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Linear backward stochastic differential equations with Gaussian Volterra processes

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Abstract Explicit solutions for a class of linear backward stochastic differential equations (BSDE) driven by Gaussian Volterra processes are given. These processes include the multifractional Brownian motion and the multifractional Ornstein-Uhlenbeck process. By an Itô formula, proven in the context of Malliavin calculus, the BSDE is associated to a linear second order partial differential equation with terminal condition whose solution is given by a Feynman-Kac type formula.

Keywords Backward stochastic differential equation, Itô formula, Malliavin calculus, partial differential equation, Gaussian Volterra process

AMS subject classifications MSC 35K10, 60G22, 60H05, 60H07, 60H10

1 Introduction

A backward stochastic differential equation (BSDE) with a generator $f : [0, T] \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, a terminal value ξ_T and driven by a stochastic process $X = (X^1, \dots, X^n)$ is given by the equation

$$Y_t = \xi_T - \int_t^T f(s, Y_s, Z_s)ds + \int_t^T Z_s dX_s, \; 0 \leqslant t \leqslant T. \tag{1}$$

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A solution is a pair of square integrable processes Y and $Z = (Z^1, \dots, Z^n)$ that are adapted to the filtration generated by X. Such equations appear especially in the context of asset pricing and hedging theory in finance and in the context of stochastic control problems. BSDEs may be considered as an alternative to the more familiar partial differential equations (PDE) since the solutions of BSDEs are closely related to classical or viscosity solutions of associated PDEs (see e.g. [17]). As a consequence, BSDEs may be used for the numerical solution of nonlinear PDEs.

BSDEs driven by Brownian motion have been studied extensively after the first general existence and uniqueness result proved by E. Pardoux and S.G. Peng [16]. For a synthesis of this research work we may refer to the recent textbooks [8, 9, 17, 18, 20, 24]. More recently BSDEs driven by fractional Brownian motions have been investigated (see, e.g., [4, 11–14, 3, 22, 23]). Since fractional Brownian motions (with Hurst index $H \in (0, 1/2) \cup (1/2, 1)$) are neither martingales nor Markov processes, new methods have been developed to show the wellposedness of BSDEs in certain function spaces. In particular the integral $\int_t^T Z_s dX_s$ has been defined in different ways, e.g. pathwise in the context of fractional analysis or as a divergence integral, and the notion of quasi-conditional expectation has been introduced. In fact, the classical notion of conditional expectation does not seem to be convenient for a proof of the existence and uniqueness of solutions to BSDEs whose driving process is not a martingale. Very few articles are concerned with BSDEs for more general Gaussian processes ([5, 6]) or in the context of the theory of rough paths [10]. In [5] the stochastic integral $\int_t^T Z_s dX_s$ is understood in the Wick-Itô sense, and the existence and uniqueness of the solution of (1) is proved for a class of Gaussian processes which includes fractional Brownian motion. The proof is based on a transfer theorem that aims to reduce the question of wellposedness to BSDEs driven by Brownian motion. In [6] it is shown that the wellposedness of linear BSDEs with general square integrable terminal condition ξ holds true if and only if X is a martingale.

This paper is concerned with linear BSDEs driven by Gaussian Volterra processes X. This class of processes contains multifractional Brownian motions and multifractional Ornstein-Uhlenbeck processes. Contrary to fractional Brownian motion where the Hurst parameter H is constant, it becomes for multifractional Brownian motion a function h which is assumed here to be differentiable and with values in $(1/2, 1)$. The aim is to obtain the solution of the linear BSDE with the associated linear PDE whose solution is given explicitely. This generalizes a result in [4] obtained for fractional Brownian motion. We define the stochastic integral $\int_t^T Z_s dX_s$ as a divergence integral, and extend an Itô formula in [2] to the multidimensional case. The Itô formula is then applied to the solution of the associated PDE in order to get a solution of the BSDE. Special attention is given to the fact that the variance of Volterra processes is not necessarily an increasing function of time, but in general only of bounded variation. The explicit solution of the associated PDE contains this variance and is given by a Feynman-Kac type formula on time intervals where it is increasing. The application of this formula to the BSDEs is therefore restricted to time intervals where this variance is increasing.

In this section we define the class of Volterra processes X we have in mind and the linear BSDEs and the associated PDE. Section 2 is concerned with complements on the Skorohod integral with respect to Volterra processes. The Itô formula is proved in Section 3 and applied in Section 4 to the linear BSDE.

1.1 Gaussian Volterra processes

Let $X = \{X_t, 0 \leqslant t \leqslant T\}$ be a zero mean continuous Gaussian process given by

$$X_t = \int_0^T K(t,s)dW_s \tag{2}$$

where $W = \{W_t, 0 \leqslant t \leqslant T\}$ is a standard Brownian motion and $K : [0,T]^2 \to \mathbb{R}$ is a square integrable kernel, i.e. $\int_{[0,T]^2} K(t,s)^2 dtds < +\infty$. We assume that K is of Volterra type, i.e, $K(t,s) = 0$ whenever $t < s$. Usually, the representation (2) is called a Volterra representation of X. Gaussian Volterra processes and their stochastic analysis have been studied e.g. in [2, 21] and [19]. In [2] K is called *regular* if it satisfies

(H) For all $s \in (0,T]$, $\int_0^T \mid K \mid ((s,T],s)^2 ds < \infty$, where $\mid K \mid ((s,T],s)$ denotes the total variation of $K(.,s)$ on $(s,T]$.

We assume the following condition on $K(t,s)$ which is more restrictive than *(H)* ([2, 21]):

(H1) $K(t,s)$ is continuous for all $0 < s \leqslant t < T$ and continuously differentiable in the variable t in $0 < s < t < T$,

(H2) For some $0 < \alpha, \beta < \frac{1}{2}$, there is a finite constant $c > 0$ such that

$$\left|\frac{\partial K}{\partial t}(t,s)\right| \leqslant c(t-s)^{\alpha-1}\left(\frac{t}{s}\right)^{\beta}, \text{ for all } 0 < s < t < T.$$

The covariance function of X is given by

$$R(t,s) := \mathbb{E}X_t X_s = \int_0^{min(t,s)} K(t,u)K(s,u)du. \tag{3}$$

We discuss shortly some examples of Gaussian Volterra processes that satisfy *(H1) and (H2)*.

Example 1. The multi-fractional Brownian motion (mBm) $(B_t^{h(t)}, 0 \leqslant t \leqslant T)$ with Hurst function $h : [0,T] \to [a,b] \subset (\frac{1}{2}, 1)$. Its kernel is given by [7]

$$K(t,s) = s^{1/2-h(t)} \int_s^t (y-s)^{h(t)-3/2} y^{h(t)-1/2} dy, \tag{4}$$

where h is assumed to be continously differentiable with bounded derivative. We get

$$\begin{aligned}\frac{\partial K}{\partial t}(t,s) = h'(t)s^{\frac{1}{2}-h(t)} \int_s^t (y-s)^{h(t)-\frac{3}{2}} y^{h(t)-\frac{1}{2}} ln\Big(\Big(\frac{y}{s}-1\Big)y\Big)dy \\ + s^{\frac{1}{2}-h(t)}(t-s)^{h(t)-\frac{3}{2}} t^{h(t)-\frac{1}{2}}.\end{aligned}$$

A straightforward calculation shows that $(H2)$ is satisfied with $\alpha = a - \frac{1}{2}$, $\beta = b + \epsilon - \frac{1}{2}$ with ϵ small enough and c depends on a, b, T and ϵ. The mBm generalizes fractional Brownian motion (fBm) with Hurst index $H > 1/2$. MBm is a more flexible model than fBm since the Hölder continuity of its trajectories varies with h. The trajectories of mBm $B^{h(\cdot)}_{\cdot}$ are in fact locally Hölder continous of order $h(t)$ at t ([7], Proposition 6).

