{2})$. The behavior of $f(H)$ is presented in Figure 1.

So, we proved that $D(0) > 0$ (the behavior of $D(0)$ as a function of H is presented in Figure 2), and it follows from Lemma 4 that $D(\gamma)$ is increasing in $\gamma > 0$ for any $H \in (0, 1/2)$. Therefore, for every $H \in (0, \frac{1}{2})$ and $\gamma > 0$ we have that the quadratic equation (5) has two roots.

More precisely, if you use standard notations for the coefficients of the quadratic equation, then coefficient a at x^2 in (5) is strongly positive, $a = 1$, coefficient at x equals $b = -2c_H B(\gamma - H + \frac{3}{2}, H + \frac{1}{2})(\gamma + 1)$ and is negative, and $c = 2H > 0$. We conclude that our quadratic equation has two positive roots $x_1 = \frac{-b-\sqrt{b^2-4ac}}{2a}$ and $x_2 = \frac{-b+\sqrt{b^2-4ac}}{2a}$, $x_1 \leq x_2$.

According to our notations, we let $t_i := x_i^{\frac{1}{\gamma-H+\frac{1}{2}}}$, $i = 1, 2$. Since $x = t^{\gamma-H+\frac{1}{2}} \in [0, 1]$ for $t \in [0, 1]$ and the left-hand side of (5) is negative for $x \in (x_1, x_2)$, we get the following cases:

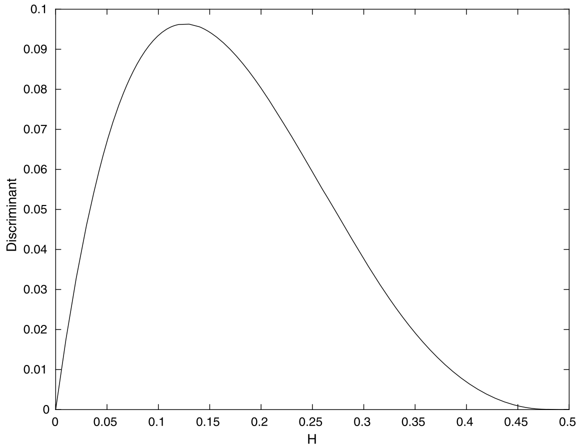

Fig. 2. The behavior of $D(0)$ as a function of $H \in (0, \frac{1}{2})$

(i) Let $x_1 < 1$ and $x_2 < 1$. Then $\max_{t\in[0,1]} h(t, \gamma)$ can be achieved at one of two points: $t = t_1$ or $t = 1$.

(ii) Let $x_1 < 1$ and $x_2 \geq 1$. Then $\max_{t\in[0,1]} h(t, \gamma)$ is achieved at point $t = t_1$.

(iii) Let $x_1 \geq 1$ (and consequently $x_2 > 1$). Then $\max_{t\in[0,1]} h(t, \gamma)$ is achieved at point $t = 1$.

Now, we rewrite the discriminant (6) in the following form:

$$\begin{aligned} D &= 4\left(c_H^2\left(B\left(\gamma - H + \frac{3}{2}, H + \frac{1}{2}\right)(\gamma + 1)\right)^2 - 2H\right) \\ &=: 4(d_H^2(\gamma) - 2H) > 0, \end{aligned} \tag{8}$$

where $d_H(\gamma) = c_H B(\gamma - H + \frac{3}{2}, H + \frac{1}{2})(\gamma + 1)$. From Lemma 5, $x_1 < \sqrt{2H} < 1$, so case (iii) never occurs.

According to Lemma 5, the biggest of two roots, $x_2 = d_H(\gamma) + \sqrt{d_H^2(\gamma) - 2H}$, is increasing in $\gamma > 0$ and $x_2 > 1 \Leftrightarrow d_H > \frac{1}{2} + H$. Moreover, it follows from Lemma 5, (iv) that $d_H(\gamma) \to +\infty$ as $\gamma \to +\infty$. Therefore, for all $H \in (0, \frac{1}{2})$ there exists $\gamma_0(H) > 0$ such that for all $\gamma > \gamma_0$ we have $x_2 > 1$.

