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Simple approximations for the ruin probability in the risk model with stochastic premiums and a constant dividend strategy

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Abstract We deal with a generalization of the risk model with stochastic premiums where dividends are paid according to a constant dividend strategy and consider heuristic approximations for the ruin probability. To be more precise, we construct five- and three-moment analogues to the De Vylder approximation. To this end, we obtain an explicit formula for the ruin probability in the case of exponentially distributed premium and claim sizes. Finally, we analyze the accuracy of the approximations for some typical distributions of premium and claim sizes using statistical estimates obtained by the Monte Carlo methods.

Keywords Risk model with stochastic premiums, constant dividend strategy, ruin probability, net profit condition, De Vylder approximation, Monte Carlo method
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1 Introduction

The ruin probability of an insurance company is one of the main risk measures considered in risk theory, and the problems of its calculation and approximation have attracted a lot of attention recently (see, e.g., [2, 16, 22, 25, 27–29] and references therein). Risk models where shareholders receive dividends from their insurance company have been of great interest to researchers since De Finetti first considered dividend strategies in insurance dealing with a binomial model [12]. The classical risk model and its various modifications with different dividend strategies are investigated in a number of papers (see, e.g., [1, 4, 8, 10, 11, 15, 20, 21, 26, 27, 30] and references therein).

It is well known that explicit formulas for the ruin probability can be derived only in a few special cases even for the classical Cramér–Lundberg risk model, so numerous heuristic approximations for this function have been proposed and studied (see [2, 3, 5, 9, 13, 16, 17, 28, 29]). So-called simple approximations, which use only some moments of the distribution of claim sizes and do not take into account its tail behavior, form a special class of approximations for the ruin probabilities.

The De Vylder approximation, which is introduced in [13] for the classical risk model, is supposed to be one of the most successful simple approximations. It is based on the heuristic idea to replace the investigated risk process by a risk process with exponentially distributed claim sizes such that the first three moments coincide (see also [16, 28, 29] for details). Thus, to apply the De Vylder approximation, we need to calculate only the first three moments of the distribution of the claim sizes. Despite its simplicity, the approximation gives surprisingly good results when the initial surplus is not too small, especially when the distribution of claim sizes is light-tailed. This fact was explained later by Grandell [17] after analyzing the simple approximations from a mathematical viewpoint. Analogues to the De Vylder approximation are constructed in the risk model with additional funds [23] and some risk models with reinsurance [7, 19].

The present paper deals with a generalization of the risk model with stochastic premiums where dividends are paid according to a constant dividend strategy. In what follows, we suppose that all stochastic objects we use below are defined on a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ satisfying the usual conditions. In the risk model with stochastic premiums (see, e.g., [6, 22]), premium sizes form a sequence $(\bar{Y}_i)_{i\geq 1}$ of non-negative independent and identically distributed (i.i.d.) random variables (r.v.'s) with cumulative distribution function (c.d.f.) $F_{\bar{Y}}(y) = \mathbb{P}[\bar{Y}_i \leq y]$, and the number of premiums on the time interval $[0, t]$ is a Poisson process $(\bar{N}_t)_{t\geq 0}$ with constant intensity $\bar{\lambda} > 0$. Similarly, claim sizes form a sequence $(Y_i)_{i\geq 1}$ of i.i.d. r.v.'s with c.d.f. $F_Y(y) = \mathbb{P}[Y_i \leq y]$, and the number of claims on the time interval $[0, t]$ is a Poisson process $(N_t)_{t\geq 0}$ with constant intensity $\lambda > 0$. Thus, the total premiums and claims on $[0, t]$ equal $\sum_{i=1}^{\bar{N}_t} \bar{Y}_i$ and $\sum_{i=1}^{N_t} Y_i$, respectively. Note that here $\sum_{i=1}^{\bar{N}_t} \bar{Y}_i = 0$ if $\bar{N}_t = 0$, and $\sum_{i=1}^{N_t} Y_i = 0$ if $N_t = 0$. In what follows, we also assume that the r.v.'s $(Y_i)_{i\geq 1}$ and $(\bar{Y}_i)_{i\geq 1}$ have finite expectations $\mu > 0$ and $\bar{\mu} > 0$, respectively, and $(Y_i)_{i\geq 1}$, $(\bar{Y}_i)_{i\geq 1}$, $(N_t)_{t\geq 0}$ and $(\bar{N}_t)_{t\geq 0}$ are mutually independent.

Moreover, we make the additional assumption that the insurance company pays dividends to its shareholders according to a constant dividend strategy, which implies that dividends are paid continuously at a rate $d > 0$. The strategy can be considered as a multi-layer dividend strategy where the number of layers is equal to one (see, e.g., [27]). Next, we denote a non-negative initial surplus of the insurance company by x, and let $X_t(x)$ be its surplus at time t provided that the initial surplus is x. Then the surplus process $\big(X_t(x)\big)_{t\geq 0}$ is defined by the equality

$$X_t(x) = x + \sum_{i=1}^{\bar{N}_t} \bar{Y}_i - \sum_{i=1}^{N_t} Y_i - dt, \quad t \geq 0. \tag{1}$$

Next, let $\tau(x) = \inf\{t \geq 0 \colon X_t(x) < 0\}$ be the ruin time for the risk process $(X_t(x))_{t\geq 0}$ defined by (1). For $x \geq 0$, the infinite-horizon ruin probability is defined by

$$\psi(x) = \mathbb{E}[\mathbb{1}(\tau(x) < \infty) \mid X_0(x) = x],$$

where $\mathbb{1}(\cdot)$ is the indicator function. Note that the ruin probability is a special case of the expected discounted penalty function, which is introduced in [14] and also called the Gerber–Shiu function.

Thus, it is easily seen that the risk model described above is a special case of the model with stochastic premiums and a multi-layer dividend strategy investigated in [27], although in that paper it is assumed that the number of layers is more than one. In [27], piecewise integro-differential equations for the Gerber–Shiu function and the expected discounted dividend payments until ruin are derived. In addition, the model is studied in detail in the case of exponentially distributed claim and premium sizes. In particular, explicit formulas for the ruin probability as well as for the expected discounted dividend payments are obtained.

The aim of the present paper is to construct analogues to the De Vylder approximation for the ruin probability in the risk model described above and analyze the accuracy of these approximations. The rest of the paper is organized as follows. In Section 2, we obtain an explicit formula for the ruin probability in the case of exponentially distributed premium and claim sizes. We use this formula in Section 3, where we derive five- and three-moment analogues to the De Vylder approximation. Finally, Section 4 is devoted to numerical illustrations. To be more precise, we deal with some typical distributions of premium and claim sizes and apply the results obtained in Section 3. To analyze the accuracy of the approximations, we use statistical estimates obtained by the Monte Carlo methods.

2 An explicit formula for the ruin probability in the case of exponentially distributed premium and claim sizes

From now on, we suppose that the net profit condition holds, which in this model means that

$$\bar{\lambda}\bar{\mu} > \lambda\mu + d. \tag{2}$$

Theorem 1 below is a special case of Theorem 1 in [27], where it is formulated and proved for the Gerber–Shiu function in the model where the number of layers is more than one. It is easy to check that the assertion of the theorem remains true if the number of layers equals one.

Theorem 1. *Let the surplus process $(X_t(x))_{t\geq 0}$ be defined by (1) under the above assumptions, and let $F_Y(y)$ be continuous on $\mathbb{R}_+$. Then the function $\psi(x)$ is differentiable on $\mathbb{R}_+$ and satisfies the integro-differential equation*

$$\begin{aligned} d\psi'(x) + (\bar{\lambda} + \lambda)\psi(x) &= \bar{\lambda}\int_0^\infty \psi(x+y)\,\mathrm{d}F_{\bar{Y}}(y) \\ &\quad + \lambda\int_0^x \psi(x-y)\,\mathrm{d}F_Y(y) + \lambda\big(1 - F_Y(x)\big), \quad x \geq 0. \end{aligned} \tag{3}$$

Remark 1. To solve equation (3), we use the following two boundary conditions. Firstly, using standard considerations (see, e.g., [22, 24, 28]) it can be easily shown that $\lim_{x\to\infty}\psi(x) = 0$ provided that the net profit condition (2) holds. Secondly, it is obvious that $\psi(0) = 1$ for this risk model. Although equation (3) is not solvable analytically in the general case, we can find explicit expressions for the corresponding ruin probability in some special cases. The uniqueness of the required solution to equation (3) should be justified in each case.

Assume now that the premium and claim sizes are exponentially distributed, i.e. their probability density functions (p.d.f.'s) are

$$f_{\bar{Y}}(y) = \frac{1}{\bar{\mu}}\, e^{-y/\bar{\mu}} \quad \text{and} \quad f_Y(y) = \frac{1}{\mu}\, e^{-y/\mu}, \quad y \ge 0,$$

respectively. In this case, the integro-differential equation (3) can be reduced to a linear differential equation with constant coefficients.

Lemma 1. *Let the surplus process $\big(X_t(x)\big)_{t\ge 0}$ be defined by (1) under the above assumptions, and let the premium and claim sizes be exponentially distributed with means $\bar{\mu}$ and μ, respectively. Then for all $x \ge 0$, $\psi(x)$ is a solution to the differential equation*

$$d\bar{\mu}\mu\psi'''(x) + \big(d(\bar{\mu}-\mu) + \bar{\mu}\mu(\bar{\lambda}+\lambda)\big)\psi''(x) + (\bar{\lambda}\bar{\mu} - \lambda\mu - d)\psi'(x) = 0. \quad (4)$$

The proof of Lemma 1 is similar to the proof of Lemma 1 in [27]. An explicit formula for the ruin probability is given in Theorem 2 below.