Example 2. The multi-fractional Ornstein-Uhlenbeck process $U = \{U_t, 0 \leqslant t \leqslant T\}$ given by $U_t = \int_0^t e^{-\theta(t-s)} dB_s^{h(s)}$, where $\theta > 0$ is a parameter and B^h is the mbm of Example 1. The kernel of U is given by

$$\mathcal{K}(t,r) = \int_r^t e^{-\theta(t-s)} \frac{\partial K}{\partial s}(s,r)ds = K(t,r) - \theta \int_r^t e^{-\theta(t-s)} K(s,r)ds.$$

In fact we have

$$\begin{aligned}\int_0^t \mathcal{K}(t,r)dW_r &= \int_0^t K(t,r)dW_r - \theta \int_0^t \left(\int_r^t e^{-\theta(t-s)} K(s,r)ds \right) dW_r \\ &= B_t^{h(t)} - \theta \int_0^t e^{-\theta(t-s)} B_s^{h(s)} ds.\end{aligned}$$

An integration by parts gives the representation of U. We notice that in the framework of the divergence integral (Section 2) the integral with respect to mbm can be reduced to an integral with respect to Brownian motion. $(H2)$ is satisfied with the same values of α and β as in Example 1.

Example 3. The Liouville multi-fractional Brownian motion $(B_t^{L,h(t)}, t \in [0,T])$ with Hurst function h as in Example 1. Its kernel is given by $\tilde{\mathcal{K}}(t,r) = (t-r)^{h(t)-\frac{1}{2}} 1_{(0,t]}(r)$. We refer to [21] for the Liouville fractional Brownian motion.

1.2 Linear backward stochastic differential equations

Let $W = (W^1, \ldots, W^n)$ a standard Brownian motion in $\mathbb{R}^n$, defined on the probability space $(\Omega, \mathcal{F}, P)$, and let $\mathbb{F} = \{\mathcal{F}_t \subset \mathcal{F}, t \in [0,T]\}$ be the filtration generated by W and augmented by the P-null sets. We consider the $\mathbb{R}^n$-valued Volterra processes $X = (X^1, \ldots, X^n)$ given by

$$X_t^j = \int_0^t K^j(t,s)dW_s^j, \ j = 1, \ldots, n, \tag{5}$$

where $K^j : [0,T]^2 \to \mathbb{R}$ satisfies the conditions *(H1)* and *(H2)*. Let σ^j, $j = 1, \ldots, n$ be bounded functions on $[0,T]$, and let $b^j \in \mathcal{C}^1((0,T),\mathbb{R}) \cap \mathcal{C}([0,T],\mathbb{R})$, $j = 1, \ldots, n$. The process $N := (N^1, \ldots, N^n)$ is defined by

$$N_t^j = b_t^j + \int_0^t \sigma_s^j \delta X_s^j, \ t \in [0,T], j = 1, \ldots, n, \tag{6}$$

where the integral $\int_t^T Z_s \delta X_s$ is defined as a divergence integral and will be studied in Sections 2 and 3. Let $t_0 \geqslant 0$ be fixed, and denote by $\mathbb{L}^2(\mathbb{F}, \mathbb{R}^n)$ the set of $\mathbb{F}$-adapted $\mathbb{R}^n$-valued processes Z such that $\mathbb{E}(\int_{t_0}^T \mid Z_t \mid^2 dt) < \infty$. We consider the linear

BDSE for the processes $Y = (Y_t, t \in [t_0, T]) \in \mathbb{L}^2(\mathbb{F}, \mathbb{R})$ and $Z = ((Z_t^1, \dots, Z_t^n), t \in [t_0, T]) \in \mathbb{L}^2(\mathbb{F}, \mathbb{R}^n)$ given by

$$Y_t = g(N_T) - \int_t^T [f(s) + A_1(s)Y_s - A_2(s)Z_s]ds + \int_t^T Z_s \delta X_s, \quad t \in [t_0, T], \quad (7)$$

where the real-valued functions g, f, A_1 and the $\mathbb{R}^n$-valued function A_2 are supposed to be known. Equation (7) is associated to the following second order linear PDE with terminal condition

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) &= -\frac{1}{2}\sum_{j=1}^n \frac{d}{dt} Var(N_t^j) \frac{\partial^2 u}{\partial x_j^2}(t,x) + \sum_{j=1}^n \Big[\sigma_t^j A_2^j(t) - \frac{d}{dt} b_t^j\Big] \frac{\partial u}{\partial x_j}(t,x) \\ &\quad + A_1(t)u(t,x) + f(t), \qquad (8) \\ u(T,x) &= g(x), \ (t,x) \in [t_0, T) \times \mathbb{R}^n. \end{aligned}$$

By means of the Itô formula of Section 3 we show that

$$Y_t := u(t, N_t) \text{ and } Z_t^j := -\sigma_t^j \frac{\partial u}{\partial x_j}(t, N_t), \ j = 1, \dots, n \quad (9)$$

is a solution of (7). Equation (8) is solved explicitely in Section 4.

2 On the divergence integral for Gaussian Volterra processes

The kernel K in (2) defines a linear operator in $L^2([0,T])$ given by $(K\sigma)_t = \int_0^t K(t,s)\sigma_s ds$, $\sigma \in L^2([0,T])$. Let $\mathcal{E}$ be the set of step functions on $[0,T]$, and let $K_T^* : \mathcal{E} \to L^2([0,T])$ be defined by

$$(K_T^*\sigma)_u := \int_u^T \sigma_s \frac{\partial K}{\partial s}(s,u)ds.$$

The operator K_T^* is the adjoint of K ([2], Lemma 1).

Remarks. a) For $s > t$, we have $(K_T^*\sigma 1_{[0,t]})_s = 0$, and we will denote $(K_T^*\sigma 1_{[0,t]})_s$ by $(K_t^*\sigma)_s$ where K_t^* is the adjoint of the operator K in the interval $[0,t]$.

b) If $K(u,u) = 0$ for all $u \in [0,T]$, $(K_T^* 1_{[0,r]})_u = K(r,u)$ for $u < r$. Indeed, if $u \leqslant r$, we have

$$(K_T^* 1_{[0,r]})_u = \int_u^T 1_{[0,r]}(s) \frac{\partial K}{\partial s}(s,u)ds = \int_u^r \frac{\partial K}{\partial s}(s,u)ds.$$

Therefore

$$\begin{aligned} R(t,s) = \mathbb{E}\Big[X_t X_s\Big] &= \int_0^{min(t,s)} (K_T^* 1_{[0,t]})_u (K_T^* 1_{[0,s]})_u du \\ &= <K_T^* 1_{[0,t]}, K_T^* 1_{[0,s]}>_{L^2([0,T])}. \end{aligned}$$

For $\sigma, \widetilde{\sigma} \in \mathcal{E}$ this equality may be extended to $X(\sigma) := \int_0^t (K_t^*\sigma)_s dW_s$ by

$$E\Big[X(\sigma)X(\widetilde{\sigma})\Big] = <K_T^*\sigma, K_T^*\widetilde{\sigma}>_{L^2([0,T])}.$$

Let $\mathcal{H}$ be the closure of the linear span of the indicator functions $1_{[0,t]}$, $t \in [0, T]$ with respect to the scalar product

$$<1_{[0,t]}, 1_{[0,s]}>_{\mathcal{H}} := <K_T^* 1_{[0,t]}, K_T^* 1_{[0,s]}>_{L^2([0,T])}.$$

