

Ergodic properties of the solution to a fractional stochastic heat equation, with an application to diffusion parameter estimation

Diana Avetisian, Kostiantyn Ralchenko*

*Department of Probability Theory, Statistics and Actuarial Mathematics,
Taras Shevchenko National University of Kyiv,
Volodymyrska 64, 01601 Kyiv, Ukraine*

diana.avetisian2017@gmail.com (D. Avetisian), k.ralchenko@gmail.com (K. Ralchenko)

Received: 11 May 2020, Revised: 2 August 2020, Accepted: 11 September 2020,
Published online: 18 September 2020

Abstract The paper deals with a stochastic heat equation driven by an additive fractional Brownian space-only noise. We prove that a solution to this equation is a stationary and ergodic Gaussian process. These results enable us to construct a strongly consistent estimator of the diffusion parameter.

Keywords Stochastic partial differential equation, fractional Brownian motion, stationary process, ergodic process, strong consistency

2010 MSC 60G22, 60H15, 62F10

1 Introduction

In the last four decades, there have been enormous advances in the study of random field solutions to stochastic partial differential equations (SPDEs) driven by general Brownian noises. The starting point of this theory was the seminal work by Walsh [38]. Most of the results published till now are mainly focused on the analysis of heat and wave equations perturbed by Gaussian white noises in time with a fairly

*Corresponding author.

general spatial correlation [3, 8, 16, 17, 31, 33]. At the same time some publications are devoted to SPDEs driven by fractional-type noises [20, 22, 28, 30, 34–36].

The interest in equations of such type is mainly due to the development of analysis and the theory of random processes on one hand, and various applications to some real phenomena and real situations on the other hand. For instance, various types of SPDEs provide suitable models in the study of population growth [18], some climate and oceanographic phenomena [1, 23], mathematical finance [7].

Consequently, statistical methods are required to calibrate SPDE models from given observations. In particular, a problem of parameter estimation based on discrete observations of a solution to an SPDE attracted considerable interest very recently. It was first investigated in [29]. Later, applying similar methods, parabolic SPDEs including the stochastic heat equation had been studied in [6, 10, 14]. In these articles estimators based on power variations of time-increments of the solution were constructed and central limit theorems were proved. In [24] an adaptive maximum likelihood type estimator of the coefficient parameter was proposed for a parabolic linear second order SPDE. However, there are still basic questions which are not settled in the statistical literature on SPDEs, see the recent review [13] for details.

This paper deals with the stochastic heat equation, which is a typical example of SPDE. This equation and its numerical approximation has been extensively studied in [11, 19, 21, 25, 37]. Also, in [5, 28] estimators for the drift parameter based on power variation of the solution to the fractional stochastic heat equation was constructed.

Specifically, this paper is devoted to the following stochastic heat equation with a fractional Brownian noise B_x^H :

$$\left(\frac{\partial u}{\partial t} - \frac{1}{2} \Delta u \right) (t, x) = \sigma \dot{B}_x^H, \quad t > 0, \quad x \in \mathbb{R}, \quad (1)$$

$$u(0, x) = 0, \quad x \in \mathbb{R}.$$

We prove the stationarity and ergodicity of its solution $u(t, x)$ as a function of the spatial variable x by analyzing the behavior of the covariance function. Then these properties are applied for construction of a strongly consistent estimator for the unknown diffusion parameter σ . We extend the results of [2], where a similar problem for SPDE with white noise was studied.

We would like to emphasize that in the present paper we study a stochastic heat equation driven by a fractional space-only noise. Unlike SPDEs with space-time noise, equations with purely spatial Gaussian noise are studied not so extensively in the statistical literature. We can mention only the papers [2, 12, 15], where the parameter estimation for equations with space-only white noise was investigated. At the same time, such type of noise is an important type of stationary perturbations, see discussion and examples in [27].

Concerning the sampling scheme, we assume that the solution $u(t, x)$ is observed at equidistant spatial points for a fixed time. On one hand, in many practical applications the solution indeed is observed only at some discrete space points; e. g. temperature of a heated body, velocity of a turbulent flow, instantaneous forward rates where the space variable corresponds to time until maturity [14], see also [29]. On the other hand, it turns out that in order to estimate the diffusion coefficient, it is enough to

observe the solution at one time instant. Such situation is quite common for statistical inference for SPDEs, see [13, Section 3]. However, it is possible to incorporate the additional information of observing the solution discretely in time by taking the (weighted) average of the estimators similarly to [6] or [14].

The paper is organized as follows. In Section 2 we recall the definition of mild solution to SPDE (1) and the required theory of stochastic integration with respect to fractional Brownian motion. In Section 3 we calculate the covariance and variance of the solution and establish its stationarity and ergodicity. Based on these results, in Section 4 we construct a strongly consistent estimator of the parameter σ by discrete observations of a trajectory of the solution. Numerical results are presented in Section 5. All proofs are collected in Appendix A.

2 Preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. Let $B^H = \{B_x^H, x \in \mathbb{R}\}$ be a two-sided fractional Brownian motion with the Hurst index $H \in (0, 1)$, that is, a centered Gaussian process, starting at 0, with the covariance function

$$\mathbb{E} \left[B_x^H B_y^H \right] = \frac{1}{2} \left(|x|^{2H} + |y|^{2H} - |x - y|^{2H} \right), \quad x, y \in \mathbb{R}. \tag{2}$$

Following [38], we define a solution to SPDE (1) by

$$u(t, x) = \sigma \int_0^t \int_{\mathbb{R}} G(t - s, x - y) dB_y^H ds, \quad t \geq 0, x \in \mathbb{R}, \tag{3}$$

where G is Green’s function of the heat equation:

$$G(t, x) = \begin{cases} \frac{1}{\sqrt{2\pi t}} \exp \left\{ -\frac{x^2}{2t} \right\}, & \text{if } t > 0, \\ \delta_0(x), & \text{if } t = 0. \end{cases}$$

Due to the Hölder properties of a fractional Brownian motion and Green’s function, the integral with respect to the fractional Brownian motion in (3) exists as the pathwise Riemann–Stieltjes integral. Let us briefly recall some basic facts concerning the integration of Hölder continuous functions.

Let $a < b$. Denote by $C^\lambda([a, b])$ the space of λ -Hölder continuous functions, $\lambda \in (0, 1]$. If $f \in C^\alpha([a, b])$ and $g \in C^\beta([a, b])$ with $\alpha + \beta > 1$, then the integral $\int_a^b f(x) dg(x)$ exists as the Riemann–Stieltjes integral

$$\int_a^b f(x) dg(x) = \lim_{|\pi| \rightarrow 0} \sum_i f(x_i^*) (g(x_{i+1}) - g(x_i)),$$

where $\pi = \{a = x_0 \leq x_0^* \leq x_1 \leq \dots \leq x_{n-1} \leq x_{n-1}^* \leq x_n = b\}$, $|\pi| = \max_i |x_{i+1} - x_i|$, see, e. g., [40, Thm. 4.2.1]. Moreover, the integral $\int_a^b g(x) df(x)$ also exists, and the integration-by-parts formula holds:

$$\int_a^b f(x) dg(x) = f(b)g(b) - f(a)g(a) - \int_a^b g(x) df(x)$$

(see [40, Thms. 4.2.1 and 3.1]). It is well known that in the case of Hölder continuous functions the Riemann–Stieltjes integral coincides with Young’s integral [39] and with the generalized Lebesgue–Stieltjes integral [40].