It means that for $\gamma > \gamma_0$ our maximum is reached at the point $t_1 = (x_1)^{\frac{1}{\gamma + \frac{1}{2} - H}}$. Finally,

$$\begin{aligned} &\max_{t\in[0,1]} \mathrm{E}\left(B_t^H - \int_0^t s^\gamma dW_s\right)^2 \\ &= t_1^{2H} - 2t_1^{\gamma + \frac{1}{2} + H} c_H B\left(\gamma - H + \frac{3}{2}, H + \frac{1}{2}\right)\frac{\gamma + 1}{\gamma + \frac{1}{2} + H} + \frac{1}{2\gamma + 1} t_1^{2\gamma + 1}, \end{aligned}$$

where $t_1 = (x_1)^{\frac{1}{\gamma+\frac{1}{2}-H}}$. □

Remark 1. The implicit equation $d_H(\gamma) = \frac{1}{2} + H$ considered as the equation for γ_0 as a function of H gives us the relation between $H \in \left(0, \frac{1}{2}\right)$ and respective $\gamma_0 > 0$ which, by virtue of the foregoing, is determined unambiguously. The form of the algebraic curve $\gamma_0 = \gamma_0(H)$ is presented in Figure 3.

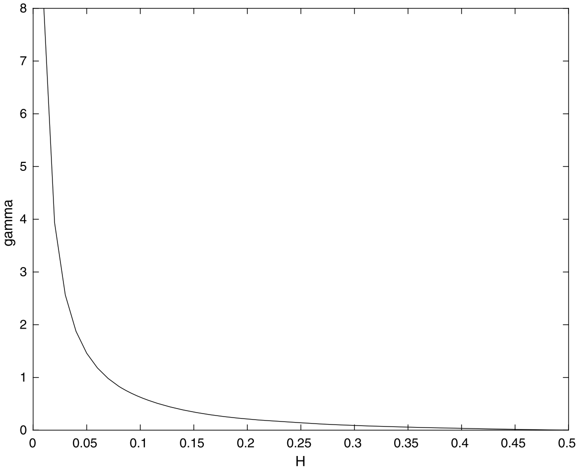

Fig. 3. The algebraic curve $\gamma_0 = \gamma_0(H)$

Consider one of the coefficients that are present in (4), namely,

$$c^1_{H,\gamma} = 2c_H B\left(\gamma - H + \frac{3}{2}, H + \frac{1}{2}\right)\frac{\gamma+1}{\gamma+H+\frac{1}{2}}$$
$$= 2\left(\frac{2H\Gamma\left(\frac{3}{2}-H\right)}{\Gamma\left(H+\frac{1}{2}\right)\Gamma(2-2H)}\right)^{1/2}\frac{\Gamma\left(\gamma-H+\frac{3}{2}\right)\Gamma\left(H+\frac{1}{2}\right)}{\Gamma(\gamma+1)\left(\gamma+H+\frac{1}{2}\right)}.$$

Let $\gamma = 0$. Then

$$c^1_{H,0} = 2\left(\frac{2H\Gamma\left(\frac{3}{2}-H\right)}{\Gamma\left(H+\frac{1}{2}\right)\Gamma(2-2H)}\right)^{1/2}\frac{\Gamma\left(\frac{3}{2}-H\right)\Gamma\left(H+\frac{1}{2}\right)}{H+\frac{1}{2}}$$
$$= \frac{2}{H+\frac{1}{2}}\left(\frac{2H\Gamma\left(\frac{3}{2}-H\right)}{\Gamma\left(H+\frac{1}{2}\right)\Gamma(2-2H)}\right)^{1/2}\frac{\pi\left(\frac{1}{2}-H\right)}{\sin\left(\pi\left(\frac{1}{2}-H\right)\right)}.$$

For $H = \frac{1}{2}$, one has $c^1_{1/2,0} = 2$. Obviously, $c^1_{H,0} \to 0$ as $H \downarrow 0$. It means that $c^1_{H,0}$ is small in some neighborhood of zero. Having this in mind, we establish some sufficient condition for $h(t, 0)$ to get its maximum at point 1.

Lemma 1. *Let $c^1_{H,0} < 1$. Then $h(t_1, 0) < h(1, 0)$.*

Proof. Indeed, $h(t_1,0) = t_1^{2H} - c_{H,0}^1 t_1^{1/2+H} + t_1$, while $h(1,0) = 2 - c_{H,0}^1$. Then the inequality $h(t_1,0) < h(1,0)$ is equivalent to the following one:

$$c_{H,0}^1\left(1 - t_1^{1/2+H}\right) < 2 - t_1 - t_1^{2H}.$$

If $c_{H,0}^1 < 1$, then $c_{H,0}^1\big(1 - t_1^{1/2+H}\big) < 1 - t_1^{1/2+H} < 1 - t_1 < 2 - t_1 - t_1^{2H}$. □

Remark 2. As it was mentioned before, $c_{1/2,0}^1 = 2$, and so for $H = \frac{1}{2}$ the condition of Lemma 1 is not satisfied. In this case $d_{1/2}(0) = 1$, and $t_1 = t_2 = 1$, so that we have the equality $h(t_1,0) = h(1,0)$.