Theorem 2. *Let the surplus process $\big(X_t(x)\big)_{t\ge 0}$ follow (1) under the above assumptions, and let premium and claim sizes be exponentially distributed with means $\bar{\mu}$ and μ, respectively. If the net profit condition (2) holds, then*

$$\psi(x) = C_1 e^{z_1 x} + C_2 e^{z_2 x} \quad \textit{for all} \quad x \ge 0, \quad (5)$$

where

$$z_1 = \frac{-\big(d(\bar{\mu}-\mu) + \bar{\mu}\mu(\bar{\lambda}+\lambda)\big) + \sqrt{\mathrm{D}}}{2d\bar{\mu}\mu},$$

$$z_2 = \frac{-\big(d(\bar{\mu}-\mu) + \bar{\mu}\mu(\bar{\lambda}+\lambda)\big) - \sqrt{\mathrm{D}}}{2d\bar{\mu}\mu},$$

$$\mathrm{D} = \big(d(\bar{\mu}+\mu) + \bar{\mu}\mu(\lambda-\bar{\lambda})\big)^2 + 4\bar{\lambda}\lambda\bar{\mu}^2\mu^2,$$

$$C_1 = \frac{\bar{\lambda}\bar{\mu}(\bar{\mu}+\mu)(dz_2+\bar{\lambda}) + d\bar{\lambda}\mu(\bar{\mu}z_1 - 1)}{d\bar{\lambda}\bar{\mu}^2(z_2 - z_1)}$$

and

$$C_2 = -\frac{\bar{\lambda}\bar{\mu}(\bar{\mu}+\mu)(dz_1+\bar{\lambda}) + d\bar{\lambda}\mu(\bar{\mu}z_2 - 1)}{d\bar{\lambda}\bar{\mu}^2(z_2 - z_1)}.$$

Proof. By Lemma 1, $\psi(x)$ is a solution to (4) for all $x \ge 0$. The characteristic equation corresponding to (4) has the form

$$d\bar{\mu}\mu z^3 + \big(d(\bar{\mu}-\mu) + \bar{\mu}\mu(\bar{\lambda}+\lambda)\big)z^2 + (\bar{\lambda}\bar{\mu} - \lambda\mu - d)z = 0. \quad (6)$$

The discriminant of the equation

$$d\bar{\mu}\mu z^2 + \big(d(\bar{\mu}-\mu) + \bar{\mu}\mu(\bar{\lambda}+\lambda)\big)z + (\bar{\lambda}\bar{\mu} - \lambda\mu - d) = 0 \tag{7}$$

is equal to

$$\begin{aligned}&\big(d(\bar{\mu}-\mu) + \bar{\mu}\mu(\bar{\lambda}+\lambda)\big)^2 - 4d\bar{\mu}\mu(\bar{\lambda}\bar{\mu} - \lambda\mu - d)\\ &\quad= \big(d(\bar{\mu}+\mu) + \bar{\mu}\mu(\lambda-\bar{\lambda})\big)^2 + 4\bar{\lambda}\lambda\bar{\mu}^2\mu^2,\end{aligned}$$

which is obviously positive and coincides with the constant D introduced above. Therefore, z_1 and z_2 defined in the assertion of the theorem are two real roots of equation (7).

By the net profit condition (2), we conclude that $\bar{\lambda}\bar{\mu} - \lambda\mu - d > 0$ and

$$d(\bar{\mu}-\mu) + \bar{\mu}\mu(\bar{\lambda}+\lambda) = \mu(\bar{\lambda}\bar{\mu} - \lambda\mu - d) + \lambda\mu^2 + \lambda\bar{\mu}\mu + d\bar{\mu} > 0,$$

which implies that $z_1 < 0$ and $z_2 < 0$ by Vieta's theorem. Consequently, $z_1 < 0$, $z_2 < 0$ and $z_3 = 0$ are roots of equation (6), from which we deduce that

$$\psi(x) = C_1 e^{z_1 x} + C_2 e^{z_2 x} + C_3 \quad \text{for all} \quad x \geq 0$$

with some constants C_1, C_2 and C_3. Since the net profit condition (2) holds, by Remark 1, we have $\lim_{x\to\infty} \psi(x) = 0$, which yields $C_3 = 0$. So we obtain (5).

To determine the constants C_1 and C_2, we use the following two conditions. Firstly, substituting (5) into the equality $\psi(0) = 1$ we get

$$C_1 + C_2 = 1. \tag{8}$$

Secondly, letting $x = 0$ in (3) we obtain

$$d\psi'(0) + (\bar{\lambda}+\lambda)\psi(0) = \bar{\lambda}\int_0^\infty \psi(y)\,\mathrm{d}F_{\bar{Y}}(y) + \lambda. \tag{9}$$

Since

$$\psi'(x) = C_1 z_1 e^{z_1 x} + C_2 z_2 e^{z_2 x} \quad \text{for all} \quad x \geq 0$$

and

$$\frac{1}{\bar{\mu}}\int_0^\infty \psi(u)e^{-u/\bar{\mu}}\,\mathrm{d}u = -\frac{C_1}{\bar{\mu}z_1 - 1} - \frac{C_2}{\bar{\mu}z_2 - 1},$$

from (9) we get

$$C_1\left(dz_1 + \frac{\bar{\lambda}}{\bar{\mu}z_1 - 1}\right) + C_2\left(dz_2 + \frac{\bar{\lambda}}{\bar{\mu}z_2 - 1}\right) = -\bar{\lambda}. \tag{10}$$

Taking into account that

$$\begin{aligned}&(\bar{\mu}z_1 - 1)(\bar{\mu}z_2 - 1) = \bar{\mu}^2 z_1 z_2 - \bar{\mu}(z_1 + z_2) + 1\\ &\quad= \bar{\mu}^2\,\frac{\bar{\lambda}\bar{\mu} - \lambda\mu - d}{d\bar{\mu}\mu} + \bar{\mu}\,\frac{d(\bar{\mu}-\mu) + \bar{\mu}\mu(\bar{\lambda}+\lambda)}{d\bar{\mu}\mu} + 1 = \frac{\bar{\lambda}\bar{\mu}(\bar{\mu}+\mu)}{d\mu},\end{aligned}$$

we find the constants C_1 and C_2 from the system of linear equations (8) and (10), which always has a unique solution. Applying arguments similar to those in the proof of Theorem 3 in [27] we can show that the function $\psi(x)$ that we found is a unique solution to (3) satisfying the required conditions, which completes the proof. □

3 Analogues to the De Vylder approximation

3.1 An auxiliary result

Let the process $\big(U_t\big)_{t\ge 0}$ be defined by

$$U_t = \sum_{i=1}^{\bar{N}_t} \bar{Y}_i - \sum_{i=1}^{N_t} Y_i - dt, \quad t \ge 0. \tag{11}$$

We construct two analogues to the De Vylder approximation replacing the process $\big(U_t\big)_{t\ge 0}$ by a process $\big(\tilde{U}_t\big)_{t\ge 0}$ with exponentially distributed premium and claim sizes. Since in this risk model the process $\big(\tilde{U}_t\big)_{t\ge 0}$ is determined by five parameters, which we denote by $\bar{\lambda}_0$, $\bar{\mu}_0$, λ_0, μ_0 and d_0, five equalities are required to determine these parameters. Consequently, we need the first five moments of $\big(U_t\big)_{t\ge 0}$.

Lemma 2. *Let the process $\big(U_t\big)_{t\ge 0}$ be defined by (11) under the above assumptions, $\mathbb{E}[\bar{Y}_i^5] < \infty$ and $\mathbb{E}[Y_i^5] < \infty$. Then for all $t \ge 0$, we have*

$$\mathbb{E}[U_t] = (\bar{\lambda}\bar{\mu} - \lambda\mu - d)t, \tag{12}$$

$$\mathbb{E}[U_t^2] = \big(\mathbb{E}[U_t]\big)^2 + \big(\bar{\lambda}\mathbb{E}[\bar{Y}^2] + \lambda\mathbb{E}[Y^2]\big)t, \tag{13}$$

$$\mathbb{E}[U_t^3] = \big(\mathbb{E}[U_t]\big)^3 + 3\mathbb{E}[U_t]\cdot\big(\bar{\lambda}\mathbb{E}[\bar{Y}^2] + \lambda\mathbb{E}[Y^2]\big)t + \big(\bar{\lambda}\mathbb{E}[\bar{Y}^3] - \lambda\mathbb{E}[Y^3]\big)t, \tag{14}$$

$$\begin{aligned}\mathbb{E}[U_t^4] = {} & \big(\mathbb{E}[U_t]\big)^4 + 6\big(\mathbb{E}[U_t]\big)^2\cdot\big(\bar{\lambda}\mathbb{E}[\bar{Y}^2] + \lambda\mathbb{E}[Y^2]\big)t \\ & + 4\mathbb{E}[U_t]\cdot\big(\bar{\lambda}\mathbb{E}[\bar{Y}^3] - \lambda\mathbb{E}[Y^3]\big)t + 3\big(\bar{\lambda}\mathbb{E}[\bar{Y}^2] + \lambda\mathbb{E}[Y^2]\big)t^2 \\ & + \big(\bar{\lambda}\mathbb{E}[\bar{Y}^4] + \lambda\mathbb{E}[Y^4]\big)t,\end{aligned} \tag{15}$$

$$\begin{aligned}\mathbb{E}[U_t^5] = {} & \big(\mathbb{E}[U_t]\big)^5 + 10\big(\mathbb{E}[U_t]\big)^3\cdot\big(\bar{\lambda}\mathbb{E}[\bar{Y}^2] + \lambda\mathbb{E}[Y^2]\big)t \\ & + 10\big(\mathbb{E}[U_t]\big)^2\cdot\big(\bar{\lambda}\mathbb{E}[\bar{Y}^3] - \lambda\mathbb{E}[Y^3]\big)t \\ & + 15\mathbb{E}[U_t]\cdot\big(\bar{\lambda}\mathbb{E}[\bar{Y}^2] + \lambda\mathbb{E}[Y^2]\big)^2 t^2 \\ & + 5\mathbb{E}[U_t]\cdot\big(\bar{\lambda}\mathbb{E}[\bar{Y}^4] + \lambda\mathbb{E}[Y^4]\big)t \\ & + 10\big(\bar{\lambda}\mathbb{E}[\bar{Y}^2] + \lambda\mathbb{E}[Y^2]\big)\big(\bar{\lambda}\mathbb{E}[\bar{Y}^3] - \lambda\mathbb{E}[Y^3]\big)t^2 \\ & + \big(\bar{\lambda}\mathbb{E}[\bar{Y}^5] - \lambda\mathbb{E}[Y^5]\big)t.\end{aligned} \tag{16}$$

