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On distributions of exponential functionals of the processes with independent increments

Lioudmila Vostrikova

Université d'Angers, Département de Mathématiques, LAREMA,
2, Bd Lavoisier 49045, Angers Cedex 01, France

Lioudmila.Vostrikova@univ-angers.fr (L. Vostrikova)

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Abstract The aim of this paper is to study the laws of exponential functionals of the processes $X = (X_s)_{s\geq 0}$ with independent increments, namely

$$I_t = \int_0^t \exp(-X_s)ds, \ \ t \geq 0,$$

and also

$$I_\infty = \int_0^\infty \exp(-X_s)ds.$$

Under suitable conditions, the integro-differential equations for the density of I_t and I_∞ are derived. Sufficient conditions are derived for the existence of a smooth density of the laws of these functionals with respect to the Lebesgue measure. In the particular case of Lévy processes these equations can be simplified and, in a number of cases, solved explicitly.

Keywords Process with independent increments, exponential functional, Kolmogorov-type equation, smoothness of the density
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1 Introduction

This study was inspired by the questions arising in mathematical finance, namely by the questions related to perpetuities containing the liabilities, perpetuities subjected to the influence of economical factors (see, for example, Kardaras, Robertson [23]), and also with the price of Asian options and similar questions (see, for instance, Jeanblanc, Yor, Chesnay [21], Vecer [38] and references there). The study of exponential functionals is also important in the insurance, since the distributions of these functionals appear very naturally in the ruin problem (see, for example, Asmussen [2], Paulsen [29], Kabanov, Pergamentshchikov [22], Spielmann, Vostrikova [37] and the references there).

In mathematical finance exponential functionals of the processes with independent increments (PII in short; in what follows, this abbreviation will also denote the property of being a process with independent increments) arise very often[1]. This fact is related to the observation that log price is usually not a homogeneous process on a relatively long time interval. For this reason several authors used for the modeling of log price the process $X = (X_t)_{t\geq 0}$ such that

$$X_t = \int_0^t g_{s-} dL_s$$

where L is a Lévy process and g is a càdlàg random process independent of L for which the integral is well defined. In this case, the conditioned process given σ-algebra generated by g, is a PII. Another important example of X is a Lévy process L with time changed by an independent increasing process $(\tau_t)_{t\geq 0}$ (cf. Carr, Wu [16]), i.e.

$$X_t = L_{\tau_t}.$$

Again, conditionally to the process τ, the process $X = (X_t)_{t\geq 0}$ is PII.

In Salminen, Vostrikova [31, 32] we proved the recurrent formulas for the Mellin transform and we used these formulas to calculate the moments of exponential functionals of the processes with independent increments. In this paper we obtain the equations for the densities, when they exist, of the laws of exponential functionals

$$I_t = \int_0^t \exp(-X_s)ds, \;\; t \geq 0, \tag{1}$$

and also

$$I_\infty = \int_0^\infty \exp(-X_s)ds,$$

where the process $X = (X_t)_{t\geq 0}$ is PII. It is not difficult to see that under the condition

$$\lim_{s\to+\infty} \frac{X_s}{s} = \mu > 0 \;\; (\mathbf{P}\text{-a.s.})$$

we have $I_\infty < +\infty$ ($\mathbf{P}$-a.s.). As it was shown in Bertoin, Yor [10], this condition is necessary and sufficient in the case of Lévy processes.

Exponential functionals of Lévy processes were studied in a large number of articles, most of them were related to the study of I_∞. In the celebrated paper by Carmona, Petit, Yor [15] the asymptotic behaviour of exponential functionals I_∞ was studied, in particular for α-stable Lévy processes. The authors also gave an integro-

[1] See for example the book of E. Eberlein and J. Kallsen *Mathematical finance* , Springer 2019, and references there.

differential equation for the density of the law of exponential functionals, when this density exists w.r.t. the Lebesgue measure. The questions related to the characterization of the law of exponential functionals by the moments were studied by Bertoin, Yor in [10].

In a more general setting, related to the Lévy case, the functional

$$I_\infty(\eta) = \int_0^\infty \exp(-X_{s-})d\eta_s, \tag{2}$$

where $X = (X_t)_{t\geq 0}$ and $\eta = (\eta_t)_{t\geq 0}$ are independent Lévy processes, was intensively studied. The conditions for finiteness of the integral (2) were obtained in Erickson, Maller [18]. The continuity properties of the law of this integral were studied in Bertoin, Lindner, Maller [9], where the authors gave the conditions for absence of atoms and also the conditions for absolute continuity of laws of integral functionals w.r.t. the Lebesgue measure. The question of smoothness of the density of the law of I_∞ in the Schwarz sense was considered in Carmona, Petit, Yor [15] and Bertoin, Yor [10]. Under the assumptions on the existence of smooth density of these functionals, the equations for the density are given in Carmona, Petit, Yor [15], Bertoin, Yor [10], Behme [4], Behme, Lindner [5], Kuznetsov, Pardo, Savov [24].

In the papers by Patie, Savov [28], Pardo, Rivero, Van Shaik [27], again for a Lévy process, the properties of exponential functionals $I_{\tau_q}(\eta)$ killed at an independent exponential time τ_q of the parameter $q > 0$, were investigated. In the article [27] the authors studied the existence of the density of the law of $I_{\tau_q}(\eta)$, they gave an integral equation for the density and the asymptotics of the law of $I_\infty(\eta)$ at zero and at infinity, when X is a positive subordinator. The results given in [28] involve the analytic Wiener–Hopf factorisation, Bernstein functions and contain the conditions for regularity, semi-explicite expression and asymptotics for the distribution function of $I_{\tau_q}(\eta)$. In Behme, Lindner, Reker, Rivero [6], Behme, Lindner, Reker [7] the authors give sufficient conditions for absolute continuity of the laws of $I_{\tau_q}(\eta)$ as well as the sufficient conditions for the absolute continuity of the laws of $I_t(\eta)$ and $I_\infty(\eta)$.

Despite numerous studies, the distributions of $I_t(\eta)$ and $I_\infty(\eta)$ are known only in a limited number of cases. When X is a Brownian motion with drift, the distributions of I_t and I_∞ were studied in Dufresne [17] and for a large number of specific processes X and η, like a Brownian motion with drift and a compound Poisson process, the distributions of $I_\infty(\eta)$ were given in Gjessing, Paulsen [19].

The exponential functionals for diffusions stopped at the first hitting time were studied in Salminen, Wallin [30], where authors derive the Laplace transform of the functionals and then, to find their laws, perform numerical inversion of the Laplace transform. The relations between hitting times and occupation times for the exponential functionals were considered in Salminen, Yor [33], where the versions of identities in law such as Dufresne's identity, Ciesielski–Taylor's identity, Biane's identity, LeGall's identity were given.

In this article we consider a real-valued process $X = (X_t)_{t\geq 0}$ with independent increments and $X_0 = 0$, which is a semimartingale with respect to its natural filtration. We denote by (B, C, ν) a semimartingale triplet of this process, which can be chosen to be deterministic (see Jacod, Shiryaev [20], Ch. II, p.106). We suppose that $B = (B_t)_{t\geq 0}$, $C = (C_t)_{t\geq 0}$ and ν are absolutely continuous with respect to the

Lebesgue measure, i.e.

$$B_t = \int_0^t b_s\,ds, \quad C_t = \int_0^t c_s\,ds, \quad \nu(dt, dx) = dt\,K_t(dx) \tag{3}$$

with càdlàg functions $b = (b_s)_{s\geq 0}$, $c = (c_s)_{s\geq 0}$, $K = (K_s(A))_{s\geq 0, A\in\mathcal{B}(\mathbb{R})}$. We assume that the compensator of the measure of jumps ν verifies the usual relation: for each $t \in \mathbb{R}^+$

$$\int_0^t \int_{\mathbb{R}} (x^2 \wedge 1) K_s(dx)\,ds < +\infty. \tag{4}$$

For the main result we will assume an additional technical condition:

$$\int_0^t \int_{|x|>1} e^{|x|}\, K_s(dx)\,ds < +\infty. \tag{5}$$

The last condition implies that $\mathbf{E}(|X_t|) < +\infty$ for $t > 0$ (cf. Sato [34], Th. 25.3, p.159) so the truncation of jumps is no more necessary.

