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Practical approaches to the estimation of the ruin probability in a risk model with additional funds

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Abstract We deal with a generalization of the classical risk model when an insurance company gets additional funds whenever a claim arrives and consider some practical approaches to the estimation of the ruin probability. In particular, we get an upper exponential bound and construct an analogue to the De Vylder approximation for the ruin probability. We compare results of these approaches with statistical estimates obtained by the Monte Carlo method for selected distributions of claim sizes and additional funds.

Keywords Risk model, survival probability, exponential bound, De Vylder approximation, Monte Carlo method
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1 Introduction

Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a probability space satisfying the usual conditions, and let all the objects be defined on it. We deal with the risk model that generalizes the classical one and was considered in [10].

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In the classical risk model (see, e.g., [1, 7, 11]), an insurance company has an initial surplus $x \ge 0$ and receives premiums with constant intensity $c > 0$. Claim sizes form a sequence $(\xi_i)_{i\ge1}$ of nonnegative i.i.d. random variables with c.d.f. $F_1(y) = \mathbb{P}[\xi_i \le y]$ and finite expectation $\mathbb{E}[\xi_i] = \mu_1$. The number of claims on the time interval $[0, t]$ is a homogeneous Poisson process $(N_t)_{t\ge0}$ with intensity $\lambda > 0$.

In addition to the classical risk model, we suppose that the insurance company gets additional funds η_i when the ith claim arrives. These funds can be considered, for instance, as additional investment income, which does not depend on the surplus of the company. We assume that $(\eta_i)_{i\ge1}$ is a sequence of nonnegative i.i.d. random variables with c.d.f. $F_2(y) = \mathbb{P}[\eta_i \le y]$ and finite expectation $\mathbb{E}[\eta_i] = \mu_2$. The sequences $(\xi_i)_{i\ge1}$, $(\eta_i)_{i\ge1}$ and the process $(N_t)_{t\ge0}$ are mutually independent. Let $(\mathfrak{F}_t)_{t\ge0}$ be the filtration generated by $(\xi_i)_{i\ge1}$, $(\eta_i)_{i\ge1}$, and $(N_t)_{t\ge0}$.

Let $X_t(x)$ be the surplus of the insurance company at time t, provided that its initial surplus is x. Then the surplus process $(X_t(x))_{t\ge0}$ is defined as

$$X_t(x) = x + ct - \sum_{i=1}^{N_t} (\xi_i - \eta_i), \quad t \ge 0. \tag{1}$$

Note that we set $\sum_{i=1}^{0} (\xi_i - \eta_i) = 0$ in (1) if $N_t = 0$.

The ruin time is defined as

$$\tau(x) = \inf\{t \ge 0\colon X_t(x) < 0\}.$$

We suppose that $\tau(x) = \infty$ if $X_t(x) \ge 0$ for all $t \ge 0$. The infinite-horizon ruin probability is given by

$$\psi(x) = \mathbb{P}\big[\inf_{t\ge0} X_t(x) < 0\big],$$

which is equivalent to

$$\psi(x) = \mathbb{P}\big[\tau(x) < \infty\big].$$

The corresponding infinite-horizon survival probability equals

$$\varphi(x) = 1 - \psi(x).$$

Note that the ruin never occurs if $\mathbb{P}[\xi_i - \eta_i \le 0] = 1$. If $\mathbb{P}[\xi_i - \eta_i \ge 0] = 1$, then we deal with the classical risk model. So in what follows, we assume that $\mathbb{P}[\xi_i - \eta_i > 0] > 0$ and $\mathbb{P}[\xi_i - \eta_i < 0] > 0$. In this case, if $c - \lambda\mu_1 + \lambda\mu_2 \le 0$, then $\varphi(x) = 0$ for all $x \ge 0$; if $c - \lambda\mu_1 + \lambda\mu_2 > 0$, then $\lim_{x\to+\infty} \varphi(x) = 1$ (see [10, Lemma 2.1]).

In this paper, we consider some practical approaches to the estimation of the ruin probability. In particular, we get an upper exponential bound and construct an analogue to the De Vylder approximation for the ruin probability. Moreover, we compare results of these approaches with statistical estimates obtained by the Monte Carlo method for selected distributions of claim sizes and additional funds.

Paper [10], where this risk model is considered, is devoted to the investigation of continuity and differentiability of the infinite-horizon survival probability and derivation of an integro-differential equation for this function. When claim sizes and additional funds are exponentially distributed, a closed-form solution to this equation can be found.

Theorem 1 ([10], Theorem 4.1)**.** *Let the surplus process* $(X_t(x))_{t\geq 0}$ *follow* (1) *under the above assumptions, the random variables* ξ_i *and* η_i*,* $i \geq 1$*, be exponentially distributed with means* μ_1 *and* μ_2 *correspondingly, and* $c - \lambda\mu_1 + \lambda\mu_2 > 0$*. Then*

$$\varphi(x) = 1 + \frac{\lambda\mu_1(1-\alpha\mu_2)}{(c\alpha - \lambda)(1-\alpha\mu_2)(\mu_1+\mu_2) + \lambda\mu_2} e^{\alpha x} \tag{2}$$

for all $x \geq 0$*, where*

$$\alpha = \frac{\lambda\mu_1\mu_2 + c\mu_1 - c\mu_2 - \sqrt{c^2(\mu_1^2+\mu_2^2) + \lambda^2\mu_1^2\mu_2^2 + 2c\mu_1\mu_2(c - \lambda\mu_1 + \lambda\mu_2)}}{2c\mu_1\mu_2}.$$

Remark 1 ([10], Remark 4.1)**.** It is justified in the proof of Theorem 1 that $\alpha < 0$ and

$$-1 < \frac{\lambda\mu_1(1-\alpha\mu_2)}{(c\alpha - \lambda)(1-\alpha\mu_2)(\mu_1+\mu_2) + \lambda\mu_2} < 0.$$

So the function $\varphi(x)$ defined by (2) satisfies all the natural properties of the survival probability. In particular, this function is nondecreasing and bounded by 0 from below and by 1 from above.

