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On infinite divisibility of a class of two-dimensional vectors in the second Wiener chaos

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Abstract Infinite divisibility of a class of two-dimensional vectors with components in the second Wiener chaos is studied. Necessary and sufficient conditions for infinite divisibility are presented as well as more easily verifiable sufficient conditions. The case where both components consist of a sum of two Gaussian squares is treated in more depth, and it is conjectured that such vectors are infinitely divisible.

Keywords Sums of Gaussian squares, infinite divisibility, second Wiener chaos
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1 Introduction

Paul Lévy raised the question of infinite divisibility of Gaussian squares, that is, for a centered Gaussian vector $(X_1, \dots, X_n)$ when can $(X_1^2, \dots, X_n^2)$ be written as a sum of m independent identically distributed random vectors for any $m \in \mathbb{N}$? (See [11].) Several authors have studied this problem. We refer to [4–8, 13, 14] and references therein. These works include several novel approaches and give a great understanding of when Gaussian squares are infinitely divisible. In this paper we will provide a characterization of infinite divisibility of sums of Gaussian squares which to the best

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of our knowledge has not been studied in the literature except in special cases. This problem is highly motivated by the fact that sums of Gaussian squares are usual limits in many limit theorems in the presence of either long range dependence, see [2] or [17], or degenerate U-statistics, see [9]. In the following we will go in more details.

Let Y be a random variable in the second (Gaussian) Wiener chaos, that is, the closed linear span in L^2 of $\{W(h)^2 - 1 : h \in H, \|h\| = 1\}$ for a real separable Hilbert space H and an isonormal Gaussian process W. For convenience, we assume H is infinite-dimensional. Then there exists a sequence of independent standard Gaussian variables (ξ_i) and a sequence of real numbers (α_i) such that

$$Y \stackrel{d}{=} \sum_{i=1}^{\infty} \alpha_i (\xi_i^2 - 1),$$

where the sum converges in L^2 (see for example [9, Theorem 6.1]). Since the ξ_i's are independent, $(\xi_1^2, \dots, \xi_d^2)$ is infinitely divisible for any $d \geq 1$ and therefore, Y is infinitely divisible. Such a sum of Gaussian squares appears as the limit of U-statistics in the degenerate case (see [9, Corollary 11.5]). In this case the α_i are certain binomial coefficients times the eigenvalues of operators associated to the U-statistics. But even though any random variable in the second Wiener chaos is infinitely divisible, it is well known (cf. Theorem 1 below) that a vector with dimension greater than two and components in the second Wiener chaos needs not be infinite divisibility. In between the case of a random variable in the second Wiener chaos and the vector case with dimension greater than two, there is the open question of infinite divisibility of a two-dimensional vector with components in the second Wiener chaos. Let $(X_1, \dots, X_{n_1+n_2})$ be a zero mean Gaussian vector for $n_1, n_2 \in \mathbb{N}$. The fact that any two-dimensional vector in the second Wiener chaos is infinitely divisible is equivalent to

$$(d_1 X_1^2 + \dots + d_{n_1} X_{n_1}^2, d_{n_1} X_{n_1+1}^2 + \dots + d_{n_1+n_2} X_{n_1+n_2}^2) \tag{1}$$

being infinitely divisible for any $d_1, \dots, d_{n_1+n_2} = \pm 1$, any covariance structure of $(X_1, \dots, X_{n_1+n_2})$, and any $n_1, n_2 \in \mathbb{N}$ (something what follows by the definition of the second Wiener chaos).

The following theorem, which is due to Griffiths [8] and Bapat [1], is an important first result related to infinite divisibility in the second Wiener chaos. We refer to [12, Theorem 13.2.1 and Lemma 14.9.4] for a proof.

Theorem 1 (Griffiths and Bapat). *Let $(X_1, \dots, X_n)$ be a zero mean Gaussian vector with a positive definite covariance matrix Σ. Then $(X_1^2, \dots, X_n^2)$ is infinitely divisible if and only if there exists an $n \times n$ matrix U of the form* $\mathrm{diag}(\pm 1, \dots, \pm 1)$ *such that $U^t \Sigma^{-1} U$ has nonpositive off-diagonal elements.*

This theorem resolved the question of infinite divisibility of Gaussian squares. For $n \geq 3$ there is an $n \times n$ positive definite matrix Σ for which there does not exist an $n \times n$ matrix U of the form $\mathrm{diag}(\pm 1, \dots, \pm 1)$ such that $U^t \Sigma^{-1} U$ has nonpositive off-diagonal elements. Consequently, there are zero mean Gaussian vectors $(X_1, \dots, X_n)$ such that $(X_1^2, \dots, X_n^2)$ is not infinite divisible whenever $n \geq 3$.

Eisenbaum [3] and Eisenbaum and Kaspi [5] found a connection between the condition of Griffiths and Bapat and the Green function of a Markov process. In particular, a Gaussian process has infinite divisible squares if and only if its covariance function (up to a constant function) can be associated with the Green function of a strongly symmetric transient Borel right Markov process.

When discussing the infinite divisibility of the Wishart distribution, [16] showed the following result.

Theorem 2 (Shanbhag)**.** *For any $n \in \mathbb{N}$ and any zero mean Gaussian vector $(X_1, \dots, X_n)$, the two-dimensional random vector $(X_1^2, X_2^2 + \dots + X_n^2)$ is infinitely divisible.*

Furthermore, it was found that infinite divisibility of any bivariate marginal of a centered Wishart distribution can be reduced to infinite divisibility of (X_1X_2, X_3X_4). By the polarization identity,

$$(X_1X_2, X_3X_4) = \tfrac{1}{4}((X_1 + X_2)^2 - (X_1 - X_2)^2, (X_3 + X_4)^2 - (X_3 - X_4)^2).$$

Consequently, infinite divisibility of any bivariate marginals of a centered Wishart distribution is again related to the question of infinite divisibility of a two-dimensional vector from the second Wiener chaos.

Key question and contributions. The main question of the paper is: Given a zero mean Gaussian vector $(X_1, \dots, X_{n_1+n_2})$

$$\textit{when is } \ (X_1^2 + \dots + X_{n_1}^2, X_{n_1+1}^2 + \dots + X_{n_1+n_2}^2) \ \textit{ infinitely divisible?} \qquad (2)$$

The vector in (2) is (1) in the case $d_1 = \dots = d_{n_1+n_2} = 1$. In Theorem 3 we give a necessary and sufficient condition for infinite divisibility of the random vector in (2); namely that (7) is satisfied for all $k, l \in \mathbb{N}_0$. Furthermore, in Theorem 4 we give sufficient conditions for infinite divisibility of (2) in the case $n_1 = n_2 = 2$, We conjecture that (2) is always infinitely divisible in the case $n_1 = n_2 = 2$, and in fact in Theorem 5 we show that our necessary and sufficient condition (7) is always satisfied for all $k, l \in \mathbb{N}_0$ with $k + l \le 7$. The general case of infinite divisibility of (1), where $d_i = -1$ for at least one i, seems to require new ideas going beyond this paper.

Note that if $(X_1^2, \dots, X_{n_1+n_2}^2)$ is infinitely divisible then so is the vector in (2). Thus, under the conditions in Theorem 1, (2) is answered in the affirmative. However, we shall see that there are cases where the vector in (2) is infinitely divisible even though $(X_1^2, \dots, X_{n_1+n_2}^2)$ is not, see the comment after Theorem 4. Besides, based on our results we can give a short new proof of Shanbhag's Theorem 2, see just below Theorem 3.

The paper is structured as follows. In Section 2 we introduce the required notation to state our results. The main results without proofs are presented in Section 3. Section 4 contains two examples and a small numerical discussion. We end with Section 5 where the proofs are given.

2 Notation

In the following two subsections we will introduce the notation used for our main results on infinite divisibility of two-dimensional vectors in the second Wiener chaos (2).

2.1 *The general case $n_1, n_2 \in \mathbb{N}$*

Let $n_1, n_2 \in \mathbb{N}$ and consider a zero mean Gaussian vector $(X_1, \ldots, X_{n_1+n_2})$ with a positive definite covariance matrix Σ. For $a > 0$, let $Q = I - (I + a\Sigma)^{-1}$ and write

$$Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} \tag{3}$$

where Q_{11} is an $n_1 \times n_1$ matrix, Q_{22} is an $n_2 \times n_2$ matrix, and $Q_{12} = Q_{21}^t$ (where Q_{21}^t is the transpose of Q_{21}) is an $n_1 \times n_2$ matrix. Note that if λ is an eigenvalue of Σ, $\frac{a\lambda}{1+a\lambda}$ is an eigenvalue of Q. Since Q is symmetric and has positive eigenvalues, it is positive definite.

The following definition is a natural extension to the present setup of the terminology used by [1].

Definition 1. Let $n_1, n_2 \in \mathbb{N}$. An $(n_1 + n_2) \times (n_1 + n_2)$ orthogonal matrix U is said to be an (n_1, n_2)-signature matrix if

$$U = \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix}$$

where U_1 is an $n_1 \times n_1$ matrix and U_2 is an $n_2 \times n_2$ matrix, both orthogonal, and for 0's of suitable dimensions.

2.2 *The special case $n_1 = n_2 = 2$*

The following notation is only used in the special case where $n_1 = n_2 = 2$ in (2). Consider a 2×2 symmetric matrix A. Let v_1 and v_2 be the eigenvectors of A, and λ_1 and λ_2 be the corresponding eigenvalues. We say that v_i is associated with the largest eigenvalue if $\lambda_i \geq \lambda_j$ for $j = 1, 2$. Furthermore, whenever A is a multiple of the identity matrix, we fix $(1, 0)$ to be the eigenvector associated with the largest eigenvalue.