We recall that the characteristic function of X_t,

$$\phi_t(\lambda) = \mathbf{E}\exp(i\lambda X_t),$$

is expressed as follows: for $\lambda \in \mathbb{R}$

$$\phi_t(\lambda) = \exp\{i\lambda B_t - \frac{1}{2}\lambda^2 C_t + \int_0^t \int_{\mathbb{R}} (e^{i\lambda x} - 1 - i\lambda x)\, K_s(dx)\,ds\}.$$

We recall also that X is a semimartingale if and only if for all $\lambda \in \mathbb{R}$ the characteristic function of X_t is of finite variation in t on finite intervals (cf. Jacod, Shiryaev [20], Ch.2, Th. 4.14, p.106). Moreover, the process X always can be written as a sum of a semimartingale and a deterministic function which is not necessarily of finite variation on finite intervals.

The article is organized as follows. Part 2 is devoted to the Kolmogorov type equation for the law of I_t. It is known that the exponential functional $(I_t)_{t>0}$ is not a Markov process with respect to the filtration generated by the process X. It is a continuous increasing process, what prevents the use of stochastic calculus in an efficient way. For these reasons we fix t and introduce a family of stochastic processes $V^{(t)} = (V_s^{(t)})_{0\leq s\leq t}$ indexed by t and such that $I_t = V_t^{(t)}$ (P-a.s.) (see Lemma 1). The construction of such processes is made via the time reversion of the process X at a fixed time t and gives a Generalised Ornstein-Uhlenbeck process process with a PII noise (GOU process in short). We give the Kolmogorov type equations for $V^{(t)}$ (see Theorem 1). Assuming the existence of the smooth density of the law of the process $V^{(t)}$ we derive the integro-differential equation for the density of the laws of $V_s^{(t)}$, $0 < s < t$. The density of the law of $V_t^{(t)}$ can be obtained just by integration of the right-hand side of the equation for the density of the law of $V_s^{(t)}$ in s on the interval $]0, t[$.

In Part 3 we consider the question of existence of the smooth density of the process $V^{(t)}$. The question of existence of the density of the law of $V_s^{(t)}$, $0 \leq s \leq t$, of the class $C^{1,2}(]0, t[\times\mathbb{R}^{+,*})$, where $\mathbb{R}^{+,*} = \mathbb{R} \setminus \{0\}$, is a rather difficult one, which was an open question in all cited papers on exponential functionals (see [15, 10, 4, 5, 24]). Since the processes $V_t^{(t)}$ and I_t coincide (P-a.s.), the results on the density of $V_t^{(t)}$ give the answer for the density of I_t. In Proposition 2 we give sufficient conditions for the existence of the density of the class $C^\infty(]0,t[\times\mathbb{R}^{+,*})$ of the law of I_t when X is a Lévy process. For a non-homogeneous PII we give a partial answer to this question in Corollary 1.

Part 4 is devoted to Lévy processes. When X is a Lévy process, the equations for the density of I_t can be simplified due to the homogeneity (see Proposition 1). We present also the equations for the distribution functions of the laws of I_t and I_∞, since for these equations we have the explicit boundary conditions (cf. Corollary 2). In Corollary 3 we consider the well-known Brownian case. In Corollary 4 we give the equations for the case of Lévy processes with integrable jumps, and in Corollary 5, we consider the case of exponential jumps. In the particular case of I_∞ and integrable jumps the equations coincide with the known ones from [15].

2 Kolmogorov type equation for the density of the law of I_t

We introduce, for fixed $t > 0$, a time reversal process $Y = (Y_s)_{0\le s\le t}$ with

$$Y_s = X_t - X_{(t-s)-}.$$

Of course, this process depends on the parameter t, but we will omit this parameter for the simplicity of notations.

For convenience of the readers we present here Lemma 1 and Lemma 2 proved in Salminen, Vostrikova [31]. The first result establishes the relation between I_t defined by (1) and the process $Y = (Y_s)_{0\le s\le t}$.

Lemma 1 (cf. [31])**.** *For $t > 0$,*

$$I_t = e^{-Y_t}\int_0^t e^{Y_s}ds \quad (\mathbf{P}\text{-}a.s.)$$

Remark 1. In the case of Lévy processes the equality in law between right and left side was proved in [15], Lemma 2.3. It should be noticed that Lemma 1 gives more even in the Lévy case, since one can define both processes in the initial probability space.

In the following lemma we claim that Y is PII and we precise its semimartingale triplet. For that we introduce the functions $\bar{b} = (\bar{b}_u)_{0\le u\le t}$, $\bar{c} = (\bar{c}_u)_{0\le u\le t}$ and $\bar{K} = (\bar{K}_u)_{0\le u\le t}$ putting

$$\bar{b}_u = \mathbf{1}_{\{t\}}(u)(b_t - b_0) + b_{t-u}, \tag{6}$$

$$\bar{c}_u = \mathbf{1}_{\{t\}}(u)(c_t - c_0) + c_{t-u}, \tag{7}$$

$$\bar{K}_u(A) = \mathbf{1}_{\{t\}}(u)(K_t(A) - K_0(A)) + K_{t-u}(A), \tag{8}$$

where $\mathbf{1}_{\{\cdot\}}$ is the indicator function and $A \in \mathcal{B}(\mathbb{R})$. It means, for instance for $\bar{b} = (\bar{b}_u)_{0\le u\le t}$, that

$$\bar{b}_u = \begin{cases} b_{t-u} & \text{if} \quad 0 \le u < t, \\ b_t & \text{if} \quad u = t. \end{cases}$$

So, the function $\bar{b}$ can have a discontinuity at t, since in general $b_0 \neq b_t$. These functions are constructed to correspond, in a way, to the derivatives in time of the semimartingale characteristics of the process Y.

Lemma 2 (cf. [31])**.** *The process Y is a process with independent increments, it is a semimartingale with respect to its natural filtration, and its semimartingale triplet $(\bar{B}, \bar{C}, \bar{\nu})$ is given as: for $0 \leq s \leq t$,*

$$\bar{B}_s = \int_0^s \bar{b}_u du, \;\; \bar{C}_s = \int_0^s \bar{c}_u du, \;\; \bar{\nu}(du, dx) = \bar{K}_u(dx)\, du. \tag{9}$$

To obtain an integro-differential equation for the density, we introduce two important processes related to the process Y, namely the process $V = (V_s)_{0 \leq s \leq t}$ and $J = (J_s)_{0 \leq s \leq t}$, with

$$V_s = e^{-Y_s} J_s, \qquad J_s = \int_0^s e^{Y_u} du. \tag{10}$$

We underline that both processes depend on the parameter t, since it is so for the process Y.

We notice that according to Lemma 1, $I_t = V_t$ (**P**-a.s.), and then they have the same laws. As we will see, the process $V = (V_s)_{0 \leq s \leq t}$ is a Markov process with respect to the natural filtration $\mathbb{F}^Y = (\mathcal{F}_s^Y)_{0 \leq s \leq t}$ of the process Y and this fact will help us very much to find the equation for the density of the law of I_t.

Lemma 3. *The process $V = (V_s)_{0 \leq s \leq t}$ is a Markov process with respect to the natural filtration $\mathbb{F}^Y = (\mathcal{F}_s^Y)_{0 \leq s \leq t}$ of the process Y.*

Proof. We write that for $h > 0$

$$V_{s+h} = e^{-Y_{s+h}} \int_0^{s+h} e^{Y_u} du = e^{-(Y_{s+h} - Y_s)}\, [V_s + \int_s^{s+h} e^{Y_u - Y_s} du].$$

Then for all measurable bounded functions f

$$\mathbf{E}(f(V_{s+h}) \,|\, \mathcal{F}_s^Y) = \mathbf{E}\left(f(e^{-(Y_{s+h}-Y_s)}[V_s + \int_s^{s+h} e^{Y_u - Y_s} du]) \,|\, \mathcal{F}_s^Y \right) =$$
$$\mathbf{E}\left(f(e^{-(Y_{s+h}-Y_s)}[x + \int_s^{s+h} e^{Y_u - Y_s} du]) \right)_{|x = V_s},$$

since Y is a process with independent increments. Hence, $\mathbf{E}(f(V_{s+h}) \,|\, \mathcal{F}_s^Y)$ is a measurable function of V_s and we conclude that V is a Markov process with respect to the filtration generated by Y. □

Remark 2. In the case of Lévy processes a similar result was proved in [15], Lemma 5.1, using the homogeneity of the processes. Since a PII process is not homogeneous in general, the proof of the Markov property here is different.