It is well known that even for the classical risk model, there are only a few cases where an analytic expression for the survival probability can be found. So numerous approximations have been considered and investigated for the classical risk model (see, e.g., [1, 2, 4, 5, 7, 11]). "Simple approximations" form a special class of approximations for the ruin or survival probabilities. They use only some moments of the distribution of claim sizes and do not take into account the detailed tail behavior of that distribution. Such approximations may be based on limit theorems or on heuristic arguments. The most successful "simple approximation" is certainly the De Vylder approximation [5], which is based on the heuristic idea to replace the risk process with a risk process with exponentially distributed claim sizes such that the first three moments coincide (see also [7, 11]). This approximation is known to work extremely well for some distributions of claim sizes. Later, Grandell analyzed the De Vylder approximation and other "simple approximations" from a more mathematical point of view and gave a possible explanation why the De Vylder approximation is so good (see [8]).

We deal with the case where the claim sizes have a light-tailed distribution. The rest of the paper is organized as follows. In Section 2, we get an upper exponential bound for the ruin probability, which is an analogue of the famous Lundberg inequality. Section 3 is devoted to the construction of an analogue of the De Vylder approximation. In Section 4, we give a simple formula that relates the accuracy and reliability of the approximation of the ruin probability by its statistical estimate obtained by the Monte Carlo method. In Section 5, we compare the results of these approaches for some distributions of claim sizes and additional funds. Section 6 concludes the paper.

2 Exponential bound

To get an upper exponential bound for the ruin probability, we use the martingale approach introduced by Gerber [6] (see also [3, 7, 11]).

Let

$$U_t = ct - \sum_{i=1}^{N_t}(\xi_i - \eta_i), \quad t \ge 0.$$

For all $R \ge 0$, we define the exponential process $(V_t(R))_{t\ge 0}$ by

$$V_t(R) = e^{-RU_t}.$$

Lemma 1. *If there is* $\hat{R} > 0$ *such that*

$$\lambda\left(\int_0^{+\infty} e^{\hat{R}y}\, dF_1(y) \cdot \int_0^{+\infty} e^{-\hat{R}y}\, dF_2(y) - 1\right) = c\hat{R}, \tag{3}$$

then $(V_t(\hat{R}))_{t\ge 0}$ *is an* $(\mathfrak{F}_t)$*-martingale.*

Proof. For all $R > 0$ such that $\mathbb{E}[e^{R\xi_i}] < \infty$, if any, we have

$$\begin{aligned}\mathbb{E}\big[V_t(R)\big] &= e^{-cRt}\, \mathbb{E}\left[\exp\left\{R\sum_{i=1}^{N_t}(\xi_i - \eta_i)\right\}\right] \\ &= e^{-cRt} \sum_{j=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^j}{j!} \big(\mathbb{E}\big[e^{R(\xi_i - \eta_i)}\big]\big)^j \\ &= \exp\big\{t\big(\lambda \mathbb{E}\big[e^{R(\xi_i-\eta_i)}\big] - \lambda - cR\big)\big\}.\end{aligned} \tag{4}$$

If there is $\hat{R} > 0$ such that (3) holds, then $\mathbb{E}[e^{\hat{R}\xi_i}] < \infty$, and for all $t_2 \ge t_1 \ge 0$, we have

$$\begin{aligned}\mathbb{E}\big[V_{t_2}(\hat{R}) \,/\, \mathfrak{F}_{t_1}\big] &= \mathbb{E}\left[\exp\left\{-\hat{R}\left(ct_2 - \sum_{i=1}^{N_{t_2}}(\xi_i - \eta_i)\right)\right\} \Big/ \mathfrak{F}_{t_1}\right] \\ &= \mathbb{E}\left[\exp\left\{-\hat{R}\left(ct_1 - \sum_{i=1}^{N_{t_1}}(\xi_i - \eta_i)\right)\right\}\right] \\ &\quad \times \mathbb{E}\left[\exp\left\{-\hat{R}\left(c(t_2 - t_1) - \sum_{i=N_{t_1}}^{N_{t_2}}(\xi_i - \eta_i)\right)\right\} \Big/ \mathfrak{F}_{t_1}\right] \\ &= \mathbb{E}\big[V_{t_1}(\hat{R})\big] \cdot \mathbb{E}\left[\exp\left\{-\hat{R}\left(c(t_2 - t_1) - \sum_{i=1}^{N_{t_2-t_1}}(\xi_i - \eta_i)\right)\right\}\right] \\ &= \mathbb{E}\big[V_{t_1}(\hat{R})\big].\end{aligned}$$

Here we used the fact that

$$\begin{aligned}&\mathbb{E}\left[\exp\left\{-\hat{R}\left(c(t_2 - t_1) - \sum_{i=1}^{N_{t_2-t_1}}(\xi_i - \eta_i)\right)\right\}\right] \\ &\quad = \exp\big\{(t_2 - t_1)\big(\lambda \mathbb{E}\big[e^{\hat{R}(\xi_i - \eta_i)}\big] - \lambda - c\hat{R}\big)\big\} = 1\end{aligned}$$

by (3) and (4).