Let W be a $(2, 2)$-signature matrix such that

$$W^t Q W = \begin{pmatrix} W_1^t Q_{11} W_1 & W_1^t Q_{12} W_2 \\ W_2^t Q_{21} W_1 & W_2^t Q_{22} W_2 \end{pmatrix} = \begin{pmatrix} q_{11} & 0 & q_{13} & q_{14} \\ 0 & q_{22} & q_{23} & q_{24} \\ q_{13} & q_{23} & q_{33} & 0 \\ q_{14} & q_{24} & 0 & q_{44} \end{pmatrix}, \tag{4}$$

where $q_{11} \geq q_{22} > 0$ and $q_{33} \geq q_{44} > 0$ which exist by Lemma 2. Note that q_{ij} is the (i, j)-th entry not of Q but of $W^t Q W$. Let (γ_1, γ_2) be the eigenvector of $W_1^t Q_{12} Q_{21} W_1$ associated with the largest eigenvalue. If $q_{11} = q_{22}$ or $q_{33} = q_{44}$, any orthogonal W_1 or W_2 gives the desired form. In this case, we may always choose W_1 or W_2 such that $\gamma_1 q_{13}(\gamma_1 q_{13} + \gamma_2 q_{23}) \geq 0$ (see the proof of Lemma 3, (ii) $\Rightarrow$ (iii)), and we fix this choice.

Write

$$\Sigma^{-1} = \begin{pmatrix} \Sigma^{11} & \Sigma^{12} \\ \Sigma^{21} & \Sigma^{22} \end{pmatrix}$$

where Σ^{ij} is a 2×2 matrix for $i, j = 1, 2$. Let W be a $(2, 2)$-signature matrix such that

$$W^t \Sigma^{-1} W = \begin{pmatrix} W_1^t \Sigma^{11} W_1 & W_1^t \Sigma^{12} W_2 \\ W_2^t \Sigma^{21} W_1 & W_2^t \Sigma^{22} W_2 \end{pmatrix} = \begin{pmatrix} \sigma_{11} & 0 & \sigma_{13} & \sigma_{14} \\ 0 & \sigma_{22} & \sigma_{23} & \sigma_{24} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} & 0 \\ \sigma_{14} & \sigma_{24} & 0 & \sigma_{44} \end{pmatrix} \tag{5}$$

where $\sigma_{11} \geq \sigma_{22} > 0$ and $\sigma_{33} \geq \sigma_{44} > 0$ which exist by Lemma 2. Note that σ_{ij} is the (i, j)-th entry not of Σ^{-1} but of $W^t \Sigma^{-1} W$. Let (ν_1, ν_2) be the eigenvector of $W_1^t \Sigma^{12} \Sigma^{21} W_1$ associated with the largest eigenvalue. If $\sigma_{11} = \sigma_{22}$ or $\sigma_{33} = \sigma_{44}$, any orthogonal W_1 or W_2 gives the desired form. In this case, we may choose W_1 or W_2 such that $\nu_2 \sigma_{24}(\nu_2 \sigma_{24} + \nu_1 \sigma_{14}) \geq 0$, and we fix this choice.

3 Main results

In this section we will first consider the general case $n_1, n_2 \in \mathbb{N}$ in Subsection 3.1, and thereafter, in Subsection 3.2, provide more specialized conditions for the case $n_1 = n_2 = 2$.

3.1 The general case

The following result gives a necessary and sufficient condition for infinite divisibility as stated in the Key question (2).

Theorem 3. *Let $n_1, n_2 \in \mathbb{N}$ and $(X_1, \ldots, X_{n_1+n_2})$ denote a zero mean Gaussian vector with a positive definite covariance matrix Σ, and let Q be defined in* (3)*. Then*

$$(X_1^2 + \cdots + X_{n_1}^2, X_{n_1+1}^2 + \cdots + X_{n_1+n_2}^2) \tag{6}$$

is infinitely divisible if and only if for all $k, m \in \mathbb{N}_0$ and for all $a > 0$ sufficiently large,

$$\begin{aligned} &\sum \text{trace } Q_{11}^{k_1} Q_{12} Q_{22}^{m_1} Q_{21} Q_{11}^{k_2} \cdots Q_{11}^{k_d} Q_{12} Q_{22}^{m_d} Q_{21} Q_{11}^{k_{d+1}} \\ &\quad + \sum \text{trace } Q_{22}^{m_1} Q_{21} Q_{11}^{k_1} Q_{12} Q_{22}^{m_2} \cdots Q_{22}^{m_{d-1}} Q_{21} Q_{11}^{k_d} Q_{12} Q_{22}^{m_{d+1}} \geq 0, \end{aligned} \tag{7}$$

where the first sum is over all $k_1, \ldots, k_{d+1}$ and $m_1, \ldots, m_d$ such that

$$k_1 + \cdots + k_{d+1} + d = k \quad \textit{and} \quad m_1 + \cdots + m_d + d = m,$$

and the second sum is over all $m_1, \ldots, m_{d+1}$ and $k_1, \ldots, k_d$ such that

$$m_1 + \cdots + m_{d+1} + d = m \quad \textit{and} \quad k_1 + \cdots + k_d + d = k.$$

Our proof of Theorem 3 relies on similar techniques used in the proof of Theorem 1, namely the series expansion of the Laplace transform together with an application of a result by Feller. By applying Theorem 3 we can give a new and simple proof of Shanbhag's result, Theorem 2, which states that $(X_1^2, X_2^2 + \cdots + X_{1+n_2}^2)$ is infinitely divisible.

Proof of Theorem 2. Consider Theorem 3 in the case $n_1 = 1$ and $n_2 \in \mathbb{N}$. Then Q_{11} is a positive number and $Q_{12}Q_{22}^m Q_{21}$ is a nonnegative number for any $m \in \mathbb{N}$. In particular, we have

$$\begin{aligned}\operatorname{trace} & Q_{11}^{k_1} Q_{12} Q_{22}^{m_1} Q_{21} \cdots Q_{12} Q_{22}^{m_d} Q_{21} Q_{11}^{k_{d+1}} \\ &= Q_{11}^{k_1} \cdots Q_{11}^{k_{d+1}} Q_{12} Q_{22}^{m_1} Q_{21} \cdots Q_{12} Q_{22}^{m_d} Q_{21} \geq 0\end{aligned}$$

for any $k_1, \ldots, k_{d+1}, m_1, \ldots, m_d \in \mathbb{N}_0$. Consequently, the first sum in (7) is a sum of nonnegative numbers. A similar argument gives that the other sum is nonnegative, too. We thus conclude that $(X_1^2, X_2^2 + \cdots + X_{1+n_2}^2)$ is infinite divisible by Theorem 3. □

Despite the fact that Theorem 3 implies Shanbhag's result, in general it can be difficult to check condition (7). The following proposition yields a sufficient condition for infinite divisibility which might be more easy to check in some cases.

Proposition 1. *Let $(X_1, \ldots, X_{n_1+n_2})$ be a zero mean Gaussian vector with a positive definite covariance matrix Σ. Then*

$$(X_1^2 + \cdots + X_{n_1}^2, X_{n_1+1}^2 + \cdots + X_{n_1+n_2}^2) \tag{8}$$

is infinitely divisible if there exists an (n_1, n_2)-signature matrix U (cf. Definition 1) such that $U^t \Sigma^{-1} U$ has nonpositive off-diagonal elements.

Proof. Write $X = (X_1, \ldots, X_{n_1})$ and $Y = (X_{n_1+1}, \ldots, X_{n_1+n_2})$, and note that

$$\begin{aligned}(X_1^2 + \cdots + X_{n_1}^2, X_{n_1+1}^2 + \cdots + X_{n_1+n_2}^2) &= (\|X\|^2, \|Y\|^2) \\ &= (\|U_1 X\|^2, \|U_2 Y\|^2)\end{aligned} \tag{9}$$

for any $n_1 \times n_1$ orthogonal matrix U_1 and $n_2 \times n_2$ orthogonal matrix U_2. Consequently, any property of the distribution of (8) is invariant under transformations of the form

$$\begin{pmatrix} U_1^t & 0 \\ 0 & U_2^t \end{pmatrix} \Sigma \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix}$$

of the covariance matrix Σ. Therefore, when there exists an (n_1, n_2)-signature matrix U such that $U^t \Sigma^{-1} U$ has nonpositive off-diagonal elements, Theorem 1 ensures infinite divisibility of (9). □

In the following subsection we will specialize our setting to obtain conditions which are easier to apply.

3.2 The $n_1 = n_2 = 2$ case

The following theorem provides *easy to check* conditions for infinite divisibility of vectors of the form $(X_1^2 + X_2^2, X_3^2 + X_4^2)$. By "easy to check" we mean that the conditions (i) and (ii) of Theorem 4 can be explicitly calculated through a finite number of standard matrix operations, what is opposite to the general condition (7).

Theorem 4. *Let (X_1, X_2, X_3, X_4) denote a zero mean Gaussian vector with a positive definite covariance matrix Σ, and let q_{ij}, γ_i, σ_{ij} and ν_i be defined as in Subsection 2.2. Then the vector $(X_1^2 + X_2^2, X_3^2 + X_4^2)$ is infinitely divisible if at least one of the following two conditions is satisfied:*

(i) The inequality $\gamma_1 q_{13}(\gamma_1 q_{13} + \gamma_2 q_{23}) \geq 0$ *holds.*

(ii) The inequality $\nu_2\sigma_{24}(\nu_2\sigma_{24} + \nu_1\sigma_{14}) \geq 0$ *holds.*

Example 2, from the next section, shows that (4) of Theorem 4 holds in some cases where $(X_1^2, X_2^2, X_3^2, X_4^2)$ is not infinitely divisible, and hence it cannot be deduced from a direct application of Griffiths and Bapat's result (Theorem 1). The following result gives some insight to the question whether infinite divisibility of the vector $(X_1^2 + X_2^2, X_3^2 + X_4^2)$ holds in general, which still remains an open problem. The result, in particular, says that the necessary and sufficient condition (7) for infinitely divisibility is always satisfied whenever k and m are not too large.