Proof. Let $M_{\bar{Y}}(s)$ and $M_Y(s)$ be the moment generating functions of the r.v.'s $\bar{Y}_i$ and Y_i, respectively, provided that they exist in some neighborhood of $s = 0$. Furthermore, we denote the moment generating function of $\big(U_t\big)_{t\ge 0}$ by $M(s)$. An easy computation shows that

$$M(s) = \mathbb{E}[e^{sU_t}] = \exp\big\{\bar{\lambda}t\big(M_{\bar{Y}}(s) - 1\big) + \lambda t\big(M_Y(-s) - 1\big) - dts\big\}.$$

Taking the first five derivatives of $M(s)$ yields

$$M'(s) = \big(\bar{\lambda}tM'_{\bar{Y}}(s) - \lambda tM'_Y(-s) - dt\big)M(s),$$

$$M''(s) = \Big(\big(\bar{\lambda}tM'_{\bar{Y}}(s) - \lambda tM'_Y(-s) - dt\big)^2 + \bar{\lambda}tM''_{\bar{Y}}(s) + \lambda tM''_Y(-s)\Big)M(s),$$

$$\begin{aligned} M'''(s) = \Big(& \big(\bar{\lambda} t M'_{\bar{Y}}(s) - \lambda t M'_Y(-s) - dt\big)^3 \\ & + 3\big(\bar{\lambda} t M'_{\bar{Y}}(s) - \lambda t M'_Y(-s) - dt\big)\big(\bar{\lambda} t M''_{\bar{Y}}(s) + \lambda t M''_Y(-s)\big) \\ & + \bar{\lambda} t M'''_{\bar{Y}}(s) - \lambda t M'''_Y(-s)\Big) M(s), \\ M^{(IV)}(s) = \Big(& \big(\bar{\lambda} t M'_{\bar{Y}}(s) - \lambda t M'_Y(-s) - dt\big)^4 \\ & + 6\big(\bar{\lambda} t M'_{\bar{Y}}(s) - \lambda t M'_Y(-s) - dt\big)^2 \big(\bar{\lambda} t M''_{\bar{Y}}(s) + \lambda t M''_Y(-s)\big) \\ & + 4\big(\bar{\lambda} t M'_{\bar{Y}}(s) - \lambda t M'_Y(-s) - dt\big)\big(\bar{\lambda} t M'''_{\bar{Y}}(s) - \lambda t M'''_Y(-s)\big) \\ & + 3\big(\bar{\lambda} t M''_{\bar{Y}}(s) + \lambda t M''_Y(-s)\big)^2 \\ & + \bar{\lambda} t M^{(IV)}_{\bar{Y}}(s) + \lambda t M^{(IV)}_Y(-s)\Big) M(s) \end{aligned}$$

and

$$\begin{aligned} M^{(V)}(s) = \Big(& \big(\bar{\lambda} t M'_{\bar{Y}}(s) - \lambda t M'_Y(-s) - dt\big)^5 \\ & + 10\big(\bar{\lambda} t M'_{\bar{Y}}(s) - \lambda t M'_Y(-s) - dt\big)^3 \big(\bar{\lambda} t M''_{\bar{Y}}(s) + \lambda t M''_Y(-s)\big) \\ & + 10\big(\bar{\lambda} t M'_{\bar{Y}}(s) - \lambda t M'_Y(-s) - dt\big)^2 \big(\bar{\lambda} t M'''_{\bar{Y}}(s) - \lambda t M'''_Y(-s)\big) \\ & + 15\big(\bar{\lambda} t M'_{\bar{Y}}(s) - \lambda t M'_Y(-s) - dt\big)\big(\bar{\lambda} t M''_{\bar{Y}}(s) + \lambda t M''_Y(-s)\big)^2 \\ & + 5\big(\bar{\lambda} t M'_{\bar{Y}}(s) - \lambda t M'_Y(-s) - dt\big)\big(\bar{\lambda} t M^{(IV)}_{\bar{Y}}(s) + \lambda t M^{(IV)}_Y(-s)\big) \\ & + 10\big(\bar{\lambda} t M''_{\bar{Y}}(s) + \lambda t M''_Y(-s)\big)\big(\bar{\lambda} t M'''_{\bar{Y}}(s) - \lambda t M'''_Y(-s)\big) \\ & + \bar{\lambda} t M^{(V)}_{\bar{Y}}(s) - \lambda t M^{(V)}_Y(-s)\Big) M(s). \end{aligned}$$

Since $\mathbb{E}[U_t^k] = M^{(k)}(0)$ for all integer $k \geq 1$, substituting $s = 0$ into the formulas above gives (12)–(16).

If the moment generating functions of $\bar{Y}_i$ and Y_i do not exist, we can obtain (12)–(16) by a direct computation of the required expectations provided that $\mathbb{E}[\bar{Y}_i^5] < \infty$ and $\mathbb{E}[Y_i^5] < \infty$, which completes the proof. □

In what follows, we use the following constants:

$$\gamma_2 = \bar{\lambda}\mathbb{E}[\bar{Y}^2] + \lambda\mathbb{E}[Y^2], \qquad \gamma_3 = \bar{\lambda}\mathbb{E}[\bar{Y}^3] - \lambda\mathbb{E}[Y^3],$$
$$\gamma_4 = \bar{\lambda}\mathbb{E}[\bar{Y}^4] + \lambda\mathbb{E}[Y^4], \qquad \gamma_5 = \bar{\lambda}\mathbb{E}[\bar{Y}^5] - \lambda\mathbb{E}[Y^5].$$

3.2 A five-moment approximation

To construct a five-moment analogue to the De Vylder approximation, we replace the process $\big(U_t\big)_{t\geq 0}$ by a process $\big(\tilde{U}_t\big)_{t\geq 0}$ with exponentially distributed premium and claim sizes such that

$$\mathbb{E}[U_t^k] = \mathbb{E}[\tilde{U}_t^k], \qquad k = 1, 2, 3, 4, 5. \tag{17}$$

Theorem 3 (a five-moment analogue to the De Vylder approximation)**.** *Let the surplus process* $\big(X_t(x)\big)_{t\geq 0}$ *be defined by* (1) *under the above assumptions,* $\mathbb{E}[\bar{Y}_i^5] < \infty$, $\mathbb{E}[Y_i^5] < \infty$, *and let the net profit condition* (2) *hold. Then the ruin probability is approximately equal to*

$$\psi_{DV5}(x) = C_1 e^{z_1 x} + C_2 e^{z_2 x} \quad \textit{for all} \quad x \geq 0, \tag{18}$$

where

$$\begin{aligned}
z_1 &= \frac{-\big(d_0(\bar{\mu}_0 - \mu_0) + \bar{\mu}_0\mu_0(\bar{\lambda}_0 + \lambda_0)\big) + \sqrt{\mathrm{D}}}{2d_0\bar{\mu}_0\mu_0},\\
z_2 &= \frac{-\big(d_0(\bar{\mu}_0 - \mu_0) + \bar{\mu}_0\mu_0(\bar{\lambda}_0 + \lambda_0)\big) - \sqrt{\mathrm{D}}}{2d_0\bar{\mu}_0\mu_0},\\
\mathrm{D} &= \big(d_0(\bar{\mu}_0 + \mu_0) + \bar{\mu}_0\mu_0(\lambda_0 - \bar{\lambda}_0)\big)^2 + 4\bar{\lambda}_0\lambda_0\bar{\mu}_0^2\mu_0^2,\\
C_1 &= \frac{\bar{\lambda}_0\bar{\mu}_0(\bar{\mu}_0 + \mu_0)(dz_2 + \bar{\lambda}_0) + d_0\bar{\lambda}_0\mu_0(\bar{\mu}_0 z_1 - 1)}{d_0\bar{\lambda}_0\bar{\mu}_0^2(z_2 - z_1)},\\
C_2 &= -\frac{\bar{\lambda}_0\bar{\mu}_0(\bar{\mu}_0 + \mu_0)(dz_1 + \bar{\lambda}_0) + d_0\bar{\lambda}_0\mu_0(\bar{\mu}_0 z_2 - 1)}{d_0\bar{\lambda}_0\bar{\mu}_0^2(z_2 - z_1)},
\end{aligned}$$

and the constants $\bar{\lambda}_0$, $\bar{\mu}_0$, λ_0, μ_0 *and* d_0 *are defined by the following equalities:*

$$\begin{aligned}
\mu_0 &= -\frac{5\gamma_3\gamma_4 - 3\gamma_2\gamma_5}{40\gamma_3^2 - 30\gamma_2\gamma_4} + \frac{\sqrt{(5\gamma_3\gamma_4 - 3\gamma_2\gamma_5)^2 + (4\gamma_3\gamma_5 - 5\gamma_4^2)(20\gamma_3^2 - 15\gamma_2\gamma_4)}}{|40\gamma_3^2 - 30\gamma_2\gamma_4|},\\
\bar{\mu}_0 &= \frac{5\gamma_3\gamma_4 - 3\gamma_2\gamma_5}{40\gamma_3^2 - 30\gamma_2\gamma_4} + \frac{\sqrt{(5\gamma_3\gamma_4 - 3\gamma_2\gamma_5)^2 + (4\gamma_3\gamma_5 - 5\gamma_4^2)(20\gamma_3^2 - 15\gamma_2\gamma_4)}}{|40\gamma_3^2 - 30\gamma_2\gamma_4|},\\
\lambda_0 &= \frac{3\bar{\mu}_0\gamma_2 - \gamma_3}{6\mu_0^2(\bar{\mu}_0 + \mu_0)}, \quad \bar{\lambda}_0 = \frac{\gamma_2 - 2\lambda_0\mu_0^2}{2\bar{\mu}_0^2},\\
d_0 &= \bar{\lambda}_0\bar{\mu}_0 - \lambda_0\mu_0 - (\bar{\lambda}\bar{\mu} - \lambda\mu - d),
\end{aligned}$$

provided that $\bar{\lambda}_0 > 0$, $\bar{\mu}_0 > 0$, $\lambda_0 > 0$, $\mu_0 > 0$, $d_0 > 0$, $4\gamma_3^2 - 3\gamma_2\gamma_4 \neq 0$ *and*

$$(5\gamma_3\gamma_4 - 3\gamma_2\gamma_5)^2 + (4\gamma_3\gamma_5 - 5\gamma_4^2)(20\gamma_3^2 - 15\gamma_2\gamma_4) > 0.$$