This theory can be easily applied to the pathwise stochastic integration with respect to a fractional Brownian motion, since the sample paths of B^H belong to $C^\beta([a, b])$ a. s., for any $\beta \in (0, H)$ (see, e. g., [32, Sec. 1.16]). On the other hand, it is not hard to see that for any fixed $t > 0, s > 0$ and $x \in \mathbb{R}$, the function $G(t - s, x - \cdot)$ is (globally) Lipschitz, i. e., it belongs to $C^\alpha([a, b])$ with $\alpha = 1$. This means that for almost all $\omega \in \Omega$, the integral $\int_a^b G(t - s, x - y) dB_y^H$ exists in the Riemann–Stieltjes sense, and

$$\int_a^b G(t - s, x - y)dB_y^H = G(t - s, x - b)B_b^H - G(t - s, x - a)B_a^H + \int_a^b B_y^H G'_2(t - s, x - y) dy,$$

where G'_2 denotes the partial derivative of G with respect to the spatial variable:

$$G'_2(t, x) = \frac{\partial}{\partial x} G(t, x) = -\frac{x}{\sqrt{2\pi t^{3/2}}} \exp\left\{-\frac{x^2}{2t}\right\}, \quad t > 0, x \in \mathbb{R}. \tag{4}$$

It is known (see, e. g., the proof of Prop. A.1 in [9]) that for all $\gamma > H, \frac{B_x^H}{|x|^\gamma} \rightarrow 0$ a. s., as $|x| \rightarrow \infty$. Therefore, $G(t - s, x - y) B_y^H \rightarrow 0$, a. s., as $y \rightarrow \pm\infty$. Hence,

$$\int_{-\infty}^{+\infty} G(t - s, x - y)dB_y^H = \int_{-\infty}^{+\infty} B_y^H G'_2(t - s, x - y) dy,$$

and the solution (3) can be written in the form

$$u(t, x) = \sigma \int_0^t \int_{\mathbb{R}} G'_2(t - s, x - y) B_y^H dy ds, \quad t > 0, x \in \mathbb{R}. \tag{5}$$

This form of the solution does not contain a stochastic integral with respect to a fractional Brownian motion, hence, it is more convenient for further calculations.

3 Properties of solution

In this section we will investigate properties of the solution u related to its covariance structure. In particular, we will establish stationarity and ergodicity.

3.1 Covariance and variance. Exact formulae

We start with deriving explicit expressions for the variance and covariance of $u(t, \cdot)$. The asymptotic behavior of the covariance function will be investigated in the next subsection.

Proposition 1. For fixed $t \in [0, T]$, $u(t, \cdot)$ is a stationary Gaussian process with covariance function

$$\begin{aligned}
 R(t, x) &:= \text{cov}(u(t, 0), u(t, x)) \\
 &= -\frac{\sigma^2}{2} \int_0^t \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} G'_2(t-s, -y) G'_2(t-r, x-v) |y-v|^{2H} dr ds dv dy.
 \end{aligned}
 \tag{6}$$

The next result gives a simpler expression for the covariance function of the solution. This expression contains a single integral over \mathbb{R} instead of double integral over \mathbb{R}^2 .

Proposition 2. The covariance function $R(t, x)$ can be represented in the form

$$\begin{aligned}
 R(t, x) &= \frac{\sigma^2 H}{\sqrt{2\pi}} \int_0^t \int_0^t (2t-s-r)^{-\frac{3}{2}} \\
 &\quad \times \int_{\mathbb{R}} w^{(2H-1)}(w-x) \exp\left\{-\frac{(w-x)^2}{2(2t-s-r)}\right\} dw ds dr,
 \end{aligned}
 \tag{7}$$

where $w^{(\alpha)} := |w|^\alpha \text{sign } w$.

Remark 1. In the case $H > \frac{1}{2}$, it is possible to integrate by parts once more and rewrite the formula for $R(t, x)$ in the form

$$\begin{aligned}
 R(t, x) &= \frac{\sigma^2 H(2H-1)}{\sqrt{2\pi}} \int_0^t \int_0^t (2t-s-r)^{-\frac{1}{2}} \\
 &\quad \times \int_{\mathbb{R}} |w|^{2H-2} \exp\left\{-\frac{(w-x)^2}{2(2t-s-r)}\right\} dw ds dr.
 \end{aligned}$$

Proposition 3. The variance of $u(t, x)$ equals

$$\mathbb{E}[u(t, x)^2] = \frac{\sigma^2 2^{H+1} (2^H - 1) \Gamma(H + \frac{1}{2}) t^{H+1}}{\sqrt{\pi} (H + 1)}, \quad t > 0, x \in \mathbb{R},
 \tag{8}$$

where Γ denotes the gamma function.

Remark 2. In the case of the standard Brownian motion ($H = \frac{1}{2}$), (8) becomes

$$\mathbb{E}[u(t, x)^2] = \frac{\sigma^2 4(2 - \sqrt{2}) t^{3/2}}{3\sqrt{\pi}}.$$

This agrees with the result of [2, Thm. 2.1].

3.2 Upper bound for covariance function and its asymptotic behavior

Our next goal is to establish ergodicity of $u(t, \cdot)$. Since $u(t, \cdot)$ is a stationary Gaussian process (by Proposition 1), it suffices to show that $R(t, x) \rightarrow 0$, as $x \rightarrow \infty$. The following proposition plays a crucial role in the proof of ergodicity.

Proposition 4. For $t > 0$ and $x > 0$, the covariance function $R(t, x)$ admits the following upper bound:

$$|R(t, x)| \leq C_H \sigma^2 t^2 x^{2H-2},$$

where C_H is a positive constant depending only on H .

Proposition 4 implies that the covariance function $R(t, x)$ of the solution $u(t, x)$ vanishes as $x \rightarrow +\infty$. Since $u(t, \cdot)$ is a stationary Gaussian process, this yields the following result.

Corollary 1. For a fixed $t > 0$, the random process $\{u(t, x), x \in \mathbb{R}\}$ is ergodic.

4 Diffusion parameter estimation

In this section we apply previous results to the following statistical problem. Assume that for a fixed time $t > 0$ and a fixed step $\delta > 0$, the random field u given by (3) is observed at the points $x_k = k\delta, k = 1, \dots, N$. Our aim is to construct a strongly consistent estimator for σ based on these observations. By the results of the previous section, the field $u(t, x), x \in \mathbb{R}$ is strictly stationary and ergodic. Therefore, for any Borel function $g: \mathbb{R} \rightarrow \mathbb{R}$ such that $\mathbf{E}|g(u(t, 0))| < \infty$, thanks to ergodic theorem, it holds that

$$\frac{1}{N} \sum_{k=1}^N g(u(t, x_k)) \rightarrow \mathbf{E}[g(u(t, 0))], \quad \text{a. s., as } N \rightarrow \infty. \tag{9}$$

This gives the idea to consider the following estimator for σ^2 :

$$\hat{\sigma}_N^2 = \frac{1}{Nv^2} \sum_{k=1}^N u(t, x_k)^2,$$

where

$$v^2 = \frac{t^{H+1} 2^{H+1} (2^H - 1) \Gamma(H + \frac{1}{2})}{\sqrt{\pi} (H + 1)},$$

see Proposition 3. Taking into account (9), we have the following theorem.

Theorem 1. $\hat{\sigma}_N^2$ is a strongly consistent estimator for the parameter σ^2 as $N \rightarrow \infty$.

Remark 3. The Hurst parameter $H \in (0, 1)$ is assumed to be known. It can be estimated independently of σ with the help of quadratic variations, see [4, 26].