Theorem 5. *Consider a zero mean Gaussian vector* (X_1, X_2, X_3, X_4) *with a positive definite covariance matrix* Σ. *Recall that* $(X_1^2+X_2^2, X_3^2+X_4^2)$ *is infinitely divisible if and only if condition (7) is satisfied for all* $k, m \in \mathbb{N}_0$. *We have that (7) is satisfied for any* $k, m \in \mathbb{N}_0$ *for which at least one of the following inequalities holds (i):* $k \leq 2$, *(ii):* $m \leq 2$, *or (iii):* $k + m \leq 7$.

Remark 1. In the proof of Theorem 4 we show that the inequality $\gamma_1 q_{13}(\gamma_1 q_{13} + \gamma_2 q_{23}) \geq 0$ holds if and only if for all integers $d \in \mathbb{N}_0$, $k_1, \ldots, k_{d+1} \in \mathbb{N}_0$ and $m_1, \ldots, m_d \in \mathbb{N}_0$ we have

$$\operatorname{trace} Q_{11}^{k_1} Q_{12} Q_{22}^{m_1} Q_{21} Q_{11}^{k_1} \cdots Q_{11}^{k_d} Q_{12} Q_{22}^{m_d} Q_{21} Q_{11}^{k_{d+1}} \geq 0.$$

Hence, when $\gamma_1 q_{13}(\gamma_1 q_{13} + \gamma_2 q_{23}) < 0$, we know that there are $k, m \in \mathbb{N}_0$ such that (7) with $n_1 = n_2 = 2$ contains negative terms. It is still an open problem to decide if the positive terms will always compensate for the negative terms such that $(X_1^2 + X_2^2, X_3^2 + X_4^2)$ is always infinitely divisible, or this is only the case when k and m is small, e.g. $k + m \leq 7$, cf. Theorem 5.

4 Examples and numerics

We begin this section by presenting two examples treating the inequalities in Theorem 3.2 in special cases. Then we calculate the sums in (7) numerically with $n_1 = n_2 = 2$ for a specific value of Q for k and m less than 60.

Example 1. Fix $a > 0$ and assume that the matrix $Q = I - (I + a\Sigma)^{-1}$ is of the form

$$Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} = \begin{pmatrix} q_1 & 0 & \varepsilon & \varepsilon \\ 0 & q_2 & \varepsilon & -\delta \\ \varepsilon & \varepsilon & q_3 & 0 \\ \varepsilon & -\delta & 0 & q_4 \end{pmatrix}$$

where $\delta, \varepsilon > 0$, $q_1 > q_2 > 0$, and $q_3 > q_4 > 0$. Let $\gamma = (\gamma_1, \gamma_2)$ be the eigenvector of

$$Q_{12}Q_{21} = \begin{pmatrix} 2\varepsilon^2 & \varepsilon(\varepsilon - \delta) \\ \varepsilon(\varepsilon - \delta) & \varepsilon^2 + \delta^2 \end{pmatrix}$$

associated with the largest eigenvalue λ_1. We will argue that the inequality in Theorem 3.2(i), which reads

$$\gamma_1(\gamma_1 + \gamma_2) \geq 0 \tag{10}$$

in this case, holds if and only if $\delta \leq \varepsilon$, and hence $(X_1^2 + X_2^2, X_3^2 + X_4^2)$ is infinitely divisible whenever $\delta \leq \varepsilon$, cf. Theorem 3.2(i).

Since $-\gamma$ also is an eigenvector of $Q_{12}Q_{21}$ associated with the largest eigenvalue, we assume $\gamma_1 \geq 0$ without loss of generality. Assume $\delta \leq \varepsilon$. If $\delta = \varepsilon$, $v = (1, 0)$ and the inequality in (10) holds. Assume $\delta < \varepsilon$. Since λ_1 is the largest eigenvalue,

$$\lambda_1 = \sup_{|\gamma|=1} \gamma^t Q_{12}Q_{21}\gamma \geq 2\varepsilon^2$$

which implies that

$$2\varepsilon^2 - \lambda_1 \leq 0 \leq \varepsilon(\varepsilon - \delta).$$

Since γ is an eigenvector, $(Q - \lambda_1)\gamma = 0$ and we therefore have that

$$0 = (2\varepsilon^2 - \lambda_1)\gamma_1 + \varepsilon(\varepsilon - \delta)\gamma_2 \leq \varepsilon(\varepsilon - \delta)(\gamma_1 + \gamma_2).$$

We conclude that (10) holds.

On the other hand, assume $\delta > \varepsilon$ and $\gamma_1 \geq 0$. Since λ_1 is the largest eigenvalue, $\lambda_1 \geq \delta^2 + \varepsilon^2 > \delta\varepsilon + \varepsilon^2$ and therefore

$$(\lambda_1 - 2\varepsilon^2) > \varepsilon(\delta - \varepsilon).$$

Note that γ_1 cannot be zero since the off-diagonal element in $Q_{12}Q_{21}$ is nonzero. We conclude that

$$0 = (\lambda_1 - 2\varepsilon^2)\gamma_1 + \varepsilon(\delta - \varepsilon)\gamma_2 > \varepsilon(\delta - \varepsilon)(\gamma_1 + \gamma_2).$$

This implies that (10) does not hold.

Example 2. Assume Σ^{-1} is of the form

$$\Sigma^{-1} = \begin{pmatrix} \Sigma^{11} & \Sigma^{12} \\ \Sigma^{21} & \Sigma^{22} \end{pmatrix} = \begin{pmatrix} \sigma_1 & 0 & -\delta & \varepsilon \\ 0 & \sigma_2 & \varepsilon & \varepsilon \\ -\delta & \varepsilon & \sigma_3 & 0 \\ \varepsilon & \varepsilon & 0 & \sigma_4 \end{pmatrix} \tag{11}$$

where $\sigma_1 > \sigma_2 > 0$, $\sigma_3 > \sigma_4 > 0$, and $\delta, \varepsilon > 0$. Let $\nu = (\nu_1, \nu_2)$ be the eigenvector of $\Sigma^{12}\Sigma^{21}$ associated with the largest eigenvalue. We will argue that the inequality in Theorem 4(ii) holds if and only if $\delta \leq \varepsilon$. Then the same theorem implies that $(X_1^2 + X_2^2, X_3^2 + X_4^2)$ is infinitely divisible whenever $\delta \leq \varepsilon$. On the other hand, Theorem 1 implies that $(X_1^2, X_2^2, X_3^2, X_4^2)$ is never infinite divisible under (11) since there does not exist a matrix D of the form $\mathrm{diag}(\pm 1, \pm 1, \pm 1, \pm 1)$ such that $D\Sigma^{-1}D$ has nonpositive off-diagonal elements. Indeed, for any two matrices D_1 and D_2 of the form $\mathrm{diag}(\pm 1, \pm 1)$, $D_1\Sigma^{12}D_2$ has either three negative and one positive or one negative and three positive entries.

In the following we will see that the inequality in Theorem 4(ii), which reads $\nu_2(\nu_1 + \nu_2) \geq 0$ in this setting, holds if and only if $\delta \leq \varepsilon$. Let

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and Q_{12} be given as in Example 1. Then $P\Sigma^{12}P = Q_{12}$, implying that (ν_2, ν_1) is the eigenvector associated with the largest eigenvalue of $Q_{12}Q_{21}$. We have argued in Example 1 that $\nu_2(\nu_1 + \nu_2) \geq 0$ holds if and only if $\delta \leq \varepsilon$, what is the desired conclusion.

Now we investigate infinite divisibility of $(X_1^2+X_2^2, X_3^2+X_4^2)$ numerically. More specifically, we consider the sums in (7) with $n_1 = n_2 = 2$ for a specific choice of a positive definite matrix and different values of k and m. We will scale Q to have its largest eigenvalue equal to one to avoid getting too close to zero. Due to Theorem 4(i) the case where $\gamma_1 q_{13}(\gamma_1 q_{13} + \gamma_2 q_{23}) < 0$ (in the notation of Theorem 4) is the only case where the question on the infinite divisibility of $(X_1^2 + X_2^2, X_3^2 + X_4^2)$ is open.

Let

$$Q = \frac{1}{\lambda}\begin{pmatrix} 0.8 & 0 & 0.01 & 0.01 \\ 0 & 0.3 & 0.01 & -0.2 \\ 0.01 & 0.01 & 0.8 & 0 \\ 0.01 & -0.2 & 0 & 0.3 \end{pmatrix}$$

where $\lambda > 0$ is chosen such that Q has its largest eigenvalue equal to 1. Note that by Example 1, $\gamma_1 q_{13}(\gamma_1 q_{13}+\gamma_2 q_{23}) < 0$. In the below Figure 1 the logarithm of the sums in (7) for k and m between 0 and 60 is plotted. It is seen that the logarithm seems stable and hence the sums in (7) seem to remain positive in this case. A similar analysis have been done for other positive definite matrices, and we have not encountered any $k, m \in \mathbb{N}_0$ such that (7) is negative. This, together with Theorem 4(i), leads us to the conjecture that $(X_1^2+X_2^2, X_3^2+X_4^2)$ is infinitely divisible for any zero mean Gaussian vector (X_1, X_2, X_3, X_4).