Thus, $(V_t(\hat{R}))_{t\ge 0}$ is an $(\mathfrak{F}_t)$-martingale, which is the desired conclusion. □

Theorem 2. *If there is $\hat{R} > 0$ such that* (3) *holds, then for all $x \geq 0$, we have*

$$\psi(x) \leq e^{-\hat{R}x}. \tag{5}$$

Proof. It is easily seen that $\tau(x)$ is an $(\mathfrak{F}_t)$-stopping time. Hence, $\tau(x) \wedge T$ is a bounded $(\mathfrak{F}_t)$-stopping time for any fixed $T \geq 0$. The process $(V_t(\hat{R}))_{t\geq 0}$ is an $(\mathfrak{F}_t)$-martingale by Lemma 1. Moreover, $(V_t(\hat{R}))_{t\geq 0}$ is positive a.s. by its definition. Consequently, applying the optional stopping theorem yields

$$\begin{aligned}
1 &= V_0(\hat{R}) = \mathbb{E}\big[V_{\tau(x)\wedge T}(\hat{R})\big] \\
&= \mathbb{E}\big[V_{\tau(x)}(\hat{R}) \cdot \mathbb{I}_{\{\tau(x)<T\}}\big] + \mathbb{E}\big[V_T(\hat{R}) \cdot \mathbb{I}_{\{\tau(x)\geq T\}}\big] \\
&\geq \mathbb{E}\big[V_{\tau(x)}(\hat{R}) \cdot \mathbb{I}_{\{\tau(x)<T\}}\big] \\
&= \mathbb{E}\Bigg[\exp\Bigg\{-\hat{R}\Bigg(c\tau(x) - \sum_{i=1}^{N_{\tau(x)}}(\xi_i - \eta_i)\Bigg)\Bigg\} \cdot \mathbb{I}_{\{\tau(x)<T\}}\Bigg] \\
&\geq e^{\hat{R}x} \cdot \mathbb{P}\big[\tau(x) < T\big],
\end{aligned}$$

where $\mathbb{I}_{\{\cdot\}}$ is the indicator of an event. This gives

$$\mathbb{P}\big[\tau(x) < T\big] \leq e^{-\hat{R}x} \tag{6}$$

for all $T \geq 0$. Letting $T \to \infty$ in (6) yields

$$\mathbb{P}\big[\tau(x) < \infty\big] \leq e^{-\hat{R}x},$$

which is our assertion. □

Example 1. Let the random variables ξ_i and η_i, $i \geq 1$, be exponentially distributed with means μ_1 and μ_2, respectively. Then (3) can be rewritten as

$$\lambda\left(\frac{1}{(1-\mu_1\hat{R})(1+\mu_2\hat{R})} - 1\right) = c\hat{R},$$

where $\hat{R} \in (0, 1/\mu_1)$. This condition is equivalent to

$$c\mu_1\mu_2\hat{R}^3 + (\lambda\mu_1\mu_2 + c\mu_1 - c\mu_2)\hat{R}^2 - (c - \lambda\mu_1 + \lambda\mu_2)\hat{R} = 0. \tag{7}$$

If $c - \lambda\mu_1 + \lambda\mu_2 > 0$, then

$$c^2\big(\mu_1^2 + \mu_2^2\big) + \lambda^2\mu_1^2\mu_2^2 + 2c\mu_1\mu_2(c - \lambda\mu_1 + \lambda\mu_2) > 0.$$

So there are three real solutions to (7). They are

$$\hat{R}_1 = 0,$$

$$\hat{R}_2 = -\frac{\lambda\mu_1\mu_2 + c\mu_1 - c\mu_2 - \sqrt{A(c,\lambda,\mu_1,\mu_2)}}{2c\mu_1\mu_2},$$

$$\hat{R}_3 = -\frac{\lambda\mu_1\mu_2 + c\mu_1 - c\mu_2 + \sqrt{A(c,\lambda,\mu_1,\mu_2)}}{2c\mu_1\mu_2},$$

where

$$A(c,\lambda,\mu_1,\mu_2) = c^2\big(\mu_1^2+\mu_2^2\big) + \lambda^2\mu_1^2\mu_2^2 + 2c\mu_1\mu_2(c-\lambda\mu_1+\lambda\mu_2).$$

Furthermore, it is easy to check that, in this case,

$$|\lambda\mu_1\mu_2 + c\mu_1 - c\mu_2| < \sqrt{A(c,\lambda,\mu_1,\mu_2)}.$$

From this we conclude that $\hat{R}_2 > 0$ and $\hat{R}_3 < 0$. Since

$$A(c,\lambda,\mu_1,\mu_2) < (\lambda\mu_1\mu_2 + c\mu_1 + c\mu_2)^2,$$

we have

$$\hat{R}_2 < \frac{(\lambda\mu_1\mu_2 + c\mu_1 + c\mu_2) - (\lambda\mu_1\mu_2 + c\mu_1 - c\mu_2)}{2c\mu_1\mu_2} < \frac{1}{\mu_1}.$$

Hence, $\hat{R}_2$ is a unique positive solution to (7), and an exponential bound (5) can be rewritten as follows:

$$\psi(x) \le e^{-\hat{R}_2 x}. \tag{8}$$

Comparing (8) with (2), we see that the exponential bound and the analytic expression for the ruin probability differ in a constant multiplier only.

If $c - \lambda\mu_1 + \lambda\mu_2 \le 0$, then $\mu_1 > \mu_2$, which gives $\lambda\mu_1\mu_2 + c\mu_1 - c\mu_2 > 0$. Let $\hat{R}_2$ and $\hat{R}_3$ be two nonzero solutions to (7). Applying Vieta's formulas yields $\hat{R}_2 + \hat{R}_3 < 0$ and $\hat{R}_2\hat{R}_3 > 0$. Consequently, if $\hat{R}_2$ and $\hat{R}_3$ are real, they are negative. Thus, (7) has no positive solution, and Theorem 2 does not give us an exponential bound for the ruin probability. Indeed, in this case, $\psi(x) = 1$ for all $x \ge 0$ (see [10, Lemma 2.1]).

3 Analogue to the De Vylder approximation

To construct an analogue to the De Vylder approximation, we replace the process $(U_t)_{t\ge0}$ with a process $(\tilde{U}_t)_{t\ge0}$ with exponentially distributed claim sizes such that

$$\mathbb{E}\big[U_t^k\big] = \mathbb{E}\big[\tilde{U}_t^k\big], \quad k = 1,2,3. \tag{9}$$

Since the process $(\tilde{U}_t)_{t\ge0}$ in this risk model is determined by the four parameters $(\tilde{c}, \tilde{\lambda}, \tilde{\mu}_1, \tilde{\mu}_2)$ in contrast to the classical risk model, where it is determined by three parameters, we use the additional condition

$$\frac{\mu_1}{\mu_2} = \frac{\tilde{\mu}_1}{\tilde{\mu}_2}. \tag{10}$$

Note that we could have used the condition $\mathbb{E}[U_t^4] = \mathbb{E}[\tilde{U}_t^4]$ instead of (10), but it would have led to tedious calculations and solving polynomial equations of higher degree.