5 Simulation

In this section we illustrate the quality of the estimator with the help of simulation experiments. We generate the trajectories of the solution $u(t, x)$ to the equation (1) by the discretization of the formula (3). Since the parameter σ enters into the right-hand side of (3) multiplicatively, it suffices to consider the case $\sigma = 1$ only. Also we choose $t = 1$ as the observation time. For each value of the Hurst index H , we simulate 100 sample paths of the solution $u(1, x)$, using the partition with the step $\Delta x = 2^{-7}$. First, we integrate Green’s function numerically with respect to s , then

Table 1. Means and standard deviations of the estimator $\hat{\sigma}_N^2$ for the observations step $\delta = 1$

	N	2^4	2^5	2^6	2^7	2^8	2^9	2^{10}
$H = 0.1$	Mean	1.0470	1.0120	1.0017	0.9981	1.0073	1.0132	1.0135
	Std. dev	0.3880	0.2619	0.1812	0.1254	0.0892	0.0658	0.0455
$H = 0.3$	Mean	0.9991	1.0161	0.9994	0.9932	0.9992	0.9964	0.9951
	Std. dev	0.4326	0.2920	0.1928	0.1467	0.1088	0.0699	0.0502
$H = 0.5$	Mean	1.0051	0.9613	0.9576	0.9883	0.9833	0.9893	0.9915
	Std. dev	0.4292	0.2601	0.2009	0.1343	0.0858	0.0731	0.0479
$H = 0.7$	Mean	0.9476	0.9532	0.9916	1.0299	1.0187	1.0038	1.0049
	Std. dev	0.5587	0.4144	0.2941	0.2363	0.1780	0.1161	0.0775
$H = 0.9$	Mean	0.9910	1.0163	0.9722	0.9800	0.9588	0.9543	0.9602
	Std. dev	0.9989	0.8766	0.7046	0.5660	0.4976	0.4934	0.3416

Table 2. Means and standard deviations of the estimator $\hat{\sigma}_N^2$ for the observations step $\delta = \frac{1}{2}$

	N	2^4	2^5	2^6	2^7	2^8	2^9	2^{10}
$H = 0.1$	Mean	1.1065	1.0451	1.0291	1.0155	1.0182	1.0325	1.0268
	Std. dev	0.6812	0.5189	0.3806	0.2639	0.1821	0.1363	0.0948
$H = 0.3$	Mean	1.0415	1.0292	1.0489	1.0109	1.0064	1.0150	1.0078
	Std. dev	0.7277	0.5332	0.3749	0.2667	0.1944	0.1325	0.0875
$H = 0.5$	Mean	1.0695	0.9728	1.0597	1.0671	1.0327	1.0205	1.0079
	Std. dev	0.9274	0.6440	0.4363	0.2916	0.1742	0.1215	0.1086
$H = 0.7$	Mean	0.9555	0.9919	0.9982	0.9838	0.9821	1.0414	1.0228
	Std. dev	0.9722	0.7945	0.5504	0.3966	0.2896	0.2088	0.1417
$H = 0.9$	Mean	0.9763	0.9302	0.9012	0.9731	0.9667	0.9365	0.9323
	Std. dev	1.3018	1.1639	0.8602	0.7813	0.7033	0.5201	0.4076

Table 3. Means and standard deviations of the estimator $\hat{\sigma}_N^2$ for the observations step $\delta = \frac{1}{4}$

	N	2^4	2^5	2^6	2^7	2^8	2^9	2^{10}
$H = 0.1$	Mean	0.8753	1.0704	1.1010	1.0429	1.0229	0.9875	0.9893
	Std. dev	0.8769	0.9028	0.6155	0.4187	0.3364	0.2298	0.1550
$H = 0.3$	Mean	1.1555	1.0970	1.0627	0.9430	0.9469	0.9717	0.9923
	Std. dev	1.0885	0.8464	0.6896	0.4895	0.3406	0.2733	0.1872
$H = 0.5$	Mean	0.9159	0.9157	0.9598	0.9836	1.0244	1.0295	1.0139
	Std. dev	1.2274	0.8237	0.6379	0.5033	0.4382	0.3087	0.2228
$H = 0.7$	Mean	0.9758	0.9743	0.9323	0.8939	0.9806	1.0009	1.0069
	Std. dev	1.3749	1.1514	0.9001	0.6895	0.5734	0.4143	0.3277
$H = 0.9$	Mean	0.6687	0.6877	0.7213	0.7765	0.8427	0.8468	0.9224
	Std. dev	0.8962	0.8343	0.7729	0.7604	0.7558	0.6637	0.6916

we calculate the stochastic integral with respect to the fractional Brownian motion B_y^H replacing it by the sum (we integrate over $[x - 4, x + 4]$ instead of \mathbb{R}).

We study the performance of the estimator $\hat{\sigma}_N^2$ for various values of the Hurst index H and three values of the observation step $\delta = 1, 0.5, 0.25$. The means and standard deviations of the estimates are reported in Tables 1–3. We see that the estimates converge to the true value of σ^2 , and their standard deviations tend to zero. Hence these simulations confirm the consistency. However the rate of convergence for $H = 0.9$ is not very high. Also we see that the best results are obtained for $\delta = 1$. Probably, the horizon of observations $T = N\delta$ is more important for the quality of the estimators than the step δ .

A Proofs

Proof of Proposition 1. Using the formula (5) for the solution, we can write the covariance in the form

$$\begin{aligned} \text{cov}(u(t, z), u(t, z + x)) &= \mathbb{E}[u(t, z)u(t, z + x)] \\ &= \sigma^2 \mathbb{E} \left[\int_0^t \int_{\mathbb{R}} G'_2(t - s, z - y) B_y^H ds dy \int_0^t \int_{\mathbb{R}} G'_2(t - r, z + x - v) B_v^H dr dv \right] \\ &= \sigma^2 \int_0^t \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} G'_2(t - s, z - y) G'_2(t - r, z + x - v) \mathbb{E} \left[B_y^H B_v^H \right] dr ds dv dy. \end{aligned}$$

Inserting the explicit expression (2) for the covariance function of a fractional Brownian motion, we obtain

$$\begin{aligned} \text{cov}(u(t, z), u(t, z + x)) &= \frac{\sigma^2}{2} \int_0^t \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} G'_2(t - s, z - y) G'_2(t - r, z + x - v) |y|^{2H} dr ds dv dy \\ &\quad + \frac{\sigma^2}{2} \int_0^t \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} G'_2(t - s, z - y) G'_2(t - r, z + x - v) |v|^{2H} dr ds dv dy \\ &\quad - \frac{\sigma^2}{2} \int_0^t \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} G'_2(t - s, z - y) G'_2(t - r, z + x - v) |y - v|^{2H} dr ds dv dy \\ &=: I_1 + I_2 + I_3. \end{aligned}$$

The term I_1 can be represented as a product of two integrals:

$$I_1 = \frac{\sigma^2}{2} \int_0^t \int_{\mathbb{R}} |y|^{2H} G'_2(t - s, z - y) dy ds \cdot \int_0^t \int_{\mathbb{R}} G'_2(t - r, z + x - v) dv dr.$$

It is not hard to see that $\int_0^t \int_{\mathbb{R}} |y|^{2H} G'_2(t - s, z - y) dy ds < \infty$ and

$$\int_0^t \int_{\mathbb{R}} G'_2(t - r, z + x - v) dv dr = \int_0^t G(t - r, x - v) \Big|_{v=-\infty}^{\infty} dr = 0,$$

because $\lim_{x \rightarrow -\infty} G(t, x) = \lim_{x \rightarrow \infty} G(t, x) = 0$. Hence, $I_1 = 0$.