5 Proofs

In this section we will prove Theorems 3, 4, and 5.

5.1 Proof of Theorem 3

The following lemma will be useful in the proof of Theorem 3. A proof can be found in [12, Lemma 13.2.2].

Lemma 1. *Let $\psi : \mathbb{R}_+^n \to (0, \infty)$ be a continuous function. Suppose that, for all $a > 0$ sufficiently large, $\log \psi(a(1-s_1), \ldots, a(1-s_n))$ has a power series expansion for $s = (s_1, \ldots, s_n) \in [0, 1]^n$ around $s = 0$ with all its coefficients nonnegative, except for the constant term. Then ψ is the Laplace transform of an infinitely divisible random variable in $\mathbb{R}_+^n$.*

We now give the proof of Theorem 3.

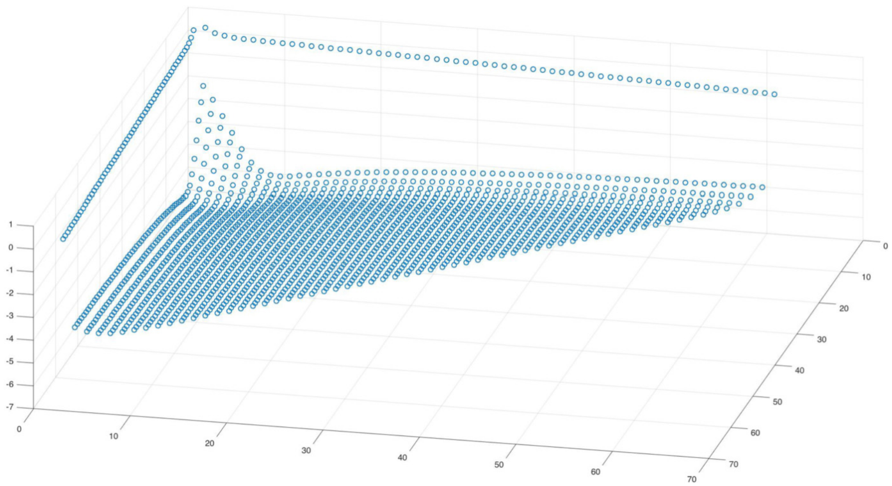

Fig. 1. The logarithm of the sums in (7) for k and m between 0 and 60

Proof of Theorem 3. By [12, Lemma 5.2.1],

$$\begin{aligned} &P(s_1, s_2) \\ &\quad = \mathbb{E}\exp\{-\tfrac{1}{2}a((1-s_1)(X_1^2+\cdots+X_{n_1}^2)+(1-s_2)(X_{n_1+1}^2+\cdots+X_{n_2}^2))\} \\ &\quad = \frac{1}{|I+\Sigma a(I-S)|^{1/2}}, \end{aligned}$$

where S is the $(n_1+n_2)\times(n_1+n_2)$ diagonal matrix with s_1 on the first n_1 diagonal entries and s_2 on the remaining n_2 diagonal entries. Recall that $Q = I-(I+a\Sigma)^{-1}$. Then

$$\begin{aligned} P(s_1,s_2)^2 &= |I+a\Sigma - a\Sigma S|^{-1} \\ &= |(I-Q)^{-1} - ((I-Q)^{-1}-I)S|^{-1} \\ &= |I-Q||I-QS|^{-1}, \end{aligned}$$

from which it follows that

$$\begin{aligned} 2\log P(s_1,s_2) &= \log|I-Q| - \log|I-QS| \\ &= \log|I-Q| + \sum_{n=1}^{\infty}\frac{\mathrm{trace}\{(QS)^n\}}{n}, \end{aligned} \tag{12}$$

where the last equality follows from [12, p. 562]. Now assume that the vector $(X_1^2+\cdots+X_{n_1}^2, X_{n_1+1}^2+\cdots+X_{n_1+n_2}^2)$ is infinitely divisible, and write

$$(X_1^2+\cdots+X_{n_1}^2, X_{n_1+1}^2+\cdots+X_{n_1+n_2}^2) \overset{d}{=} Y_1^n+\cdots+Y_n^n$$

where $Y_1^n, \dots Y_n^n$ are 2-dimensional independent identically distributed stochastic vectors. Let Y_{ij}^n be the j-th component of Y_i^n and note that $Y_{ij}^n \geq 0$ a.s. for all i, j, n. Then

$$P(s_1, s_2)^{1/n} = \mathbb{E} \exp\{-\tfrac{1}{2}a((1-s_1)Y_{11}^n + (1-s_2)Y_{12}^n)\}.$$

From the expression

$$\exp\{-\tfrac{1}{2}a((1-s_j)Y_{1j}^n)\} = \exp\{-\tfrac{1}{2}aY_{1j}^n\} \sum_{k=0}^{\infty} \frac{(s_j a Y_{1j}^n)^k}{2^k k!},$$

it follows that $P^{1/n}(s_1, s_2)$ has a power series expansion with all coefficient nonnegative. We have that

$$\log P(s_1, s_2) = \lim_{n\to\infty} (n(P^{1/n}(s_1, s_2) - 1)). \tag{13}$$

Note that $(s_1, s_2) \mapsto n(P^{1/n}(s_1, s_2) - 1)$ and all its derivatives converge uniformly on $[0, 1) \times [0, 1)$ by a Weierstrass M-test (see for example [15, Theorem 7.10]). Consequently, we may use [15, Theorem 7.17] to conclude that

$$\frac{\partial^{\alpha+\beta}}{\partial s_1^{\alpha} \partial s_2^{\beta}} \lim_{n\to\infty} (n(P^{1/n}(s_1, s_2) - 1)) = \lim_{n\to\infty} \frac{\partial^{\alpha+\beta}}{\partial s_1^{\alpha} \partial s_2^{\beta}} (n(P^{1/n}(s_1, s_2) - 1))$$

for any $\alpha, \beta \in \mathbb{N}_0$. Thus, the fact that all the terms in the power series expansion of $P^{1/n}(s_1, s_2)$ are nonnegative implies that all the terms in the power series representation of $\log P(s_1, s_2)$ except the constant term are nonnegative by (13). By (12) we conclude that any coefficient in front of $s_1^k s_2^m$ in trace$\{(QS)^{k+m}\}$ has to be nonnegative for all $k, m \in \mathbb{N}$ and $a > 0$. Expanding the trace then gives that this is equivalent to nonnegativity of the sum in (7) for all $k, m \in \mathbb{N}_0$.

On the other hand, if the sum in (7) is nonnegative for all $k, m \in \mathbb{N}_0$ and $a > 0$ sufficiently large, (12) and Lemma 1 imply that

$$(X_1^2 + \cdots + X_{n_1}^2, X_{n_1+1}^2 + \cdots + X_{n_1+n_2}^2)$$

is infinitely divisible. □

5.2 *Proof of Theorem 4*

We start this section with two lemmas on linear algebra. Lemma 3 will be very useful in the proofs that make up the rest of this section.

Lemma 2. *Let A be an $n \times n$ positive definite matrix. Let $n_1, n_2 \in \mathbb{N}$ be such that $n_1 + n_2 = n$ and write*

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

where A_{11} is an $n_1 \times n_1$ matrix, A_{22} is an $n_2 \times n_2$ matrix, and $A_{12} = A_{21}^t$ is an $n_1 \times n_2$ matrix. Then there exists an (n_1, n_2)-signature matrix W such that $W^t AW$ has the form

$$\begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{pmatrix}$$

where $\tilde{A}_{11} = \text{diag}(a_1, \dots, a_{n_1})$ and $\tilde{A}_{22} = \text{diag}(a_{n_1+1}, \dots, a_{n_1+n_2})$ with $a_i > 0$ for $i = 1, \dots, n_1+n_2$, and where $\tilde{A}_{12} = \tilde{A}_{21}^t$. Furthermore, we may choose W such that $a_1 \geq a_2 \geq \cdots \geq a_{n_1}$ and $a_{n_1+1} \geq a_{n_1+2} \geq \cdots \geq a_{n_1+n_2}$.

Proof. Since A is positive definite, A_{11} and A_{22} are positive definite. Consequently, by the spectral theorem (see for example [10, Corollary 6.4.7]), there exist an $n_1 \times n_1$ matrix W_1 and an $n_2 \times n_2$ matrix W_2, both orthogonal, such that $W_1^t A_{11} W_1$ and $W_2^t A_{22} W_2$ are diagonal with positive diagonal entries. Since permutation matrices are orthogonal matrices, we may assume the diagonal is ordered by size in both $W_1^t A_{11} W_1$ and $W_2^t A_{22} W_2$. Consequently, letting

$$W = \begin{pmatrix} W_1 & 0 \\ 0 & W_2 \end{pmatrix},$$

implies that $W^t AW$ has the right form. □

For a fixed eigenvector v_i we will say that the system $Av_i = \lambda_i v_i$ is the system of eigenequations. The k-th equation in this system will be called the k-th eigenequation associated with v_i. Let A be a 4×4 positive definite matrix, and let W be a (2, 2)-signature such that

$$W^t AW = \begin{pmatrix} W_1^t A_{11} W_1 & W_1^t A_{12} W_2 \\ W_2^t A_{21} W_1 & W_2^t A_{22} W_2 \end{pmatrix} = \begin{pmatrix} a_{11} & 0 & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ a_{13} & a_{23} & a_{33} & 0 \\ a_{14} & a_{24} & 0 & a_{44} \end{pmatrix},$$

where $a_{11} \geq a_{22} > 0$ and $a_{33} \geq a_{44} > 0$ which exist by Lemma 2. Note that a_{ij} is the (i, j)-th entry not of A but of $W^t AW$. Let $v_1 = (v_{11}, v_{21})$ be the eigenvector associated with the largest eigenvalue of $W_1^t A_{12} A_{21} W_1$. If $a_{11} = a_{22}$ or $a_{33} = a_{44}$, any orthogonal W_1 or W_2 give the desired form. In this case, we may choose W_1 or W_2 such that $v_{11}a_{13}(v_{11}a_{13} + v_{21}a_{23}) \geq 0$, and we fix this choice. Then the lemma below will play a central role in the proofs of the previously stated results.