We define the set of functions

$$\mathcal{C} = \{ f \in \mathcal{C}_b^2(\mathbb{R}^+) \,|\, \sup_{y \in \mathbb{R}^+} |f'(y) y| < \infty, \;\; \sup_{y \in \mathbb{R}^+} |f''(y) y^2| < \infty \}$$

such that $f(0) = f'(0) = 0$. Here we use the usual notation $\mathcal{C}_b^2(\mathbb{R}^+)$ for the set of twice continuously differentiable functions with bounded derivatives on $\mathbb{R}^+$.

For $0 \le s \le t$ we put

$$\overline{a}_s = -\overline{b}_s + \frac{1}{2}\overline{c}_s + \int_{\mathbb{R}} (e^{-x} - 1 + x)\bar{K}_s(dx). \tag{11}$$

We notice that the assumptions (3) and (4) imply that (λ-a.s.)

$$\int_{\mathbb{R}} |e^{-x} - 1 + x|\, \bar{K}_s(dx) < +\infty,$$

so that $\overline{a}_s$ is (λ-a.s.) well defined. We introduce also for $f \in \mathcal{C}$ the generator $(\mathcal{A}_s^V)_{0\le s<t}$ of the process V via

$$\mathcal{A}_s^V(f)(y) = (1 + y\,\overline{a}_s)\, f'(y) + \frac{1}{2}\overline{c}_s\, f''(y)\, y^2 + \tag{12}$$
$$\int_{\mathbb{R}} \left[f(ye^{-x}) - f(y) - f'(y)y(e^{-x} - 1)\right] \bar{K}_s(dx).$$

Theorem 1. *Under the assumptions* (3)*,* (4) *and* (5)*, the infinitesimal generator* $(\mathcal{A}_s^V)_{0\le s<t}$ *of the Markov process* V *is given by* (12)*. In addition, for* $0 \le s \le t$ *and* $f \in \mathcal{C}$

$$\mathbf{E}(f(V_s)) = \int_0^s \mathbf{E}(\,\mathcal{A}_u^V(f)(V_u)\,)du \tag{13}$$

where $\mathcal{A}_t^V = \lim_{s\to t-} \mathcal{A}_s^V$*. If for* $0 < s < t$ *the density* p_s *w.r.t. the Lebesgue measure* λ *of the law of* V_s *exists and belongs to the class* $\mathcal{C}^{1,2}(]0,t[\times\mathbb{R}^{+,*})$*, then* λ*-a.s.*

$$\frac{\partial}{\partial s} p_s(y) = \frac{1}{2}\bar{c}_s \frac{\partial^2}{\partial y^2}(y^2\, p_s(y)) - \frac{\partial}{\partial y}((\bar{a}_s\, y + 1)\, p_s(y)) + \tag{14}$$
$$\int_{\mathbb{R}} \left[e^x\, p_s(ye^x) - p_s(y) + (e^{-x} - 1)\frac{\partial}{\partial y}(yp_s(y))\right] \bar{K}_s(dx)$$

and the density p_t *of the law of* I_t *verifies*

$$p_t(y) = \int_0^t \Bigg\{\frac{1}{2}\bar{c}_s \frac{\partial^2}{\partial y^2}(y^2\, p_s(y)) - \frac{\partial}{\partial y}((\bar{a}_s\, y + 1)\, p_s(y)) + \tag{15}$$
$$\int_{\mathbb{R}} \left[e^x\, p_s(ye^x) - p_s(y) + (e^{-x} - 1)\frac{\partial}{\partial y}(yp_s(y))\right] \bar{K}_s(dx)\Bigg\}\, ds.$$

The proof of Theorem 1 will be divided in four parts. In Lemma 4 we give a semimartingale decomposition of the process $(e^{-Y_s})_{0\le s\le t}$, then in Lemma 5 we prove a semimartingale decomposition of $(f(V_s))_{0\le s\le t}$ with f belonging to $\mathcal{C}$ and in Lemma 6 we prove (13), and finally in Lemma 7 we obtain the equation (14). Then we combine all results together to get (15).

Lemma 4. *For* $0 \le s \le t$

$$e^{-Y_s} = e^{-Y_0} + A_s + N_s \tag{16}$$

where $A = (A_s)_{0\le s\le t}$ *is a process with locally bounded variation and* $N = (N_s)_{0\le s\le t}$ *is a local martingale.*

Proof. By Ito's formula we get

$$e^{-Y_s} = 1 - \int_0^s e^{-Y_{u-}} dY_u + \frac{1}{2}\int_0^s e^{-Y_{u-}} d\langle Y^c\rangle_u + \int_0^s \int_{\mathbb{R}} e^{-Y_{u-}}(e^{-x} - 1 + x)\mu_Y(du, dx) \tag{17}$$

where μ_Y is the measure of jumps of the process Y.

Let us write the expressions for A and N. From Ito's formula obtained previously we find that the process $(A_s)_{s\geq 0}$ is given by

$$A_s = \int_0^s e^{-Y_{u-}}[-\bar{b}_u + \frac{1}{2}\bar{c}_u + \int_{\mathbb{R}} (e^{-x} - 1 + x)\bar{K}_u(dx)]\, du \tag{18}$$

and it is a process of locally bounded variation on bounded intervals. In fact, let us introduce a sequence of stopping times: for $n \geq 1$

$$\tau_n = \inf\{0 \leq s \leq t \,|\, e^{-Y_s} \geq n\} \tag{19}$$

with $\inf\{\emptyset\} = \infty$. We notice that this sequence of stopping times tends to $+\infty$ as $n \to \infty$. Then, since $e^{-Y_{s-}} < n$ on the stochastic interval $[0, \tau_n[$, we get from (3), (4) and (5) that

$$\mathrm{Var}(A)_{s\wedge\tau_n} \leq n \int_0^t \left[|\bar{b}_u| + \frac{1}{2}\bar{c}_u + \int_{\mathbb{R}} |e^{-x} - 1 + x|\, \bar{K}_u(dx)\right] du < \infty.$$

In (16) the process $N = (N_s)_{s\geq 0}$ is defined by

$$N_s = -\int_0^s e^{-Y_{u-}} d\bar{M}_u + \int_0^s \int_{\mathbb{R}} e^{-Y_{u-}}(e^{-x} - 1 + x)(\mu_Y(du, dx) - \bar{K}_u(dx)du). \tag{20}$$

In the relation (20), the process $\bar{M}$ is the local martingale component of the semi-martingale decomposition of Y, $Y_s = \bar{B}_s + \bar{M}_s$, and μ_Y is the measure of jumps of the process Y. It should be noticed that since Y is a process with independent increments and $\bar{B}$ is deterministic, $\bar{M}$ is a martingale (see [36], Th. 58, p. 45) as well as its pure discontinuous part $\bar{M}^d$. Then, the process $(N_{s\wedge\tau_n})_{0\leq s\leq t}$ is a local martingale as a stochastic integral of a bounded function w.r.t. a martingale. □

Lemma 5. *For $f \in \mathcal{C}$ we have*

$$f(V_s) = f(V_0) + B_s^V + N_s^V \tag{21}$$

where B^V is a process with locally bounded variation and N^V is a local martingale.

