

Averaging principle for a stochastic cable equation

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Abstract We consider the cable equation in the mild form driven by a general stochastic measure. The averaging principle for the equation is established. The rate of convergence is estimated. The regularity of the mild solution is also studied. The orders in time and space variables in the Hölder condition for the solution are improved in comparison with previous results in the literature on this topic.

Keywords Averaging principle, stochastic cable equation, stochastic measure, mild solution, Hölder regularity

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1 Introduction

Averaging methods are important for describing and investigating the asymptotic behavior of dynamical systems. Therefore, the theory of the averaging principle for stochastic differential equations is a fascinating modern topic and many mathematicians work quite actively within this field. For instance, a weak order in averaging for wave equations with L^2 -valued Wiener processes is studied in [14]. Bao et al. [3] considered two-time-scale equations with α -stable noises. Strong and weak orders in averaging for stochastic partial differential equations with Wiener processes are given in [10, 11, 13].

The averaging principle for fractional differential equations driven by Lévy noise is established by Shen et al. [31]. Wang and Xu [34] investigated the stochastic av-

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eraging method for neutral stochastic delay equations driven by fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$. Other interesting examples of studying of averaging for stochastic differential equations can be found in papers [1, 12, 15, 18].

The averaging principle for equations driven by general stochastic measures is considered in [6, 22, 23, 26, 30].

Our aim here is to establish the averaging principle for the cable equation driven by stochastic measure studied in [25]. For this purpose we also improve the orders of the Hölder condition for the mild solution with respect to space and time variables obtained in [25, Theorem 5.1] (see Theorem 1 below).

Properties of the mild solutions to stochastic partial differential equations are studied in a number of papers. In particular, the existence and uniqueness of a solution for the class of non-autonomous parabolic stochastic partial differential equations defined on a bounded open subset $D \subset \mathbb{R}^d$ and driven by an $L^2(D)$ -valued fractional Brownian motion with the Hurst index $H > 1/2$ are proved in [28]. In [2] the ergodic property of the solution to a fractional stochastic heat equation is established. Wave equations with general stochastic measures and α -stable distributions are investigated in the papers [5, 7, 8, 24] and [20, 29, 19], respectively.

Let $L_0(\Omega, \mathcal{F}, P)$ be the set of all real-valued random variables defined on a complete probability space (Ω, \mathcal{F}, P) , X be an arbitrary set and $\mathcal{B}(X)$ be a σ -algebra of Borel subsets of X . Let μ be a stochastic measure on $\mathcal{B}(X)$, i.e. a σ -additive mapping $\mu : \mathcal{B}(X) \rightarrow L_0(\Omega, \mathcal{F}, P)$. Such μ is also called a general stochastic measure (see, for example, [17, Section 7]). Examples of stochastic measures can be found in [17, 21, 30].

Consider the mild solution to the following equation:

$$\begin{cases} \frac{\partial u_\varepsilon(t, x)}{\partial t} = \frac{\partial^2 u_\varepsilon(t, x)}{\partial x^2} - u_\varepsilon(t, x) + \sigma(t/\varepsilon, x) \dot{\mu}(x), \\ u_\varepsilon(0, x) = u_0(x), \quad \frac{\partial u_\varepsilon(t, 0)}{\partial x} = \frac{\partial u_\varepsilon(t, L)}{\partial x} = 0, \end{cases} \tag{1}$$

where $(t, x) \in [0, T] \times [0, L]$, $T > 0$, $L > 0$, $\varepsilon > 0$, and μ is a stochastic measure defined on the Borel σ -algebra $\mathcal{B}([0, L])$.

Let G be the fundamental solution of the homogeneous cable equation, that is

$$G(t, x, y) = \frac{e^{-t}}{\sqrt{4\pi t}} \sum_{n=-\infty}^{\infty} \left(e^{-\frac{(y-x-2nL)^2}{4t}} + e^{-\frac{(y+x-2nL)^2}{4t}} \right) \tag{2}$$

(see, for example, [33, p. 312] or [32, equality (5.69B)]). Then the mild solution of problem (1) is given by the formula

$$u_\varepsilon(t, x) = \int_0^L G(t, x, y) u_0(y) dy + \int_{[0, L]} d\mu(y) \int_0^t G(t-s, x, y) \sigma(s/\varepsilon, y) ds. \tag{3}$$