However, the question what will be for $\gamma = 0$ and $H_0 < H < \frac{1}{2}$, where H_0 is such a value that for $0 < H < H_0$, $c_{H,0}^1 < 1$, is open. In order to fill this gap, we provide the numerical results with some comments.

Consider two fuctions $h(t_1,\gamma)$ and $h(1,\gamma)$ as functions of γ and H. We already know that $\max\limits_{t\in[0,1]} h(t,\gamma) = \max\{h(t_1,\gamma), h(1,\gamma)\}$. The projection of the surface of $\max\{h(t_1,\gamma), h(1,\gamma)\}$ on the (H,γ)-plane is presented in Figure 4. Points, where $h(t,\gamma)$ reaches its maximum at $t = 1$ are represented in green color, and points where $h(t,\gamma)$ reaches its maximum at $t = t_1$ are represented in brown color. The black curve is the algebraic curve $\gamma_0 = \gamma_0(H)$, which is presented in Figure 3.

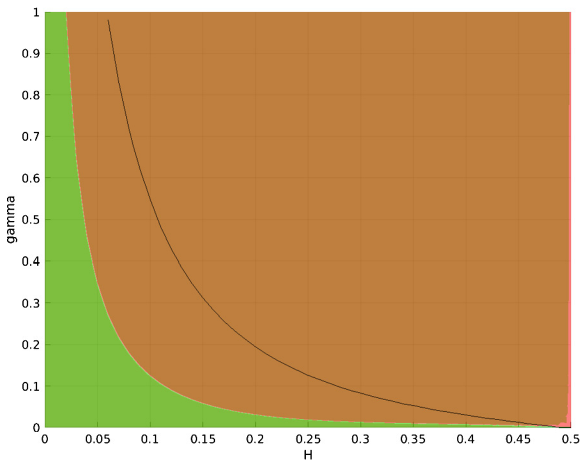

Fig. 4. The projection of the surface of $\max\limits_{t\in[0,1]} h(t,\gamma)$

Appendix section

In the proof of Theorem 1, we use these auxiliary results.

Lemma 2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a strictly convex function of one variable. Take the function $g(x_1,x_2) = \frac{f(x_1)-f(x_2)}{x_1-x_2}$, $x_1 \neq x_2$, $x_1,x_2 \in \mathbb{R}$. Then $g(x_1+\alpha, x_2+\alpha)$ is strictly increasing in $\alpha > 0$.*

Lemma 3. *Let $\Gamma(x)$ be the Gamma function. Then*

$$(\ln \Gamma(x))' = \frac{\Gamma'(x)}{\Gamma(x)} = \int_0^1 \frac{1-t^{x-1}}{1-t} dt - C,$$

where C is a fixed constant.

Proofs of Lemma 2 and Lemma 3 could be found in [6].

Lemma 4. *If $H \in \left(0, \frac{1}{2}\right)$, then the function $z(\gamma) := \frac{\Gamma(\gamma - H + \frac{3}{2})}{\Gamma(\gamma+1)}$ is increasing in $\gamma > 0$.*

Proof. For every $\gamma > 0$ we have $z(\gamma) > 0$. Let us calculate

$$\ln z(\gamma) = \ln \frac{\Gamma\left(\gamma - H + \frac{3}{2}\right)}{\Gamma(\gamma+1)} =$$
$$= \left(\frac{1}{2} - H\right) \frac{\ln \Gamma\left(\gamma - H + \frac{3}{2}\right) - \ln \Gamma(\gamma+1)}{\left(\gamma - H + \frac{3}{2}\right) - (\gamma+1)} =: \left(\frac{1}{2} - H\right) \omega(\gamma).$$

According to Lemma 2 and the fact that $\ln(\Gamma(x))$ is strictly convex we have that $\omega(\gamma)$ is increasing. Since $\left(\frac{1}{2} - H\right) > 0$, it is clear that $z(\gamma)$ is increasing in $\gamma > 0$. □

Lemma 5. *Let $d_H(\gamma) = c_H B(\gamma - H + \frac{3}{2}, H + \frac{1}{2})(\gamma+1)$, and $x_1 = d_H(\gamma) - \sqrt{d_H^2(\gamma) - 2H}$, $x_2 = d_H(\gamma) + \sqrt{d_H^2(\gamma) - 2H}$ be roots of the quadratic equation (5). Then for all $\gamma > 0$, $H \in \left(0, \frac{1}{2}\right)$ the following statements hold.*

i) $d_H(\gamma)$ is increasing in $\gamma > 0$.

ii) $x_1 < \sqrt{2H}$ and $x_2 > \sqrt{2H}$.

iii) $d_H(\gamma) > \frac{1}{2} + H \Leftrightarrow x_2 > 1$ and $x_1 < 2H$.

iv) $d_H(\gamma) \to \infty$ as $\gamma \to \infty$.