The operator K_T^* is an isometry between $\mathcal{H}$ and a closed subspace of $L^2([0, T])$, and $\| \cdot \|_{\mathcal{H}}$ is a semi-norm on $\mathcal{H}$. Furthermore, for $\varphi, \psi \in \mathcal{H}$,

$$<K_T^*\varphi, K_T^*\psi>_{L^2([0,T])} = \int_0^T \int_0^T \Big(\int_0^{min(r,s)} \frac{\partial K}{\partial r}(r,t) \frac{\partial K}{\partial s}(s,t) dt \Big) \varphi_r \psi_s ds dr.$$

For further use let

$$\phi(r,s) := \int_0^{min(r,s)} \frac{\partial K}{\partial r}(r,t) \frac{\partial K}{\partial s}(s,t) dt, \ r \neq s,$$
$$\widetilde{\phi}(r,s) := \int_0^{min(r,s)} \Big| \frac{\partial K}{\partial r}(r,t) \Big| \Big| \frac{\partial K}{\partial s}(s,t) \Big| dt, \ r \neq s.$$

Note that $\phi(r,s) = \frac{\partial^2 R}{\partial s \partial r}(r,s)$ $(r \neq s)$ (ϕ may be infinite on the diagonal $r = s$). Let $| \mathcal{H} |$ be the closure of the linear span of indicator functions with respect to the semi-norm given by

$$\| \varphi \|_{|\mathcal{H}|}^2 = \int_0^T \Big(\int_t^T | \varphi_r | \Big| \frac{\partial K}{\partial r}(r,t) \Big| dr \Big)^2 dt$$
$$= 2 \int_0^T dr \int_0^r ds \widetilde{\phi}(r,s) | \varphi_r || \varphi_s | .$$

We briefly recall some basic elements of the stochastic calculus of variations with respect to X. We refer to [15] for a more complete presentation. Let $\mathcal{S}$ be the set of random variables of the form $F = f(X(\varphi_1), \dots, X(\varphi_n))$, where $n \geqslant 1$, $f \in \mathcal{C}_b^\infty(\mathbb{R}^n)$ (f and its derivatives are bounded) and $\varphi_1, \dots, \varphi_n \in \mathcal{H}$. The derivative of F

$$D^X F := \sum_{j=1}^n \frac{\partial f}{\partial x_j}(X(\varphi_1), \dots, X(\varphi_n)) \varphi_j,$$

is an $\mathcal{H}$-valued random variable, and D^X is a closable operator from $L^p(\Omega)$ to $L^p(\Omega; \mathcal{H})$ for all $p \geqslant 1$. We denote by $\mathbb{D}_{1,p}^X$ the closure of $\mathcal{S}$ with respect to the semi-norm

$$\|F\|_{1,p}^p = \mathbb{E}|F|^p + \mathbb{E}\|D^X F\|_{\mathcal{H}}^p. \quad (10)$$

We denote by $Dom(\delta^X)$ the subset of $L^2(\Omega, \mathcal{H})$ composed of those elements u for which there exists a positive constant c such that

$$\Big| \mathbb{E}\Big[<D^X F, u>_{\mathcal{H}} \Big] \Big| \leqslant c\sqrt{\mathbb{E}[F^2]}, \text{ for all } F \in \mathbb{D}_{1,2}^X. \quad (11)$$

For $u \in L^2(\Omega; \mathcal{H})$ in $Dom(\delta^X)$, $\delta^X(u)$ is the element in $L^2(\Omega)$ defined by the duality relationship

$$\mathbb{E}\Big[F\delta^X(u) \Big] = \mathbb{E}\Big[<D_\cdot^X F, u_\cdot>_{\mathcal{H}} \Big], \ F \in \mathbb{D}_{1,2}^X. \quad (12)$$

We also use the notation $\int_0^T u_t \delta X_t$ for $\delta^X(u)$. A class of processes that belong to the domain of δ^X is given as follows: let $\mathcal{S}^{\mathcal{H}}$ be the class of $\mathcal{H}$-valued random variables $u = \sum_{j=1}^n F_j h_j$ ($F_j \in \mathcal{S}, h_j \in \mathcal{H}$).

In the same way $\mathbb{D}^X_{1,p}(|\ \mathcal{H}\ |)$ is defined as the completion of $\mathcal{S}^{|\mathcal{H}|}$ under the semi-norm

$$\| u \|^p_{1,p,|\mathcal{H}|} := \mathbb{E} \| u \|^p_{|\mathcal{H}|} + \mathbb{E} \| D^X u \|^p_{|\mathcal{H}|\otimes|\mathcal{H}|},$$

where

$$\| D^X u \|^2_{|\mathcal{H}|\otimes|\mathcal{H}|} = \int_{[0,T]^4} | D^X_s u_t || D^X_{t'} u_{s'} | \widetilde{\phi}(s,s')\widetilde{\phi}(t,t')dsdtds'dt'. \tag{13}$$

The space $\mathbb{D}^X_{1,2}(|\ \mathcal{H}\ |)$ is included in the domain of δ^X, and we have, for $u \in \mathbb{D}^X_{1,2}(|\ \mathcal{H}\ |)$,

$$\mathbb{E}\Big(\delta^X(u)^2\Big) \leqslant \mathbb{E} \| u \|^2_{|\mathcal{H}|} + \mathbb{E} \| D^X u \|^2_{|\mathcal{H}|\otimes|\mathcal{H}|} .$$

For $F \in \mathcal{S}$, let

$$\mathbb{D}^X_s F := \int_0^T \phi(s,t) D^X_t F dt,\ s \in [0,T]. \tag{14}$$

Then (14) implies

$$\int_{[0,T]^2} \mathbb{D}^X_s u_t \mathbb{D}^X_t v_s dsdt = \int_{[0,T]^4} D^X_s u_t D^X_{t'} v_{s'} \phi(s,s')\phi(t,t')dsdtds'dt'. \tag{15}$$

Proposition 1. *Let $f, g \in \mathbb{D}^X_{1,2}(|\ \mathcal{H}\ |)$. Then the integrals $\delta^X(f)$ and $\delta^X(g)$ exist in $L^2(\Omega)$ and*

$$\mathbb{E}\Big[\delta^X(f)\delta^X(g)\Big] = \mathbb{E}\langle f, g\rangle_{\mathcal{H}} + \int_0^T ds \int_0^T dt \mathbb{E}\Big[\mathbb{D}^X_t f_s \mathbb{D}^X_s g_t\Big]. \tag{16}$$

Remark. With the choice $f = g$ Proposition 1 implies $\mathbb{D}^X_{1,2}(|\ \mathcal{H}\ |) \subset dom(\delta^X)$. In fact, for $f \in \mathbb{D}^X_{1,2}(|\ \mathcal{H}\ |)$,

$$\|f\|_{\mathcal{H}} \leqslant \|f\|_{|\mathcal{H}|} \text{ and } \int_{[0,T]^2} \mathbb{D}^X_s f_t \mathbb{D}^X_t f_s dsdt \leqslant \| D^X f \|^2_{|\mathcal{H}|\otimes|\mathcal{H}|} .$$

Since (16) is a standard property of the divergence integral (adapted to the actual framework), we omit here its proof.