Lemma 3. *In the notation above, the following are equivalent.*

(i) There exists a (2, 2)-signature matrix U such that $U^t AU$ has all entries non-negative.

(ii) For any $d \in \mathbb{N}$ and $k_1, \dots, k_{d+1}, m_1, \dots m_d \in \mathbb{N}_0$,

$$\text{trace}\, A_{11}^{k_1} A_{12} A_{22}^{m_1} A_{21} A_{11}^{k_2} \cdots A_{11}^{k_d} A_{12} A_{22}^{m_d} A_{21} A_{11}^{k_{d+1}} \geq 0.$$

(iii) The inequality $v_{11}a_{13}(v_{11}a_{13} + v_{21}a_{23}) \geq 0$ holds.

Proof. (i) $\Rightarrow$ (ii). Let

$$U = \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix}$$

be such that $B_{ij} = U_i^t A_{ij} U_j$ has nonnegative entries for $i, j = 1, 2$. Then

$$\begin{aligned} &\text{trace} A_{11}^{k_0} A_{12} A_{22}^{m_1} A_{21} A_{11}^{k_1} \cdots A_{11}^{k_{d-1}} A_{12} A_{22}^{m_d} A_{21} A_{11}^{k_d} \\ &\quad = \text{trace } B_{11}^{k_0} B_{12} B_{22}^{m_1} B_{21} B_{11}^{k_1} \cdots B_{11}^{k_{d-1}} B_{12} B_{22}^{m_d} B_{21} B_{11}^{k_d}. \end{aligned}$$

This trace is nonnegative since all matrices in the product only contain nonnegative entries.

(ii) $\Rightarrow$ (iii). By the spectral theorem, we may write $W_1^t A_{12} A_{21} W_1 = V \Lambda V^t$ where V is a 2×2 orthogonal matrix and $\Lambda = \text{diag}(\lambda_1, \lambda_2)$ with $\lambda_1 \geq \lambda_2 \geq 0$. Note that v_1, the eigenvector associated with the largest eigenvalue of $W_1^t A_{12} A_{21} W_1$, is the first column of V. If $\lambda_1 = \lambda_2$, $v_1 = (1, 0)$ and the inequality holds. If $a_{11} = a_{22}$ or $a_{33} = a_{44}$, $W_1^t A_{11} W_1 = A_{11}$ or $W_2^t A_{22} W_2 = A_{22}$, and choosing W_1 or W_2 such that $a_{23} = 0$ then ensures that the inequality in (iii) holds.

Assume now that $\lambda_1 > \lambda_2$, $a_{11} > a_{22}$, and $a_{33} > a_{44}$. It follows by assumption that

$$\begin{aligned} 0 &\leq \frac{1}{a_{11}^k} \frac{1}{a_{33}^k} \frac{1}{\lambda_1^k} \text{trace } A_{11}^k A_{12} A_{22}^k A_{21} (A_{12} A_{21})^k \\ &= \text{trace} \begin{pmatrix} 1 & 0 \\ 0 & (\frac{a_{22}}{a_{11}})^k \end{pmatrix} W_1^t A_{12} W_2 \begin{pmatrix} 1 & 0 \\ 0 & (\frac{a_{44}}{a_{33}})^k \end{pmatrix} W_2^t A_{21} W_1 V \begin{pmatrix} 1 & 0 \\ 0 & (\frac{\lambda_1}{\lambda_2})^k \end{pmatrix} V^t \\ &\to \text{trace} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} W_1^t A_{12} W_2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} W_2^t A_{21} W_1 V \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} V^t \end{aligned}$$

as $k \to \infty$. This gives the inequality in (iii) since

$$\begin{aligned} &\text{trace} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} W_1^t A_{12} W_2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} W_2^t A_{21} W_1 V \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} V^t \\ &\quad = v_{11} a_{13} (v_{11} a_{13} + v_{21} a_{23}). \end{aligned}$$

(iii) $\Rightarrow$ (i). To ease the notation and without loss of generality assume that $W = I$. We are then pursuing two 2×2 orthogonal matrices U_1 and U_2 such that $U_1^t A_{11} U_1$, $U_1^t A_{12} U_2$, and $U_2^t A_{22} U_2$ all have nonnegative entries. Initially consider D_1 and D_2 of the form $\text{diag}(\pm 1, \pm 1)$. Then clearly, $D_1 A_{11} D_1 = A_{11}$ and $D_2 A_{22} D_2 = A_{22}$ since A_{11} and A_{22} are diagonal matrices. Next, note that it is possible to find D_1 and D_2 such that either $D_1 A_{12} D_2$ has all entries nonnegative or

$$D_1 A_{12} D_2 = \begin{pmatrix} a_{13} & a_{14} \\ a_{23} & -a_{24} \end{pmatrix} \tag{14}$$

where $a_{13}, a_{23}, a_{14}, a_{24} > 0$. Consequently, we will assume A_{12} is of the form in (14) since otherwise choosing $U_1 = D_1$ and $U_2 = D_2$ would be sufficient.

As one of two cases, assume $a_{13}a_{23} - a_{14}a_{24} \geq 0$, and define

$$U_2 = \begin{pmatrix} \alpha \frac{a_{14}a_{24}}{a_{23}} & \beta a_{23} \\ \alpha a_{14} & -\beta a_{24} \end{pmatrix}$$

where $\alpha, \beta > 0$ are chosen such that each column in U_2 has norm one. Then U_2 is orthogonal,

$$A_{12}U_2 = \begin{pmatrix} \alpha(a_{14}^2 + \frac{a_{13}a_{14}a_{24}}{a_{23}}) & \beta(a_{13}a_{23} - a_{14}a_{24}) \\ 0 & \beta(a_{23}^2 + a_{24}^2) \end{pmatrix},$$

and

$$U_2^t A_{22} U_2 = \begin{pmatrix} \alpha^2 \left(a_{33} \left(\frac{a_{14}a_{24}}{a_{23}} \right)^2 + a_{44}a_{14}^2 \right) & \alpha\beta a_{14}a_{24}(a_{33} - a_{44}) \\ \alpha\beta a_{14}a_{24}(a_{33} - a_{44}) & \beta^2 a_{23}^2 + \beta^2 a_{24}^2 \end{pmatrix}.$$

Since $a_{33} \geq a_{44}$, all entries in $A_{12}U_2$ and $U_2^t A_{22} U_2$ are nonnegative. Choosing $U_1 = I$ then gives a pair of orthogonal matrices with the desired property.

Now assume $a_{13}a_{23} - a_{14}a_{24} < 0$. Note that A_{12} of the form (14) cannot be singular and consequently, there exist $\lambda_1 \geq \lambda_2 > 0$ and an orthogonal matrix V such that $A_{12}A_{21} = V \Lambda V^t$, where $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2)$. Furthermore, since V contains the eigenvectors of $A_{12}A_{21}$ we may assume v_{11} and v_{12} have the same sign where v_{ij} is the (i, j)-th component of V. Define

$$W = A_{21} V (\Lambda^{1/2})^{-1}, \tag{15}$$

and note that this is an orthogonal matrix which, together with V, decomposes A_{12} into its singular value decomposition, that is, $V^t A_{12} W = \Lambda^{1/2}$. Then

$$V^t A_{11} V = \begin{pmatrix} a_{11}v_{11}^2 + a_{22}v_{21}^2 & v_{11}v_{12}(a_{11} - a_{22}) \\ v_{11}v_{12}(a_{11} - a_{22}) & a_{11}v_{12}^2 + a_{22}v_{22}^2 \end{pmatrix}.$$

All entries in $V^t A_{11} V$ are nonnegative since we chose v_{11} and v_{12} to have the same sign, and since $a_{11} \geq a_{22} > 0$.