Similarly, one can prove that $I_2 = 0$. Thus, we have

$$\begin{aligned} \text{cov}(u(t, z), u(t, z + x)) &= -\frac{\sigma^2}{2} \int_0^t \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} G'_2(t - s, z - y) G'_2(t - r, z + x - v) |y - v|^{2H} dr ds dv dy. \end{aligned}$$

Finally, by the change of variables $y' = y - z, v' = v - z$, we obtain

$$\begin{aligned} \text{cov}(u(t, z), u(t, z + x)) &= -\frac{\sigma^2}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_0^t \int_0^t G'_2(t - s, -y') G'_2(t - r, x - v') |y' - v'|^{2H} dr ds dv' dy'. \end{aligned}$$

Since the expression does not depend on z , the process $u(t, \cdot)$ is stationary. Moreover, we see that the formula (6) holds. \square

Proof of Proposition 2. In order to obtain (7), we need to transform the inner integral in (6), namely

$$I(s, r) = \int_{\mathbb{R}} \int_{\mathbb{R}} |y - v|^{2H} G'_2(t - s, -y) G'_2(t - r, x - v) dy dv. \tag{10}$$

By the substitution $v - y = w$, we have

$$I(s, r) = \int_{\mathbb{R}} \int_{\mathbb{R}} |w|^{2H} G'_2(t - s, w - v) G'_2(t - r, x - v) dw dv.$$

Using the explicit expression for G'_2 , see (4), we get

$$I(s, r) = \frac{1}{2\pi(t-s)^{3/2}(t-r)^{3/2}} \times \int_{\mathbb{R}} |w|^{2H} \int_{\mathbb{R}} (w-v)(x-v) \exp \left\{ - \left(\frac{(w-v)^2}{2(t-s)} + \frac{(x-v)^2}{2(t-r)} \right) \right\} dv dw. \tag{11}$$

Let us transform the sum under the exponent:

$$\begin{aligned} \frac{(w-v)^2}{2(t-s)} + \frac{(x-v)^2}{2(t-r)} &= \frac{(w^2 - 2wv + v^2)(t-r) + (x^2 - 2xv + v^2)(t-s)}{2(t-s)(t-r)} \\ &= \frac{v^2(2t-s-r) - 2v(w(t-r) + x(t-s)) + w^2(t-r) + x^2(t-s)}{2(t-s)(t-r)} = \\ &= \frac{v^2 - 2v \frac{w(t-r) + x(t-s)}{2t-s-r} + \frac{w^2(t-r)}{2t-s-r} + \frac{x^2(t-s)}{2t-s-r}}{2 \frac{(t-s)(t-r)}{2t-s-r}} \\ &= \frac{\left(v - \frac{w(t-r) + x(t-s)}{2t-s-r} \right)^2 - \frac{(w(t-r) + x(t-s))^2}{(2t-s-r)^2} + \frac{w^2(t-r)}{2t-s-r} + \frac{x^2(t-s)}{2t-s-r}}{2 \frac{(t-s)(t-r)}{2t-s-r}} \\ &= \frac{\left(v - \frac{w(t-r) + x(t-s)}{2t-s-r} \right)^2}{2 \frac{(t-s)(t-r)}{2t-s-r}} + \frac{(w-x)^2}{2(2t-s-r)}. \end{aligned}$$

Hence, (11) is rewritten in the form

$$I(s, r) = \frac{1}{2\pi(t-s)^{3/2}(t-r)^{3/2}} \int_{\mathbb{R}} |w|^{2H} \exp \left\{ - \frac{(w-x)^2}{2(2t-s-r)} \right\} \times \int_{\mathbb{R}} (v^2 - v(w+x) + wx) \exp \left\{ - \frac{\left(v - \frac{w(t-r) + x(t-s)}{2t-s-r} \right)^2}{2 \frac{(t-s)(t-r)}{2t-s-r}} \right\} dv dw. \tag{12}$$

The inner integral can be computed by applying the following equalities for the moments of a normal random variable: if $\xi \sim \mathcal{N}(\mu, \zeta^2)$, then

$$\frac{1}{\sqrt{2\pi}\zeta} \int_{\mathbb{R}} v^2 \exp \left\{ - \frac{(v-\mu)^2}{2\zeta^2} \right\} dv = \mathbb{E}\xi^2 = \zeta^2 + \mu^2,$$

$$\frac{1}{\sqrt{2\pi}\zeta} \int_{\mathbb{R}} v \exp \left\{ -\frac{(v-\mu)^2}{2\zeta^2} \right\} dv = \mathbb{E}\xi = \mu,$$

$$\frac{1}{\sqrt{2\pi}\zeta} \int_{\mathbb{R}} \exp \left\{ -\frac{(v-\mu)^2}{2\zeta^2} \right\} dv = 1.$$

In our case, $\mu = \frac{w(t-r)+x(t-s)}{2t-r-s}$, $\zeta = \sqrt{\frac{(t-s)(t-r)}{2t-s-r}}$. Hence,

$$\begin{aligned} & \int_{\mathbb{R}} (v^2 - v(w+x) + wx) \exp \left\{ -\frac{\left(v - \frac{w(t-r)+x(t-s)}{2t-s-r}\right)^2}{2\frac{(t-s)(t-r)}{2t-s-r}} \right\} dv \\ &= \sqrt{\frac{2\pi(t-s)(t-r)}{2t-s-r}} \left(\frac{(t-s)(t-r)}{2t-s-r} + \left(\frac{w(t-r)+x(t-s)}{2t-s-r} \right)^2 \right. \\ & \quad \left. - (w+x) \frac{w(t-r)+x(t-s)}{2t-s-r} + wx \right) \\ &= \frac{\sqrt{2\pi(t-s)(t-r)}}{(2t-s-r)^{5/2}} \left((t-s)(t-r)(2t-s-r) + w^2(t-r)^2 \right. \\ & \quad \left. + 2wx(t-r)(t-s) + x^2(t-s)^2 \right. \\ & \quad \left. - (w-x)(w(t-r)+x(t-s))(2t-s-r) + wx(2t-s-r)^2 \right) \\ &= \frac{\sqrt{2\pi}(t-s)^{3/2}(t-r)^{3/2}}{(2t-s-r)^{5/2}} (2t-s-r - (w-x)^2). \end{aligned}$$

Combining this with (12), we obtain

$$\begin{aligned} I(s, r) &= \frac{1}{\sqrt{2\pi}(2t-s-r)^{5/2}} \\ & \quad \times \int_{\mathbb{R}} |w|^{2H} \left(2t-s-r - (w-x)^2 \right) \exp \left\{ -\frac{(w-x)^2}{2(2t-s-r)} \right\} dw \\ &= \frac{1}{\sqrt{2\pi}(2t-s-r)^{5/2}} \\ & \quad \times \int_{-\infty}^0 |w|^{2H} \left(2t-s-r - (w-x)^2 \right) \exp \left\{ -\frac{(w-x)^2}{2(2t-s-r)} \right\} dw \\ & \quad + \frac{1}{\sqrt{2\pi}(2t-s-r)^{5/2}} \\ & \quad \times \int_0^{+\infty} |w|^{2H} \left(2t-s-r - (w-x)^2 \right) \exp \left\{ -\frac{(w-x)^2}{2(2t-s-r)} \right\} dw \\ &=: I_-(s, r) + I_+(s, r). \end{aligned}$$