3 Itô formula

Let $F \in \mathcal{C}^{1,2}([0,T]\times\mathbb{R}^n)$ and suppose that

$$\max\left(\Big|F(t,x)\Big|, \Big|\frac{\partial F}{\partial t}(t,x)\Big|, \Big|\frac{\partial F}{\partial x_j}(t,x)\Big|, \Big|\frac{\partial^2 F}{\partial x_j^2}(t,x)\Big|, j=1,\dots,n\right) \leqslant ce^{\lambda|x|^2} \tag{17}$$

for all $t \in [0, T]$ and $x \in \mathbb{R}^n$, where c, λ are positive constants such that $\lambda < \frac{1}{4} \min_i (\sup_{t\in[0,T]} Var(N_t^i))^{-1}$. This implies

$$\begin{aligned}
&\mathbb{E}\Big|F(t, N_t)\Big|^2 \\
&\leqslant c^2 \mathbb{E} \exp(2\lambda \mid N_t \mid^2) \\
&= \frac{c^2}{(2\pi)^{n/2}} \prod_{j=1}^{n} \frac{1}{(Var(N_t^j))^{1/2}} \int exp\left(2\lambda x_j^2 - \frac{1}{2}\frac{(x_j - b_t^j)^2}{Var(N_t^j)}\right) dx_j \\
&= \frac{c^2}{(2\pi)^{n/2}} \prod_{j=1}^{n} \frac{1}{(Var(N_t^j))^{1/2}} \\
&\times \int exp\left(-x_j^2 \frac{(1 - 4\lambda Var N_t^j)}{2Var N_t^j} + \frac{b_t^j}{Var N_t^j} x_j - \frac{(b_t^j)^2}{2Var N_t^j}\right) dx_j \\
&= \frac{c^2}{(2\pi)^{n/2}} \prod_{j=1}^{n} \frac{1}{(Var(N_t^j))^{1/2}} \sqrt{\frac{2\pi Var N_t^j}{1 - 4\lambda Var N_t^j}} \\
&\times exp\left(\frac{(b_t^j)^2}{2Var N_t^j (1 - 4\lambda Var N_t^j)} - \frac{(b_t^j)^2}{2Var N_t^j}\right) \\
&= \frac{c^2}{(2\pi)^{(n-1)/2}} \prod_{j=1}^{n} \frac{1}{\sqrt{1 - 4\lambda Var N_t^j}} exp\left(\frac{2\lambda (b_t^j)^2}{1 - 4\lambda Var N_t^j}\right) < \infty, \qquad (18)
\end{aligned}$$

and the same inequalities holds for $\frac{\partial F}{\partial t}(t, x)$, $\frac{\partial F}{\partial x_j}(t, x)$ and $\frac{\partial^2 F}{\partial x_j^2}(t, x)$, $j = 1, \ldots, n$.

Theorem 1. *Let N be given by (6), and suppose that, for $j = 1, \ldots, n$, the kernels K^j of X^j satisfy (H1) and (H2), $b^j \in \mathcal{C}^1((0, T), \mathbb{R}) \cap \mathcal{C}([0, T], \mathbb{R})$, and $\sigma = \{\sigma_t^j, t \in [0, T], j = 1, \ldots, n\}$ is bounded. If $F \in \mathcal{C}^{1,2}([0, T] \times \mathbb{R}^n)$ satisfies (17), $\frac{\partial F}{\partial x_j}(\cdot, N.) \in \mathbb{D}_{1,2}^{X^j}(\mid \mathcal{H}^j \mid)$, $j = 1, \ldots, n$ and, for all $t \in [0, T]$,*

$$\begin{aligned}
F(t, N_t) = F(0, 0) &+ \int_0^t \frac{\partial F}{\partial s}(s, N_s) ds + \sum_{j=1}^{n} \int_0^t \frac{\partial F}{\partial x_j}(s, N_s)\Big(\frac{d}{ds} b_s^j ds + \sigma_s^j \delta X_s^j\Big) \\
&+ \frac{1}{2} \sum_{j=1}^{n} \int_0^t \frac{\partial^2 F}{\partial x_j^2}(s, N_s) \frac{d}{ds} Var(N_s^j) ds. \qquad (19)
\end{aligned}$$

Remarks. a) The growth assumption (17) on F may be unexpected for the proof an Itô formula. In fact, in [1] an Itô formula is shown for fractional Brownian motion for any $F \in \mathcal{C}^{1,2}$ by means of a method of localization. The reason for hypothesis (17) is that it implies the finiteness of the second moment of $F(t, N_t)$, shown in (18). When applied to the solution u of the PDE (8), it implies, together with Theorem 2, the finiteness of the second moment of the solution (Y, Z) of the BSDE, and this seems to be an important ingredient for the proof of the uniqueness of the solution ([11, 14]

for BSDE with fBm). Moreover, Lemma 1 shows that (17) is also reasonable from the point of view of the PDE (8).

b) A more general model for N than in Section 1.2 is

$$\widetilde{N}_t^i = b_t^i + \sum_{j=1}^{n} \int_0^t \widetilde{\sigma}_s^{i,j} \delta X_s^j, \; i = 1, \ldots, n, \tag{20}$$

where $\widetilde{\sigma} = (\widetilde{\sigma}_t^{i,j}, i, j = 1, \ldots, n)$ is a matrix of bounded functions $\widetilde{\sigma}^{i,j}$ defined on $[0, T]$. Let $\widetilde{N} = (\widetilde{N}^1, \ldots, \widetilde{N}^n)$. The components of $\widetilde{N}$ are dependent since $\widetilde{N}^i$ depends not only on a single random perturbation X^i, but on the others X^j $(j \neq i)$ as well. The model of Section 1.2 may be recovered by choosing the matrix $\widetilde{\sigma}$ diagonal with functions $\sigma^j := \widetilde{\sigma}^{j,j}$ in the diagonal. An Itô formula can be shown for $F(t, \widetilde{N_t})$ too, but, instead of the variances of N, the covariances of $\widetilde{N}$ appear now in the second order term. It reads

$$\begin{aligned} F(t, \widetilde{N}_t) &= F(0,0) + \int_0^t \frac{\partial F}{\partial s}(s, \widetilde{N}_s) ds \\ &+ \sum_{i=1}^{n} \int_0^t \frac{\partial F}{\partial x_i}(s, \widetilde{N}_s) \Big(\frac{d}{ds} b_s^i ds + \sum_{j=1}^{n} \widetilde{\sigma}_s^{i,j} \delta X_s^j \Big) \\ &+ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \int_0^t \frac{\partial^2 F}{\partial x_i \partial x_j}(s, \widetilde{N}_s) \frac{d}{ds} Cov(\widetilde{N}_s^i, \widetilde{N}_s^j) ds. \end{aligned} \tag{21}$$

The derivatives of the variances of N in the second order term of (8) have therefore to be replaced by the derivatives of the covariances of $\widetilde{N}$, and one has to assume that the matrix $(\frac{d}{dt} Cov(\widetilde{N}_t^i, \widetilde{N}_t^j), i, j = 1, \ldots, n)$ is positive definite. We notice that

$$\begin{aligned} Var(N_t^j) &= \mathbb{E}\left(\int_0^t \left(K_t^{*,j} \sigma^j \right)_s \delta W_s^j \right)^2 = \int_0^t \left(K_t^{*,j} \sigma^j \right)_s^2 ds, \text{ and} \\ \frac{d}{dt} Var(N_t^j) &= 2\sigma_t^j \mathbb{D}_t^j N_t^j, \end{aligned}$$

c) In the model where $\widetilde{\sigma}$ is diagonal, X^i and X^j are defined with independent Brownian motions W^i and W^j if $i \neq j$. This differs from the model where all the Volterra processes $\overline{X}^j$ are defined with the same Brownian motion $\overline{W}$ as follows:

$$\overline{X}_t^j = \int_0^t K^j(t,s) \delta \overline{W}_s, \; \overline{N}_t^j = b_t^j + \int_0^t \sigma_s^j \delta \overline{X}_s^j, \; t \in [0, T], \; j = 1, \ldots, n.$$

In this case the processes $\overline{N}^j$ are again correlated, and the matrix $(\frac{d}{dt} Cov(\overline{N}_t^i, \overline{N}_t^j),$ $i, j = 1, \ldots, n)$ is not diagonal and not necessarily positive semidefinite.