Let $(\tilde{\xi}_i)_{i\geq 1}$ be a sequence of i.i.d. random variables exponentially distributed with mean $\tilde{\mu}_1$. Similarly, let $(\tilde{\eta}_i)_{i\geq 1}$ be a sequence of i.i.d. random variables exponentially distributed with mean $\tilde{\mu}_2$. An easy computation shows that

$$\mathbb{E}\big[\tilde{\xi}_i^k\big] = k!\tilde{\mu}_1^k \quad \text{and} \quad \mathbb{E}\big[\tilde{\eta}_i^k\big] = k!\tilde{\mu}_2^k. \tag{11}$$

Let $\mathbb{E}[\xi_i^3] < \infty$ and $\mathbb{E}[\eta_i^3] < \infty$. Then we have

$$\mathbb{E}[U_t] = ct - \mathbb{E}\Bigg[\sum_{i=1}^{N_t}(\xi_i - \eta_i)\Bigg] = ct - \lambda t\,\mathbb{E}[\xi_i - \eta_i],$$

$$\begin{aligned}
\mathbb{E}\big[U_t^2\big] &= (ct)^2 - 2ct\,\mathbb{E}\Bigg[\sum_{i=1}^{N_t}(\xi_i - \eta_i)\Bigg] + \mathbb{E}\Bigg[\Bigg(\sum_{i=1}^{N_t}(\xi_i - \eta_i)\Bigg)^2\Bigg] \\
&= (ct)^2 - 2ct\cdot\lambda t\,\mathbb{E}[\xi_i - \eta_i] + \lambda t\,\mathbb{E}\big[(\xi_i - \eta_i)^2\big] + (\lambda t)^2\big(\mathbb{E}[\xi_i - \eta_i]\big)^2 \\
&= \lambda t\,\mathbb{E}\big[(\xi_i - \eta_i)^2\big] + \big(ct - \lambda t\,\mathbb{E}[\xi_i - \eta_i]\big)^2 \\
&= \lambda t\,\mathbb{E}\big[(\xi_i - \eta_i)^2\big] + \big(\mathbb{E}[U_t]\big)^2,
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}\big[U_t^3\big] &= (ct)^3 - 3(ct)^2\,\mathbb{E}\Bigg[\sum_{i=1}^{N_t}(\xi_i - \eta_i)\Bigg] \\
&\quad + 3ct\,\mathbb{E}\Bigg[\Bigg(\sum_{i=1}^{N_t}(\xi_i - \eta_i)\Bigg)^2\Bigg] - \mathbb{E}\Bigg[\Bigg(\sum_{i=1}^{N_t}(\xi_i - \eta_i)\Bigg)^3\Bigg] \\
&= (ct)^3 - 3(ct)^2\cdot\lambda t\,\mathbb{E}[\xi_i - \eta_i] + 3ct(\lambda t\,\mathbb{E}\big[(\xi_i - \eta_i)^2\big] \\
&\quad + (\lambda t)^2\big(\mathbb{E}[\xi_i - \eta_i]\big)^2 - \lambda t\,\mathbb{E}\big[(\xi_i - \eta_i)^3\big] - 3(\lambda t)^2\,\mathbb{E}\big[(\xi_i - \eta_i)^2\big] \\
&\quad + \lambda t\,\mathbb{E}\big[(\xi_i - \eta_i)^2\big]\,\mathbb{E}\big[(\xi_i - \eta_i)^2\big] - (\lambda t)^3\big(\mathbb{E}[\xi_i - \eta_i]\big)^3 \\
&= -\lambda t\,\mathbb{E}\big[(\xi_i - \eta_i)^3\big] + \big(ct - \lambda t\,\mathbb{E}[\xi_i - \eta_i]\big)^3 \\
&\quad + 3\lambda t\,\mathbb{E}\big[(\xi_i - \eta_i)^2\big]\big(ct - \lambda t\,\mathbb{E}[\xi_i - \eta_i]\big) \\
&= -\lambda t\,\mathbb{E}\big[(\xi_i - \eta_i)^3\big] + \big(\mathbb{E}[U_t]\big)^3 + 3\big(\mathbb{E}\big[U_t^2\big] - \big(\mathbb{E}[U_t]\big)^2\big)\,\mathbb{E}[U_t].
\end{aligned}$$

Applying similar arguments to the process $(\tilde{U}_t)_{t\geq 0}$, we conclude that (9) is equivalent to

$$\begin{cases}
ct - \lambda t\,\mathbb{E}[\xi_i - \eta_i] = \tilde{c}t - \tilde{\lambda}t\,\mathbb{E}[\tilde{\xi}_i - \tilde{\eta}_i], \\
\lambda t\,\mathbb{E}\big[(\xi_i - \eta_i)^2\big] = \tilde{\lambda}t\,\mathbb{E}\big[(\tilde{\xi}_i - \tilde{\eta}_i)^2\big], \\
-\lambda t\,\mathbb{E}\big[(\xi_i - \eta_i)^3\big] = -\tilde{\lambda}t\,\mathbb{E}\big[(\tilde{\xi}_i - \tilde{\eta}_i)^3\big],
\end{cases} \tag{12}$$

By (11) we can rewrite (12) as

$$\begin{cases}
c - \lambda(\mu_1 - \mu_2) = \tilde{c} - \tilde{\lambda}(\tilde{\mu}_1 - \tilde{\mu}_2), \\
\lambda\,\mathbb{E}\big[(\xi_i - \eta_i)^2\big] = 2\tilde{\lambda}\big(\tilde{\mu}_1^2 - \tilde{\mu}_1\tilde{\mu}_2 + \tilde{\mu}_2^2\big), \\
\lambda\,\mathbb{E}\big[(\xi_i - \eta_i)^3\big] = 6\tilde{\lambda}\big(\tilde{\mu}_1^3 - \tilde{\mu}_1^2\tilde{\mu}_2 + \tilde{\mu}_1\tilde{\mu}_2^2 - \tilde{\mu}_2^3\big).
\end{cases} \tag{13}$$