Let us transform each integral separately, applying integration by parts. First, we rewrite $I_-(s, r)$ as follows:

$$\begin{aligned}
 I_-(s, r) &= \frac{1}{\sqrt{2\pi}(2t-s-r)^{3/2}} \\
 &\quad \times \int_{-\infty}^0 (-w)^{2H} \left(1 - \frac{(w-x)^2}{2t-s-r}\right) \exp\left\{-\frac{(w-x)^2}{2(2t-s-r)}\right\} dw \\
 &= \frac{1}{\sqrt{2\pi}(2t-s-r)^{3/2}} \int_{-\infty}^0 (-w)^{2H} d\left((w-x) \exp\left\{-\frac{(w-x)^2}{2(2t-s-r)}\right\}\right) \\
 &= \frac{1}{\sqrt{2\pi}(2t-s-r)^{3/2}} \left((-w)^{2H}(w-x) \exp\left\{-\frac{(w-x)^2}{2(2t-s-r)}\right\} \Big|_{w=-\infty}^0 \right. \\
 &\quad \left. + 2H \int_{-\infty}^0 (-w)^{2H-1}(w-x) \exp\left\{-\frac{(w-x)^2}{2(2t-s-r)}\right\} dw \right) \\
 &= \frac{2H}{\sqrt{2\pi}(2t-s-r)^{3/2}} \int_{-\infty}^0 (-w)^{2H-1}(w-x) \exp\left\{-\frac{(w-x)^2}{2(2t-s-r)}\right\} dw.
 \end{aligned}$$

Now let us transform the second integral in the same way:

$$\begin{aligned}
 I_+(s, r) &= \frac{1}{\sqrt{2\pi}(2t-s-r)^{\frac{3}{2}}} \\
 &\quad \times \int_0^{+\infty} w^{2H} \left(1 - \frac{(w-x)^2}{2t-s-r}\right) \exp\left\{-\frac{(w-x)^2}{2(2t-s-r)}\right\} dw \\
 &= \frac{1}{\sqrt{2\pi}(2t-s-r)^{\frac{3}{2}}} \int_0^{+\infty} w^{2H} d\left((w-x) \exp\left\{-\frac{(w-x)^2}{2(2t-s-r)}\right\}\right) \\
 &= -\frac{2H}{\sqrt{2\pi}(2t-s-r)^{\frac{3}{2}}} \int_0^{+\infty} w^{2H-1}(w-x) \exp\left\{-\frac{(w-x)^2}{2(2t-s-r)}\right\} dw.
 \end{aligned}$$

Thus,

$$\begin{aligned}
 I(s, r) &= I_-(s, r) + I_+(s, r) \\
 &= -\frac{2H}{\sqrt{2\pi}(2t-s-r)^{\frac{3}{2}}} \int_{-\infty}^{+\infty} w^{(2H-1)}(w-x) \exp\left\{-\frac{(x-w)^2}{2(2t-s-r)}\right\} dw.
 \end{aligned}$$

Taking into account that, by (6) and (10), $R(t, x) = -\frac{\sigma^2}{2} \int_0^t \int_0^t I(s, r) ds dr$, we conclude the proof. □

Proof of Proposition 3. By Propositions 1 and 2, we have that

$$\begin{aligned}
 \mathbb{E} \left[u(t, x)^2 \right] &= R(t, 0) \\
 &= \frac{\sigma^2 H}{\sqrt{2\pi}} \int_0^t \int_0^t (2t-s-r)^{-\frac{3}{2}} \int_{\mathbb{R}} |w|^{2H} \exp\left\{-\frac{w^2}{2(2t-s-r)}\right\} dw ds dr. \quad (13)
 \end{aligned}$$

Let us consider the inner integral. Taking into account that the integrand is an even function, and using the substitution $\frac{w^2}{2(2t-s-r)} = z$, $w = \sqrt{2(2t-s-r)z}$, $dw = \sqrt{2(2t-s-r)} \frac{1}{2\sqrt{z}} dz$, we obtain

$$\begin{aligned} \int_{\mathbb{R}} |w|^{2H} \exp \left\{ -\frac{w^2}{2(2t-s-r)} \right\} dw &= 2 \int_0^\infty w^{2H} \exp \left\{ -\frac{w^2}{2(2t-s-r)} \right\} dw \\ &= 2^{H+\frac{1}{2}} (2t-s-r)^{H+\frac{1}{2}} \int_0^\infty z^{H-\frac{1}{2}} e^{-z} dz \\ &= 2^{H+\frac{1}{2}} (2t-s-r)^{H+\frac{1}{2}} \Gamma \left(H + \frac{1}{2} \right). \end{aligned}$$

Inserting this expression into (13), we arrive at

$$\mathbb{E} \left[u(t, x)^2 \right] = \frac{\sigma^2 H 2^H \Gamma \left(H + \frac{1}{2} \right)}{\sqrt{\pi}} \int_0^t \int_0^t (2t-s-r)^{H-1} ds dr. \tag{14}$$

It remains to compute the integral

$$\begin{aligned} \int_0^t \int_0^t (2t-s-r)^{H-1} ds dr &= \frac{1}{H} \int_0^t \left((2t-r)^H - (t-r)^H \right) dr \\ &= \frac{1}{H(H+1)} \left(-(2t-r)^{H+1} + (t-r)^{H+1} \right) \Big|_{r=0}^t \\ &= \frac{(2t)^{H+1} - 2t^{H+1}}{H(H+1)} = \frac{2(2^H - 1)t^{H+1}}{H(H+1)}. \end{aligned}$$

Combining this with (14), we get the announced equality for $\mathbb{E} [u(t, x)^2]$. □

In order to prove Proposition 4, we start with an auxiliary result.

Lemma 1. *For any $\alpha > 0$ and $\beta > 0$, there exists a constant $C > 0$ such that for all $t > 0$ and $x > 0$,*

$$\int_0^t \int_0^t (2t-s-r)^{-\alpha} \exp \left\{ -\frac{\beta x^2}{2t-s-r} \right\} ds dr \leq C t^2 x^{-2\alpha}.$$

Proof. By the substitution $\frac{t-s}{x^2} = \eta$, $\frac{t-r}{x^2} = \rho$, we obtain

$$\begin{aligned} \int_0^t \int_0^t (2t-s-r)^{-\alpha} \exp \left\{ -\frac{\beta x^2}{2t-s-r} \right\} ds dr &= \int_0^{t/x^2} \int_0^{t/x^2} \left(\eta x^2 + \rho x^2 \right)^{-\alpha} \exp \left\{ -\frac{\beta x^2}{\eta x^2 + \rho x^2} \right\} x^4 d\eta d\rho \\ &= x^{4-2\alpha} \int_0^{t/x^2} \int_0^{t/x^2} (\eta + \rho)^{-\alpha} \exp \left\{ -\frac{\beta}{\eta + \rho} \right\} d\eta d\rho. \end{aligned}$$

Note that the function $f(x) = x^{-\alpha} \exp \left\{ -\frac{\beta}{x} \right\}$ is continuous on $(0, +\infty)$, and $\lim_{x \downarrow 0} f(x) = \lim_{x \rightarrow +\infty} f(x) = 0$. Hence, f is bounded: $x^{-\alpha} \exp \left\{ -\frac{\beta}{x} \right\} \leq C$ for

all $x > 0$ (one can choose $C = \sup_{x>0} f(x) = f(\frac{\alpha}{\beta}) = (\frac{\alpha}{\beta})^{-\alpha} e^{-\alpha}$). Consequently,

$$x^{4-2\alpha} \int_0^{t/x^2} \int_0^{t/x^2} (\eta + \rho)^{-\alpha} \exp \left\{ -\frac{\beta}{\eta + \rho} \right\} d\eta d\rho \leq C x^{4-2\alpha} \frac{t}{x^2} \cdot \frac{t}{x^2} = C t^2 x^{-2\alpha}. \quad \square$$

Proof of Proposition 4. In this proof C will denote a generic positive constant that may depend on H ; its value is not important and may change from one line to another.