Let us prove now Theorem 1.

Proof. 1. First we show that $\frac{\partial F}{\partial x_j}(\cdot, N.) \in Dom(\delta^{X^j})$ for all $j = 1, \ldots, n$. For this we show that $\frac{\partial F}{\partial x_j}(\cdot, N.) \in \mathbb{D}_{1,2}^{X^j}(|\, \mathcal{H}^j \,|)$, where $|\, \mathcal{H}^j \,|$ is the space defined in Section 2 with X replaced by X^j. The terms ϕ^j and $\widetilde{\phi^j}$ refer now to the kernel K^j of X^j. The constants in the inequalities below may vary from line to line.

$$\mathbb{E}\Big\|\frac{\partial F}{\partial x_j}(.,N_.)\Big\|^2_{|\mathcal{H}^j|} = \mathbb{E}\int_0^T\Big(\int_s^T\Big|\frac{\partial F}{\partial x_j}(t,N_t)\Big|\Big|\frac{\partial K^j}{\partial t}(t,s)\Big|dt\Big)^2ds$$
$$\leqslant c\mathbb{E}\int_0^T\Big(\int_s^T\exp(\lambda\mid N_t\mid^2)(t-s)^{\alpha_j-1}\Big(\frac{t}{s}\Big)^{\beta_j}dt\Big)^2ds.$$

Applying Hölder's inequality to $1<p_j<\frac{1}{1-\alpha_j}<2$ and q_j its conjugate, we get

$$\Big(\int_s^T\exp(\lambda\mid N_t\mid^2)(t-s)^{\alpha_j-1}\Big(\frac{t}{s}\Big)^{\beta_j}dt\Big)^2$$
$$\leqslant\Big(\int_s^T\exp(q_j\lambda\mid N_t\mid^2)dt\Big)^{\frac{2}{q_j}}\Big(\int_s^T(t-s)^{p_j(\alpha_j-1)}\Big(\frac{t}{s}\Big)^{p_j\beta_j}dt\Big)^{\frac{2}{p_j}}.$$

Then

$$\mathbb{E}\Big(\int_0^T\exp(q_j\lambda\mid N_t\mid^2)dt\Big)^{\frac{2}{q_j}}\leqslant\mathbb{E}\Big(\int_0^T\exp(q_j\lambda\sup_{t\in[0,T]}\mid N_t\mid^2)dt\Big)^{\frac{2}{q_j}}$$
$$\leqslant T^{2/q_j}\prod_{i=1}^n\mathbb{E}\exp(2\lambda\sup_{t\in[0,T]}(N_t^i)^2)dt.$$

The right side of the inequality above is finite for $\lambda<\frac{1}{4}\min_i(\sup_{t\in[0,T]}Var(N_t^i))^{-1}$, see [2]. Moreover,

$$\int_s^T(t-s)^{p_j(\alpha_j-1)}\Big(\frac{t}{s}\Big)^{p_j\beta_j}dt=s^{-p_j\beta_j}\int_s^T(t-s)^{p_j(\alpha_j-1)}t^{p_j\beta_j}dt$$
$$\leqslant s^{-p_j\beta_j}T^{p_j\beta_j}\frac{(T-s)^{p_j(\alpha_j-1)+1}}{p_j(\alpha_j-1)+1},$$

and

$$\int_0^T\Big(\int_s^T(t-s)^{p_j(\alpha_j-1)}\Big(\frac{t}{s}\Big)^{p_j\beta_j}dt\Big)^{\frac{2}{p_j}}ds$$
$$\leqslant\frac{T^{2\beta_j}}{\Big(p_j(\alpha_j-1)+1\Big)^{\frac{2}{p_j}}}\int_0^T\Big(s^{-p_j\beta_j}(T-s)^{p_j(\alpha_j-1)+1}\Big)^{\frac{2}{p_j}}ds$$
$$=\frac{T^{2\beta_j}}{\Big(p_j(\alpha_j-1)+1\Big)^{\frac{2}{p_j}}}\int_0^T s^{-2\beta_j}T^{2(\alpha_j-1)+\frac{2}{p_j}}\Big(1-\frac{s}{T}\Big)^{2(\alpha_j-1)+\frac{2}{p_j}}ds$$
$$=\frac{T^{2(\alpha_j-1)+\frac{2}{p_j}+1}}{\Big(p_j(\alpha_j-1)+1\Big)^{\frac{2}{p_j}}}\mathcal{B}\Big(1-2\beta_j,2(\alpha_j-1)+\frac{2}{p_j}+1\Big),$$

where $\mathcal{B}$ is the beta function. It remains to show that $\mathbb{E}\left\| D^{X^j} \frac{\partial F}{\partial x_j}(., N_.)\right\|_{|\mathcal{H}^j|\otimes|\mathcal{H}^j|} < \infty$.

$$\begin{aligned}
&\mathbb{E}\left\| D^{X^j} \frac{\partial F}{\partial x_j}(., N_.)\right\|^2_{|\mathcal{H}^j|\otimes|\mathcal{H}^j|}\\
&= \mathbb{E}\int_{[0,T]^4} \left|D_s^{X^j} \frac{\partial F}{\partial x_j}(t, N_t)\right|\left|D_{t'}^{X^j} \frac{\partial F}{\partial x_j}(s', N_{s'})\right|\widetilde{\phi^j}(s,s')\widetilde{\phi^j}(t,t')dsdtds'dt'\\
&= \mathbb{E}\int_{[0,T]^4} \left|\frac{\partial^2 F}{\partial x_j^2}(t, N_t)\sigma_s^j\right|\left|\frac{\partial^2 F}{\partial x_j^2}(s', N_{s'})\sigma_{t'}^j\right|\widetilde{\phi^j}(s,s')\widetilde{\phi^j}(t,t')dsdtds'dt'\\
&\leqslant c\int_0^T dt\int_0^T ds'\left[\mathbb{E}\left(\frac{\partial^2 F}{\partial x_j^2}(t, N_t)\right)^2 + \mathbb{E}\left(\frac{\partial^2 F}{\partial x_j^2}(s', N_{s'})\right)^2\right]\\
&\times\int_0^T dt'\widetilde{\phi^j}(t,t')\int_0^T ds\widetilde{\phi^j}(s',s).
\end{aligned}$$

By (18) with F replaced by $\frac{\partial^2 F}{\partial x_j^2}$, $\mathbb{E}\big(\frac{\partial^2 F}{\partial x_j^2}(t, N_t)\big)^2$ stays bounded in $t \in [0, T]$. The finiteness of the remaining integrals follows from *(H1), (H2)* applied to K^j.