To see that $W^t A_{22} W$ also have all entries nonnegative, consider the first line in the eigenequations for $A_{12}A_{21}$ associated with the eigenvector (v_{12}, v_{22}), the eigenvector associated with the smallest eigenvalue λ_2,

$$(a_{13}^2 + a_{14}^2 - \lambda_2)v_{12} + (a_{13}a_{23} - a_{14}a_{24})v_{22} = 0. \tag{16}$$

Since λ_2 is the smallest eigenvalue of $A_{12}A_{21}$,

$$\lambda_2 = \inf_{|v|=1} v^t A_{12} A_{21} v,$$

and since the off-diagonal elements in $A_{12}A_{21}$ are nonzero, $(1, 0)$ and $(0, 1)$ cannot be eigenvectors. Consequently, λ_2 is strictly smaller than any diagonal element of $A_{12}A_{21}$, and in particular $a_{13}^2 + a_{14}^2 - \lambda_2 > 0$. Since we also have $a_{13}a_{23} - a_{14}a_{24} < 0$,

(16) gives that v_{12} and v_{22} need to have the same sign for the sum to equal zero. Let w_{ij} be the (i, j)-th component of W and note that by (15),

$$w_{11}w_{12} = \frac{v_{11}a_{13} + v_{21}a_{23}}{\lambda_1^{1/2}} \frac{v_{12}a_{13} + v_{22}a_{23}}{\lambda_2^{1/2}}.$$

The assumption $v_{11}a_{13}(v_{11}a_{13} + v_{21}a_{23}) \geq 0$ implies that $v_{11}a_{13} + v_{21}a_{23}$ and v_{11} have the same sign. Since v_{11} and v_{12} were chosen to have the same sign, and v_{12} and v_{22} have the same sign, we conclude that $(v_{11}a_{13} + v_{21}a_{23})(v_{12}a_{13} + v_{22}a_{23})$ is nonnegative and, therefore, $w_{11}w_{12}$ is nonnegative, too. Then writing

$$W^t A_{22} W = \begin{pmatrix} a_{33}w_{11}^2 + a_{44}w_{21}^2 & w_{11}w_{12}(a_{33} - a_{44}) \\ w_{11}w_{12}(a_{33} - a_{44}) & a_{33}w_{12}^2 + a_{44}w_{22}^2 \end{pmatrix}$$

makes it clear that $W^t A_{22} W$ has nonnegative elements. Thus, letting $U_1 = V$ and $U_2 = W$ completes the proof. □

Corollary 1. *Let A and v_1 be given as in Lemma 3. Then there exists a $(2, 2)$-signature matrix U such that $U^t AU$ has nonpositive off-diagonal elements if and only if*

$$v_{21}a_{24}(v_{21}a_{24} + v_{11}a_{14}) \geq 0. \tag{17}$$

Proof. Let W be defined as in Lemma 3. Define

$$P_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix} P_1 & 0 \\ 0 & P_1 \end{pmatrix}.$$

Then $P_1 v_1 = (v_{21}, v_{11})$ is the eigenvector of $P_1 W_1^t A_{12} A_{21} W_1 P_1$ associated with the largest eigenvalue. Let

$$\tilde{A} = \begin{pmatrix} W_1^t A_{11} W_1 & P_1 W_1^t A_{12} W_2 P_1 \\ P_1 W_2^t A_{21} W_1 P_1 & W_2^t A_{22} W_2 \end{pmatrix} = \begin{pmatrix} a_{11} & 0 & a_{24} & a_{23} \\ 0 & a_{22} & a_{14} & a_{13} \\ a_{24} & a_{14} & a_{33} & 0 \\ a_{23} & a_{13} & 0 & a_{44} \end{pmatrix}.$$

By Lemma 3, there exists a $(2, 2)$-signature matrix

$$\tilde{U} = \begin{pmatrix} \tilde{U}_1 & 0 \\ 0 & \tilde{U}_2 \end{pmatrix}$$

such that $\tilde{U}^t \tilde{A} \tilde{U}$ has nonnegative entries if and only if $v_{21}a_{24}(v_{21}a_{24} + v_{11}a_{14}) \geq 0$. Define now the $(2, 2)$-signature matrix U as

$$U = \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix} = \begin{pmatrix} -W_1 P_1 \tilde{U}_1 & 0 \\ 0 & W_2 P_1 \tilde{U}_2 \end{pmatrix}.$$

Let $\tilde{u}_{ij}$ be the (i, j)-th component of $\tilde{U}_1$. Since $\tilde{U}_1$ is orthogonal, $\tilde{u}_{12}\tilde{u}_{22} = -\tilde{u}_{11}\tilde{u}_{21}$ implying that

$$U_1^t A_{11} U_1 = \begin{pmatrix} \tilde{u}_{11}^2 a_{22} + \tilde{u}_{21}^2 a_{11} & \tilde{u}_{11}\tilde{u}_{12}(a_{22} - a_{11}) \\ \tilde{u}_{11}\tilde{u}_{12}(a_{22} - a_{11}) & \tilde{u}_{12}^2 a_{22} + \tilde{u}_{22}^2 a_{11} \end{pmatrix}$$

and

$$\tilde{U}_1^t W_1^t A_{11} W_1 \tilde{U}_1 = \begin{pmatrix} \tilde{u}_{11}^2 a_{11} + \tilde{u}_{21}^2 a_{22} & \tilde{u}_{11}\tilde{u}_{12}(a_{11} - a_{22}) \\ \tilde{u}_{11}\tilde{u}_{12}(a_{11} - a_{22}) & \tilde{u}_{12}^2 a_{11} + \tilde{u}_{22}^2 a_{22} \end{pmatrix}.$$

Consequently $\tilde{U}_1^t W_1^t A_{11} W_1 \tilde{U}_1$ has nonnegative elements if and only if $U_1^t A_{11} U_1$ has nonpositive off-diagonal elements. Similarly, $\tilde{U}_2^t W_2^t A_{22} W_2 \tilde{U}_2$ has nonnegative elements if and only if $U_2^t A_{22} U_2$ has nonpositive off-diagonal elements by a similar argument. Finally we note that

$$U_1^t A_{12} U_2 = -\tilde{U}_1^t P_1 W_1^t A_{12} W_2 P_1 \tilde{U}_2,$$

and it follows that $U^t AU$ has nonpositive off-diagonal elements if and only if

$$\tilde{U}_1^t P_1 W_1^t A_{12} W_2 P_1 \tilde{U}_2, \quad \tilde{U}_1^t W_1^t A_{11} W_1 \tilde{U}_1 \quad \text{and,} \quad \tilde{U}_2^t W_2 A_{22} W_2 \tilde{U}_2$$

have all entries nonnegative. We conclude that we can find a (2, 2)-signature matrix U such that $U^t AU$ has nonpositive off-diagonal element if and only if (17) holds. □

Proof of Theorem 4. To prove (i) set $Q = I - (I - a\Sigma)^{-1}$ for sufficiently large $a > 0$. The implication (iii)⇒ (ii) of Lemma 3 used on $A = Q$, together with Theorem 3, show that (i) implies infinite divisibility of $(X_1^2 + X_2^2, X_3^2 + X_4^2)$. To prove (ii) we use Corollary 1 on $A = \Sigma^{-1}$, which together with Proposition 1, show that (ii) implies infinite divisibility of $(X_1^2 + X_2^2, X_3^2 + X_4^2)$. □

5.3 Proof of Theorem 5

Now we set out to show that the sum in Theorem 3 is nonnegative for $k, m \in \mathbb{N}_0$ such that $k \le 2$, $m \le 2$, or $k + m \le 7$ in the case $n_1 = n_2 = 2$. To this end, consider a 4×4 positive definite matrix Q and write

$$Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}$$

where Q_{ij} is a 2×2 matrix for $i, j = 1, 2$. Let W_1 and W_2 be two 2×2 orthogonal matrices and define $P_{ij} = W_i Q_{ij} W_j$. Then

$$\begin{aligned} &\text{trace } Q_{11}^{k_1} Q_{12} Q_{22}^{m_1} Q_{21} \cdots Q_{12} Q_{22}^{m_d} Q_{21} Q_{11}^{k_{d+1}} \\ &\quad = \text{trace } P_{11}^{k_1} P_{12} P_{22}^{m_1} P_{21} \cdots P_{12} P_{22}^{m_d} P_{21} P_{11}^{k_{d+1}}. \end{aligned} \tag{18}$$

Consequently (see Lemma 2), we may assume, without loss of generality, that Q_{11} and Q_{22} are diagonal with the first diagonal element greater than or equal to the other and all entries nonnegative.

There exist D_1 and D_2 of the form diag$(\pm 1, \pm 1)$ such that either $D_1 Q_{12} D_2$ has all entries nonnegative or such that

$$D_1 Q_{12} D_2 = \begin{pmatrix} q_{13} & q_{23} \\ q_{14} & -q_{24} \end{pmatrix}$$

where $q_{13}, q_{23}, q_{14}, q_{24} > 0$. If $D_1 Q_{12} D_2$ has all entries nonnegative, writing as in (18) with W_i replaced by D_i implies nonnegativity of each individual trace. We conclude that we may assume

$$Q = \begin{pmatrix} \lambda_1 & 0 & q_{13} & q_{14} \\ 0 & \lambda_2 & q_{23} & -q_{24} \\ q_{13} & q_{23} & \lambda_3 & 0 \\ q_{14} & -q_{24} & 0 & \lambda_4 \end{pmatrix},$$

where $\lambda_1 \geq \lambda_2 \geq 0$ and $\lambda_3 \geq \lambda_4 \geq 0$ and $q_{13}, q_{23}, q_{14}, q_{24} > 0$, without loss of generality.

We now write out the traces in (7) for specific values of k and m and show nonnegativity in each case.

$k = 0$ or $m = 0$

Assume $k = 0$ and fix some $m \in \mathbb{N}$. Then the terms in the sum in Theorem 3 reduce to trace Q_{22}^m. Since Q_{22} is positive definite, Q_{22}^m is positive definite. Consequently, trace $Q_{22}^m > 0$. Similarly, when $m = 0$ and $k \in \mathbb{N}$, the terms in the sum in Theorem 3 reduce to trace Q_{11}^k, which again is positive since Q_{11} is positive definite.

$k = 1$ or $m = 1$

Assume $k = 1$ and fix some $m \in \mathbb{N}$. Then (7) reduces to

$$\operatorname{trace} Q_{12} Q_{22}^m Q_{21} + \sum_{m_1=0}^{m-1} \operatorname{trace} Q_{22}^{m_1} Q_{21} Q_{12} Q_{22}^{m-1-m_1},$$

which equals

$$(m+1) \operatorname{trace} Q_{12} Q_{22}^m Q_{21}.$$

Since $Q_{12} = Q_{21}^t$ and Q_{22} is positive definite, $Q_{12} Q_{22}^m Q_{21}$ is positive semidefinite. We conclude that trace $Q_{12} Q_{22}^m Q_{21} \geq 0$.