We study the convergence

$$u_\varepsilon(t, x) \rightarrow \bar{u}(t, x), \quad \varepsilon \rightarrow 0,$$

where $\bar{u}(t, x)$ is the mild solution of the averaged equation, that is,

$$\bar{u}(t, x) = \int_0^L G(t, x, y)u_0(y) dy + \int_{[0,L]} d\mu(y) \int_0^t G(t - s, x, y)\bar{\sigma}(y) ds, \quad (4)$$

and

$$\bar{\sigma}(x) = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \sigma(s, x) ds. \quad (5)$$

The rest of the paper is organized as follows. Section 2 contains some basic facts concerning the estimates of stochastic integrals with respect to general stochastic measures. In Section 3, we study the Hölder regularity of the mild solution of the cable equation with respect to the set of all variables. The averaging principle for the cable equation is established in Section 4.

2 Preliminaries

To prove the convergence of solutions, we will apply an estimate of a stochastic integral using the norm of the Besov space $B_{22}^\alpha([b, c])$, $\alpha \in (1/2, 1)$, $b, c \in \mathbb{R}$ (see, for example, [16]). $B_{22}^\alpha([b, c])$ is the space of functions $g : [b, c] \rightarrow \mathbb{R}$ such that the norm

$$\|g\|_{B_{22}^\alpha([b,c])} = \|g\|_{L_2([b,c])} + \left(\int_0^{c-b} (w_{2,[b,c]}(g, r))^2 r^{-2\alpha-1} dr \right)^{1/2}, \quad (6)$$

is finite. Here

$$w_{2,[b,c]}(g, r) = \sup_{0 \leq h \leq r} \left(\int_b^{c-h} |g(s+h) - g(s)|^2 ds \right)^{1/2}.$$

Denote

$$\Delta_{kn}^{(L)} = ((k-1)2^{-n}L, k2^{-n}L), \quad n \geq 0, \quad 1 \leq k \leq 2^n.$$

Let Z be an arbitrary set and a function $g(y, z) : [0, L] \times Z \rightarrow \mathbb{R}$ be such that $g(\cdot, z)$ is continuous on $[0, L]$ for all $z \in Z$. Put

$$g_n(y, z) = g(0, z)\mathbb{1}_{\{0\}}(y) + \sum_{1 \leq k \leq 2^n} g((k-1)2^{-n}L, z)\mathbb{1}_{\Delta_{kn}^{(L)}}(y).$$

By [27, Lemma 3], the random function

$$\eta(z) = \int_{[0,L]} g(y, z)d\mu(y), \quad z \in Z,$$

has a version

$$\begin{aligned} \tilde{\eta}(z) &= \int_{[0,L]} g_0(y, z)d\mu(y) \\ &+ \sum_{n \geq 1} \left(\int_{[0,L]} g_n(y, z)d\mu(y) - \int_{[0,L]} g_{n-1}(y, z)d\mu(y) \right), \end{aligned}$$

such that, for all $\varepsilon > 0, \omega \in \Omega, z \in Z,$

$$\begin{aligned}
 |\tilde{\eta}(z)| &\leq |g(0, z)\mu([0, L])| \\
 &+ \left\{ \sum_{n \geq 1} 2^{n\varepsilon} \sum_{1 \leq k \leq 2^n} |g(k2^{-n}L, z) - g((k-1)2^{-n}L, z)|^2 \right\}^{\frac{1}{2}} \\
 &\times \left\{ \sum_{n \geq 1} 2^{-n\varepsilon} \sum_{1 \leq k \leq 2^n} \left| \mu \left(\Delta_{kn}^{(L)} \right) \right|^2 \right\}^{\frac{1}{2}}.
 \end{aligned}$$

This version is the same for all $z \in Z.$

According to [16, Theorem 1.2], [21, Lemma 3.2] and [9, inequality (6)], we have

$$\begin{aligned}
 |\tilde{\eta}(z)| &\leq |g(0, z)\mu([0, L])| \\
 &+ C \|g(\cdot, z)\|_{B_{22}^\alpha([0, L])} \left\{ \sum_{n \geq 1} 2^{-n\varepsilon} \sum_{1 \leq k \leq 2^n} \left| \mu \left(\Delta_{kn}^{(L)} \right) \right|^2 \right\}^{\frac{1}{2}}, \tag{7}
 \end{aligned}$$

where $\alpha = \varepsilon/2 + 1/2,$ the constant C depends on α, t and does not depend on $z, \omega.$

Here and in what follows the same symbol C denotes some positive constants that may be different in different places of the paper. The precise values of these constants are not important for our purposes.