2. We proceed now to the outline of the proof of the Itô formula. Let

$$\begin{aligned}
N_t^{j,\varepsilon} &= b_t^j + \int_0^t \Big(\int_s^t \sigma_r^j \frac{\partial K^j}{\partial r}(r+\varepsilon, s)dr\Big)\delta W_s^j\\
&= b_t^j + \int_0^t \Big(\int_0^r \sigma_r^j \frac{\partial K^j}{\partial r}(r+\varepsilon, s)\delta W_s^j\Big)dr,
\end{aligned}$$

for $t \leqslant T - \varepsilon$. Then, $F(t, N_t^{1,\varepsilon}, \ldots, N_t^{n,\varepsilon})$ has locally bounded variation, and we can write

$$\begin{aligned}
dF(t, N_t^{1,\varepsilon}, \ldots, N_t^{n,\varepsilon}) &= \frac{\partial F}{\partial t}(t, N_t^{1,\varepsilon}, \ldots, N_t^{n,\varepsilon})dt\\
&+ \sum_{i=1}^n \frac{\partial F}{\partial x_i}(t, N_t^{1,\varepsilon}, \ldots, N_t^{n,\varepsilon})dN_t^{i,\varepsilon}\\
&= \left(\frac{\partial F}{\partial t}(t, N_t^\varepsilon) + \sum_{j=1}^n \frac{\partial F}{\partial x_j}(t, N_t^\varepsilon)\frac{d}{dt}b_t^j\right)dt\\
&+ \sum_{j=1}^n \frac{\partial F}{\partial x_j}(t, N_t^\varepsilon)\sigma_t^j\int_0^t \frac{\partial K^j}{\partial t}(t+\varepsilon, s)\delta W_s^j dt
\end{aligned}$$

with the notation $N^\varepsilon = (N^{1,\varepsilon}, \ldots, N^{n,\varepsilon})$. Furthermore,

$$\begin{aligned}
\frac{\partial F}{\partial x_j}(t, N_t^\varepsilon)\sigma_t^j\int_0^t \frac{\partial K^j}{\partial t}(t+\varepsilon, s)\delta W_s^j &= \sigma_t^j\Big[\int_0^t \frac{\partial F}{\partial x_j}(t, N_t^\varepsilon)\frac{\partial K^j}{\partial t}(t+\varepsilon, s)\delta W_s^j\\
&+ \int_0^t D_s^{W^j}\Big(\frac{\partial F}{\partial x_j}(t, N_t^\varepsilon)\Big)\frac{\partial K^j}{\partial t}(t+\varepsilon, s)ds\Big],
\end{aligned}$$

where

$$D_s^{W^j}\Big(\frac{\partial F}{\partial x_j}(t,N_t^\varepsilon)\Big)=\frac{\partial^2 F}{\partial x_j^2}(t,N_t^\varepsilon)D_s^{W^j}N_t^{j,\varepsilon}=\frac{\partial^2 F}{\partial x_j^2}(t,N_t^\varepsilon)\int_s^t\sigma_r^j\frac{\partial K^j}{\partial r}(r+\varepsilon,s)dr.$$

Therefore

$$\begin{aligned}F(t,N_t^\varepsilon)=F(0,0)&+\int_0^t\left(\frac{\partial F}{\partial s}(s,N_s^\varepsilon)+\sum_{j=1}^n\frac{\partial F}{\partial x_j}(s,N_s^\varepsilon)\frac{d}{ds}b_s^j\right)ds\\&+\sum_{j=1}^n\int_0^t\Big(\int_s^t\sigma_r^j\frac{\partial F}{\partial x_j}(r,N_r^\varepsilon)\frac{\partial K^j}{\partial r}(r+\varepsilon,s)dr\Big)\delta W_s^j\\&+\frac{1}{2}\sum_{j=1}^n\int_0^t\frac{\partial^2 F}{\partial x_j^2}(r,N_r^\varepsilon)\frac{\partial}{\partial r}\Big(\int_0^r\Big(K_r^{*,\varepsilon,j}\sigma^j\Big)_s^2ds\Big)dr.\end{aligned}$$

The divergence integral coincides, up to ε, with the integral that appears in the statement of the theorem. The last term coincides, up to ε, with the term at the end of Remark b) after the theorem. It remains to show that the terms above converge in $L^2(\Omega)$ towards the terms in the statement of the theorem as $\varepsilon\to 0$. This can be done for each integral similarly as in the proof of Theorem 4 in [2]. □

4 Solvability of linear BSDEs

As mentioned in the introduction the aim is to apply the Itô formula (19) for F replaced by the solution u of the PDE (8) and to show that Y and Z defined by (9) satisfy the BSDE (7). We will show later in this section that u in fact satisfies the growth condition (17) under a suitable growth condition on the final condition in (7). This implies by (18) that Y and Z are square integrable.

The Itô formula (19), with F replaced by u reads

$$\begin{aligned}u(t,N_t)=u(T,N_T)&-\int_t^T\frac{\partial u}{\partial s}(s,N_s)ds-\sum_{j=1}^n\int_t^T\frac{\partial u}{\partial x_j}(s,N_s)\Big(\frac{d}{ds}b_s^jds+\sigma_s^j\delta X_s^j\Big)\\&-\frac{1}{2}\sum_{j=1}^n\int_t^T\frac{\partial^2 u}{\partial x_j^2}(s,N_s)\frac{d}{ds}Var(N_s^j)ds.\end{aligned}\tag{22}$$

An application of (8) to the second term on the right hand side of (22) yields

$$\begin{aligned}u(t,N_t)=u(T,N_T)&-\int_t^T\Big(f(s)+A_1(s)u(s,N_s)+\sum_{j=1}^n A_2^j(s)\sigma_s^j\frac{\partial u}{\partial x_j}(s,N_s)\Big)ds\\&-\sum_{j=1}^n\int_t^T\sigma_s^j\frac{\partial u}{\partial x_j}(s,N_s)\delta X_s^j.\end{aligned}\tag{23}$$

We get (7) by setting $Y_t := u(t, N_t)$ and $Z_t^j := -\sigma_t^j \frac{\partial u}{\partial x_j}(t, N_t)$, i.e. $(Y, (Z^1, \dots, Z^n))$ solves (7) and is adapted to $\mathbb{F}$. As in Section 1.2 we consider this equation for $t \in [t_0, T]$, for some fixed $t_0 \geqslant 0$.

In order to solve (8) explicitely, we have to assume, in addition to *(H1)* and *(H2)* for the kernels K^j, some regularity and integrability conditions. *(H5)* will be discussed later.

(H3) There exist constants $c, C > 0$ such that $c < \sigma^j < C$, $j = 1, \dots, n$, and $A_1, f, A_2 := (A_2^1, \dots, A_2^n)$ are bounded.

(H4) g is continuous, and there exist positive constants c' and $\lambda' < \min_{j=1,\dots,n}(16 \sup_{t\in[0,T]} Var(N_t^j))^{-1}$ such that $| g(x) | \leqslant c' e^{\lambda'|x|^2}$ for all $x \in \mathbb{R}^n$.

(H5) $\frac{d}{dt} Var(N_t^j) > 0$ for all $t \in [t_0, T]$ and $\int_{t_0}^T (Var(N_T^j) - Var(N_t^j))^{-1/2} dt < \infty$, $j = 1, \dots, n$.

Theorem 2. *Assume that (H1)–(H5) hold, and let* $v(t, z) = (2\pi t)^{-1/2} \exp(-z^2/2t)$. *Then* $(Y, (Z^1, \dots, Z^n))$ *given by* (9) *solves* (7)*, where*

$$\begin{aligned} u(t,x) = &-\int_t^T \exp\Big(\int_s^t A_1(r)dr\Big) f(s)ds + \exp\Big(-\int_t^T A_1(s)ds\Big) \\ &\times \int_{\mathbb{R}^n} g(y) \prod_{j=1}^n v\Big(Var(N_T^j) - Var(N_t^j), x_j \\ &- \int_t^T (\sigma_s^j A_2^j(s) - \frac{d}{ds} b_s^j)ds - y_j\Big) dy \end{aligned} \tag{24}$$

solves (8).