Proof. Taking into account that the kth moments of the r.v.'s that are exponentially distributed with means $\bar{\mu}_0$ and μ_0 equal $k!\bar{\mu}_0^k$ and $k!\mu_0^k$, respectively, from Lemma 2 we conclude that (17) is equivalent to the system of equations (19)–(23):

$$\bar{\lambda}_0\bar{\mu}_0 - \lambda_0\mu_0 - d_0 = \bar{\lambda}\bar{\mu} - \lambda\mu - d, \tag{19}$$

$$2\bar{\lambda}_0\bar{\mu}_0^2 + 2\lambda_0\mu_0^2 = \gamma_2, \tag{20}$$

$$6\bar{\lambda}_0\bar{\mu}_0^3 - 6\lambda_0\mu_0^3 = \gamma_3, \tag{21}$$

$$24\bar{\lambda}_0\bar{\mu}_0^4 + 24\lambda_0\mu_0^4 = \gamma_4, \tag{22}$$

$$120\bar{\lambda}_0\bar{\mu}_0^5 - 120\lambda_0\mu_0^5 = \gamma_5. \tag{23}$$

Now our aim is to find the constants $\bar{\lambda}_0$, $\bar{\mu}_0$, λ_0, μ_0 and d_0 from this system. From (20) we have $2\bar{\lambda}_0\bar{\mu}_0^2 = \gamma_2 - 2\lambda_0\mu_0^2$. Substituting this into equations (21)–(23) we get

$$3\gamma_2\bar{\mu}_0 - 6\lambda_0\mu_0^2(\bar{\mu}_0 + \mu_0) = \gamma_3, \tag{24}$$

$$12\gamma_2\bar{\mu}_0^2 - 24\lambda_0\mu_0^2(\bar{\mu}_0^2 - \mu_0^2) = \gamma_4, \tag{25}$$

$$60\gamma_2\bar{\mu}_0^3 - 120\lambda_0\mu_0^2(\bar{\mu}_0^3 + \mu_0^3) = \gamma_5. \tag{26}$$

Next, from (24) we have $6\lambda_0\mu_0^2(\bar{\mu}_0 + \mu_0) = 3\gamma_2\bar{\mu}_0 - \gamma_3$. Substituting this into equations (25)–(26) we obtain

$$12\gamma_2\bar{\mu}_0\mu_0 + 4\gamma_3(\bar{\mu}_0 - \mu_0) = \gamma_4 \tag{27}$$

and

$$60\gamma_2\bar{\mu}_0\mu_0(\bar{\mu}_0 - \mu_0) + 20\gamma_3\big((\bar{\mu}_0 - \mu_0)^2 + \bar{\mu}_0\mu_0\big) = \gamma_5. \tag{28}$$

Multiplying (27) by $\big(-5(\bar{\mu}_0 - \mu_0)\big)$ and adding (28) we get

$$20\gamma_3\bar{\mu}_0\mu_0 + 5\gamma_4(\bar{\mu}_0 - \mu_0) = \gamma_5. \tag{29}$$

Note that (27) and (29) form a system of two equations, which are linear with respect to variables $\bar{\mu}_0\mu_0$ and $\bar{\mu}_0 - \mu_0$. Solving this system we obtain

$$\bar{\mu}_0\mu_0 = \frac{4\gamma_3\gamma_5 - 5\gamma_4^2}{80\gamma_3^2 - 60\gamma_2\gamma_4} \tag{30}$$

and

$$\bar{\mu}_0 - \mu_0 = \frac{5\gamma_3\gamma_4 - 3\gamma_2\gamma_5}{20\gamma_3^2 - 15\gamma_2\gamma_4} \tag{31}$$

provided that $4\gamma_3^2 - 3\gamma_2\gamma_4 \neq 0$.

Substituting the expression for $\bar{\mu}_0$ from (31) into (30) gives

$$\mu_0^2 + \frac{5\gamma_3\gamma_4 - 3\gamma_2\gamma_5}{20\gamma_3^2 - 15\gamma_2\gamma_4}\mu_0 - \frac{4\gamma_3\gamma_5 - 5\gamma_4^2}{80\gamma_3^2 - 60\gamma_2\gamma_4} = 0, \tag{32}$$

from which we have

$$\mu_0 = -\frac{5\gamma_3\gamma_4 - 3\gamma_2\gamma_5}{40\gamma_3^2 - 30\gamma_2\gamma_4} \pm \frac{\sqrt{(5\gamma_3\gamma_4 - 3\gamma_2\gamma_5)^2 + (4\gamma_3\gamma_5 - 5\gamma_4^2)(20\gamma_3^2 - 15\gamma_2\gamma_4)}}{|40\gamma_3^2 - 30\gamma_2\gamma_4|}$$

provided that $(5\gamma_3\gamma_4 - 3\gamma_2\gamma_5)^2 + (4\gamma_3\gamma_5 - 5\gamma_4^2)(20\gamma_3^2 - 15\gamma_2\gamma_4) > 0$ and $4\gamma_3^2 - 3\gamma_2\gamma_4 \neq 0$. Hence, taking into account (31) we get

$$\bar{\mu}_0 = \frac{5\gamma_3\gamma_4 - 3\gamma_2\gamma_5}{40\gamma_3^2 - 30\gamma_2\gamma_4} \pm \frac{\sqrt{(5\gamma_3\gamma_4 - 3\gamma_2\gamma_5)^2 + (4\gamma_3\gamma_5 - 5\gamma_4^2)(20\gamma_3^2 - 15\gamma_2\gamma_4)}}{|40\gamma_3^2 - 30\gamma_2\gamma_4|}.$$

Since both $\bar{\mu}_0$ and μ_0 must be positive, from the expressions for $\bar{\mu}_0$ and μ_0 we deduce that we can take only the values of the parameters given in the assertion of the theorem (otherwise, if we take the values with "−", at least one of the parameters $\bar{\mu}_0$ or μ_0 will be negative). Finally, we obtain the corresponding values of the parameters λ_0, $\bar{\lambda}_0$ and d_0 from (24), (20) and (19), respectively, provided that all the values are positive.

Thus, we have a new process $(\tilde{U}_t)_{t\ge 0}$ with exponentially distributed premium and claim sizes, which is completely determined by $\bar{\lambda}_0$, $\bar{\mu}_0$, λ_0, μ_0 and d_0. Now the assertion of the theorem follows immediately from Theorem 2. □

3.3 A three-moment approximation

From the assertion of Theorem 3 it is clear that its conditions are quite restrictive. In particular, some of the parameters $\bar{\lambda}_0$, $\bar{\mu}_0$, λ_0, μ_0 and d_0 can be negative, and numerical computations show that such situations happen quite often. Hence, it is impossible to construct the five-moment approximation in those cases. Namely for this reason we also consider a simplified three-moment analogue to the De Vylder approximation. To construct it, we replace the process $(U_t)_{t\ge 0}$ by the process $(\tilde{U}_t)_{t\ge 0}$ with exponentially distributed premium and claim sizes such that

$$\mathbb{E}[U_t^k] = \mathbb{E}[\tilde{U}_t^k], \qquad k = 1, 2, 3, \tag{33}$$

and the following proportionality conditions hold:

$$\frac{\bar{\mu}}{\mu} = \frac{\bar{\mu}_0}{\mu_0} \quad \text{and} \quad \frac{\bar{\lambda}}{\lambda} = \frac{\bar{\lambda}_0}{\lambda_0}. \tag{34}$$

In particular, condition (34) implies that $\bar{\lambda}\bar{\mu}/\lambda\mu = \bar{\lambda}_0\bar{\mu}_0/\lambda_0\mu_0$. This means that the ratio between the expected premiums and the expected claims per unit time remains the same, which seems to be natural.