Proof. For $f \in \mathcal{C}$ and $0 \leq s \leq t$ we write Ito's formula:

$$f(V_s) = f(V_0) + \int_0^s f'(V_{u-})dV_u + \frac{1}{2}\int_0^s f''(V_{u-})d\langle V^c\rangle_u + \int_0^s \int_{\mathbb{R}} \big(f(V_{u-} + x) - f(V_{u-}) - f'(V_{u-})x\big)\, \mu_V(du, dx) \tag{22}$$

where μ_V is the measure of jumps of the process V. From the definition of the process V we can easily find that

$$dV_s = ds + J_s\, d(\,e^{-Y_s}\,). \tag{23}$$

Combining this formula with Ito's decomposition (17) we find that

$$dV_s^c = -e^{-Y_{s-}} J_s\, dY_s^c = -V_{s-}\, dY_s^c.$$

Moreover,

$$d\langle V^c\rangle_s = V_{s-}^2\, d\langle Y^c\rangle_s,$$

and

$$\Delta V_s = V_s - V_{s-} = e^{-Y_{s-}} J_s(e^{-\Delta Y_s} - 1) = V_{s-}(e^{-\Delta Y_s} - 1).$$

We use the relations (23), (16), (18), (20) and (22) to obtain a final decomposition for $f(V_s)$. To present this final decomposition, we put for $y \geq 0$ and $x \in \mathbb{R}$

$$F(y,x) = f(ye^{-x}) - f(y) - f'(y)y(e^{-x} - 1),$$

and also

$$B_s^V = \int_0^s \left[f'(V_{u-})(1 + \bar{a}_u\, V_{u-}) + \frac{1}{2} f''(V_{u-}) V_{u-}^2\, \bar{c}_u + \int_{\mathbb{R}} F(V_{u-}, x)\bar{K}_u(dx) \right] du$$

and

$$N_s^V = \int_0^s f'(V_{u-})\, V_{u-}[-d\bar{M}_u + \int_{\mathbb{R}} (e^{-x} - 1 + x)(\mu_Y(du,dx) - \bar{K}_u(dx)\, du] +$$
$$\int_0^s \int_{\mathbb{R}} F(V_{u-}, x)(\mu_Y(du,dx) - \bar{K}_u(dx)\, du).$$

It remains to show that the process B^V is of locally bounded variation and the process N^V is a local martingale. Let us use the sequence of stopping times τ_n defined by (19) and let

$$D = \sup_{y\in\mathbb{R}} \max(|f(y)|, |f'(y)|, |f'(y)y|, |f''(y)y^2|).$$

Then, for $0 \leq s \leq t$,

$$\mathrm{Var}(B^V)_{s\wedge\tau_n} \leq\; D \int_0^t \left(1 + |\bar{b}_u| + \bar{c}_u + \int_{\mathbb{R}} |e^{-x} - 1 + x|\, \bar{K}_u(x)dx \right) du$$
$$+ \int_0^t \int_{\mathbb{R}} |F(V_{u-}, x)| \bar{K}_u(dx)\, du.$$

The first term of the r.h.s. is finite since the functions $(\bar{B}_s)_{0\leq s\leq t}$ and $(\bar{C}_s)_{0\leq s\leq t}$ have finite variation on finite intervals and since (5) holds. Now, using the Taylor–Lagrange formula of the second order, we find that for $y > 0$ and $|x| \leq 1$

$$|F(y,x)| = \frac{1}{2} |f''(y(1 + \theta(e^{-x} - 1)))| y^2 (e^{-x} - 1)^2 \leq \frac{D(e^{-x} - 1)^2}{2[1 + \theta(e^{-x} - 1)]^2}$$

where $0 < \theta < 1$. Since for $|x| \leq 1$, $1 + \theta(e^{-x} - 1) \geq \frac{1}{e}$ and $|e^{-x} - 1| \leq e|x|$, we find that $|F(V_{u-}, x)| \leq \frac{1}{2} De^4 x^2$.

For $|x| > 1$ we use the Taylor–Lagrange formula of the first order to get

$$|F(y,x)| = |(f'(y(1+\theta(e^{-x}-1))) - f'(y))y(e^{-x}-1)| \leq \\ D\left[|1+\theta(e^{-x}-1)|^{-1} + 1\right] |e^{-x}-1|.$$

Again, for $x > 1$, $1 + \theta(e^{-x} - 1) \geq e^{-x}$, and for $x < -1$, $1 + \theta(e^{-x} - 1) \geq 1$. Moreover, for $x > 1$, $|e^{-x} - 1| \leq 1$ and for $x < -1$, $|e^{-x} - 1| \leq e^{-x}$. Finally,

$$|F(y,x)| \leq C\left(e^{|x|}\mathbf{1}_{\{|x|>1\}} + x^2\,\mathbf{1}_{\{|x|\leq 1\}}\right)$$

with some positive constant C. Then, the assumptions (4) and (5) imply that

$$\int_0^t \int_{\mathbb{R}} |F(V_{u-}, x)| \bar{K}_u(dx)\, du < +\infty.$$

So, the process B^V is of locally bounded variation.

Using the above results we see also that $(N^V_{s\wedge\tau_n})_{0\leq s\leq t}$ is a local martingale as an integral of a bounded function w.r.t. a martingale. □

Lemma 6. *Under the assumptions of Theorem 1, for* $0 \leq s \leq t$ *and* $f \in \mathcal{C}$ *we have the equation* (13).

Proof. Let $(\tau'_n)_{n\in\mathbb{N}}$ be a localizing sequence for N_V and $\bar{\tau_n} = \tau_n \wedge \tau'_n$ where τ_n is defined by (19). Let $s \in [0, t[$ and $\delta > 0$ such that $s + \delta \leq t$. Then, from the previous decomposition using the localisation we get

$$\mathbf{E}(f(V_{(s+\delta)\wedge\bar{\tau_n}}) - f(V_{s\wedge\bar{\tau_n}}) \,|\, \mathcal{F}^Y_s) = \mathbf{E}(B^V_{(s+\delta)\wedge\bar{\tau_n}} - B^V_{s\wedge\bar{\tau_n}} \,|\, \mathcal{F}^Y_s).$$

Since f is a bounded function and $\lim_{n\to\infty}\bar{\tau_n} = +\infty$, we can pass to the limit in the l.h.s. by the Lebesgue convergence theorem. The same can be done on the r.h.s. since the process $B^V = (B^V_s)_{0\leq s\leq t}$ is a process of bounded variation on bounded intervals, uniformly in s and n. After taking a limit as $n \to +\infty$ we get that

$$\mathbf{E}(f(V_{s+\delta}) - f(V_s) \,|\, \mathcal{F}^Y_s) = \mathbf{E}(B^V_{s+\delta} - B^V_s \,|\, \mathcal{F}^Y_s).$$

Now, we write the expression for $B^V_{s+\delta} - B^V_s$:

$$B^V_{s+\delta} - B^V_s = \int_s^{s+\delta} [f'(V_{u-})(1 + \bar{a}_u\, V_{u-}) + \\ \frac{1}{2} f''(V_{u-})V^2_{u-}\,\bar{c}_u + \int_{\mathbb{R}} F(V_{u-}, x)\bar{K}_u(dx)\,]du.$$

We remark that

$$\lim_{\delta\to 0} \frac{B^V_{s+\delta} - B^V_s}{\delta} = f'(V_{s-})(1+\bar{a}_s\,V_{s-}) + \frac{1}{2} f''(V_{s-})V^2_{s-}\,\bar{c}_s + \int_{\mathbb{R}} F(V_{s-}, x)\bar{K}_s(dx).$$

We show that the quantities $\frac{B^V_{s+\delta}-B^V_s}{\delta}$ are uniformly bounded, for small $\delta > 0$, by a constant. In fact, we can write that

$$\frac{\mid B^V_{s+\delta} - B^V_s \mid}{\delta} \leq \frac{C}{\delta} \int_s^{s+\delta} \left[\left(1 + \bar{a}_u + \frac{1}{2} \bar{c}_u \right) + \int_{\mathbb{R}} |F(V_{u-}, x)| \bar{K}_u(dx) \right] du.$$

We use the estimations for $|F(V_{s-}, x)|$ obtained previously, and the fact that the quantities $\frac{1}{\delta}\int_s^{s+\delta} \bar{a}_u du$, $\frac{1}{\delta}\int_s^{s+\delta} \bar{c}_u du$ and

$$\frac{1}{\delta} \int_s^{s+\delta} \int_{\mathbb{R}} \left(x^2 I_{\{|x|\leq 1\}} + e^{|x|} I_{\{|x|>1\}} \right) \bar{K}_u(dx) du$$

are uniformly bounded by a constant for $s \in [0, t[$ and small values of $\delta > 0$. This conclusion follows from the fact that derivatives of the semimartingale characteristics of the process Y are deterministic càdlàg functions. So, we deduce that the quantities $\frac{|B^V_{s+\delta}-B^V_s|}{\delta}$ are uniformly bounded for $s \in [0, t[$ for small $\delta > 0$ by a constant, too.