Remark. An explicit calculation of the partial derivatives of u shows that u is in fact a classical solution of (8). Sufficient conditions for the uniqueness of the solution of (8) (even in the general nonlinear case) can be found in [12], Theorem 2.4. The question of uniqueness of the solution $(Y, (Z^1, \dots, Z^n))$ of (7) is more delicate for equations with Volterra processes than for equations with fractional Brownian motion and will be adressed in a separate paper. Here we notice that (7) has a unique solution of the form (9) if the solution of (8) is unique. We show now that u verifies the growth condition (17).

Lemma 1. *Let* u *be given by* (24)*. Then there are positive constants* M *and* $\lambda < \min_{j=1,\dots n}(4 \sup_{t\in[0,T]} Var(N_t^j))^{-1}$ *such that*

1) $| u(t,x) | \leqslant M e^{\lambda|x|^2}$, $(t,x) \in [t_0, T] \times \mathbb{R}^n$,

2) $\left|\frac{\partial u}{\partial x_j}(t,x)\right| \leqslant M(Var(N_T^j) - Var(N_t^j))^{-1/2} e^{\lambda|x|^2}$, $(t,x) \in [t_0, T) \times \mathbb{R}^n$,

3) $\left|\frac{\partial^2 u}{\partial x_j^2}(t,x)\right| \leqslant M(Var(N_T^j) - Var(N_t^j))^{-1} e^{\lambda|x|^2}$, $(t,x) \in [t_0, T) \times \mathbb{R}^n$.

Proof. We prove 2), the proofs of 1) and 3) are simpler or similar. Let us write r_s^i for $\sigma_s^i A_2^i(s) - \frac{d}{ds} b_s^i$ and D_t^i for $Var(N_T^i) - Var(N_t^i)$.

$$\frac{\partial u}{\partial x_i}(t,x) = e^{-\int_t^T A_1(s)ds} \int_{\mathbb{R}^n} g(y) \frac{\partial}{\partial \xi_i} v(D_t^i, \xi_i)\Big|_{\xi_i = x_i - \int_t^T r_s^i ds - y_i}$$

$$\times \prod_{j\neq i}^{n} v\Big(D_t^j, x_j - \int_t^T r_s^j ds - y_j\Big) dy$$

$$= e^{-\int_t^T A_1(s)ds} \int_{\mathbb{R}^n} g(y) \frac{1}{\sqrt{2\pi D_t^i}} \Big(- \frac{x_i - \int_t^T r_s^i ds - y_i}{D_t^i} \Big)$$

$$\times exp\Big(- \frac{(x_i - \int_t^T r_s^i ds - y_i)^2}{2D_t^i} \Big)$$

$$\times \prod_{j\neq i}^{n} \frac{1}{\sqrt{2\pi D_t^j}} exp\Big(- \frac{(x_j - \int_t^T r_s^j ds - y_j)^2}{2D_t^j} \Big) dy.$$

By $(H4)$ and the change of variables $z_j = x_j - \int_t^T r_s^j ds - y_j$, $j = 1, \ldots, n$ we get

$$\Big| \frac{\partial u}{\partial x_i}(t,x) \Big| \leqslant C \int_{\mathbb{R}^n} \frac{1}{\sqrt{2\pi D_t^i}} \Big| \frac{z_i}{D_t^i} \Big| exp\Big(- \frac{z_i^2}{2D_t^i} \Big) exp\Big(\lambda' \Big| x - \int_t^T r_s ds - z \Big|^2 \Big)$$

$$\times \prod_{j\neq i}^{n} \frac{1}{\sqrt{2\pi D_t^j}} exp\Big(- \frac{z_j^2}{2D_t^j} \Big) dz.$$

Let us prove that

$$exp\Big(\lambda' \Big| x - \int_t^T r_s ds - z \Big|^2 \Big) \prod_{j\neq i} \frac{exp\Big(- \frac{z_j^2}{2D_t^j} \Big)}{\sqrt{2\pi D_t^j}}$$

$$\leqslant exp\Big(2\lambda' \Big| x - \int_t^T r_s ds \Big|^2 \Big) exp(2\lambda' z_i^2)$$

$$\times \prod_{j\neq i} \frac{exp\Big[- \frac{z_j^2}{D_t^j} \Big(\frac{1}{2} - 2\lambda' Var N_T^j \Big) \Big]}{\sqrt{2\pi D_t^j}}.$$

In fact,

$$exp\Big(\lambda' \Big| x - \int_t^T r_s ds - z \Big|^2 \Big)$$

$$\leqslant exp\Big(2\lambda' \Big| x - \int_t^T r_s ds \Big|^2 \Big) exp\Big(2\lambda' \mid z \mid^2 \Big)$$

$$= exp\Big(2\lambda' \Big| x - \int_t^T r_s ds \Big|^2 \Big) exp(2\lambda' z_i^2) \prod_{j\neq i} exp(2\lambda' z_j^2)$$

$$\leqslant exp\Big(2\lambda' \Big| x - \int_t^T r_s ds \Big|^2 \Big) exp(2\lambda' z_i^2) \prod_{j\neq i} exp\Big(2\lambda' \frac{Var N_T^j}{D_t^j} z_j^2 \Big).$$

Consequently,

$$\Big|\frac{\partial u}{\partial x_i}(t,x)\Big|$$
$$\leqslant Cexp\Big(2\lambda'\Big|x-\int_t^T r_s ds\Big|^2\Big)\int_{\mathbb{R}^n}\frac{1}{\sqrt{2\pi D_t^i}}\frac{1}{\sqrt{D_t^i}}\frac{\big|z_i\big|}{\sqrt{D_t^i}}exp(2\lambda' z_i^2)exp\Big(-\frac{z_i^2}{2D_t^i}\Big)$$
$$\times\prod_{j\neq i}\frac{exp\Big[-\frac{z_j^2}{D_t^j}\Big(\frac{1}{2}-2\lambda' VarN_T^j\Big)\Big]}{\sqrt{2\pi D_t^j}}dz.$$

Moreover, for any $\epsilon > 0$, there is a constant K_ϵ such that $\frac{|z_i|}{\sqrt{D_t^i}} \leqslant K_\epsilon exp\big(\frac{\epsilon}{D_t^i}z_i^2\big)$. Therefore

$$\Big|\frac{\partial u}{\partial x_i}(t,x)\Big|$$
$$\leqslant CK_\epsilon exp\Big(2\lambda'\Big|x-\int_t^T r_s ds\Big|^2\Big)\int_{\mathbb{R}^n}\frac{1}{\sqrt{2\pi D_t^i}}\frac{1}{\sqrt{D_t^i}}exp\Big(\frac{\epsilon}{D_t^i}z_i^2\Big)$$
$$\times exp(2\lambda' z_i^2)exp\Big(-\frac{z_i^2}{2D_t^i}\Big)\prod_{j\neq i}\frac{exp\Big[-\frac{z_j^2}{D_t^j}\Big(\frac{1}{2}-2\lambda' VarN_T^j\Big)\Big]}{\sqrt{2\pi D_t^j}}dz$$
$$\leqslant CK_\epsilon exp\Big(2\lambda'\Big|x-\int_t^T r_s ds\Big|^2\Big)$$
$$\times\int_{\mathbb{R}^{n-1}}\Big[\int_{\mathbb{R}}\frac{1}{\sqrt{2\pi}D_t^i}exp\Big(\frac{-(\frac{1}{2}-\epsilon-2\lambda' VarN_T^i)}{D_t^i}z_i^2\Big)dz_i\Big]$$
$$\times\prod_{j\neq i}\frac{exp\Big[-\frac{z_j^2}{D_t^j}\Big(\frac{1}{2}-2\lambda' VarN_T^j\Big)\Big]}{\sqrt{2\pi D_t^j}}dz',\quad z'=(z_1,\dots,z_{i-1},z_{i+1},\dots,z_n)$$
$$\leqslant CK_\epsilon exp\Big(2\lambda'\Big|x-\int_t^T r_s ds\Big|^2\Big)\int_{\mathbb{R}^{n-1}}\Big[\frac{1}{\sqrt{2\pi}D_t^i}\sqrt{\frac{2\pi D_t^i}{1-2\epsilon-4\lambda' VarN_T^i}}\Big]$$
$$\times\prod_{j\neq i}\frac{exp\Big[-\frac{z_j^2}{D_t^j}\Big(\frac{1}{2}-2\lambda' VarN_t^j\Big)\Big]}{\sqrt{2\pi D_t^j}}dz'$$
$$=CK_\epsilon\frac{1}{\sqrt{1-2\epsilon-4\lambda' VarN_T^i}}\frac{1}{\sqrt{D_t^i}}exp\Big(2\lambda'\Big|x-\int_t^T r_s ds\Big|^2\Big)$$
$$\times\prod_{j\neq i}\frac{1}{\sqrt{2\pi D_t^j}}\sqrt{\frac{2\pi D_t^j}{1-4\lambda' VarN_T^j}}$$