According to Proposition 2,

$$\begin{aligned} R(t, x) &= \frac{\sigma^2 H}{\sqrt{2\pi}} \int_0^t \int_0^t (2t - s - r)^{-\frac{3}{2}} \\ &\quad \times \int_{\mathbb{R}} w^{(2H-1)}(w - x) \exp \left\{ -\frac{(w - x)^2}{2(2t - s - r)} \right\} dw ds dr \\ &= J_1 + J_2 + J_3, \end{aligned}$$

where

$$\begin{aligned} J_1 &= -\frac{\sigma^2 H}{\sqrt{2\pi}} \int_0^t \int_0^t (2t - s - r)^{-\frac{3}{2}} \\ &\quad \times \int_{-\infty}^0 |w|^{2H-1} (w - x) \exp \left\{ -\frac{(w - x)^2}{2(2t - s - r)} \right\} dw ds dr, \end{aligned} \quad (15)$$

$$\begin{aligned} J_2 &= \frac{\sigma^2 H}{\sqrt{2\pi}} \int_0^t \int_0^t (2t - s - r)^{-\frac{3}{2}} \\ &\quad \times \int_0^{x/2} w^{2H-1} (w - x) \exp \left\{ -\frac{(w - x)^2}{2(2t - s - r)} \right\} dw ds dr \end{aligned} \quad (16)$$

$$\begin{aligned} J_3 &= \frac{\sigma^2 H}{\sqrt{2\pi}} \int_0^t \int_0^t (2t - s - r)^{-\frac{3}{2}} \\ &\quad \times \int_{x/2}^{\infty} w^{2H-1} (w - x) \exp \left\{ -\frac{(w - x)^2}{2(2t - s - r)} \right\} dw ds dr. \end{aligned} \quad (17)$$

Let us consider each term separately. In order to bound J_1 , we write

$$\begin{aligned} &\int_{-\infty}^0 |w|^{2H-1} |w - x| \exp \left\{ -\frac{(w - x)^2}{2(2t - s - r)} \right\} dw \\ &= \exp \left\{ -\frac{x^2}{2(2t - s - r)} \right\} \int_{-\infty}^0 |w|^{2H-1} (x - w) \\ &\quad \times \exp \left\{ -\frac{w^2}{2(2t - s - r)} \right\} \exp \left\{ \frac{2wx}{2(2t - s - r)} \right\} dw \\ &\leq \exp \left\{ -\frac{x^2}{2(2t - s - r)} \right\} \int_{-\infty}^0 |w|^{2H-1} (x - w) \exp \left\{ -\frac{w^2}{2(2t - s - r)} \right\} dw \\ &= \exp \left\{ -\frac{x^2}{2(2t - s - r)} \right\} \int_0^{+\infty} w^{2H-1} (x + w) \exp \left\{ -\frac{w^2}{2(2t - s - r)} \right\} dw, \end{aligned} \quad (18)$$

because $\exp\left\{\frac{2wx}{2(2t-s-r)}\right\} \leq 1$ when $w \leq 0$ and $x \geq 0$. Further, we transform the integral in the right-hand side of (18), using the substitution $z = \frac{w^2}{2(2t-s-r)}$, $w = \sqrt{2(2t-s-r)z}$, $dw = \sqrt{2(2t-s-r)}\frac{1}{2\sqrt{z}}dz$. We obtain

$$\begin{aligned} & \int_0^{+\infty} w^{2H-1}(x+w) \exp\left(-\frac{w^2}{2(2t-s-r)}\right)dw \\ &= 2^{H-1}(2t-s-r)^H \int_0^{+\infty} z^{H-1}\left(x+\sqrt{2(2t-s-r)z}\right)e^{-z}dz \\ &= 2^{H-1}(2t-s-r)^H\left(x\Gamma(H)+\sqrt{2(2t-s-r)}\Gamma\left(H+\frac{1}{2}\right)\right) \\ &= 2^{H-1}\Gamma(H)(2t-s-r)^Hx+2^{H-1/2}\Gamma\left(H+\frac{1}{2}\right)(2t-s-r)^{H+1/2}. \end{aligned}$$

Hence, by (18),

$$\begin{aligned} & \int_{-\infty}^0 |w|^{2H-1}|w-x| \exp\left\{-\frac{(w-x)^2}{2(2t-s-r)}\right\}dw \leq \exp\left\{-\frac{x^2}{2(2t-s-r)}\right\} \\ & \quad \times \left(2^{H-1}\Gamma(H)(2t-s-r)^Hx+2^{H-1/2}\Gamma\left(H+\frac{1}{2}\right)(2t-s-r)^{H+1/2}\right). \end{aligned}$$

Inserting this bound into (15), we arrive at

$$\begin{aligned} |J_1| &\leq \frac{\sigma^2H2^{H-1}\Gamma(H)x}{\sqrt{2\pi}} \int_0^t \int_0^t (2t-s-r)^{H-\frac{3}{2}} \exp\left\{-\frac{x^2}{2(2t-s-r)}\right\} ds dr \\ & \quad + \frac{\sigma^2H2^{H-1/2}\Gamma\left(H+\frac{1}{2}\right)}{\sqrt{2\pi}} \\ & \quad \times \int_0^t \int_0^t (2t-s-r)^{H-1} \exp\left\{-\frac{x^2}{2(2t-s-r)}\right\} ds dr. \end{aligned}$$

The integrals in the right-hand side can be bounded with the help of Lemma 1. We obtain

$$|J_1| \leq C\sigma^2t^2x^{2H-2}.$$

Now let us consider J_2 . If $0 \leq w \leq \frac{x}{2}$, then $|x-w| \leq x$ and $(x-w)^2 \geq \frac{x^2}{4}$. Therefore,

$$\begin{aligned} |J_2| &\leq \frac{\sigma^2H}{\sqrt{2\pi}} \int_0^t \int_0^t (2t-s-r)^{-\frac{3}{2}} \\ & \quad \times \int_0^{\frac{x}{2}} w^{2H-1}|w-x| \exp\left\{-\frac{(w-x)^2}{2(2t-s-r)}\right\} dw ds dr \\ &\leq \frac{\sigma^2H}{\sqrt{2\pi}} \int_0^t \int_0^t (2t-s-r)^{-\frac{3}{2}} \int_0^{\frac{x}{2}} w^{2H-1}x \exp\left\{-\frac{x^2}{8(2t-s-r)}\right\} dw ds dr. \end{aligned}$$

Integrating with respect to w , we get

$$|J_2| \leq \frac{\sigma^2}{\sqrt{2\pi}2^{2H+1}}x^{2H+1} \int_0^t \int_0^t (2t-s-r)^{-\frac{3}{2}} \exp\left\{-\frac{x^2}{8(2t-s-r)}\right\} ds dr.$$