Under these conditions we can exchange the limit and the conditional expectation and it gives us the expression for the generator of V at $0 \leq s < t$. As a conclusion, we get that for $0 \leq s < t$

$$\frac{d}{ds} \mathbf{E}(f(V_s)) = \mathbf{E} \mathcal{A}^V_s(f)(V_{s-}). \tag{24}$$

Let us prove that we can replace V_{s-} by V_s in the above expression. In fact, for $\lambda \in \mathbb{R}$

$$\mathbf{E}(e^{i\lambda \ln(\frac{V_s}{V_{s-}})}) = \mathbf{E}(e^{-i\lambda \Delta Y_s}) = \lim_{h \to 0+} \mathbf{E}(e^{-i\lambda(Y_{s+h}-Y_s)}) = 1,$$

since the characteristic function of Y is continuous in time and Y is PII. Hence, $V_{s-} = V_s$ ($\mathbf{P}$-a.s.) and they have the same laws. Then after the replacement of V_{s-} by V_s in (24) and the integration w.r.t. s we obtain (13). For $s = t$ we take $\lim_{s \to t-}$ in (13). $\square$

Lemma 7. *Under the assumptions of Theorem 1, for* $0 < s < t$ *we have the relation* (14).

Proof. We denote by P_s the law of V_s. Then from (13) we get that for $0 < s < t$

$$\int_0^s \int_0^\infty \left[f'(y)(1 + y\,\bar{a}_u) + \frac{1}{2} f''(y) y^2\, \bar{c}_u + \right. \tag{25}$$

$$\left. \int_{\mathbb{R}} \left(f(ye^{-x}) - f(y) - f'(y)y(e^{-x} - 1) \right) \bar{K}_u(x)dx \right] P_u(dy)\, du = \int_0^\infty f(y) P_s(dy).$$

Moreover, since we assume that the law P_s of V_s has a density p_s w.r.t. the Lebesgue measure, it gives

$$\int_0^s \int_0^\infty \left[f'(y)(1 + y\,\bar{a}_u) + \frac{1}{2} f''(y) y^2\, \bar{c}_u + \right. \tag{26}$$

$$\int_{\mathbb{R}} \left(f(ye^{-x}) - f(y) - f'(y)y(e^{-x}-1)\right) \bar{K}_u(x)dx\Big] p_u(y)dy\,du$$
$$= \int_0^\infty f(y)p_s(y)dy.$$

To obtain the equation for the density, we consider the set of continuously differentiable functions on compact support $\mathcal{C}_K^2 \subseteq \mathcal{C}$. We take the right-hand partial derivative in s of the above equation to get

$$\int_0^\infty \Big[f'(y)(1 + y\,\bar{a}_s) + \frac{1}{2} f''(y)y^2\,\bar{c}_s + \tag{27}$$
$$\int_{\mathbb{R}} \left(f(ye^{-x}) - f(y) - f'(y)y(e^{-x}-1)\right) \bar{K}_s(x)dx\Big] p_s(y)$$
$$= \int_0^\infty f(y)\,\frac{\partial}{\partial s} p_s(y)\,dy.$$

Using the integration by parts formula we deduce that

$$\int_0^\infty f'(y)\,p_s(y)dy = -\int_0^\infty \frac{\partial}{\partial y}(p_s(y))\,f(y)dy,$$
$$\int_0^\infty f'(y)\,y\,p_s(y)dy = -\int_0^\infty \frac{\partial}{\partial y}(y\,p_s(y))\,f(y)dy,$$
$$\int_0^\infty f''(y)\,y^2\,p_s(y)dy = \int_0^\infty \frac{\partial^2}{\partial y^2}(y^2\,p_s(y))\,f(y)dy.$$

By the change of variables and by the integration by parts we obtain

$$\int_0^\infty \int_{\mathbb{R}} p_s(y)\left[f(ye^{-x}) - f(y) - f'(y)y(e^{-x}-1)\right]\bar{K}_s(dx)dy =$$
$$\int_0^\infty \left(\int_{\mathbb{R}} [e^x p_s(ye^x) - p_s(y) + (e^{-x}-1)\frac{\partial}{\partial y}(yp_s(y)]\,\bar{K}_s(dx)\right) f(y)dy.$$

The mentioned relations together with the equation (27) give that for all $f \in \mathcal{C}_K^2$

$$\int_0^\infty f(y)\Big[-\frac{\partial}{\partial s}p_s(y) + \frac{1}{2}\bar{c}_s\frac{\partial}{\partial y}(y^2 p_s(y)) - \frac{\partial}{\partial y}((\bar{a}_s y + 1)p_s(y))$$
$$+ \int_{\mathbb{R}} e^x p_s(ye^x) - p_s(y) + (e^{-x}-1)\frac{\partial}{\partial y}(yp_s(y))\bar{K}_s(dx)\Big] = 0,$$

and it proves our claim about the equation for p_s. □

Proof of Theorem 1. We use the result of Lemma 7. Then we integrate the equation for p_s on the interval $]0, t-\delta[$ for $\delta > 0$ and we pass to the limit as $\delta \to 0$. Since the laws of V_{t-} and V_t coincide, we get in this way the equation for p_t. Since $I_t = V_t$ ($\mathbf{P}$-a.s.), p_t is a density of the law of I_t. □

3 Some results about the existence of smooth densities

In the case of Lévy processes, the question of existence of a smooth density of I_∞ in the Schwarz sense was considered in Carmona, Petit, Yor [15] and Bertoin, Yor [10]. Sufficient conditions for the smoothness of the one-dimensional law of an Ornstein–Uhlenbeck process driven by a pure discontinuous Lévy process was obtained in Bondarchuk, Kulik [14], namely under the Kallenberg condition the density of the one-dimensional distribution of an Ornstein–Uhlenbeck process is of the class $C_b^\infty(\mathbb{R})$.

The question of existence of the density of the law of V_s, $0 < s < t$, of the class $C^{1,2}(]0, t[, \mathbb{R})$ is a rather difficult one, which was, at our knowledge, an open question in all cited works on exponential functionals. We give here a partial answer to this question via the known result on the Malliavin calculus from Bichteler, Gravereaux, Jacod [11]. For the convenience of the readers we present this result here in the one-dimensional setting.

We consider the stochastic differential equation

$$Z_t^x = x + \int_0^t a(Z_{s-}^x)\,ds + \int_0^t b(Z_{s-}^x)dW_s + \int_0^t \int_{\mathbb{R}} c(Z_{s-}^x, z)(\mu - \nu)(ds, dz)$$

where $x \in \mathbb{R}$, a, b, c are measurable functions on $\mathbb{R}$ and $\mathbb{R}^2$ respectively, W is the standard Brownian motion, and μ and ν are the jump measure and its compensator of Z^x. It is assumed that the solution of this equation exists and is unique, and also that the following assumptions hold.

Assumption (A-r):

(i) a and b are r-times differentiable with bounded derivatives of all orders from 1 to r,

(ii) $c(\cdot, z)$ is r-times differentiable and there exists a sigma-finite measure G on $\mathbb{R}$ such that

  (a) $c(0, \cdot) \in \bigcap_{2\le p<\infty} L^p(\mathbb{R}^*, G)$,

  (b) for $1 \le n \le r$, $\sup_y \mid \frac{\partial^n}{\partial y^n}(c(y, \cdot) \mid \in \bigcap_{2\le p<\infty} L^p(\mathbb{R}^*, G)$.