3 Regularity of the mild solution of a cable equation

Regularity of the mild solution

$$u(t, x) = \int_0^L G(t, x, y)u_0(y) dy + \int_{[0, L]} d\mu(y) \int_0^t G(t - s, x, y)\sigma(s, y) ds, \tag{8}$$

of a cable equation driven by a general stochastic measure is studied in [25]. It was proved there that the paths of the solution are Hölder continuous. The following conditions was considered.

Condition 1. The function $\sigma(s, y) : [0, T] \times [0, L] \rightarrow \mathbb{R}$ has the derivative $\frac{\partial^2 \sigma}{\partial t \partial x^2},$ which is continuous with respect to the pair of arguments.

Condition 2. The function $u_0(y) = u_0(y, \omega) : [0, L] \times \Omega \rightarrow \mathbb{R}$ is measurable and has the derivative $\frac{\partial u_0}{\partial y},$ which is continuous with respect to y and bounded for all fixed $\omega \in \Omega.$

By [25, Theorem 5.1], if Conditions 1–2 hold, then for all fixed $\delta > 0, \gamma_2 < 1/18,$ and $\gamma_1 < 1/6,$ function (8) has a version $\tilde{u}(t, x)$ such that

$$|\tilde{u}(t_1, x_1) - \tilde{u}(t_2, x_2)| \leq L_{\tilde{u}}(\omega) (|t_1 - t_2|^{\gamma_2} + |x_1 - x_2|^{\gamma_1}), \tag{9}$$

for all $t_1, t_2 \in [\delta, T], x_1, x_2 \in [0, L]$ and for some $L_{\tilde{u}}(\omega) > 0.$

The proof of Theorem 5.1 ([25]) uses the Hölder regularity of the mild solution of a heat equation that is established in [21]. Restrictions $\gamma_1 < 1/6$ and $\gamma_2 < 1/18$

for relation (9) are due to the restrictions for the corresponding orders of the Hölder condition for the stochastic integrals from the heat equation (see [21, Lemma 5.1 and Lemma 6.1]). But now we can use the results of [4], which improve Hölder continuity orders obtained in [21]. Namely, the Hölder regularity with $\gamma_1 < 1/2$ and $\gamma_2 < 1/4$ for the mild solution of a parabolic equation with a stochastic measure is proved in [4, Lemma 1 and Lemma 2].

The following assumptions are used in the rest of the paper.

A1. The function $u_0(y) = u_0(y, \omega) : [0, L] \times \Omega \rightarrow \mathbb{R}$ is measurable and Hölder continuous, that is

$$|u_0(y_1) - u_0(y_2)| \leq L_{u_0}(\omega) |y_1 - y_2|^{\beta(u_0)}, \quad \beta(u_0) > 0.$$

A2. $\sigma(s, y) : \mathbb{R}_+ \times [0, L] \rightarrow \mathbb{R}$ is measurable, bounded and Hölder continuous in $s \in \mathbb{R}_+, y \in [0, L]$, that is

$$|\sigma(s, y)| \leq C_\sigma, \\ |\sigma(s_1, y_1) - \sigma(s_2, y_2)| \leq L_\sigma \left(|s_1 - s_2|^{\beta(\sigma)} + |y_1 - y_2|^{\beta(\sigma)} \right), \quad 1/2 < \beta(\sigma) < 1.$$

A3. Limit (5) exists and $\Sigma(r, y) = \int_0^r [\sigma(s, y) - \bar{\sigma}(y)] ds$ is bounded for all $t \in \mathbb{R}_+, y \in [0, L]$.