Theorem 4 (a three-moment analogue to the De Vylder approximation)**.** *Let the surplus process $(X_t(x))_{t\ge 0}$ is defined by (1) under the above assumptions, $\mathbb{E}[\bar{Y}_i^3] < \infty$, $\mathbb{E}[Y_i^3] < \infty$, and let the net profit condition (2) hold. Then the ruin probability is approximately equal to*

$$\psi_{DV3}(x) = C_1 e^{z_1 x} + C_2 e^{z_2 x} \quad \textit{for all} \quad x \ge 0, \tag{35}$$

where z_1, z_2, C_1 and C_2 are defined as in Theorem 3 and the constants $\bar{\lambda}_0$, $\bar{\mu}_0$, λ_0, μ_0 and d_0 are defined by the following equalities:

$$\mu_0 = \frac{\gamma_3\mu(\bar{\lambda}\bar{\mu}^2 + \lambda\mu^2)}{3\gamma_2(\bar{\lambda}\bar{\mu}^3 - \lambda\mu^3)}, \quad \lambda_0 = \frac{9\gamma_2^3\lambda(\bar{\lambda}\bar{\mu}^3 - \lambda\mu^3)^2}{2\gamma_3^2(\bar{\lambda}\bar{\mu}^2 + \lambda\mu^2)^3},$$

$$\bar{\mu}_0 = \frac{\gamma_3\bar{\mu}(\bar{\lambda}\bar{\mu}^2 + \lambda\mu^2)}{3\gamma_2(\bar{\lambda}\bar{\mu}^3 - \lambda\mu^3)}, \quad \bar{\lambda}_0 = \frac{9\gamma_2^3\bar{\lambda}(\bar{\lambda}\bar{\mu}^3 - \lambda\mu^3)^2}{2\gamma_3^2(\bar{\lambda}\bar{\mu}^2 + \lambda\mu^2)^3},$$

$$d_0 = \bar{\lambda}_0\bar{\mu}_0 - \lambda_0\mu_0 - (\bar{\lambda}\bar{\mu} - \lambda\mu - d),$$

provided that $\gamma_3(\bar{\lambda}\bar{\mu}^3 - \lambda\mu^3) > 0$ and $d_0 > 0$.

Proof. From Lemma 2 we conclude that (33) is equivalent to the system of equations (19)–(21), and from (34) we get

$$\bar{\lambda}_0\bar{\mu}_0^2 = \lambda_0\mu_0^2\,\frac{\bar{\lambda}\bar{\mu}^2}{\lambda\mu^2} \quad \text{and} \quad \bar{\lambda}_0\bar{\mu}_0^3 = \lambda_0\mu_0^3\,\frac{\bar{\lambda}\bar{\mu}^3}{\lambda\mu^3}. \tag{36}$$

Substituting (36) into (20) and (21) we obtain

$$2\lambda_0\mu_0^2\left(\frac{\bar{\lambda}\bar{\mu}^2}{\lambda\mu^2} + 1\right) = \gamma_2 \tag{37}$$

and

$$6\lambda_0\mu_0^3\left(\frac{\bar{\lambda}\bar{\mu}^3}{\lambda\mu^3} - 1\right) = \gamma_3. \tag{38}$$

Dividing (38) by (37) we easily get the expression for μ_0 provided that $\gamma_3(\bar{\lambda}\bar{\mu}^3 - \lambda\mu^3) > 0$. Next, substituting this expression into (37) we obtain the expression for λ_0. The constants $\bar{\mu}_0$ and $\bar{\lambda}_0$ are determined from (34). Finally, we can find d_0 from (19). Applying Theorem 2 for exponentially distributed premium and claim sizes yields the assertion of the theorem. □

Comparing the assertions of Theorems 3 and 4 we deduce that the conditions of Theorem 4 are much less restrictive.

Remark 2. Instead of conditions (34), we can consider the following more general conditions:

$$\frac{\bar{\mu}}{\mu} = \nu_1\,\frac{\bar{\mu}_0}{\mu_0} \quad \text{and} \quad \frac{\bar{\lambda}}{\lambda} = \nu_2\,\frac{\bar{\lambda}_0}{\lambda_0}, \tag{39}$$

where $\nu_1 > 0$ and $\nu_2 > 0$. The corresponding approximation for the ruin probability is calculated using the same formula (35), but the constants $\bar{\lambda}_0$, $\bar{\mu}_0$, λ_0, μ_0 and d_0 are defined by the following equalities:

$$\mu_0 = \frac{\gamma_3\mu\nu_1(\bar{\lambda}\bar{\mu}^2 + \lambda\mu^2\nu_1^2\nu_2)}{3\gamma_2(\bar{\lambda}\bar{\mu}^3 - \lambda\mu^3\nu_1^3\nu_2)}, \qquad \lambda_0 = \frac{9\gamma_2^3\lambda\nu_2(\bar{\lambda}\bar{\mu}^3 - \lambda\mu^3\nu_1^3\nu_2)^2}{2\gamma_3^2(\bar{\lambda}\bar{\mu}^2 + \lambda\mu^2\nu_1^3\nu_2)^3},$$

$$\bar{\mu}_0 = \frac{\gamma_3\bar{\mu}(\bar{\lambda}\bar{\mu}^2 + \lambda\mu^2\nu_1^2\nu_2)}{3\gamma_2(\bar{\lambda}\bar{\mu}^3 - \lambda\mu^3\nu_1^3\nu_2)}, \qquad \bar{\lambda}_0 = \frac{9\gamma_2^3\bar{\lambda}(\bar{\lambda}\bar{\mu}^3 - \lambda\mu^3\nu_1^3\nu_2)^2}{2\gamma_3^2(\bar{\lambda}\bar{\mu}^2 + \lambda\mu^2\nu_1^2\nu_2)^3},$$

$$d_0 = \bar{\lambda}_0\bar{\mu}_0 - \lambda_0\mu_0 - (\bar{\lambda}\bar{\mu} - \lambda\mu - d),$$

provided that $\gamma_3(\bar{\lambda}\bar{\mu}^3 - \lambda\mu^3\nu_1^3\nu_2) > 0$ and $d_0 > 0$. Conditions (39) enable us to consider different cases by changing the values of the coefficients ν_1 and ν_2 and choose those ones that approximate the ruin probability more accurately. Nevertheless, note that for some values of ν_1 and ν_2, the corresponding approximations give not so good results, and consequently, should not be applied.

4 Numerical illustrations

4.1 A statistical estimate for the ruin probability

To analyze the accuracy of the approximations proposed in Section 3, we will need a statistical estimate for the ruin probability obtained by the direct simulation of the surplus process $\big(X_t(x)\big)_{t\ge 0}$ using the Monte Carlo methods. To this end, we use the approach described in [23]. Let N be the total number of simulations of $\big(X_t(x)\big)_{t\ge 0}$. To get the corresponding statistical estimate $\hat{\psi}(x)$ for the ruin probability $\psi(x)$, we divide the number of simulations leading to ruin by the total number of simulations N. To find the number of simulations N, which is necessary in order to calculate the ruin probability with the required accuracy and reliability, we apply the following proposition, which follows immediately from Hoeffding's inequality (see [18]).

Proposition 1. *Let the surplus process $\big(X_t(x)\big)_{t\ge 0}$ be defined by (1) under the above assumptions. Then for any $\varepsilon > 0$, we have*

$$\mathbb{P}\big[\big|\psi(x) - \hat{\psi}(x)\big| > \varepsilon\big] \le 2e^{-2\varepsilon^2 N}.$$

In all examples below, we set $\varepsilon = 0.005$ and $2e^{-2\varepsilon^2 N} = 0.005$. Therefore, we get $N = 119\,830$. Moreover, let $\bar{\lambda} = 2.3$, $\bar{\mu} = 0.2$, $\lambda = 0.1$, $\mu = 3$ and $d = 0.05$.

4.2 Gamma distributions for the premium and claim sizes

Let the p.d.f. of $\bar{Y}_i$ be

$$f_{\bar{Y}}(y) = \frac{1}{\Gamma(\bar{\alpha})\bar{\beta}^{\bar{\alpha}}}\, y^{\bar{\alpha}-1} e^{-y/\bar{\beta}}, \quad y \ge 0,$$

where $\bar{\alpha} > 0$, $\bar{\beta} > 0$ and $\bar{\alpha}\bar{\beta} = \bar{\mu}$, and let the p.d.f. of Y_i be

$$f_Y(y) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}}\, y^{\alpha-1} e^{-y/\beta}, \quad y \ge 0,$$

where $\alpha > 0$, $\beta > 0$ and $\alpha\beta = \mu$.

Then

$$\begin{gathered}
\mathbb{E}[\bar{Y}_i] = \bar{\alpha}\bar{\beta} = \bar{\mu}, \quad \mathbb{E}[\bar{Y}_i^2] = \bar{\alpha}(\bar{\alpha}+1)\bar{\beta}^2, \\
\mathbb{E}[\bar{Y}_i^3] = \bar{\alpha}(\bar{\alpha}+1)(\bar{\alpha}+2)\bar{\beta}^3, \\
\mathbb{E}[\bar{Y}_i^4] = \bar{\alpha}(\bar{\alpha}+1)(\bar{\alpha}+2)(\bar{\alpha}+3)\bar{\beta}^4, \\
\mathbb{E}[\bar{Y}_i^5] = \bar{\alpha}(\bar{\alpha}+1)(\bar{\alpha}+2)(\bar{\alpha}+3)(\bar{\alpha}+4)\bar{\beta}^5,
\end{gathered}$$

and analogous formulas hold for the moments of Y_i.

Therefore, we get

$$\begin{gathered}
\gamma_2 = \bar{\lambda}\bar{\alpha}(\bar{\alpha}+1)\bar{\beta}^2 + \lambda\alpha(\alpha+1)\beta^2, \\
\gamma_3 = \bar{\lambda}\bar{\alpha}(\bar{\alpha}+1)(\bar{\alpha}+2)\bar{\beta}^3 - \lambda\alpha(\alpha+1)(\alpha+2)\beta^3, \\
\gamma_4 = \bar{\lambda}\bar{\alpha}(\bar{\alpha}+1)(\bar{\alpha}+2)(\bar{\alpha}+3)\bar{\beta}^4 + \lambda\alpha(\alpha+1)(\alpha+2)(\alpha+3)\beta^4, \\
\gamma_5 = \bar{\lambda}\bar{\alpha}(\bar{\alpha}+1)(\bar{\alpha}+2)(\bar{\alpha}+3)(\bar{\alpha}+4)\bar{\beta}^5 - \lambda\alpha(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)\beta^5.
\end{gathered}$$

A number of numerical examples show that the conditions of Theorem 3 hold provided that α is very close to 1. So the five-moment approximation can be constructed only in those cases. We now consider two examples.