Substituting $\tilde{\mu}_2 = \mu_2\tilde{\mu}_1/\mu_1$ into the second and third equations of system (13) yields

$$\lambda\,\mathbb{E}\big[(\xi_i-\eta_i)^2\big] = 2\tilde{\lambda}\tilde{\mu}_1^2\left(1-\frac{\mu_2}{\mu_1}+\frac{\mu_2^2}{\mu_1^2}\right), \tag{14}$$

$$\lambda\,\mathbb{E}\big[(\xi_i-\eta_i)^3\big] = 6\tilde{\lambda}\tilde{\mu}_1^3\left(1-\frac{\mu_2}{\mu_1}+\frac{\mu_2^2}{\mu_1^2}-\frac{\mu_2^3}{\mu_1^3}\right). \tag{15}$$

Dividing (15) by (14) gives

$$\tilde{\mu}_1 = \frac{\mu_1(\mu_1^2-\mu_1\mu_2+\mu_2^2)\,\mathbb{E}[(\xi_i-\eta_i)^3]}{3(\mu_1^3-\mu_1^2\mu_2+\mu_1\mu_2^2-\mu_2^3)\,\mathbb{E}[(\xi_i-\eta_i)^2]}. \tag{16}$$

Consequently, we have

$$\tilde{\mu}_2 = \frac{\mu_2(\mu_1^2-\mu_1\mu_2+\mu_2^2)\,\mathbb{E}[(\xi_i-\eta_i)^3]}{3(\mu_1^3-\mu_1^2\mu_2+\mu_1\mu_2^2-\mu_2^3)\,\mathbb{E}[(\xi_i-\eta_i)^2]}. \tag{17}$$

Substituting (16) into (14), we get

$$\tilde{\lambda} = \frac{9\lambda(\mu_1^3-\mu_1^2\mu_2+\mu_1\mu_2^2-\mu_2^3)^2\,(\mathbb{E}[(\xi_i-\eta_i)^2])^3}{2(\mu_1^2-\mu_1\mu_2+\mu_2^2)^3\,(\mathbb{E}[(\xi_i-\eta_i)^3])^2}. \tag{18}$$

Substituting (16)–(18) into the first equation of system (13), we obtain

$$\tilde{c} = c-\lambda(\mu_1-\mu_2)\left(1-\frac{3(\mu_1^3-\mu_1^2\mu_2+\mu_1\mu_2^2-\mu_2^3)\,(\mathbb{E}[(\xi_i-\eta_i)^2])^2}{2(\mu_1^2-\mu_1\mu_2+\mu_2^2)^2\,\mathbb{E}[(\xi_i-\eta_i)^3]}\right). \tag{19}$$

Note that since $F_1(y)$ and $F_2(y)$ are known, it is easy to find $\mathbb{E}[(\xi_i-\eta_i)^2]$ and $\mathbb{E}[(\xi_i-\eta_i)^3]$ if $\mathbb{E}[\xi_i^3]<\infty$ and $\mathbb{E}[\eta_i^3]<\infty$.

By (16) and (17), $\tilde{\mu}_1$ and $\tilde{\mu}_2$ are positive, provided that

$$\big(\mu_1^3-\mu_1^2\mu_2+\mu_1\mu_2^2-\mu_2^3\big)\,\mathbb{E}\big[(\xi_i-\eta_i)^3\big]>0. \tag{20}$$

If (20) holds, then $\mu_1\neq\mu_2$. So $\tilde{\lambda}$ is also positive. Moreover, $\tilde{c}$ is positive, provided that

$$c-\lambda(\mu_1-\mu_2)\left(1-\frac{3(\mu_1^3-\mu_1^2\mu_2+\mu_1\mu_2^2-\mu_2^3)\,(\mathbb{E}[(\xi_i-\eta_i)^2])^2}{2(\mu_1^2-\mu_1\mu_2+\mu_2^2)^2\,\mathbb{E}[(\xi_i-\eta_i)^3]}\right)>0. \tag{21}$$

Thus, we get the following result.

Proposition 1 (An analogue to the De Vylder approximation)**.** *Let the surplus process $(X_t(x))_{t\geq0}$ follow (1) under the above assumptions, $c-\lambda\mu_1+\lambda\mu_2>0$, $\mathbb{E}[\xi_i^3]<\infty$, $\mathbb{E}[\eta_i^3]<\infty$, and let conditions (20) and (21) hold. Then the ruin probability is approximately equal to*

$$\psi_{DV}(x) = \frac{\tilde{\lambda}\tilde{\mu}_1(\tilde{\alpha}\tilde{\mu}_2-1)}{(\tilde{c}\tilde{\alpha}-\tilde{\lambda})(1-\tilde{\alpha}\tilde{\mu}_2)(\tilde{\mu}_1+\tilde{\mu}_2)+\tilde{\lambda}\tilde{\mu}_2}\,e^{\tilde{\alpha}x}$$

for all $x\geq0$, where

$$\tilde{\alpha} = \frac{\tilde{\lambda}\tilde{\mu}_1\tilde{\mu}_2 + \tilde{c}\tilde{\mu}_1 - \tilde{c}\tilde{\mu}_2 - \sqrt{\tilde{c}^2(\tilde{\mu}_1^2 + \tilde{\mu}_2^2) + \tilde{\lambda}^2\tilde{\mu}_1^2\tilde{\mu}_2^2 + 2\tilde{c}\tilde{\mu}_1\tilde{\mu}_2(\tilde{c} - \tilde{\lambda}\tilde{\mu}_1 + \tilde{\lambda}\tilde{\mu}_2)}}{2\tilde{c}\tilde{\mu}_1\tilde{\mu}_2}$$

and the parameters $(\tilde{c}, \tilde{\lambda}, \tilde{\mu}_1, \tilde{\mu}_2)$ *are defined by* (16)–(19).