Theorem 1. *Let Assumptions A1–A2 hold. Then there exists a version $\tilde{u}(t, x)$ of the random function $u(t, x)$ defined by (8) such that for all fixed $\delta > 0, \tilde{\gamma}_2 < 1/4 \wedge (\beta(\sigma) - 1/2)$ and $\tilde{\gamma}_1 < 1 - 1/(2\beta(\sigma)), \tilde{\gamma}_1 \leq \beta(u_0)$, we have*

$$|\tilde{u}(t_1, x_1) - \tilde{u}(t_2, x_2)| \leq C_{\tilde{u}}(\omega) (|t_1 - t_2|^{\tilde{\gamma}_2} + |x_1 - x_2|^{\tilde{\gamma}_1}), \quad (10)$$

for all $t_1, t_2 \in [\delta, T]$ and $x_1, x_2 \in [0, L]$ and for some constant $C_{\tilde{u}}(\omega) > 0$.

Proof. The reasoning is the same as that used in [25] with some differences. We have divided the proof into 3 steps: the Hölder continuity of the stochastic integral with respect to the space variable; the Hölder continuity of the stochastic integral with respect to the time variable and the Hölder continuity of the function $u(t, x)$.

Step 1. By obtaining the Hölder condition for the stochastic integral with respect to the space variable we get the analogues of [25, inequalities (3.1) and (3.2)]. We use Assumption A2 instead of Condition 1.

For any fixed $x_1, x_2 \in [0, L]$, and for $y \in [0, L], h \in [0, L - y]$, put

$$F = |f_n(s, x_2, y + h) - f_n(s, x_1, y + h) - f_n(s, x_2, y) + f_n(s, x_1, y)| \\ \leq |f_n(s, x_2, y + h) - f_n(s, x_1, y + h)| + |f_n(s, x_2, y) - f_n(s, x_1, y)| \\ = F_1 + F_2,$$

where notation

$$f_n(s, x, y) = \frac{e^{-(t-s)}}{\sqrt{4\pi(t-s)}} e^{-\frac{(y-x-2nL)^2}{4(t-s)}} \sigma(s, y)$$

is used (see [25]).

Consider $n \neq 0, 1$. By A2, we have

$$F_2 \leq \frac{C_\sigma}{\sqrt{\pi}(4(t-s))^{3/2}} |x_1 - x_2| 2(|n| + 1) L e^{-\frac{\min\{(y-x_1-2n)^2, (y-x_2-2n)^2\} L^2}{4(t-s)}} \\ \leq \frac{C}{(t-s)^{3/2}} |x_1 - x_2| \cdot |n| e^{-\frac{(2|n|-1)^2 L^2}{4(t-s)}}.$$

For details see the method used for obtaining bound (35) below. The same estimation holds for term F_1 . Therefore,

$$F \leq \frac{C}{(t-s)^{3/2}} |x_1 - x_2| \cdot |n| e^{-\frac{(2|n|-1)^2 L^2}{4(t-s)}}. \tag{11}$$

On the other hand,

$$F \leq |f_n(s, x_1, y+h) - f_n(s, x_1, y)| + |f_n(s, x_2, y+h) - f_n(s, x_2, y)| \\ = \tilde{F}_1 + \tilde{F}_2.$$

Since the function σ is bounded and Hölder continuous (by Assumption A2), and $(t-s)^{-1/2} \leq T(t-s)^{-3/2}$, we obtain

$$\tilde{F}_1 \leq \frac{1}{\sqrt{4\pi}(t-s)} e^{-\frac{(y-x_1-2nL)^2}{4(t-s)}} |\sigma(s, y+h) - \sigma(s, y)| \\ + \frac{1}{\sqrt{4\pi}(t-s)} \left| e^{-\frac{(y+h-x_1-2nL)^2}{4(t-s)}} - e^{-\frac{(y-x_1-2nL)^2}{4(t-s)}} \right| |\sigma(s, y+h)| \\ \leq \frac{C}{(t-s)^{3/2}} \left(L_\sigma h^{\beta(\sigma)} + C_\sigma h 2(|n| + 1) L e^{-\frac{\min\{(y-x_1-2n)^2, (y+h-x_1-2n)^2\} L^2}{4(t-s)}} \right) \\ \leq \frac{C}{(t-s)^{3/2}} h^{\beta(\sigma)} \cdot |n| e^{-\frac{(2|n|-1)^2 L^2}{4(t-s)}}.$$

The same estimate holds for term \tilde{F}_2 . Hence,

$$F \leq \frac{C}{(t-s)^{3/2}} h^{\beta(\sigma)} \cdot |n| e^{-\frac{(2|n|-1)^2 L^2}{4(t-s)}}. \tag{12}$$

Raise the inequality (12) to the power θ_1 and multiply by inequality (11) raised to the power $1 - \theta_1$, for an arbitrary $\theta_1 \in (0, 1)$. We have

$$F \leq \frac{C}{(t-s)^{3/2}} |x_1 - x_2|^{1-\theta_1} h^{\theta_1 \beta(\sigma)} \cdot |n| e^{-\frac{(2|n|-1)^2 L^2}{4(t-s)}}, \tag{13}$$

for all $n \neq 0$.