Table 1. Results of computations: the gamma distributions for the premium and claim sizes, $\bar{\alpha} = 2$, $\bar{\beta} = 0.1$, $\alpha = 1$ and $\beta = 3$

x	$\hat{\psi}(x)$	$\psi_{DV5}(x)$	$\left(\frac{\psi_{DV5}(x)}{\hat{\psi}(x)} - 1\right) \cdot 100\%$	$\psi_{DV3}(x)$	$\left(\frac{\psi_{DV3}(x)}{\hat{\psi}(x)} - 1\right) \cdot 100\%$
1	0.6912	0.6832	−1.15%	0.6766	−2.11%
2	0.6205	0.6270	1.05%	0.6210	0.08%
3	0.5681	0.5754	1.29%	0.5700	0.34%
5	0.4870	0.4846	−0.50%	0.4802	−1.41%
7	0.4040	0.4081	1.01%	0.4045	0.12%
10	0.3149	0.3154	0.17%	0.3128	−0.66%
15	0.2024	0.2053	1.42%	0.2037	0.66%
20	0.1374	0.1336	−2.73%	0.1327	−3.38%
30	0.0584	0.0566	−3.09%	0.0563	−3.58%
50	0.0098	0.0102	3.63%	0.0101	3.45%

Example 1. Let now $\bar{\alpha} = 2$, $\bar{\beta} = 0.1$, $\alpha = 1$ and $\beta = 3$.

If we construct the five-moment analogue to the De Vylder approximation, by Theorem 3, we get $\bar{\lambda}_0 \approx 3.923743$, $\bar{\mu}_0 \approx 0.132632$, $\lambda_0 \approx 0.099996$, $\mu_0 \approx 3.000027$, $d_0 \approx 0.110423$, and consequently,

$$\psi_{DV5}(x) \approx 0.255492\, e^{-29.147189x} + 0.744508\, e^{-0.085895x} \quad \text{for all} \quad x \geq 0.$$

For the corresponding three-moment approximation using conditions (34), by Theorem 4, we have $\bar{\lambda}_0 \approx 2.129067$, $\bar{\mu}_0 \approx 0.205450$, $\lambda_0 \approx 0.092568$, $\mu_0 \approx 3.081744$, $d_0 \approx 0.042145$, and hence,

$$\psi_{DV3}(x) \approx 0.262882\, e^{-48.085872x} + 0.737118\, e^{-0.085730x} \quad \text{for all} \quad x \geq 0.$$

Table 1 presents the results of computations for some values of x. Next, Table 2 shows the values of the relative approximation errors $\left(\frac{\psi_{DV3}(x)}{\hat{\psi}(x)} - 1\right) \cdot 100\%$ for the three-moment approximations constructed using conditions (39) for different ν_1 and ν_2. Note that here we chose some values of ν_1 and ν_2 that give more or less good

Table 2. Values of $\left(\frac{\psi_{DV3}(x)}{\hat{\psi}(x)} - 1\right) \cdot 100\%$ for different ν_1 and ν_2: the gamma distributions for the premium and claim sizes, $\bar{\alpha} = 2$, $\bar{\beta} = 0.1$, $\alpha = 1$ and $\beta = 3$

x	ν_1=0.7 ν_2=1.5	ν_1=0.9 ν_2=1.2	ν_1=1.1 ν_2=0.7	ν_1=1.5 ν_2=0.7	ν_1=2 ν_2=0.2	ν_1=5 ν_2=0.05	ν_1=10 ν_2=0.05
1	−3.48%	−2.36%	−2.21%	−1.00%	−1.41%	−0.35%	0.52%
2	−1.29%	−0.17%	0.00%	1.20%	0.70%	1.84%	2.72%
3	−1.02%	0.09%	0.26%	1.43%	0.97%	2.08%	2.93%
5	−2.69%	−1.65%	−1.46%	−0.38%	−0.76%	0.28%	1.05%
7	−1.13%	−0.12%	0.09%	1.12%	0.80%	1.80%	2.52%
10	−1.82%	−0.89%	−0.67%	0.26%	0.04%	0.95%	1.57%
15	−0.39%	0.44%	0.70%	1.48%	1.42%	2.21%	2.67%
20	−4.26%	−3.58%	−3.29%	−2.70%	−2.60%	−1.97%	−1.68%
30	−4.21%	−3.75%	−3.39%	−3.13%	−2.70%	−2.33%	−2.36%
50	3.31%	3.31%	3.86%	3.45%	4.61%	4.45%	3.76%

results. Choosing some other values of the coefficients leads to extremely bad approximations. In addition, analyzing the relative approximation errors in Table 2 we conclude that it is difficult to decide which of the approximations is better: choosing ν_1 and ν_2 that yield smaller errors for some values of the initial surplus results in larger errors for other values.

Example 2. Let now $\bar{\alpha} = 4$, $\bar{\beta} = 0.05$, $\alpha = 3$ and $\beta = 1$.

In this case, the conditions of Theorem 3 do not hold, so we can construct only the three-moment analogue to the De Vylder approximation. By Theorem 4, we obtain $\bar{\lambda}_0 \approx 4.871659$, $\bar{\mu}_0 \approx 0.111879$, $\lambda_0 \approx 0.211811$, $\mu_0 \approx 1.678181$, $d_0 \approx 0.079577$, and therefore,

$$\psi_{DV3}(x) \approx 0.221130\, e^{-55.405586x} + 0.778870\, e^{-0.132881x} \quad \text{for all} \quad x \geq 0.$$

Table 3 presents the results of computations for some values of x, whereas Table 4 shows the values of the relative approximation errors $\left(\frac{\psi_{DV3}(x)}{\hat{\psi}(x)} - 1\right) \cdot 100\%$ for the three-moment approximations constructed using conditions (39) for different ν_1 and ν_2.

Table 3. Results of computations: the gamma distributions for the premium and claim sizes, $\bar{\alpha} = 4$, $\bar{\beta} = 0.05$, $\alpha = 3$ and $\beta = 1$

x	$\hat{\psi}(x)$	$\psi_{DV3}(x)$	$\left(\frac{\psi_{DV3}(x)}{\hat{\psi}(x)} - 1\right) \cdot 100\%$
1	0.6690	0.6820	1.94%
2	0.6046	0.5971	−1.24%
3	0.5301	0.5228	−1.38%
5	0.4039	0.4008	−0.77%
7	0.3039	0.3073	1.10%
10	0.2065	0.2062	−0.13%
15	0.1056	0.1061	0.50%
20	0.0535	0.0546	2.17%
30	0.0150	0.0145	−3.60%

Table 4. Values of $\left(\frac{\psi_{DV3}(x)}{\hat{\psi}(x)} - 1\right) \cdot 100\%$ for different ν_1 and ν_2: the gamma distributions for the premium and claim sizes, $\bar{\alpha} = 4$, $\bar{\beta} = 0.05$, $\alpha = 3$ and $\beta = 1$

x	ν_1=0.7 ν_2=1.5	ν_1=0.9 ν_2=1.2	ν_1=1.1 ν_2=0.6	ν_1=1.5 ν_2=0.7	ν_1=2 ν_2=0.2	ν_1=5 ν_2=0.05	ν_1=10 ν_2=0.03
1	0.82%	1.73%	1.68%	2.85%	2.51%	3.42%	3.97%
2	−2.30%	−1.44%	−1.47%	−0.39%	−0.68%	0.17%	0.68%
3	−2.41%	−1.58%	−1.58%	−0.55%	−0.81%	0.01%	0.50%
5	−1.75%	−0.96%	−0.93%	0.01%	−0.17%	0.59%	1.02%
7	0.16%	0.91%	0.99%	1.84%	1.73%	2.45%	2.84%
10	−0.97%	−0.31%	−0.17%	0.53%	0.53%	1.15%	1.45%
15	−0.21%	0.33%	0.58%	1.03%	1.22%	1.69%	1.85%
20	1.60%	2.01%	2.37%	2.58%	2.96%	3.30%	3.31%
30	−3.86%	−3.72%	−3.17%	−3.46%	−2.74%	−2.71%	−2.96%

4.3 Hyperexponential distributions for the premium and claim sizes

Let

$$F_{\bar{Y}}(y) = \bar{p}_1 F_{\bar{Y},1}(y) + \bar{p}_2 F_{\bar{Y},2}(y) + \cdots + \bar{p}_{\bar{k}} F_{\bar{Y},\bar{k}}(y), \quad y \geq 0,$$

where $\bar{k} \geq 1$, $\bar{p}_j > 0$, $F_{\bar{Y},j}$ is the c.d.f. of the exponential distribution with mean $\bar{\mu}_j$ for all $1 \leq j \leq \bar{k}$, $\sum_{j=1}^{\bar{k}} \bar{p}_j = 1$, $\sum_{j=1}^{\bar{k}} \bar{p}_j \bar{\mu}_j = \bar{\mu}$, and let

$$F_Y(y) = p_1 F_{Y,1}(y) + p_2 F_{Y,2}(y) + \cdots + p_k F_{Y,k}(y), \quad y \geq 0,$$

where $k \geq 1$, $p_j > 0$, $F_{Y,j}$ is the c.d.f. of the exponential distribution with mean μ_j for all $1 \leq j \leq k$, $\sum_{j=1}^{k} p_j = 1$, $\sum_{j=1}^{k} p_j \mu_j = \mu$.