Remark 2. Note that $\tilde{\alpha} < 0$ and

$$\frac{\tilde{\lambda}\tilde{\mu}_1(\tilde{\alpha}\tilde{\mu}_2 - 1)}{(\tilde{c}\tilde{\alpha} - \tilde{\lambda})(1 - \tilde{\alpha}\tilde{\mu}_2)(\tilde{\mu}_1 + \tilde{\mu}_2) + \tilde{\lambda}\tilde{\mu}_2} > 0$$

in Proposition 1. Indeed, the parameters $(\tilde{c}, \tilde{\lambda}, \tilde{\mu}_1, \tilde{\mu}_2)$ are positive. Moreover, since $c - \lambda\mu_1 + \lambda\mu_2 > 0$, we have $\tilde{c} - \tilde{\lambda}\tilde{\mu}_1 + \tilde{\lambda}\tilde{\mu}_2 > 0$ by the first equation of system (13). Hence, Theorem 1 and Remark 1 give us the desired conclusion.

Remark 3. If claim sizes and additional funds are exponentially distributed, then it is easily seen from (16)–(19) that $\psi(x) = \psi_{DV}(x)$.

4 Statistical estimate obtained by the Monte Carlo method

Let N be the total number of simulations of the surplus process $X_t(x)$, and let $\hat{\psi}(x)$ be the corresponding statistical estimate obtained by the Monte Carlo method. To get it, we divide the number of simulations that lead to the ruin by the total number of simulations.

Proposition 2. *Let the surplus process* $(X_t(x))_{t\geq 0}$ *follow* (1) *under the above assumptions. Then for any* $\varepsilon > 0$*, we have*

$$\mathbb{P}\big[\big|\psi(x) - \hat{\psi}(x)\big| > \varepsilon\big] \leq 2e^{-2\varepsilon^2 N}. \tag{22}$$

The assertion of Proposition 2 follows immediately from Hoeffding's inequality (see [9]).

Remark 4. Formula (22) relates the accuracy and reliability of the approximation of the ruin probability by its statistical estimate obtained by the Monte Carlo method. It enables us to find the number of simulations N, which is necessary in order to calculate the ruin probability with the required accuracy and reliability. An obvious shortcoming of the Monte Carlo method is a too large number of simulations N. In all examples in Section 5, we assume that $\varepsilon = 0.001$ and $2e^{-2\varepsilon^2 N} = 0.001$. Consequently, $N = 3\,800\,452$.

5 Comparison of results

5.1 Erlang distributions for claim sizes and additional funds

Let the probability density functions of ξ_i and η_i be

$$f_1(y) = \frac{k_1^{k_1} y^{k_1-1} e^{-k_1 y/\mu_1}}{\mu_1^{k_1}(k_1 - 1)!} \quad \text{and} \quad f_2(y) = \frac{k_2^{k_2} y^{k_2-1} e^{-k_2 y/\mu_2}}{\mu_2^{k_2}(k_2 - 1)!}$$

for $y \geq 0$, respectively, where k_1 and k_2 are positive integers.

In what follows, $h_1(R)$ and $h_2(R)$, where $R \ge 0$, denote the moment generating functions of ξ_i and η_i, respectively, that is,

$$h_1(R) = \mathbb{E}\big[e^{R\xi_i}\big] \quad \text{and} \quad h_2(R) = \mathbb{E}\big[e^{R\eta_i}\big].$$

An easy computation shows that

$$h_1(R) = \int_0^{+\infty} e^{Ry}\, dF_1(y) = \left(\frac{k_1}{k_1 - \mu_1 R}\right)^{k_1}, \quad 0 \le R < \frac{k_1}{\mu_1},$$

$$h_2(R) = \int_0^{+\infty} e^{Ry}\, dF_2(y) = \left(\frac{k_2}{k_2 - \mu_2 R}\right)^{k_2}, \quad 0 \le R < \frac{k_2}{\mu_2}.$$

Moreover, for all $R \ge 0$, we have

$$\int_0^{+\infty} e^{-Ry}\, dF_2(y) = \left(\frac{k_2}{k_2 + \mu_2 R}\right)^{k_2}.$$

Thus, condition (3) can be rewritten as

$$\lambda \left(\frac{k_1}{k_1 - \mu_1 \hat{R}}\right)^{k_1} \left(\frac{k_2}{k_2 + \mu_2 \hat{R}}\right)^{k_2} = \lambda + c\hat{R}, \tag{23}$$

where $0 < \hat{R} < k_1/\mu_1$. Furthermore, we have

$$\mathbb{E}[\xi_i] = h_1'(0) = \mu_1, \qquad \mathbb{E}[\eta_i] = h_2'(0) = \mu_2,$$

$$\mathbb{E}\big[\xi_i^2\big] = h_1''(0) = \frac{(k_1+1)\mu_1^2}{k_1}, \qquad \mathbb{E}\big[\eta_i^2\big] = h_2''(0) = \frac{(k_2+1)\mu_2^2}{k_2},$$

$$\mathbb{E}\big[\xi_i^3\big] = h_1'''(0) = \frac{(k_1+1)(k_1+2)\mu_1^3}{k_1^2}, \qquad \mathbb{E}\big[\eta_i^3\big] = h_2'''(0) = \frac{(k_2+1)(k_2+2)\mu_2^3}{k_2^2}.$$