For the same reason,

$$|f_n(s, -x_2, y+h) - f_n(s, -x_1, y+h) - f_n(s, -x_2, y) + f_n(s, -x_1, y)| \\ \leq \frac{C}{(t-s)^{3/2}} |x_1 - x_2|^{1-\theta_1} h^{\theta_1 \beta(\sigma)} \cdot |n| e^{-\frac{\min\{(2n)^2, (2n-2)^2\} L^2}{4(t-s)}}, \tag{14}$$

for all $n \neq 0, 1$.

Further, we can repeat the proof of Theorem 3.1 ([25]). Instead of using bounds (3.1) and (3.2) of [25], we can use inequalities (13) and (14) above. The integral from (6) is finite if $\theta_1\beta(\sigma) > 1/2$. Thus we get the Hölder condition with respect to the space variable of the order

$$\tilde{\gamma}_1 = 1 - \theta_1 < 1 - \frac{1}{2\beta(\sigma)} < \frac{1}{2}.$$

Besides, as mentioned above, we use [4, Lemma 1 and Lemma 2] instead of [21, Lemma 5.1 and Lemma 6.1] in the case of $n = 0, 1$.

Step 2. The reasoning is the same in the case of the Hölder property with respect to the time variable. Given x and $t_1 < t_2$ use the notation of paper [25], that is

$$\begin{aligned} \bar{f}_n(t, s, y) &= \frac{e^{-(t-s)}}{\sqrt{4\pi(t-s)}} \left(e^{-\frac{(y-x-2nL)^2}{4(t-s)}} + e^{-\frac{(y+x-2nL)^2}{4(t-s)}} \right) \sigma(s, y), \quad n \in \mathbb{Z}, \\ \bar{G}^{(1)}(t, s, y) &= \sum_{n \neq 0,1} \bar{f}_n(t, s, y), \end{aligned}$$

and

$$\bar{g}(y) = \int_0^{t_2} \bar{G}^{(1)}(t_2, s, y) ds - \int_0^{t_1} \bar{G}^{(1)}(t_1, s, y) ds. \tag{15}$$

Moreover, denote

$$\bar{G}_1(t, s, y) = \sum_{n \neq 0,1} \frac{e^{-(t-s)}}{\sqrt{4\pi(t-s)}} \left(e^{-\frac{(y-x-2nL)^2}{4(t-s)}} + e^{-\frac{(y+x-2nL)^2}{4(t-s)}} \right).$$

Now we consider the case of $n \neq 0, 1$, since the case of $n = 0, 1$ is the same as in the [25] with reference to [4, Lemma 1 and Lemma 2] instead of [21, Lemma 5.1 and Lemma 6.1] respectively. Namely, we get Hölder continuity of the order 1/4 for these terms.

We use the change of variables $s \rightarrow s + t_2 - t_1$ in the first integral of (15) and obtain

$$\begin{aligned} \bar{g}(y) &= \int_{t_1-t_2}^{t_1} \bar{G}_1(t_1, s, y) \sigma(s + t_2 - t_1, y) ds - \int_0^{t_1} \bar{G}_1(t_1, s, y) \sigma(s, y) ds \\ &= \int_{t_1-t_2}^0 \bar{G}^{(1)}(t_2, s + t_2 - t_1, y) \sigma(s + t_2 - t_1, y) ds \\ &\quad + \int_0^{t_1} \bar{G}_1(t_1, s, y) (\sigma(s + t_2 - t_1, y) - \sigma(s, y)) ds \\ &= \bar{g}_1(y) + \bar{g}_2(y). \end{aligned}$$