Then

$$\mathbb{E}[\bar{Y}_i] = \sum_{j=1}^{\bar{k}} \bar{p}_j \bar{\mu}_j = \bar{\mu}, \quad \mathbb{E}[\bar{Y}_i^2] = 2\sum_{j=1}^{\bar{k}} \bar{p}_j \bar{\mu}_j^2, \quad \mathbb{E}[\bar{Y}_i^3] = 6\sum_{j=1}^{\bar{k}} \bar{p}_j \bar{\mu}_j^3,$$

$$\mathbb{E}[\bar{Y}_i^4] = 24\sum_{j=1}^{\bar{k}} \bar{p}_j \bar{\mu}_j^4, \quad \mathbb{E}[\bar{Y}_i^5] = 120\sum_{j=1}^{\bar{k}} \bar{p}_j \bar{\mu}_j^5,$$

and analogous formulas hold for the moments of Y_i.

Hence, we obtain

$$\gamma_2 = 2\left(\bar{\lambda}\sum_{j=1}^{\bar{k}} \bar{p}_j \bar{\mu}_j^2 + \lambda \sum_{j=1}^{k} p_j \mu_j^2\right), \quad \gamma_3 = 6\left(\bar{\lambda}\sum_{j=1}^{\bar{k}} \bar{p}_j \bar{\mu}_j^3 - \lambda \sum_{j=1}^{k} p_j \mu_j^3\right),$$

$$\gamma_4 = 24\left(\bar{\lambda}\sum_{j=1}^{\bar{k}} \bar{p}_j \bar{\mu}_j^4 + \lambda \sum_{j=1}^{k} p_j \mu_j^4\right), \quad \gamma_5 = 120\left(\bar{\lambda}\sum_{j=1}^{\bar{k}} \bar{p}_j \bar{\mu}_j^5 - \lambda \sum_{j=1}^{k} p_j \mu_j^5\right).$$

Examples 3 and 4 below present some numerical results.

Example 3. Let now $\bar{k} = 2$, $\bar{p}_1 = 0.75$, $\bar{p}_2 = 0.25$, $\bar{\mu}_1 = 0.1$, $\bar{\mu}_2 = 0.5$, $k = 2$, $p_1 = 0.8$, $p_2 = 0.2$, $\mu_1 = 2.8$ and $\mu_2 = 3.8$.

If we construct the five-moment analogue to the De Vylder approximation, by Theorem 3, we obtain $\bar{\lambda}_0 \approx 10.626422$, $\bar{\mu}_0 \approx 0.141004$, $\lambda_0 \approx 0.082185$, $\mu_0 \approx 3.245591$, $d_0 \approx 1.121624$, and consequently,

$$\psi_{DV5}(x) \approx 0.236453\, e^{-2.683647x} + 0.763547\, e^{-0.079854x} \quad \text{for all} \quad x \geq 0.$$

For the corresponding three-moment approximation using conditions (34), by Theorem 4, we have $\bar{\lambda}_0 \approx 2.738661$, $\bar{\mu}_0 \approx 0.190975$, $\lambda_0 \approx 0.119072$, $\mu_0 \approx 2.864627$, $d_0 \approx 0.071919$ (here we use conditions (34)), and therefore,

$$\psi_{DV3}(x) \approx 0.228569\, e^{-34.768023x} + 0.771431\, e^{-0.080413x} \quad \text{for all} \quad x \geq 0.$$

Table 5 presents the results of computations for some values of x, whereas Table 6 shows the values of $\left(\frac{\psi_{DV3}(x)}{\hat{\psi}(x)} - 1\right) \cdot 100\%$ for the three-moment approximations constructed using conditions (39) for different ν_1 and ν_2.

Table 5. Results of computations: the hyperexponential distributions for the premium and claim sizes, $\bar{k} = 2$, $\bar{p}_1 = 0.75$, $\bar{p}_2 = 0.25$, $\bar{\mu}_1 = 0.1$, $\bar{\mu}_2 = 0.5$, $k = 2$, $p_1 = 0.8$, $p_2 = 0.2$, $\mu_1 = 2.8$ and $\mu_2 = 3.8$

x	$\hat{\psi}(x)$	$\psi_{DV5}(x)$	$\left(\frac{\psi_{DV5}(x)}{\hat{\psi}(x)} - 1\right) \cdot 100\%$	$\psi_{DV3}(x)$	$\left(\frac{\psi_{DV3}(x)}{\hat{\psi}(x)} - 1\right) \cdot 100\%$
1	0.6988	0.7211	3.20%	0.7118	1.87%
2	0.6365	0.6519	2.43%	0.6568	3.19%
3	0.5909	0.6010	1.71%	0.6061	2.58%
5	0.5023	0.5122	1.97%	0.5160	2.74%
7	0.4253	0.4366	2.67%	0.4394	3.32%
10	0.3367	0.3436	2.06%	0.3452	2.54%
15	0.2210	0.2305	4.31%	0.2309	4.51%
20	0.1542	0.1546	0.26%	0.1545	0.17%
30	0.0670	0.0696	3.84%	0.0691	3.17%
50	0.0141	0.0141	−0.09%	0.0138	−1.84%

Table 6. Values of $\left(\frac{\psi_{DV3}(x)}{\hat{\psi}(x)} - 1\right) \cdot 100\%$ for different ν_1 and ν_2: the hyperexponential distributions for the premium and claim sizes, $\bar{k} = 2$, $\bar{p}_1 = 0.75$, $\bar{p}_2 = 0.25$, $\bar{\mu}_1 = 0.1$, $\bar{\mu}_2 = 0.5$, $k = 2$, $p_1 = 0.8$, $p_2 = 0.2$, $\mu_1 = 2.8$ and $\mu_2 = 3.8$

x	$\nu_1=0.4$ $\nu_2=2.7$	$\nu_1=0.5$ $\nu_2=2.1$	$\nu_1=0.7$ $\nu_2=0.7$	$\nu_1=0.9$ $\nu_2=0.6$	$\nu_1=1.1$ $\nu_2=0.5$	$\nu_1=1.5$ $\nu_2=0.15$	$\nu_1=2$ $\nu_2=0.1$
1	−2.81%	−1.14%	0.28%	0.73%	1.37%	4.60%	4.25%
2	−1.47%	0.19%	−0.01%	1.72%	2.59%	2.18%	2.77%
3	−1.98%	−0.36%	−0.60%	1.16%	2.01%	1.12%	1.93%
5	−1.67%	−0.10%	−0.26%	1.41%	2.22%	1.35%	2.17%
7	−0.95%	0.57%	0.49%	2.08%	2.86%	2.09%	2.87%
10	−1.47%	−0.04%	0.01%	1.45%	2.17%	1.57%	2.27%
15	0.83%	2.14%	2.41%	3.65%	4.27%	3.94%	4.54%
20	−2.97%	−1.85%	−1.38%	−0.42%	0.08%	0.03%	0.50%
30	0.73%	1.59%	2.51%	3.04%	3.35%	3.86%	4.11%
50	−2.62%	−2.36%	−0.62%	−1.02%	−1.12%	0.43%	0.22%

Example 4. Let now $\bar{k} = 3$, $\bar{p}_1 = 0.2$, $\bar{p}_2 = 0.5$, $\bar{p}_3 = 0.3$, $\bar{\mu}_1 = 0.1$, $\bar{\mu}_2 = 0.15$, $\bar{\mu}_3 = 0.35$, $k = 3$, $p_1 = 0.1$, $p_2 = 0.4$, $p_3 = 0.5$, $\mu_1 = 1$, $\mu_2 = 2.7$ and $\mu_3 = 3.64$.

In this case, the conditions of Theorem 3 do not hold, so we can construct only the three-moment analogue to the De Vylder approximation. By Theorem 4, we get $\bar{\lambda}_0 \approx 2.112044$, $\bar{\mu}_0 \approx 0.217677$, $\lambda_0 \approx 0.091828$, $\mu_0 \approx 3.265162$, $d_0 \approx 0.049911$ (here we use conditions (34)), and therefore,

$$\psi_{DV3}(x) \approx 0.252988\, e^{-39.790359x} + 0.747012\, e^{-0.077929x} \quad \text{for all} \quad x \geq 0.$$

Table 7 presents the results of computations for some values of x, whereas Table 8 shows the values of $\left(\frac{\psi_{DV3}(x)}{\hat{\psi}(x)} - 1\right) \cdot 100\%$ for the three-moment approximations constructed using conditions (39) for different ν_1 and ν_2.

4.4 Lomax distributions for the premium and claim sizes

Let

$$F_{\bar{Y}}(y) = 1 - \left(\frac{\bar{\beta}}{y + \bar{\beta}}\right)^{\bar{\alpha}}, \quad y \geq 0,$$

Table 7. Results of computations: the hyperexponential distributions for the premium and claim sizes, $\bar{k} = 3$, $\bar{p}_1 = 0.2$, $\bar{p}_2 = 0.5$, $\bar{p}_3 = 0.3$, $\bar{\mu}_1 = 0.1$, $\bar{\mu}_2 = 0.15$, $\bar{\mu}_3 = 0.35$, $k = 3$, $p_1 = 0.1$, $p_2 = 0.4$, $p_3 = 0.5$, $\mu_1 = 1$, $\mu_2 = 2.7$ and $\mu_3 = 3.64$

x	$\hat{\psi}(x)$	$\psi_{DV3}(x)$	$\left(\frac{\psi_{DV3}(x)}{\hat{\psi}(x)} - 1\right) \cdot 100\%$
1	0.6949	0.6910	−0.56%
2	0.6368	0.6392	0.38%
3	0.5861	0.5913	0.88%
5	0.5035	0.5059	0.49%
7	0.4306	0.4329	0.55%
10	0.3446	0.3427	−0.56%
15	0.2299	0.2321	0.95%
20	0.1594	0.1572	−1.35%
30	0.0694	0.0721	3.91%