Assume $m = 1$ and fix some $k \in \mathbb{N}$. Similar to above, (7) reduces to

$$\operatorname{trace} Q_{21} Q_{11}^k Q_{12} + \sum_{k_1=0}^{k-1} \operatorname{trace} Q_{11}^{k_1} Q_{12} Q_{21} Q_{11}^{k-1-k_1}.$$

Nonnegativity of this trace follows by arguments similar to those above.

$k = 2$ or $m = 2$

Assume that $k = 2$ and let $m \in \mathbb{N}$. The case $m = 1$ is discussed above. Assume $m \geq 2$. Then (7) reduces to

$$\operatorname{trace} Q_{11} Q_{12} Q_{22}^{m-1} Q_{21} + \sum_{m_1+m_2+1=m} \operatorname{trace} Q_{22}^{m_1} Q_{21} Q_{11} Q_{12} Q_{22}^{m_2}$$

$$+ \sum_{m_1+m_2+2=m} \text{trace } Q_{12} Q_{22}^{m_1} Q_{21} Q_{12} Q_{22}^{m_2} Q_{21}$$
$$+ \sum_{m_1+m_2+m_3+2=m} \text{trace } Q_{22}^{m_1} Q_{21} Q_{12} Q_{22}^{m_2} Q_{21} Q_{12} Q_{22}^{m_3}.$$

All the traces above are nonnegative. To see this, consider for example

$$\text{trace } Q_{22}^{m_1} Q_{21} Q_{12} Q_{22}^{m_2} Q_{21} Q_{12} Q_{22}^{m_3}$$

for some $m_1, m_2, m_3 \in \mathbb{N}_0$. Since Q_{22} is positive definite it has a unique positive definite square root $Q_{22}^{1/2}$. We conclude that

$$\begin{aligned} &\text{trace } Q_{22}^{m_1} Q_{21} Q_{12} Q_{22}^{m_2} Q_{21} Q_{12} Q_{22}^{m_3} \\ &\quad = \text{trace } Q_{22}^{(m_1+m_3)/2} Q_{21} Q_{12} Q_{22}^{m_2} Q_{21} Q_{12} Q_{22}^{(m_1+m_3)/2}. \end{aligned} \tag{19}$$

Note that

$$Q_{22}^{(m_1+m_3)/2} Q_{21} Q_{12} = (Q_{21} Q_{12} Q_{22}^{(m_1+m_3)/2})^t,$$

which implies that (19) is the trace of a positive semidefinite matrix and therefore nonnegative.

Nonnegativity of the traces when $m = 2$ and $k \in \mathbb{N}$ follows by symmetry.

$k = 3$ *and* $m = 3$

In the following we will need to expand traces, and we therefore note that

$$\text{trace } Q_{11}^k Q_{12} Q_{22}^m Q_{21} = \lambda_1^k \lambda_3^m q_{13}^2 + \lambda_1^k \lambda_4^m q_{14}^2 + \lambda_2^k \lambda_3^m q_{23}^2 + \lambda_2^k \lambda_4^m q_{24}^2 \tag{20}$$

for any $k, m \in \mathbb{N}$, and

$$\begin{aligned} &\text{trace } Q_{11}^{k_1} Q_{12} Q_{22}^{m_1} Q_{21} Q_{11}^{k_2} Q_{12} Q_{22}^{m_2} Q_{21} \\ &\quad = \lambda_1^{k_1+k_2} \lambda_3^{m_1+m_2} q_{13}^4 + \lambda_1^{k_1+k_2} \lambda_4^{m_1+m_2} q_{14}^4 \\ &\qquad + \lambda_2^{k_1+k_2} \lambda_3^{m_1+m_2} q_{23}^4 + \lambda_2^{k_1+k_2} \lambda_4^{m_1+m_2} q_{24}^4 \\ &\qquad + \lambda_1^{k_1+k_2} (\lambda_3^{m_1} \lambda_4^{m_2} + \lambda_4^{m_1} \lambda_3^{m_2}) q_{13}^2 q_{14}^2 \\ &\qquad + \lambda_2^{k_1+k_2} (\lambda_3^{m_1} \lambda_4^{m_2} + \lambda_3^{m_2} \lambda_4^{m_1}) q_{23}^2 q_{24}^2 \\ &\qquad + \lambda_3^{m_1+m_2} (\lambda_1^{k_1} \lambda_2^{k_2} + \lambda_1^{k_2} \lambda_2^{k_1}) q_{13}^2 q_{23}^2 \\ &\qquad + \lambda_4^{m_1+m_2} (\lambda_1^{k_1} \lambda_2^{k_2} + \lambda_1^{k_2} \lambda_2^{k_1}) q_{14}^2 q_{24}^2 \\ &\qquad - (\lambda_1^{k_1} \lambda_2^{k_2} + \lambda_1^{k_2} \lambda_2^{k_1})(\lambda_3^{m_1} \lambda_4^{m_2} + \lambda_3^{m_2} \lambda_4^{m_1}) q_{13} q_{23} q_{14} q_{24} \end{aligned} \tag{21}$$

for any $k_1, k_2, m_1, m_2 \in \mathbb{N}$.

Assume now $k = 3$ and $m = 3$ and consider the sum in Theorem 3. The sum contains all terms of the form

$$\text{trace } Q_{11}^{k_1} Q_{12} Q_{22}^2 Q_{21} Q_{11}^{k_2}$$

where $k_1 + k_2 = 2$ and

$$\text{trace } Q_{22}^{m_1} Q_{21} Q_{11}^2 Q_{12} Q_{22}^{m_2}$$

where $m_1 + m_2 = 2$. All these traces equal

$$\text{trace } Q_{11}^2 Q_{12} Q_{22}^2 Q_{21},$$

and there are alltogether 6 of these terms. Next, the sum in Theorem 3 also contains all terms of the form

$$\text{trace } Q_{11}^{k_1} Q_{12} Q_{22}^{m_1} Q_{21} Q_{11}^{k_2} Q_{12} Q_{22}^{m_2} Q_{21} Q_{11}^{k_3}$$

where $k_1 + k_2 + k_3 = 1$ and $m_1 + m_2 = 1$, and

$$\text{trace } Q_{22}^{m_1} Q_{21} Q_{11}^{k_1} Q_{12} Q_{22}^{m_2} Q_{21} Q_{11}^{k_2} Q_{12} Q_{22}^{m_3}$$

where $m_1 + m_2 + m_3 = 1$ and $k_1 + k_2 = 1$. Using both that trace $AB =$ trace BA and trace $A^t =$ trace A for any two square matrices A and B of the same dimensions we get that all these traces share the common trace

$$\text{trace } Q_{11} Q_{12} Q_{21} Q_{12} Q_{22} Q_{21}.$$

Alltogether there are 12 of these terms. Finally, the sum in Theorem 3 contains the two terms

$$\text{trace}(Q_{12} Q_{21})^3 \quad \text{and} \quad \text{trace}(Q_{21} Q_{12})^3,$$

which share a common trace. We conclude that the sum in Theorem 3 reads

$$\text{trace } \{6 Q_{11}^2 Q_{12} Q_{22}^2 Q_{21} + 12 Q_{11} Q_{12} Q_{21} Q_{12} Q_{22} Q_{21} + 2(Q_{12} Q_{21})^3\}. \tag{22}$$

Since $Q_{12} = Q_{21}^t$, the matrix $Q_{12} Q_{21}$ is positive semidefinite and consequently, $\text{trace}(Q_{12} Q_{21})^3 \geq 0$. Furthermore, we have

$$\text{trace } Q_{11}^2 Q_{12} Q_{22}^2 Q_{21} = \text{trace } Q_{11} Q_{12} Q_{22}^2 Q_{21} Q_{11} \geq 0.$$

Contrarily, there exists a positive definite matrix Q such that

$$\text{trace } Q_{11} Q_{12} Q_{21} Q_{12} Q_{22} Q_{21} < 0.$$

(To see this, consider Q of the form in Example 1 with ε small and δ large relative to ε.) We will now argue that despite this, (22) remains nonnegative. Initially we note that

$$Q_{11}^{k_i} Q_{12} Q_{22}^{m_i} Q_{21} = \begin{pmatrix} \lambda_1^{k_i}(\lambda_3^{m_i} q_{13}^2 + \lambda_4^{m_i} q_{14}^2) & \lambda_1^{k_i}(\lambda_3^{m_i} q_{13} q_{23} - \lambda_4^{m_i} q_{14} q_{24}) \\ \lambda_2^{k_i}(\lambda_3^{m_i} q_{13} q_{23} - \lambda_4^{m_i} q_{14} q_{24}) & \lambda_2^{k_i}(\lambda_3^{m_i} q_{23}^2 + \lambda_4^{m_i} q_{24}^2) \end{pmatrix}$$

and

$$Q_{22}^{m_i} Q_{21} Q_{11}^{k_i} Q_{12} = \begin{pmatrix} \lambda_3^{m_i}(\lambda_1^{k_i} q_{13}^2 + \lambda_2^{k_i} q_{23}^2) & \lambda_3^{m_i}(\lambda_1^{k_i} q_{13}q_{14} - \lambda_2^{k_i} q_{23}q_{24}) \\ \lambda_4^{m_i}(\lambda_1^{k_i} q_{13}q_{14} - \lambda_2^{k_i} q_{23}q_{24}) & \lambda_4^{m_i}(\lambda_1^{k_i} q_{14}^2 + \lambda_2^{k_i} q_{24}^2) \end{pmatrix}.$$