Hence, we get

$$\begin{aligned}
\mathbb{E}\big[(\xi_i - \eta_i)^2\big] &= \mathbb{E}\big[\xi_i^2\big] - 2\mathbb{E}[\xi_i]\mathbb{E}[\eta_i] + \mathbb{E}\big[\eta_i^2\big] \\
&= \frac{(k_1+1)\mu_1^2}{k_1} - 2\mu_1\mu_2 + \frac{(k_2+1)\mu_2^2}{k_2}, \\
\mathbb{E}\big[(\xi_i - \eta_i)^3\big] &= \mathbb{E}\big[\xi_i^3\big] - 3\mathbb{E}\big[\xi_i^2\big]\mathbb{E}[\eta_i] + 3\mathbb{E}[\xi_i]\mathbb{E}\big[\eta_i^2\big] - \mathbb{E}\big[\eta_i^3\big] \\
&= \frac{(k_1+1)(k_1+2)\mu_1^3}{k_1^2} - \frac{3(k_1+1)\mu_1^2\mu_2}{k_1} \\
&\quad + \frac{3(k_2+1)\mu_2^2\mu_1}{k_2} - \frac{(k_2+1)(k_2+2)\mu_2^3}{k_2^2}.
\end{aligned}$$

Substituting $\mathbb{E}[(\xi_i - \eta_i)^2]$ and $\mathbb{E}[(\xi_i - \eta_i)^3]$ into (16)–(19), we obtain the parameters $(\tilde{c}, \tilde{\lambda}, \tilde{\mu}_1, \tilde{\mu}_2)$.

Example 2. Let $c = 10$, $\lambda = 4$, $\mu_1 = 2$, $\mu_2 = 0.5$, $k_1 = 3$, $k_2 = 2$. Then $\hat{R} \approx 0.349093$, which may not be an unique positive solution to (23), and

$$\psi_{DV}(x) = 0.612268\, e^{-0.332472\, x}.$$

The results of computations are given in Table 1.

Table 1. Results of computations: Erlang distributions for claim sizes and additional funds

x	$\hat{\psi}(x)$	$\psi_{DV}(x)$	$\left(\frac{\psi_{DV}(x)}{\hat{\psi}(x)} - 1\right) \cdot 100\%$	$e^{-\hat{R}x}$	$\left(\frac{e^{-\hat{R}x}}{\hat{\psi}(x)} - 1\right) \cdot 100\%$
0	0.634149	0.612268	−3.45%	1.000000	57.69%
1	0.492768	0.439087	−10.89%	0.705327	43.14%
2	0.355769	0.314891	−11.49%	0.497487	39.83%
5	0.137737	0.116142	−15.68%	0.174564	26.74%
10	0.023224	0.022031	−5.14%	0.030473	31.21%

5.2 Hyperexponential distributions for claim sizes and additional funds

Let

$$F_1(y) = p_{1,1}F_{1,1}(y) + p_{1,2}F_{1,2}(y) + \cdots + p_{1,k_1}F_{1,k_1}(y), \quad y \geq 0,$$

where $k_1 \geq 1$, $p_{1,j} > 0$, $\sum_{j=1}^{k_1} p_{1,j} = 1$, $\sum_{j=1}^{k_1} p_{1,j}\,\mu_{1,j} = \mu_1$, and $F_{1,j}$ is the c.d.f. of the exponential distribution with mean $\mu_{1,j}$;

$$F_2(y) = p_{2,1}F_{2,1}(y) + p_{2,2}F_{2,2}(y) + \cdots + p_{2,k_2}F_{2,k_2}(y), \quad y \geq 0,$$

where $k_2 \geq 1$, $p_{2,j} > 0$, $\sum_{j=1}^{k_2} p_{2,j} = 1$, $\sum_{j=1}^{k_2} p_{2,j}\,\mu_{2,j} = \mu_2$, and $F_{2,j}$ is the c.d.f. of the exponential distribution with mean $\mu_{2,j}$.

It is easy to check that

$$h_1(R) = \sum_{j=1}^{k_1} \frac{p_{1,j}}{1 - \mu_{1,j}R}, \quad 0 \leq R < \min\left\{\frac{1}{\mu_{1,1}}, \frac{1}{\mu_{1,2}}, \ldots, \frac{1}{\mu_{1,k_1}}\right\},$$

$$h_2(R) = \sum_{j=1}^{k_2} \frac{p_{2,j}}{1 - \mu_{2,j}R}, \quad 0 \leq R < \min\left\{\frac{1}{\mu_{2,1}}, \frac{1}{\mu_{2,2}}, \ldots, \frac{1}{\mu_{2,k_2}}\right\}.$$

Furthermore, for all $R \geq 0$, we have

$$\int_0^{+\infty} e^{-Ry}\, dF_2(y) = \sum_{j=1}^{k_2} \frac{p_{2,j}}{1 + \mu_{2,j}R}.$$

Hence, condition (3) can be rewritten as

$$\lambda\left(\sum_{j=1}^{k_1} \frac{p_{1,j}}{1 - \mu_{1,j}\hat{R}} \cdot \sum_{j=1}^{k_2} \frac{p_{2,j}}{1 + \mu_{2,j}\hat{R}}\right) = \lambda + c\hat{R}, \tag{24}$$

where $0 < \hat{R} < \min\{1/\mu_{1,1}, 1/\mu_{1,2}, \ldots, 1/\mu_{1,k_1}\}$. Moreover, we have

$$\mathbb{E}[\xi_i] = h_1'(0) = \sum_{j=1}^{k_1} p_{1,j}\,\mu_{1,j} = \mu_1, \qquad \mathbb{E}[\eta_i] = h_2'(0) = \sum_{j=1}^{k_2} p_{2,j}\,\mu_{2,j} = \mu_2,$$