Assumption (SB): there exist $\gamma > 0$ and $\delta > 0$ such that

$$b^2(y) \ge \frac{\gamma}{1 + |y|^\delta}.$$

Assumption (SC-bis): there exists $\zeta > 0$ such that for all $u \in [0, 1]$

$$|1 + u\frac{\partial}{\partial y}c(y, z)| > \zeta.$$

Theorem 2.29 (cf. [11], p. 15) *Suppose that the assumptions* (A-(2r+10))*,* (SB) *and* (SC-bis) *are satisfied. Then for* $t > 0$ *the law of* Z_t^x *has a density* $p_t(x, y)$ *w.r.t. the Lebesgue measure and the map* $(t, x, y) \to p_t(x, y)$ *is of the class* $C^r(]0, t] \times \mathbb{R} \times \mathbb{R})$.

To apply this theorem, let us write a stochastic differential equation for $(V_s)_{0\leq s\leq t}$. For that we put for $0 \leq s \leq t$

$$\begin{cases} a_s(y) = y(-\bar{b}_s + \frac{1}{2}\bar{c}_s + \int_{\mathbb{R}}(e^{-z} - 1 + z)\bar{K}_s(dz) + 1, \\ b_s(y) = y\sqrt{\bar{c}_s}, \\ c_s(y, z) = y(e^{-z} - 1). \end{cases}$$

Proposition 1. *Suppose that*

$$\int_0^t \int_{\mathbb{R}} |e^{-z} - 1 + z|\bar{K}_s(dz) < +\infty$$

and that $\bar{c}_s > 0$ for $0 < s \leq t$. Then the process $(V_s)_{0\leq s\leq t}$ satisfies the stochastic differential equation

$$V_s = \int_0^s a_u(V_{u-})du - \int_0^s b_u(V_{u-})dW_u + \int_0^s \int_{\mathbb{R}} c_u(V_{u-}, z)(d\mu_Y - \bar{K}_u(dz)du) \tag{28}$$

where μ_Y is the jump measure of Y and W is the Dubins–Dambis–Schwarz Brownian motion corresponding to the continuous martingale part Y^c of Y.

Proof. We recall that V_s is defined by (10). Let us introduce the process $\hat{Y}$ via the relation: for $0 \leq s \leq t$,

$$e^{-Y_s} = \mathcal{E}(\hat{Y})_s \tag{29}$$

where $\mathcal{E}(\cdot)$ is the Doléans-Dade exponential. Then,

$$V_s = \mathcal{E}(\hat{Y})_s \int_0^s \frac{du}{\mathcal{E}(\hat{Y})_u}$$

and we can see by the integration by parts formula that $(V_s)_{0\leq s\leq t}$ is the unique strong solution of the equation

$$dV_s = V_{s-}d\hat{Y}_s + ds \tag{30}$$

with the initial condition $V_0 = 0$. Using the definition of the Doléans-Dade exponential we see that (29) is equivalent to

$$e^{-Y_s} = e^{\hat{Y}_s - \frac{1}{2}<\hat{Y}^c>} \prod_{0<u\leq s} (1 + \Delta\hat{Y}_u)\, e^{-\Delta\hat{Y}_u}$$

where $\hat{Y}^c$ is a continuous martingale part of $\hat{Y}$. From this equality we find that $\hat{Y}^c_s = -Y^c_s$, $\ln(1 + \Delta\hat{Y}_s) = -\Delta Y_s$ and that the semimartingale characteristics $(\hat{B}, \hat{C}, \hat{\nu})$ of $\hat{Y}$ are:

$$\begin{cases} \hat{B}_s = -\bar{B}_s + \frac{1}{2}\bar{C}_s + \int_0^s \int_{\mathbb{R}}(e^{-z} - 1 + z)\bar{K}_u(dz)du, \\ \hat{C}_s = \bar{C}_s, \\ \hat{\nu}(ds, dz) = (e^{-z} - 1)\bar{K}_s(dz)ds. \end{cases}$$

Since $(\bar{B}, \bar{C}, \bar{\nu})$ are absolutely continuous w.r.t. the Lebesgue measure with the derivatives $(\bar{b}, \bar{c}, \bar{K})$, we get that

$$\hat{Y}_s = \int_0^s (-\bar{b}_u + \frac{1}{2}\bar{c}_u + \int_{\mathbb{R}}(e^{-z} - 1 + z)\bar{K}_u(dz))du - \int_0^s \sqrt{\bar{c}_u}\, dW_u +$$

$$\int_0^s \int_{\mathbb{R}} (e^{-z} - 1)(\mu_Y(du, dz) - \bar{K}_u(dz)du)$$

where W is the Dubins–Dambis–Schwarz Brownian motion corresponding to the continuous martingale part of Y. Let us put this decomposition into (30), and we obtain (28). □

To apply Theorem 2.29 from [11] we suppose that the process X, and hence also the process Y, are Lévy processes. We introduce a supplementary process

$$V_s^\epsilon = \epsilon + \mathcal{E}(\hat{Y})_s \int_0^s \frac{du}{\mathcal{E}(\hat{Y})_u}$$

with $\epsilon > 0$. We see that $V_s^\epsilon - \epsilon = V_s$, and $V_s^0 = V_s$, and also that the density $p_s(\epsilon, y)$ of the law of V_s^ϵ w.r.t. the Lebesgue measure and the density $p_s(y)$ of the law of V_s exist or not at the same time and they are related: for all $x > 0$ and $y > 0$

$$p_s(\epsilon, y + \epsilon) = p_s(y).$$

So, both densities are of the same regularity w.r.t. (s, y). In addition we have: for all $s \geq 0$ and all $\omega \in \Omega$, $V_s^\epsilon(\omega) \geq \epsilon$.

Proposition 2. *Suppose that X is a Lévy process with a triplet (b_0, c_0, K_0) and the following conditions are satisfied:*

1. $c_0 > 0$,
2. $\int_{z<-1} e^{-pz} K_0(dz) < +\infty$ *for* $p \geq 2$,
3. *there exists a constant* $A > 0$ *such that* $K_0(]A, +\infty[) = 0$.

Then, for $s > 0$, the law of V_s has a density p_s and the map $(s, y) \to p_s(y)$ is of the class $C^\infty(]0, t] \times \mathbb{R}^{+,*})$.

Proof. When X is a Lévy process, and hence Y is a Lévy process with the same parameters, the functions a_s, b_s, c_s figured in (28) are independent of s and are equal to:

$$\begin{cases} a(y) = y(-b_0 + \frac{1}{2}c_0 - \int_{\mathbb{R}}(e^{-z} - 1 + z)K_0(dz)) + 1, \\ b(y) = y\sqrt{c_0}, \\ c(y, z) = y(e^{-z} - 1). \end{cases}$$

We consider now the process V^ϵ with $\epsilon > 0$. This process satisfy the same equation (28) as the process V does with the replacement of V_{u-} and V_s by V_{u-}^ϵ and V_s^ϵ, respectively, and also with the replacement of the functions $a_s(y)$, $b_s(y)$ and $c_s(y, z)$ by $a(y)$, $b(y)$, $c(y, s)$ with $y \geq \epsilon$.

We see that the Assumption (A-r) is satisfied for all $r \geq 1$ with $G = K_0$. The Assumption (SB) is valid for $\gamma = \epsilon^2 c_0$ and any $\delta > 0$, and the Assumption (SC-bis) is satisfied with $\zeta = \frac{1}{2}e^{-A}$. As a conclusion, the map $(s, x, y) \to p_s(x, y)$ is of the class C^∞ $(]0, t] \times [\epsilon, +\infty[\times[\epsilon, +\infty[)$ for all $\epsilon > 0$, and, hence, the map $(s, y) \to p_s(x, y+x)$ is of the class $C^\infty(]0, t] \times \mathbb{R}^{+,*} \times \mathbb{R}^{+,*})$. Finally, the map $(s, y) \to p_s(y)$ is of the class $C^\infty(]0, t], \mathbb{R}^{+,*})$. □

For the PII case we obtain the following partial result.

Corollary 1. *Let $s \in]0, t[$ be fixed. Suppose that*

1. $\int_0^s \bar{c}_u du > 0$,
2. $\int_0^s \int_{z<-1} e^{-pz} K_s(dz) < +\infty$ *for* $p \geq 2$,
3. *there exists a constant* $A > 0$ *such that* $K_s(]A, +\infty[) = 0$ *for all* $0 < s < t$.

Then, the law of V_s has a density p_s such that the map $y \to p_s(y)$ is of the class $C^\infty(\mathbb{R}^{+,*})$.