Since $\lambda_1 \geq \lambda_2$ and $\lambda_3 \geq \lambda_4$, we see that if $q_{13}q_{14} \geq q_{23}q_{24}$ or $q_{13}q_{23} \geq q_{14}q_{24}$, then one of two matrices above have only nonnegative entrances for any $k_i, m_i \in \mathbb{N}_0$. Consequently,

$$\text{trace } Q_{11}^{k_1} Q_{12} Q_{22}^{m_1} Q_{21} Q_{11}^{k_2} Q_{12} Q_{22}^{m_2} Q_{21} = \text{trace } Q_{22}^{m_1} Q_{21} Q_{11}^{k_1} Q_{12} Q_{22}^{m_2} Q_{21} Q_{11}^{k_2} Q_{12}$$

would be nonnegative if this was the case. Especially, we would have

$$\text{trace } Q_{11} Q_{12} Q_{21} Q_{12} Q_{22} Q_{21} \geq 0.$$

Assume now that $q_{13}q_{14} \leq q_{23}q_{24}$ and $q_{13}q_{23} \leq q_{14}q_{24}$. By (20) and (21),

$$\begin{aligned} &\text{trace } \{\tfrac{1}{2} Q_{11}^2 Q_{12} Q_{22}^2 Q_{21} + Q_{11} Q_{12} Q_{22} Q_{21} Q_{12} Q_{21}\} \\ &= \tfrac{1}{2}\lambda_1^2\lambda_3^2 q_{13}^2 + \tfrac{1}{2}\lambda_1^2\lambda_4^2 q_{14}^2 + \tfrac{1}{2}\lambda_2^2\lambda_3^2 q_{23}^2 + \tfrac{1}{2}\lambda_2^2\lambda_4^2 q_{24}^2 \\ &\quad + \lambda_1\lambda_3 q_{13}^4 + \lambda_1\lambda_4 q_{14}^4 + \lambda_2\lambda_3 q_{23}^4 + \lambda_2\lambda_4 q_{24}^4 \\ &\quad + \lambda_1(\lambda_3 + \lambda_4) q_{13}^2 q_{14}^2 + \lambda_2(\lambda_3 + \lambda_4) q_{23}^2 q_{24}^2 \\ &\quad + \lambda_3(\lambda_1 + \lambda_2) q_{13}^2 q_{23}^2 + \lambda_4(\lambda_1 + \lambda_2) q_{14}^2 q_{24}^2 \\ &\quad - (\lambda_1 + \lambda_2)(\lambda_3 + \lambda_4) q_{13}q_{23}q_{14}q_{24}. \end{aligned} \tag{23}$$

We are going to bound the term $(\lambda_1 + \lambda_2)(\lambda_3 + \lambda_4)q_{13}q_{23}q_{14}q_{24}$ by the positive terms to show nonnegativity of this trace. We recall that $\lambda_1 \geq \lambda_2 > 0$ and $\lambda_3 \geq \lambda_4 > 0$. Initially, note that

$$\begin{aligned} \lambda_2\lambda_3 q_{13}q_{23}q_{14}q_{24} &\leq \lambda_2\lambda_3 q_{13}^2 q_{23}^2 \\ \lambda_2\lambda_4 q_{13}q_{23}q_{14}q_{24} &\leq \lambda_1\lambda_4 q_{13}^2 q_{14}^2 \\ \lambda_1\lambda_4 q_{13}q_{23}q_{14}q_{24} &\leq \lambda_1\lambda_3 q_{13}^2 q_{14}^2. \end{aligned}$$

This leaves only $\lambda_1\lambda_3 q_{13}q_{23}q_{14}q_{24}$ to be bounded. If $\lambda_1\lambda_3 q_{13}q_{23}q_{14}q_{24} \leq \frac{1}{2}\lambda_1^2\lambda_3^2 q_{13}^2$, we have a bounding term in (23). Therefore, assume $2q_{23}q_{14}q_{24} \geq \lambda_1\lambda_3 q_{13}$. Since Q was assumed positive definite, $\lambda_2\lambda_4 \geq q_{24}^2$. Consequently,

$$\begin{aligned} \lambda_1\lambda_3 q_{13}q_{23}q_{14}q_{24} &\leq 2q_{23}^2 q_{14}^2 q_{24}^2 \\ &\leq 2\lambda_2\lambda_4 q_{23}^2 q_{13}^2 \\ &\leq \lambda_2\lambda_4 (q_{23}^4 + q_{13}^4) \\ &\leq \lambda_2\lambda_3 q_{23}^4 + \lambda_1\lambda_3 q_{13}^4. \end{aligned}$$

We conclude that (23) and hence (22) is nonnegative.

$k + m = 7$

Now consider $k, m \in \mathbb{N}$ such that $k + m = 7$. Whenever $k, m = 1, 2$, we already know that the sum in Theorem 3 is nonnegative. Let $k = 3$ and $m = 4$. Then the sum in Theorem 3 reads

$$\begin{aligned}\text{trace}\,\{&14Q_{11}Q_{12}Q_{21}Q_{12}Q_{22}^2Q_{21}+7Q_{11}^2Q_{12}Q_{22}^3Q_{21}\\&7Q_{11}(Q_{12}Q_{22}Q_{21})^2+7Q_{12}Q_{22}Q_{21}(Q_{12}Q_{21})^2\}.\end{aligned}\tag{24}$$

Initially we note that

$$\text{trace}\,Q_{11}(Q_{12}Q_{22}Q_{21})^2\geq 0\quad\text{and}\quad\text{trace}\,Q_{12}Q_{22}Q_{21}(Q_{12}Q_{21})^2\geq 0$$

since they both can be written as the trace of positive semidefinite matrices (see above for more details). Next, by (20) and (21),

$$\begin{aligned}\text{trace}\,&\{\tfrac{1}{2}Q_{11}^2Q_{12}Q_{22}^3Q_{21}+Q_{11}Q_{12}Q_{22}^2Q_{21}Q_{12}Q_{21}\}\\&=\tfrac{1}{2}\lambda_1^2\lambda_3^3q_{13}^2+\tfrac{1}{2}\lambda_1^2\lambda_4^3q_{14}^2+\tfrac{1}{2}\lambda_2^2\lambda_3^3q_{23}^2+\tfrac{1}{2}\lambda_2^2\lambda_4^3q_{24}^2\\&\quad+\lambda_1(\lambda_3^2+\lambda_4^2)q_{13}^2q_{14}^2+\lambda_2(\lambda_3^2+\lambda_4^2)q_{23}^2q_{24}^2\\&\quad+\lambda_3^2(\lambda_1+\lambda_2)q_{13}^2q_{23}^2+\lambda_4^2(\lambda_1+\lambda_2)q_{14}^2q_{24}^2\\&\quad+\lambda_1\lambda_3^2q_{13}^4+\lambda_1\lambda_4^2q_{14}^4+\lambda_2\lambda_3^2q_{23}^4+\lambda_2\lambda_4^2q_{24}^4\\&\quad-(\lambda_1+\lambda_2)(\lambda_3^2+\lambda_4^2)q_{13}q_{23}q_{14}q_{24}.\end{aligned}\tag{25}$$

Again we bound the negative term by positive terms. Recall that $\lambda_1\geq\lambda_2$ and $\lambda_3\geq\lambda_4$, and that we may assume $q_{23}q_{24}\geq q_{13}q_{14}$ and $q_{14}q_{24}\geq q_{13}q_{23}$ without loss of generality. Consequently,

$$\begin{aligned}\lambda_1\lambda_4^2q_{13}q_{23}q_{14}q_{24}&\leq\lambda_1\lambda_4^2q_{14}^2q_{24}^2\\\lambda_2\lambda_3^2q_{13}q_{23}q_{14}q_{24}&\leq\lambda_2\lambda_3^2q_{23}^2q_{24}^2\\\lambda_2\lambda_4^2q_{13}q_{23}q_{14}q_{24}&\leq\lambda_2\lambda_4^2q_{14}^2q_{24}^2,\end{aligned}$$

leaving $\lambda_1\lambda_3^2q_{13}q_{23}q_{14}q_{24}$ to be bounded. First note that

$$\tfrac{1}{2}\lambda_1^2\lambda_3^3q_{13}^2-\lambda_1\lambda_3^2q_{13}q_{23}q_{14}q_{24}=\lambda_1\lambda_3^2q_{13}(\tfrac{1}{2}\lambda_1\lambda_3q_{13}-q_{23}q_{14}q_{24}),$$

so that nonnegativity holds if $\frac{1}{2}\lambda_1\lambda_3q_{13}\geq q_{23}q_{14}q_{24}$. Assume $\lambda_1\lambda_3q_{13}\leq 2q_{23}q_{14}q_{24}$ and recall that $\lambda_2\lambda_4\geq q_{24}^2$ since Q is positive definite. Then

$$\begin{aligned}\lambda_1\lambda_3^2q_{13}q_{23}q_{14}q_{24}&\leq 2\lambda_3q_{23}^2q_{14}^2q_{24}^2\\&\leq 2\lambda_2\lambda_3\lambda_4q_{23}^2q_{14}^2\\&\leq\lambda_2\lambda_3^2q_{23}^4+\lambda_2\lambda_4^2q_{14}^4\\&\leq\lambda_2\lambda_3^2q_{23}^4+\lambda_1\lambda_4^2q_{14}^4\end{aligned}$$

so we have found bounding terms for the last expression. We conclude that (25) is nonnegative and therefore, (24) is nonnegative, too. The case $k=4$ and $m=3$ follows by symmetry. It follows that the sum in Theorem 3 is nonnegative for $k+m=7$. □

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