Applying Lemma 1, we obtain the upper bound

$$|J_2| \leq C\sigma^2 t^2 x^{2H-2}.$$

In order to bound the third term J_3 , we integrate the inner integral by parts:

$$\begin{aligned} J_3 &= \frac{\sigma^2 H}{\sqrt{2\pi}} \int_0^t \int_0^t (2t - s - r)^{-\frac{3}{2}} \\ &\quad \times \int_{\frac{x}{2}}^{\infty} w^{2H-1} (w - x) \exp\left\{-\frac{(w - x)^2}{2(2t - s - r)}\right\} dw ds dr \\ &= -\frac{\sigma^2 H}{\sqrt{2\pi}} \int_0^t \int_0^t (2t - s - r)^{-\frac{1}{2}} \\ &\quad \times \int_{\frac{x}{2}}^{+\infty} w^{2H-1} d\left(\exp\left\{-\frac{(w - x)^2}{2(2t - s - r)}\right\}\right) ds dr \\ &= -\frac{\sigma^2 H}{\sqrt{2\pi}} \int_0^t \int_0^t (2t - s - r)^{-\frac{1}{2}} \left(w^{2H-1} \exp\left\{-\frac{(w - x)^2}{2(2t - s - r)}\right\} \Big|_{w=\frac{x}{2}}^{+\infty} \right. \\ &\quad \left. - (2H - 1) \int_{\frac{x}{2}}^{+\infty} w^{2H-2} \exp\left\{-\frac{(w - x)^2}{2(2t - s - r)}\right\} dw \right) ds dr \\ &= \frac{\sigma^2 H}{\sqrt{2\pi}} \int_0^t \int_0^t (2t - s - r)^{-\frac{1}{2}} \left(\frac{x^{2H-1}}{2^{2H-1}} \exp\left\{-\frac{x^2}{8(2t - s - r)}\right\} \right. \\ &\quad \left. + (2H - 1) \int_{\frac{x}{2}}^{+\infty} w^{2H-2} \exp\left\{-\frac{(w - x)^2}{2(2t - s - r)}\right\} dw \right) ds dr \\ &= \frac{\sigma^2 H}{\sqrt{2\pi} 2^{2H-1}} x^{2H-1} \int_0^t \int_0^t (2t - s - r)^{-\frac{1}{2}} \exp\left\{-\frac{x^2}{8(2t - s - r)}\right\} ds dr \\ &\quad + \frac{\sigma^2 H (2H - 1)}{\sqrt{2\pi}} \int_0^t \int_0^t (2t - s - r)^{-\frac{1}{2}} \\ &\quad \times \int_{\frac{x}{2}}^{+\infty} w^{2H-2} \exp\left\{-\frac{(w - x)^2}{2(2t - s - r)}\right\} dw ds dr \\ &=: J_{3,1} + J_{3,2}. \end{aligned}$$

As above, the term $J_{3,1}$ can be bounded with the help of Lemma 1. We obtain the bound $|J_{3,1}| \leq C\sigma^2 t^2 x^{2H-2}$.

It remains to estimate $J_{3,2}$. If $w \geq \frac{x}{2}$, then $w^{2H-2} \leq \left(\frac{x}{2}\right)^{2H-2}$, hence

$$\begin{aligned} |J_{3,2}| &\leq \sigma^2 H |2H - 1| \left(\frac{x}{2}\right)^{2H-2} \\ &\quad \times \int_0^t \int_0^t \frac{1}{\sqrt{2\pi(2t - s - r)}} \int_{x/2}^{+\infty} \exp\left(-\frac{(w - x)^2}{2(2t - s - r)}\right) dw ds dr. \end{aligned}$$

Since the integrand is a normal probability density function, we see that

$$\frac{1}{\sqrt{2\pi(2t - s - r)}} \int_{x/2}^{+\infty} \exp\left(-\frac{(w - x)^2}{2(2t - s - r)}\right) dw \leq 1.$$

Therefore,

$$|J_{3,2}| \leq \frac{H |2H - 1|}{2^{2H-2}} \sigma^2 t^2 x^{2H-2}.$$

Thus, $|J_3| \leq C \sigma^2 t^2 x^{2H-2}$. This completes the proof. \square

Acknowledgement

The authors are grateful to the referees for their valuable suggestions that helped to improve the paper's presentation.

Funding

The second author acknowledges that the present research is carried through within the frame and support of the ToppForsk project nr. 274410 of the Research Council of Norway with title STORM: Stochastics for Time-Space Risk Models.

References

- [1] Adler, R.J., Müller, P. (eds.): Stochastic Modelling in Physical Oceanography. Progress in Probability, vol. 39, p. 467. Birkhäuser Boston, Inc., Boston, MA (1996). [MR1383868](#)
- [2] Avetisian, D.A., Shevchenko, G.M.: Estimation of diffusion parameter for stochastic heat equation with white noise. Bulletin of Taras Shevchenko National University of Kyiv. Series: Physics & Mathematics (3), 9–16 (2018).
- [3] Bally, V., Gyöngy, I., Pardoux, E.: White noise driven parabolic SPDEs with measurable drift. J. Funct. Anal. **120**(2), 484–510 (1994). [MR1266318](#). <https://doi.org/10.1006/jfan.1994.1040>
- [4] Berzin, C., Latour, A., León, J.R.: Inference on the Hurst Parameter and the Variance of Diffusions Driven by Fractional Brownian Motion. Lecture Notes in Statistics, vol. 216. Springer, Cham (2014). [MR3289986](#). <https://doi.org/10.1007/978-3-319-07875-5>
- [5] Bibinger, M., Trabs, M.: On central limit theorems for power variations of the solution to the stochastic heat equation. In: Stochastic Models, Statistics and Their Applications. Springer Proc. Math. Stat., vol. 294, pp. 69–84. Springer (2019). [MR4043170](#)
- [6] Bibinger, M., Trabs, M.: Volatility estimation for stochastic PDEs using high-frequency observations. Stochastic Process. Appl. **130**(5), 3005–3052 (2020). [MR4080737](#). <https://doi.org/10.1016/j.spa.2019.09.002>
- [7] Björk, T.: A geometric view of interest rate theory. In: Option Pricing, Interest Rates and Risk Management. Handb. Math. Finance, pp. 241–277. Cambridge Univ. Press, Cambridge (2001). [MR1848554](#). <https://doi.org/10.1017/CBO9780511569708.008>
- [8] Carmona, R., Nualart, D.: Random nonlinear wave equations: smoothness of the solutions. Probab. Theory Related Fields **79**(4), 469–508 (1988). [MR0966173](#). <https://doi.org/10.1007/BF00318783>
- [9] Cheridito, P., Kawaguchi, H., Maejima, M.: Fractional Ornstein–Uhlenbeck processes. Electron. J. Probab. **8**, 3–14 (2003). [MR1961165](#). <https://doi.org/10.1214/EJP.v8-125>
- [10] Chong, C.: High-frequency analysis of parabolic stochastic PDEs. Ann. Statist. **48**(2), 1143–1167 (2020). doi:10.1214/19-AOS1841. [MR4102691](#). <https://doi.org/10.1214/19-AOS1841>