Proof. We notice that the law of Y_s coincide at the time s with the law of a Lévy process $\tilde{Y}$ with the triplet $(\frac{1}{s}\bar{B}_s, \frac{1}{s}\bar{C}_s, \frac{1}{s}\int_0^s \int_{\mathbb{R}} \bar{K}_u(dz)du)$. Then we consider a GOU process driven by $\tilde{Y}$. The previous proposition can be applied, and this gives the claim. $\square$

4 When X is a Lévy process

In this section we consider a particular case of Lévy processes. Namely, let X be a Lévy process with the parameters (b_0, c_0, K_0). As before, we suppose that

$$\int_{\mathbb{R}} (x^2 \wedge 1) K_0(dx) < +\infty \text{ and } \int_{|x|>1} e^{|x|} K_0(dx) < +\infty \tag{31}$$

and we put

$$a_0 = -b_0 + \frac{1}{2}c_0 + \int_{\mathbb{R}} (e^{-x} - 1 + x) K_0(dx).$$

Due to the homogeneity of a Lévy process, the equation for the density can be simplified as we can see from the following proposition.

Proposition 3. *Suppose that (31) holds and the density p_t of the law of I_t exists and belongs to the class $\mathcal{C}^{1,2}(]0, t] \times \mathbb{R}^{+,*})$. Then this density satisfies the equation*

$$\begin{aligned} \frac{\partial}{\partial t} p_t(y) = \frac{1}{2} c_0 \frac{\partial^2}{\partial y^2}(y^2\, p_t(y)) - \frac{\partial}{\partial y}((a_0 y + 1)\, p_t(y)) + \\ \int_{\mathbb{R}} \left[e^x p_t(ye^x) - p_t(y) + (e^{-x} - 1)\frac{\partial}{\partial y}(y p_t(y)) \right] K_0(dx). \end{aligned} \tag{32}$$

In the particular case, when $I_\infty < +\infty$ ($\mathbf{P}$-a.s.) and the density p_∞ of the law of I_∞ exists and belongs to the class $\mathcal{C}^2(\mathbb{R}^{+,*})$, we have

$$\begin{aligned} \frac{1}{2} c_0 \frac{d^2}{dy^2}(y^2\, p_\infty(y)) - \frac{d}{dy}((a_0 y + 1)\, p_\infty(y)) + \\ \int_{\mathbb{R}} \left[e^x p_\infty(ye^x) - p_\infty(y) + (e^{-x} - 1)\frac{d}{dy}(y p_\infty(y)) \right] K_0(dx) = 0. \end{aligned} \tag{33}$$

Remark 3. A similar equation for the density of I_∞ in the case when $\int_{\mathbb{R}}(|x| \wedge 1)\, K_0(dx) < \infty$ was obtained in [15]. We recall that the condition on K_0 in [15] is stronger at zero than our condition. This explains the fact that our equation has a slightly different form. Namely, under the condition in [15] one can separate third term in the integral part and combine it with the second term of the above equation. Moreover, similar equations for I_∞ can be found in [24] and [26]. It should be mentioned that the cited authors did not obtain the equation for the density of I_t.

Proof. In the case of Lévy processes we write that (**P**-a.s.)

$$V_s = e^{-Y_s}\, J_s = e^{-X_t + X_{(t-s)-}} \int_0^s e^{X_t - X_{(t-u)-}} du =$$
$$\int_0^s e^{X_{(t-s)-} - X_{(t-u)-}} du = \int_0^s e^{X_{(t-s)} - X_{(t-u)}} du.$$

Due to the homogeneity of Lévy processes we have the following identity in law:

$$\mathcal{L}((X_{t-u} - X_{t-s})_{0 \le u \le s}) = \mathcal{L}((X_{s-u})_{0 \le u \le s}).$$

Then,

$$\mathcal{L}(\int_0^s e^{X_{(t-s)} - X_{(t-u)}} du) = \mathcal{L}(\int_0^s e^{-X_{(s-u)}} du) = \mathcal{L}(\int_0^s e^{-X_u} du)$$

where the last equality is obtained by the time change. As a conclusion, $\mathcal{L}(V_s) = \mathcal{L}(I_s)$ for $0 \le s \le t$, and, hence, $(p_s)_{0<s\le t}$ are the densities of the laws of $(I_s)_{0<s\le t}$. In addition, again due to the homogeneity, for all $0 \le s \le t$, $\bar{b}_s = b_{t-s} = b_0$, $\bar{c}_s = c_{t-s} = c_0$, $\bar{K}_s(dx) = K_{t-s}(dx) = K_0(dx)$. Then, from Theorem 1 we obtain (32).

Again due to the homogeneity, for $0 < s \le t$, the generator $\mathcal{A}^V_s(f) = \mathcal{A}(f)$, where

$$\mathcal{A}(f)(y) = (1 + y\, a_0)\, f'(y) +$$
$$\frac{1}{2} c_0\, f''(y)\, y^2 + \int_{\mathbb{R}} \big[f(ye^{-x}) - f(y) - f'(y) y (e^{-x} - 1)\big]\, K_0(dx)$$

and it does not depend on s. Moreover, since $\mathcal{L}(V_s) = \mathcal{L}(I_s)$ for $0 \le s \le t$, the equality (13) becomes

$$\mathbf{E} f(I_s) = \int_0^s \mathbf{E}\mathcal{A}(f)(I_u) du.$$

We suppose now that $I_\infty < +\infty$ (**P**-a.s.). We divide both sides of the above equality by s and let s go to infinity. Since f is bounded, we get zero as a limit on the left-hand side. Since $I_s \to I_\infty$ as $s \to +\infty$ for each $\omega \in \Omega$, we also get for $f \in \mathcal{C}$

$$\lim_{s \to \infty} \mathbf{E}\mathcal{A}(f)(I_s) = \mathbf{E}\mathcal{A}(f)(I_\infty).$$

Then, $\mathbf{E}\mathcal{A}(f)(I_\infty) = 0$ and we obtain (33) in the same way as in Theorem 1, by the integration by parts and the time change. □

Corollary 2. *Under the assumptions of Proposition 3, the distribution function F_t of I_t satisfies the second order integro-differential equation*

$$\frac{\partial}{\partial t}F_t(y) = \frac{1}{2}c_0\frac{\partial}{\partial y}(y^2\,\frac{\partial}{\partial y}F_t(y)) - (a_0 y + 1)\,\frac{\partial}{\partial y}F_t(y) + \int_{\mathbb{R}}\left[F_t(ye^x) - F_t(y) + (e^{-x} - 1)\,y\,\frac{\partial}{\partial y}F_t(y))\right]K_0(dx) \tag{34}$$

with the boundary conditions

$$F_t(0) = 0, \quad \lim_{y\to+\infty} F_t(y) = 1.$$

When $I_\infty < +\infty$, for the distribution function F_∞ of the law of I_∞, the similar equation

$$\frac{1}{2}c_0\frac{d}{dy}(y^2\,F'_\infty(y)) - (a_0 y + 1)\,F'_\infty(y) + \int_{\mathbb{R}}\left[F_\infty(ye^x) - F_\infty(y) + (e^{-x} - 1)\,y\,F'_\infty(y))\right]K_0(dx) = 0 \tag{35}$$

is valid with the similar boundary conditions

$$F_\infty(0) = 0, \quad \lim_{y\to+\infty} F_\infty(y) = 1.$$

Proof. We integrate each term of the equation (32) of Proposition 3 on $[0, y]$ and use the fact that

$$\int_0^y p_t(u)du = F_t(y) - F_t(0) = F_t(y)$$

since $F_t(0) = 0$. We take into account the fact that the map $(t, u) \to p_t(u)$ is of the class $C^{1,2}(\mathbb{R}^{+,*} \times \mathbb{R}^{+,*})$ what allows to exchange the integration and the derivation. We do the same for $F_\infty(y)$. □