Table 8. Values of $\left(\frac{\psi_{DV3}(x)}{\hat{\psi}(x)} - 1\right) \cdot 100\%$ for different ν_1 and ν_2: the hyperexponential distributions for the premium and claim sizes, $\bar{k} = 3$, $\bar{p}_1 = 0.2$, $\bar{p}_2 = 0.5$, $\bar{p}_3 = 0.3$, $\bar{\mu}_1 = 0.1$, $\bar{\mu}_2 = 0.15$, $\bar{\mu}_3 = 0.35$, $k = 3$, $p_1 = 0.1$, $p_2 = 0.4$, $p_3 = 0.5$, $\mu_1 = 1$, $\mu_2 = 2.7$ and $\mu_3 = 3.64$

x	ν_1=0.5 ν_2=2.1	ν_1=0.7 ν_2=1.5	ν_1=0.9 ν_2=1.1	ν_1=1.1 ν_2=1	ν_1=1.5 ν_2=0.5	ν_1=2 ν_2=0.2	ν_1=3 ν_2=0.1
1	−3.93%	−1.89%	−0.94%	−0.16%	0.19%	0.19%	0.69%
2	−2.96%	−0.94%	0.00%	0.77%	1.12%	0.98%	1.47%
3	−2.42%	−0.42%	0.52%	1.27%	1.63%	1.50%	1.98%
5	−2.69%	−0.77%	0.13%	0.86%	1.21%	1.12%	1.58%
7	−2.51%	−0.66%	0.21%	0.90%	1.26%	1.20%	1.65%
10	−3.41%	−1.69%	−0.87%	−0.24%	0.12%	0.11%	0.53%
15	−1.65%	−0.09%	0.67%	1.23%	1.61%	1.68%	2.06%
20	−3.61%	−2.26%	−1.59%	−1.12%	−0.75%	−0.60%	−0.26%
30	2.14%	3.18%	3.72%	4.05%	4.47%	4.79%	5.07%

where $\bar{\alpha} > 1$, $\bar{\beta} > 0$ and $\bar{\beta}/(\bar{\alpha} - 1) = \bar{\mu}$, and let

$$F_Y(y) = 1 - \left(\frac{\beta}{y + \beta}\right)^{\alpha}, \quad y \geq 0,$$

where $\alpha > 1$, $\beta > 0$ and $\beta/(\alpha - 1) = \mu$. In what follows, we assume that $\bar{\alpha} > 5$, $\alpha > 5$ and both $\bar{\alpha}$ and α are integer. Then

$$\mathbb{E}[\bar{Y}_i] = \frac{\bar{\beta}}{\bar{\alpha} - 1} = \bar{\mu}, \quad \mathbb{E}[\bar{Y}_i^2] = \frac{2\bar{\beta}^2}{(\bar{\alpha} - 2)(\bar{\alpha} - 1)},$$
$$\mathbb{E}[\bar{Y}_i^3] = \frac{6\bar{\beta}^3}{(\bar{\alpha} - 3)(\bar{\alpha} - 2)(\bar{\alpha} - 1)}, \quad \mathbb{E}[\bar{Y}_i^4] = \frac{24\bar{\beta}^4}{(\bar{\alpha} - 4)(\bar{\alpha} - 3)(\bar{\alpha} - 2)(\bar{\alpha} - 1)},$$
$$\mathbb{E}[\bar{Y}_i^5] = \frac{120\bar{\beta}^5}{(\bar{\alpha} - 5)(\bar{\alpha} - 4)(\bar{\alpha} - 3)(\bar{\alpha} - 2)(\bar{\alpha} - 1)},$$

and analogous formulas hold for the moments of Y_i.

Consequently, we have

$$\gamma_2 = 2\left(\frac{\bar{\lambda}\bar{\beta}^2}{(\bar{\alpha}-2)(\bar{\alpha}-1)} + \frac{\lambda\beta^2}{(\alpha-2)(\alpha-1)}\right),$$
$$\gamma_3 = 6\left(\frac{\bar{\lambda}\bar{\beta}^3}{(\bar{\alpha}-3)(\bar{\alpha}-2)(\bar{\alpha}-1)} - \frac{\lambda\beta^3}{(\alpha-3)(\alpha-2)(\alpha-1)}\right),$$
$$\gamma_4 = 24\left(\frac{\bar{\lambda}\bar{\beta}^4}{(\bar{\alpha}-4)(\bar{\alpha}-3)(\bar{\alpha}-2)(\bar{\alpha}-1)} + \frac{\lambda\beta^4}{(\alpha-4)(\alpha-3)(\alpha-2)(\alpha-1)}\right),$$
$$\gamma_5 = 120\left(\frac{\bar{\lambda}\bar{\beta}^5}{(\bar{\alpha}-5)(\bar{\alpha}-4)(\bar{\alpha}-3)(\bar{\alpha}-2)(\bar{\alpha}-1)} - \frac{\lambda\beta^5}{(\alpha-5)(\alpha-4)(\alpha-3)(\alpha-2)(\alpha-1)}\right).$$

Note that the Lomax distribution, which is also called the Pareto type II distribution, is a heavy-tailed distribution in contrast to the gamma and hyperexponential distributions. This can be the reason why the conditions of Theorem 3 do not hold for this distribution, at least as a number of numerical examples indicate. However, the three-moment approximation can be applied provided that the conditions of Theorem 4 hold.

Example 5. Let now $\bar{\alpha} = 6$, $\bar{\beta} = 1$, $\alpha = 6$ and $\beta = 15$.

In this case, we can construct only the three-moment analogue to the De Vylder approximation. By Theorem 4, we get $\bar{\lambda}_0 = 1.035$, $\bar{\mu}_0 \approx 0.333333$, $\lambda_0 = 0.045$, $\mu_0 = 5$, $d_0 = 0.01$ (here we use conditions (34)), and therefore,

$$\psi_{DV3}(x) \approx 0.313466\, e^{-105.137225x} + 0.686534\, e^{-0.062775x} \quad \text{for all} \quad x \geq 0.$$

Table 9 presents the results of computations for some values of x, whereas Table 10 shows the values of $\left(\frac{\psi_{DV3}(x)}{\hat{\psi}(x)} - 1\right) \cdot 100\%$ for the three-moment approximations constructed using conditions (39) for different ν_1 and ν_2.

Table 9. Results of computations: the Lomax distributions for premium and claim sizes, $\bar{\alpha} = 6$, $\bar{\beta} = 1$, $\alpha = 6$ and $\beta = 15$

x	$\hat{\psi}(x)$	$\psi_{DV3}(x)$	$\left(\frac{\psi_{DV3}(x)}{\hat{\psi}(x)} - 1\right) \cdot 100\%$
1	0.6881	0.6448	−6.30%
2	0.6391	0.6055	−5.25%
3	0.5899	0.5687	−3.59%
5	0.5086	0.5016	−1.37%
7	0.4429	0.4424	−0.10%
10	0.3638	0.3665	0.73%
15	0.2643	0.2677	1.32%
20	0.1887	0.1956	3.67%
30	0.1025	0.1044	1.92%
50	0.0301	0.0298	−1.16%

Table 10. Values of $\left(\frac{\psi_{DV3}(x)}{\hat{\psi}(x)} - 1\right) \cdot 100\%$ for different ν_1 and ν_2: the Lomax distributions for the premium and claim sizes, $\bar{\alpha} = 6$, $\bar{\beta} = 1$, $\alpha = 6$ and $\beta = 15$

x	ν_1=0.9 ν_2=1.1	ν_1=1.1 ν_2=0.9	ν_1=1.5 ν_2=0.6	ν_1=2 ν_2=0.4	ν_1=5 ν_2=0.2	ν_1=7 ν_2=0.15	ν_1=10 ν_2=0.1
1	−6.78%	−5.94%	−5.11%	−4.56%	−3.24%	−3.03%	−2.89%
2	−5.73%	−4.90%	−4.07%	−3.52%	−2.21%	−2.00%	−1.86%
3	−4.07%	−3.24%	−2.40%	−1.85%	−0.55%	−0.33%	−0.20%
5	−1.84%	−1.02%	−0.19%	0.36%	1.64%	1.85%	1.99%
7	−0.56%	0.24%	1.06%	1.60%	2.85%	3.05%	3.19%
10	0.29%	1.06%	1.85%	2.37%	3.55%	3.75%	3.88%
15	0.92%	1.62%	2.36%	2.84%	3.90%	4.08%	4.20%
20	3.30%	3.94%	4.63%	5.09%	6.04%	6.20%	6.32%
30	1.65%	2.14%	2.69%	3.06%	3.74%	3.86%	3.95%
50	−1.26%	−1.06%	−0.76%	−0.55%	−0.38%	−0.34%	−0.29%

5 Conclusion

The results of computations presented in Tables 1, 3, 5, 7 and 9 indicate that both approximations yield very small relative errors. Although the existence of exponential moments of the distributions of the premium and claim sizes is not required to construct the approximations, it is easily seen that the relative errors are smaller when those distributions do not have heavy tails.

The construction of the five-moment approximation is based on the classical approach, where we only require that the first five moments of the processes coincide without any additional assumptions. The numerical illustrations show that this approach gives very good results, but unfortunately, the conditions that are necessary for its construction are too restrictive. A definite advantage of the three-moment approximation is that the corresponding conditions are much less restrictive, but the construction of this approximation requires two additional conditions, which are also based on some heuristic assumptions. Nevertheless, there is no reason to assert that one of the approximations is more accurate than the other one: the corresponding relative errors vary for different values of the initial surplus.

The relative errors for some three-moment approximations using more general conditions (39) are given in Tables 2, 4, 6, 8 and 10. The analysis of the errors shows that the accuracy of those approximations is more or less the same: choosing ν_1 and ν_2 that yield smaller errors for some values of the initial surplus leads to larger errors for other values. Nonetheless, for some other values of ν_1 and ν_2, the corresponding approximations can give not so good results. Therefore, the choice of coefficients ν_1 and ν_2 should be controlled using other methods that enable us to approximate the ruin probability.

Finally, note that although the numerical examples considered above are not sufficient to make conclusions about the accuracy of the suggested approximations in general and it would be highly desirable to have a tool to control the accuracy in terms of parameter values, those illustrations enable us to outline some general tendencies.

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