Table 2. Results of computations: hyperexponential distributions for claim sizes and additional funds

x	$\hat{\psi}(x)$	$\psi_{DV}(x)$	$\big(\frac{\psi_{DV}(x)}{\hat{\psi}(x)} - 1\big) \cdot 100\%$	$e^{-\hat{R}x}$	$\big(\frac{e^{-\hat{R}x}}{\hat{\psi}(x)} - 1\big) \cdot 100\%$
0	0.647560	0.581428	−10.21%	1.000000	54.43%
1	0.540924	0.521454	−3.60%	0.895291	65.51%
2	0.488597	0.467667	−4.28%	0.801545	64.05%
5	0.346390	0.337363	−2.61%	0.575201	66.06%
10	0.202323	0.195749	−3.25%	0.330856	63.53%
20	0.067802	0.065903	−2.80%	0.109466	61.45%
25	0.038194	0.038239	0.12%	0.062965	64.86%

$$\mathbb{E}\big[\xi_i^2\big] = h_1''(0) = \sum_{j=1}^{k_1} 2p_{1,j}\,\mu_{1,j}^2, \qquad \mathbb{E}\big[\eta_i^2\big] = h_2''(0) = \sum_{j=1}^{k_2} 2p_{2,j}\,\mu_{2,j}^2,$$

$$\mathbb{E}\big[\xi_i^3\big] = h_1'''(0) = \sum_{j=1}^{k_1} 6p_{1,j}\,\mu_{1,j}^3, \qquad \mathbb{E}\big[\eta_i^3\big] = h_2'''(0) = \sum_{j=1}^{k_2} 6p_{2,j}\,\mu_{2,j}^3.$$

Consequently, we get

$$\mathbb{E}\big[(\xi_i - \eta_i)^2\big] = 2\Bigg(\sum_{j=1}^{k_1} p_{1,j}\,\mu_{1,j}^2 - \mu_1\mu_2 + \sum_{j=1}^{k_2} p_{2,j}\,\mu_{2,j}^2\Bigg),$$

$$\mathbb{E}\big[(\xi_i - \eta_i)^3\big] = 6\Bigg(\sum_{j=1}^{k_1} p_{1,j}\,\mu_{1,j}^3 - \mu_2\sum_{j=1}^{k_1} p_{1,j}\,\mu_{1,j}^2 + \mu_1\sum_{j=1}^{k_2} p_{2,j}\,\mu_{2,j}^2 - \sum_{j=1}^{k_2} p_{2,j}\,\mu_{2,j}^3\Bigg).$$

Example 3. Let $c = 10$, $\lambda = 4$, $\mu_1 = 2$, $\mu_2 = 0.5$, $k_1 = 3$, $k_2 = 2$, $p_{1,1} = 0.4$, $\mu_{1,1} = 0.5$, $p_{1,2} = 0.3$, $\mu_{1,2} = 2$, $p_{1,3} = 0.3$, $\mu_{1,3} = 4$, $p_{2,1} = 0.75$, $\mu_{2,1} = 0.4$, $p_{2,2} = 0.25$, $\mu_{2,2} = 0.8$. Then $\hat{R} \approx 0.110607$, which may not be a unique positive solution to (24), and

$$\psi_{DV}(x) = 0.581428\,e^{-0.108865\,x}.$$

The results of computations are given in Table 2.

5.3 *Exponential distribution for claim sizes and degenerate distribution for additional funds*

Let ξ_i be exponentially distributed with mean μ_1 and $\mathbb{E}[\eta_i = \mu_2] = 1$. Then condition (3) can be rewritten as

$$\frac{\lambda e^{-\mu_2\hat{R}}}{1 - \mu_1\hat{R}} = \lambda + c\hat{R},$$

where $\hat{R} \in (0, 1/\mu_1)$, which is equivalent to

$$\lambda e^{-\mu_2\hat{R}} = -c\mu_1\hat{R}^2 + (c - \lambda\mu_1)\hat{R} + \lambda. \tag{25}$$

Table 3. Results of computations: exponential distribution for claim sizes and degenerate distribution additional funds

x	$\hat{\psi}(x)$	$\psi_{DV}(x)$	$\left(\frac{\psi_{DV}(x)}{\hat{\psi}(x)} - 1\right) \cdot 100\%$	$e^{-\hat{R}x}$	$\left(\frac{e^{-\hat{R}x}}{\hat{\psi}(x)} - 1\right) \cdot 100\%$
0	0.637998	0.582498	−8.70%	1.000000	56.74%
1	0.549737	0.482780	−12.18%	0.822610	49.64%
2	0.465171	0.400133	−13.98%	0.676687	45.47%
5	0.277026	0.227808	−17.77%	0.376678	37.97%
10	0.113399	0.089093	−21.43%	0.141886	25.12%

If $c - \lambda\mu_1 + \lambda\mu_2 > 0$, then it is easy to check that (25) has a unique solution $\hat{R} \in (0, 1/\mu_1)$.

Since

$$\mathbb{E}[\xi_i] = \mu_1, \qquad \mathbb{E}\big[\xi_i^2\big] = 2\mu_1^2, \qquad \mathbb{E}\big[\xi_i^3\big] = 6\mu_1^3,$$
$$\mathbb{E}[\eta_i] = \mu_2, \qquad \mathbb{E}\big[\eta_i^2\big] = \mu_2^2, \qquad \mathbb{E}\big[\eta_i^3\big] = \mu_2^3,$$

we get

$$\mathbb{E}\big[(\xi_i - \eta_i)^2\big] = 2\mu_1^2 - 2\mu_1\mu_2 + \mu_2^2,$$
$$\mathbb{E}\big[(\xi_i - \eta_i)^3\big] = 6\mu_1^3 - 6\mu_1^2\mu_2 + 3\mu_1\mu_2^2 - \mu_2^3.$$

Example 4. Let $c = 10$, $\lambda = 4$, $\mu_1 = 2$, $\mu_2 = 0.5$. Then $\hat{R} \approx 0.195273$ and

$$\psi_{DV}(x) = 0.582498\, e^{-0.187764\, x}.$$

The results of computations are given in Table 3.

6 Conclusion

Tables 1–3 provide results of computations when the initial surplus is not too large. In this case, the statistical estimates obtained by the Monte Carlo method can be used instead of the exact ruin probabilities to compare an accuracy of the exponential bound and the analogue to the De Vylder approximation. To get appropriate statistical estimates by the Monte Carlo method for large initial surpluses, the number of simulations must be exceeding. The results of computations show that the exponential bound is very rough. The analogue to the De Vylder approximation gives much more accurate estimations, especially in the case of hyperexponential distributions for claim sizes and additional funds. Nevertheless, it is heuristic, and its real accuracy is unknown.

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