$$\leqslant CK_\epsilon \frac{1}{\sqrt{1-2\epsilon-4\lambda' Var N_T^i}} \frac{1}{\sqrt{D_t^i}} exp\Big(4\lambda' \mid x \mid^2\Big) exp\Big(4\lambda' \Big| \int_t^T r_s ds \Big|^2\Big)$$
$$\times \frac{1}{\Big[\sqrt{1-4\lambda' \max_j Var N_T^j}\Big]^{n-1}}$$
$$\leqslant \frac{MK'_\varepsilon}{\sqrt{D_t^i}} exp\Big(\lambda \mid x \mid^2\Big)$$

with a $M = exp(4\lambda' \max\{| \int_t^T r_s ds|^2, t \in [t_0, T]\})$, $\lambda = 4\lambda'$ and a suitable constant K'_ε. □

Let us prove now Theorem 2.

Proof. It remains to show that (Y, Z) satisfies the BSDE (7). By the preceding lemma u satisfies (23) with T replaced by $T-\varepsilon$. Since $\mathbb{E}\exp(2\lambda \mid N_t \mid^2)$ is bounded on $[0, T]$ for $\lambda < \min_{j=1,\dots n}(4 \sup_{t\in[0,T]} Var(N_t^j))^{-1}$, we obtain

$$\mathbb{E}\int_t^T \left| f(s) + A_1(s)u(s, N_s) + \sum_{j=1}^n A_2^j(s)\sigma_s^j \frac{\partial u}{\partial x_j}(s, N_s)\right| ds$$
$$\leqslant \sum_{j=1}^n \int_t^T (Var(N_T^j) - Var(N_t^j))^{-1/2} dt < \infty$$

for all $t < T$ by *(H5)*. By continuity of u and N, the terms in the first line of (23) with T replaced by $T - \varepsilon$ converge to the terms in (23) as $\varepsilon \to 0$. The divergence integrals converge too in the sense

$$\mathbb{E}\left[\int_t^{T-\varepsilon} \sigma_s^j \frac{\partial u}{\partial x_j}(s, N_s)\delta X_s^j F\right] \underset{\varepsilon\to 0}{\to} \mathbb{E}\left[\int_t^T \sigma_s^j \frac{\partial u}{\partial x_j}(s, N_s)\delta X_s^j F\right]$$

for all $j = 1, \dots, n$ and $F \in \mathcal{S}$. □

Remark. We discuss now the hypothesis *(H5)*. The positivity of $\frac{d}{dt}Var(N_t^j)$ means that $Var(N_t^j)$ is (strictly) increasing on $[t_0, T]$. We note that

$$\frac{d}{dt}Var(N_t^j) = \frac{d}{dt}\int_0^t (K_t^{*,j}\sigma^j)_s^2 ds = 2\int_0^t \sigma_t^j \sigma_u^j \int_0^u \frac{\partial K^j}{\partial t}(t, s)\frac{\partial K^j}{\partial u}(u, s) ds du$$
$$= 2\sigma_t^j \int_0^t \sigma_u^j \phi^j(t, u) du.$$

Since $\sigma^j > 0$ by *(H3)*, a sufficient condition for $\frac{d}{dt}Var(N_t^j) > 0$ for all $t \in [t_0, T]$ is $\phi^j > 0$ on $[0, T]^2 \setminus [0, t_0]^2$. This is the case if $\frac{\partial K^j}{\partial u}(u, s) > 0$ for all $(u, s) \in [0, T]^2$, $u > s$, but the explicit calculation of $D_t^j := Var(N_T^j) - Var(N_t^j)$ below shows that this is a sufficient but not a necessary condition. We note that for fractional Brownian motion B^H with Hurst index $H > 1/2$

$$\phi^H(t,u) = \frac{\partial^2}{\partial t \partial u}\mathbb{E}B_t^H B_u^H = C_H \mid t-u \mid^{2H-2} > 0,$$

where $C_H > 0$ is a constant depending on H.

Let us comment now on the hypothesis of integrability of $(D_t^j)^{-1/2}$ near T. We have

$$\begin{aligned}
D_t^j &= \mathbb{E}\Big[\Big(\int_0^T \delta W_s^j \int_s^T \sigma_r^j \frac{\partial K^j}{\partial r}(r,s)dr\Big)^2 - \Big(\int_0^t \delta W_s^j \int_s^t \sigma_r^j \frac{\partial K^j}{\partial r}(r,s)dr\Big)^2\Big] \\
&= \mathbb{E}\Big[\Big(\int_0^T \delta W_s^j \int_s^T \sigma_r^j \frac{\partial K^j}{\partial r}(r,s)dr + \int_0^t \delta W_s^j \int_s^t \sigma_r^j \frac{\partial K^j}{\partial r}(r,s)dr\Big) \\
&\times \Big(\int_0^T \delta W_s^j \int_s^T \sigma_r^j \frac{\partial K^j}{\partial r}(r,s)dr - \int_0^t \delta W_s^j \int_s^t \sigma_r^j \frac{\partial K^j}{\partial r}(r,s)dr\Big)\Big].
\end{aligned}$$

An explicit calculation shows

$$\begin{aligned}
D_t^j &= \int_0^t dr \int_t^T dr' \sigma_r^j \sigma_{r'}^j \phi^j(r,r') + \int_0^t ds \left(\int_t^T dr \sigma_r^j \frac{\partial K^j}{\partial r}(r,s)\right)^2 \\
&+ \int_t^T ds \left(\int_s^T dr \sigma_r^j \frac{\partial K^j}{\partial r}(r,s)\right)^2 \\
&=: A_t^1 + A_t^2 + A_t^3.
\end{aligned}$$

Under the hypothesis *(H3)* for σ^j and if $\phi^j > 0$, a sufficient condition for $\int_{t_0}^T (D_t^j)^{-1/2} dt < \infty$ is

$$A_t^3 = \int_t^T (K_T^{*,j}\sigma^j)_s^2 ds \geqslant c(T-t)^a$$

for some constant $c > 0$ and $a \in (0,2)$ as $t \nearrow T$. For fractional Brownian motion B^H with $H > 1/2$ this condition is satisfied with $a = H + 1/2$. For the Volterra processes in Examples 1–3 this condition is satisfied with $a = 2h(T)$ if h is such that $\frac{\partial K}{\partial u}(u,s) > 0$, for $(u,s) \in (t_0,T)^2$, $u > s$.

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