- [11] Chow, P.-L.: Stochastic Partial Differential Equations. Chapman & Hall/CRC Applied Mathematics and Nonlinear Science Series, p. 281. Chapman & Hall/CRC, Boca Raton, FL (2007). [MR2295103](#)
- [12] Cialenco, I., Kim, H.-J.: Parametric estimation for a parabolic linear SPDE model based on sampled data. arXiv preprint [arXiv:2003.08920](#) (2020).
- [13] Cialenco, I.: Statistical inference for SPDEs: an overview. *Stat. Inference Stoch. Process.* **21**(2), 309–329 (2018). [MR3824970](#). <https://doi.org/10.1007/s11203-018-9177-9>
- [14] Cialenco, I., Huang, Y.: A note on parameter estimation for discretely sampled SPDEs. *Stoch. Dyn.* **20**(3), 2050016 (2020). [MR4101083](#). <https://doi.org/10.1142/S0219493720500161>
- [15] Cialenco, I., Kim, H.-J., Lototsky, S.V.: Statistical analysis of some evolution equations driven by space-only noise. *Stat. Inference Stoch. Process.* **23**(1), 83–103 (2020). [MR4072253](#). <https://doi.org/10.1007/s11203-019-09205-0>
- [16] Conus, D., Dalang, R.C.: The non-linear stochastic wave equation in high dimensions. *Electron. J. Probab.* **13**, 22–629670 (2008). [MR2399293](#). <https://doi.org/10.1214/EJP.v13-500>
- [17] Dalang, R.C.: Extending the martingale measure stochastic integral with applications to spatially homogeneous s.p.d.e.'s. *Electron. J. Probab.* **4**, 6–29 (1999). [MR1684157](#). <https://doi.org/10.1214/EJP.v4-43>
- [18] Dawson, D.A., Salehi, H.: Spatially homogeneous random evolutions. *J. Multivariate Anal.* **10**(2), 141–180 (1980). [MR0575923](#). [https://doi.org/10.1016/0047-259X\(80\)90012-3](https://doi.org/10.1016/0047-259X(80)90012-3)
- [19] Geissert, M., Kovács, M., Larsson, S.: Rate of weak convergence of the finite element method for the stochastic heat equation with additive noise. *BIT* **49**(2), 343–356 (2009). [MR2507605](#). <https://doi.org/10.1007/s10543-009-0227-y>
- [20] Gubinelli, M., Lejay, A., Tindel, S.: Young integrals and SPDEs. *Potential Anal.* **25**(4), 307–326 (2006). [MR2255351](#). <https://doi.org/10.1007/s11118-006-9013-5>
- [21] Gyöngy, I.: Approximations of stochastic partial differential equations. In: *Stochastic Partial Differential Equations and Applications (Trento, 2002)*. Lecture Notes in Pure and Appl. Math., vol. 227, pp. 287–307. Dekker, New York (2002). [MR1919514](#)
- [22] Hu, Y., Nualart, D.: Stochastic heat equation driven by fractional noise and local time. *Probab. Theory Related Fields* **143**(1-2), 285–328 (2009). [MR2449130](#). <https://doi.org/10.1007/s00440-007-0127-5>
- [23] Imkeller, P.: Energy balance models—viewed from stochastic dynamics. In: *Stochastic Climate Models (Chorin, 1999)*. *Progr. Probab.*, vol. 49, pp. 213–240. Birkhäuser, Basel (2001). [MR1948298](#)
- [24] Kaino, Y., Uchida, M.: Parametric estimation for a parabolic linear SPDE model based on discrete observations. *Journal of Statistical Planning and Inference* (2020). <https://doi.org/10.1016/j.jspi.2020.05.004>
- [25] Kovács, M., Larsson, S., Lindgren, F.: Strong convergence of the finite element method with truncated noise for semilinear parabolic stochastic equations with additive noise. *Numer. Algorithms* **53**(2-3), 309–320 (2010). [MR2600932](#). <https://doi.org/10.1007/s11075-009-9281-4>
- [26] Kubilius, K., Mishura, Y., Ralchenko, K.: *Parameter Estimation in Fractional Diffusion Models*. Bocconi & Springer Series, vol. 8. Bocconi University Press, [Milan]; Springer, Cham (2017). [MR3752152](#). <https://doi.org/10.1007/978-3-319-71030-3>

- [27] Lototsky, S.V., Rozovskii, B.L.: Stochastic partial differential equations driven by purely spatial noise. *SIAM J. Math. Anal.* **41**(4), 1295–1322 (2009). [MR2540267](#). <https://doi.org/10.1137/070698440>
- [28] Mahdi Khalil, Z., Tudor, C.: Estimation of the drift parameter for the fractional stochastic heat equation via power variation. *Mod. Stoch. Theory Appl.* **6**(4), 397–417 (2019). [MR4047392](#)
- [29] Markussen, B.: Likelihood inference for a discretely observed stochastic partial differential equation. *Bernoulli* **9**(5), 745–762 (2003). [MR2047684](#). <https://doi.org/10.3150/bj/1066418876>
- [30] Maslowski, B., Nualart, D.: Evolution equations driven by a fractional Brownian motion. *J. Funct. Anal.* **202**(1), 277–305 (2003). [MR1994773](#). [https://doi.org/10.1016/S0022-1236\(02\)00065-4](https://doi.org/10.1016/S0022-1236(02)00065-4)
- [31] Millet, A., Sanz-Solé, M.: A stochastic wave equation in two space dimension: smoothness of the law. *Ann. Probab.* **27**(2), 803–844 (1999). [MR1698971](#). <https://doi.org/10.1214/aop/1022677387>
- [32] Mishura, Y.S.: *Stochastic Calculus for Fractional Brownian Motion and Related Processes*. Lecture Notes in Mathematics, vol. 1929. Springer (2008). [MR2378138](#). <https://doi.org/10.1007/978-3-540-75873-0>
- [33] Pospíšil, J., Tribe, R.: Parameter estimates and exact variations for stochastic heat equations driven by space-time white noise. *Stoch. Anal. Appl.* **25**(3), 593–611 (2007). [MR2321899](#). <https://doi.org/10.1080/07362990701282849>
- [34] Quer-Sardanyons, L., Tindel, S.: The 1-d stochastic wave equation driven by a fractional Brownian sheet. *Stochastic Process. Appl.* **117**(10), 1448–1472 (2007). [MR2353035](#). <https://doi.org/10.1016/j.spa.2007.01.009>
- [35] Ralchenko, K., Shevchenko, G.: Existence and uniqueness of mild solution to fractional stochastic heat equation. *Mod. Stoch. Theory Appl.* **6**(1), 57–79 (2019). [MR3935427](#). <https://doi.org/10.15559/18-vmsta122>
- [36] Tindel, S., Tudor, C.A., Viens, F.: Stochastic evolution equations with fractional Brownian motion. *Probab. Theory Related Fields* **127**(2), 186–204 (2003). [MR2013981](#). <https://doi.org/10.1007/s00440-003-0282-2>
- [37] Tudor, C.A.: *Analysis of Variations for Self-similar Processes*. Probability and its Applications (New York). Springer (2013). [MR3112799](#). <https://doi.org/10.1007/978-3-319-00936-0>
- [38] Walsh, J.B.: An introduction to stochastic partial differential equations. In: *École D’été de Probabilités de Saint-Flour, XIV—1984*. Lecture Notes in Math., vol. 1180, pp. 265–439. Springer (1986). [MR0876085](#). <https://doi.org/10.1007/BFb0074920>
- [39] Young, L.C.: An inequality of the Hölder type, connected with Stieltjes integration. *Acta Math.* **67**(1), 251–282 (1936). [MR1555421](#). <https://doi.org/10.1007/BF02401743>
- [40] Zähle, M.: Integration with respect to fractal functions and stochastic calculus. I. *Probab. Theory Related Fields* **111**(3), 333–374 (1998). [MR1640795](#). <https://doi.org/10.1007/s004400050171>