Proof. (i) Note that $d_H(\gamma)$ is increasing in $\gamma > 0$ since

$$d_H(\gamma) = c_H \Gamma\left(H + \frac{1}{2}\right) \frac{\Gamma\left(\gamma - H + \frac{3}{2}\right)}{\Gamma(\gamma+1)},$$

where $c_H \Gamma\left(H + \frac{1}{2}\right) > 0$ for all $H \in \left(0, \frac{1}{2}\right)$, and according to Lemma 4, $\frac{\Gamma(\gamma - H + \frac{3}{2})}{\Gamma(\gamma+1)}$ is increasing in $\gamma > 0$.

(ii) Discriminant (6) satisfies the following relation:

$$0 < D(0) = 4\left(d_H^2(0) - 2H\right).$$

Therefore, $d_H(\gamma) > \sqrt{2H}$ for all $\gamma > 0$. Also, we can rewrite $x_1 = d_H(\gamma) - \sqrt{d_H^2(\gamma) - 2H}$ and $x_2 = d_H(\gamma) + \sqrt{d_H^2(\gamma) - 2H}$. Hence $x_2 > \sqrt{2H}$. Transform the value x_1 to the following form:

$$x_1 = \frac{d_H^2(\gamma) - (d_H^2(\gamma) - 2H)}{d_H(\gamma) + \sqrt{d_H^2(\gamma) - 2H}} = \frac{2H}{x_2} < \sqrt{2H}.$$

(*iii*) Let us assume that $x_2 > 1$ (or, what is equivalent, $x_1 = \frac{2H}{x_2} < 2H$). In turn, this is equivalent to the relation $d_H(\gamma) + \sqrt{d_H^2(\gamma) - 2H} > 1$, or $\sqrt{d_H^2(\gamma) - 2H} > 1 - d_H(\gamma)$. The latter inequality can be realized in one of two cases:

$$d_H^2(\gamma) - 2H > 1 - 2d_H(\gamma) + d_H^2(\gamma), \quad 1 - d_H(\gamma) \geq 0; \tag{9}$$

or

$$1 - d_H(\gamma) < 0. \tag{10}$$

The couple of inequalities (9) is equivalent to $1 \geq d_H(\gamma) > \frac{1}{2} + H$. Therefore, inequalities (9) and (10), taken together, indicate that $d_H(\gamma) > \frac{1}{2} + H$ if and only if $x_2 > 1$.

(*iv*) The value $d_H(\gamma)$ can be presented as

$$d_H(\gamma) = c_H \frac{\Gamma\left(\gamma - H + \frac{3}{2}\right)\Gamma\left(H + \frac{1}{2}\right)}{\Gamma(\gamma + 2)}(\gamma + 1),$$

where $c_H \Gamma\left(H + \frac{1}{2}\right) > 0$ is a fixed constant, and for all $H \in \left(0, \frac{1}{2}\right)$,

$$\begin{aligned}
\frac{\Gamma\left(\gamma - H + \frac{3}{2}\right)}{\Gamma(\gamma + 2)}(\gamma + 1) &= \frac{\Gamma\left(\gamma - H + \frac{3}{2}\right)}{\Gamma(\gamma + 1)} \\
&= \frac{\sqrt{\frac{2\pi}{\gamma - H + \frac{3}{2}}}\left(\frac{\gamma - H + \frac{3}{2}}{e}\right)^{\gamma - H + \frac{3}{2}}\left(1 + O\left(\frac{1}{\gamma - H + \frac{3}{2}}\right)\right)}{\sqrt{\frac{2\pi}{\gamma + 1}}\left(\frac{\gamma + 1}{e}\right)^{\gamma + 1}\left(1 + O\left(\frac{1}{\gamma + 1}\right)\right)} \\
&\sim \frac{1}{e^{\frac{1}{2} - H}} \frac{\left(\gamma - H + \frac{3}{2}\right)^{\gamma - H + 1}}{(\gamma + 1)^{\gamma + \frac{1}{2}}} \to \infty, \quad \gamma \to \infty,
\end{aligned}$$

which follows from the Stirling's approximation for Gamma function. $\square$

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