Corollary 3. *(cf. [17, 13]) Let us consider a Brownian motion with drift, i.e.*

$$dX_t = b_0 dt + \sqrt{c_0}dW_t$$

where $c_0 \neq 0$ and $b_0 \in \mathbb{R}$. Then the law of the exponential functional I_t associated with X has a density which satisfies

$$\frac{\partial}{\partial t}p_t(y) = \frac{1}{2}c_0\frac{\partial^2}{\partial y^2}(y^2\,p_t(y)) - \frac{\partial}{\partial y}((a_0 y + 1)\,p_t(y)).$$

*In particular, when $b_0 > 0$ we have $I_\infty < +\infty$ (**P**-a.s.) and*

$$p_\infty(x) = \frac{1}{\Gamma(\frac{2b_0}{c_0})\,x}\left(\frac{2}{c_0 x}\right)^{\frac{2b_0}{c_0}}\exp\left(-\frac{2}{c_0 x}\right). \tag{36}$$

Proof. From Proposition 3 we find the equation for p_t. From Corollary 1 we get the equation for F_∞:

$$\frac{1}{2}c_0\frac{d}{dy}(y^2 F'_\infty(y)) - (a_0 y + 1) F'_\infty(y) = 0.$$

This equation is equivalent to

$$\frac{1}{2}c_0 y^2 F''_\infty(y) - ((a_0 - c_0)y + 1) F'_\infty(y) = 0.$$

By the reduction of the order of the equation, we find that

$$F'_\infty(y) = C\, y^{2(\frac{a_0}{c_0}-1)} \exp\left(-\frac{2}{c_0 y}\right)$$

with some positive constant C. Using boundary conditions we calculate a constant C. We get that $C = \frac{1}{\Gamma\left(1-\frac{2a_0}{c_0}\right)}\left(\frac{c_0}{2}\right)^{2\frac{a_0}{c_0}-1}$ where $\Gamma(\cdot)$ is the gamma function. Since $1-\frac{2a_0}{c_0} = \frac{2b_0}{c_0}$, this gives us the final result. □

Remark 4. The formula for p_∞ is the well-known result probably for the first time mentioned in the book by Ibragimov, Khasminsky "Statistical Estimation – Asymptotic Theory". This formula appears also in [15], Example 1, but in the form of the identity in law. This result can be find also in [13].

Remark 5. Some rather complicated formulas concerning p_t were given in [17] and in [13], formula 1.10.4, p. 264. Recently, in [12], based on the derived differential equation for the distribution function, we obtained the Laplace transform for the density of the exponential integral functional of a Brownian motion with drift,

$$\hat{p}_\lambda(y) = \int_0^{+\infty} e^{-\lambda s} p_s(y)ds$$

where $\lambda > 0$. Namely,

$$\begin{aligned}\hat{p}_\lambda(y) &= \frac{1}{\lambda}\left(y\frac{c_0}{2}\right)^{-k}\frac{\Gamma\left(1-\frac{2b_0}{c_0}+k\right)}{\Gamma\left(1-\frac{2b_0}{c_0}+2k\right)}\left\{\frac{k}{y^{k+1}}M\left(k, 1-\frac{2b_0}{c_0}+2k, -\frac{2}{yc_0}\right)\right.\\ &\quad\left. -\frac{2k}{c_0\, y^{k+2}(1-\frac{2b_0}{c_0}+2k)}M\left(k+1, 2-\frac{2b_0}{c_0}+2k, -\frac{2}{yc_0}\right)\right\}\end{aligned}$$

where $k = \frac{b_0+\sqrt{b_0^2+2\lambda c_0}}{c_0}$ and M is the confluent hypergeometric function of the first kind known as Kummer's function. The Laplace transform in this case can be inverted in the usual way by the Bromwich–Mellin formula

$$p_s(y) = \int_{\lambda_0-i\infty}^{\lambda_0+\infty} e^{\lambda s}\, \hat{p}_\lambda(y)d\lambda$$

for any fixed $\lambda_0 > 0$ and $s > 0$.

Let us denote by ν^+ and ν^- the Lévy measures of positive and negative jumps, respectively; namely, for $x > 0$

$$\nu^+([x, +\infty[) = \int_x^{+\infty} K_0(du), \quad \nu^-(]-\infty, -x]) = \int_{-\infty}^{-x} K_0(du).$$

To simplify the notations, we put also

$$\nu^+(x) = \nu^+([x, +\infty[), \; \nu^-(x) = \nu^-(]-\infty, -x]).$$

Let us suppose in addition that

$$\int_{\mathbb{R}} |x| \, K_0(dx) < \infty.$$

Corollary 4. *Suppose that X is a Lévy process with integrable jumps. Then, under the conditions of Proposition 3, the density p_t of I_t satisfies the equation*

$$\frac{\partial}{\partial t} p_t(y) = \frac{1}{2} c_0 \frac{\partial^2}{\partial y^2}(y^2\, p_t(y)) - \frac{\partial}{\partial y}((r_0 y + 1)\, p_t(y)) +$$
$$\int_y^{+\infty} p_t(z) \nu^+(\ln(\frac{z}{y}))\, dz + \int_0^y p_t(z) \nu^-(-\ln(\frac{z}{y}))\, dz$$

where $r_0 = a_0 - \int_{\mathbb{R}} (e^{-x} - 1)\, K_0(dx) = -b_0 + \frac{1}{2} c_0 + \int_{\mathbb{R}} x\, K_0(dx)$.

*In the particular case, when $I_\infty < +\infty$ (**P**-a.s.), we get*

$$\frac{1}{2} c_0 \frac{\partial^2}{\partial y^2}(y^2\, p_\infty(y)) - \frac{\partial}{\partial y}((r_0 y + 1)\, p_\infty(y)) +$$
$$\int_y^{+\infty} p_\infty(z) \nu^+(\ln(\frac{z}{y}))\, dz + \int_0^y p_\infty(z) \nu^-(-\ln(\frac{z}{y}))\, dz = 0.$$

Proof. We take the equation (34) and rewrite it in the form

$$\frac{\partial}{\partial t} F_t(y) = \frac{1}{2} c_0 \frac{\partial}{\partial y}(y^2 \frac{\partial}{\partial y} F_t(y)) - (r_0 y + 1) \frac{\partial}{\partial y} F_t(y) +$$
$$\int_{\mathbb{R}} \left[F_t(y e^x) - F_t(y) \right] K_0(dx).$$

Then we divide the integral over $\mathbb{R}$ in two parts, integrating on $]0, +\infty[$ and $]-\infty, 0[$. We integrate by parts,

$$\int_{\mathbb{R}} [F_t(y e^x) - F_t(y)] K_0(dx) =$$
$$\int_0^{+\infty} \frac{\partial}{\partial x} F_t(y e^x)\, y e^x\, \nu^+(x) dx + \int_{-\infty}^0 \frac{\partial}{\partial x} F_t\,(y e^x)\, y e^x\, \nu^-(-x) dx,$$

and change the variables $z = y e^x$. We differentiate the result w.r.t. t, and this gives the claim. $\square$

Remark 6. The equation for p_∞ mentioned in Corollary 4 was also obtained in [15], example E.

Corollary 5. *Suppose that for* $x \in \mathbb{R}$

$$K_0(x) = e^{-\mu x} I_{\{x>0\}}.$$

Then, under the assumptions of Proposition 3, the density p_t *of* I_t *satisfies*

$$\frac{\partial}{\partial t} p_t(y) = \frac{1}{2} c_0 \frac{\partial^2}{\partial y^2}(y^2\, p_t(y)) - \frac{\partial}{\partial y}((r_0 y + 1)\, p_t(y)) + \frac{y^\mu}{\mu}\int_y^\infty \frac{p_t(z)}{z^\mu}\, dz.$$

In particular, when $I_\infty < +\infty$ *(*$\mathbf{P}$*-a.s.), we have*

$$\frac{1}{2} c_0 \frac{\partial^2}{\partial y^2}(y^2\, p_\infty(y)) - \frac{\partial}{\partial y}((r_0 y + 1)\, p_\infty(y)) + \frac{y^\mu}{\mu}\int_y^\infty \frac{p_\infty(z)}{z^\mu}\, dz = 0.$$

Proof. We take into account that $\nu^+(x) = \frac{1}{\mu} e^{-\mu x}$ and $\nu^-(x) = 0$ for all $x > 0$, and this gives us the equation for p_t and p_∞ in this particular case. $\square$

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