Similarly to (12), by A2, we have

$$|\bar{f}_n(t, s, y + h) - \bar{f}_n(t, s, y)| \leq \frac{C}{(t-s)^{3/2}} h^{\beta(\sigma)} \cdot |n| e^{-\frac{\min\{(2n+1)^2, (2n-2)^2\} L^2}{4(t-s)}}, \tag{16}$$

and

$$|\bar{G}^{(1)}(t_2, s + t_2 - t_1, y + h) - \bar{G}^{(1)}(t_2, s + t_2 - t_1, y)| \leq \frac{Ch^{\beta(\sigma)}}{(t_1 - s)^{3/2}} e^{-\frac{L^2}{4(t_1 - s)}}.$$

Therefore,

$$|\bar{g}_1(y + h) - \bar{g}_1(y)| \leq Ch^{\beta(\sigma)} \left| \int_{t_1 - t_2}^0 \frac{e^{-\frac{L^2}{4(t_1 - s)}}}{(t_1 - s)^{3/2}} ds \right| \leq Ch^{\beta(\sigma)} |t_2 - t_1|, \quad (17)$$

where we used the estimate

$$\frac{e^{-\frac{L^2}{4(t_1 - s)}}}{(t_1 - s)^{3/2}} \leq \frac{1}{t_1^{3/2}} \leq \frac{1}{\delta^{3/2}} \leq C,$$

for s in the domain of integration.

Then, by assumption A2, we get

$$|\bar{g}_2(y)| \leq \left| \int_0^{t_1} \bar{G}_1(t_1, s, y) (\sigma(s + t_2 - t_1, y) - \sigma(s, y)) ds \right| \leq C |t_2 - t_1|^{\beta(\sigma)}. \quad (18)$$

On the other hand, the same reasoning as that used in obtaining bound (12) proves

$$|\bar{g}_2(y + h) - \bar{g}_2(y)| \leq Ch^{\beta(\sigma)}. \quad (19)$$

Now we raise the inequality (19) to the power θ_2 and multiply by inequality (18) raised to the power $1 - \theta_2$, for an arbitrary $\theta_2 \in (0, 1)$. Thus we see that

$$|\bar{g}(y + h) - \bar{g}(y)| \leq C |t_2 - t_1|^{1 - \theta_2} h^{\theta_2 \beta(\sigma)}. \quad (20)$$

Consequently, taking to consideration (17),

$$w_2(\bar{g}, r) \leq C |t_2 - t_1|^{(1 - \theta_2)\beta(\sigma)} r^{\theta_2 \beta(\sigma)}. \quad (21)$$

From this point we can repeat the proof of Theorem 4.1 ([25]). Instead of using (4.2) of [25], we use bounds (20) and (21) above. Since the integral from (6) is finite for $\theta_2 \beta(\sigma) > 1/2$,

$$0 < 1 - \theta_2 < 1 - \frac{1}{2\beta(\sigma)} < \frac{1}{2}.$$

Therefore, we obtain the Hölder condition with respect to the time variable of the order

$$\tilde{\gamma}_2 = \min \left\{ \beta(\sigma) - \frac{1}{2}, \frac{1}{4} \right\},$$

where we also count the case $n = 0, 1$.

Step 3. The last part of the proof is analogous to the proof of Theorem 5.1 in [25]. The integral

$$\int_0^L G(t, x, y)u_0(y) dy$$

satisfies condition (10). To prove the Hölder regularity with respect to the space variable we consider $x_1 < x_2$ and denote

$$G_{\pm}(t, x, y) = \frac{e^{-t}}{\sqrt{4\pi t}} \sum_{n=-\infty}^{\infty} e^{-\frac{(y \pm x - 2nL)^2}{4t}}.$$

Then we use the change of variables $y \rightarrow y + x_2 - x_1$ and $y \rightarrow y - x_2 + x_1$ in the integrals involving x_2 and $G_-(t, x, y)$, $G_+(t, x, y)$ respectively. Thus

$$\begin{aligned} & \left| \int_0^L G(t, x_1, y)u_0(y) dy - \int_0^L G(t, x_2, y)u_0(y) dy \right| \\ &= \left| \int_0^{L-x_2+x_1} G_-(t, x_1, y)(u_0(y) - u_0(y + x_2 - x_1)) dy \right. \\ &+ \int_{x_2-x_1}^L G_+(t, x_1, y)(u_0(y) - u_0(y - x_2 + x_1)) dy \\ &- \int_{-x_2+x_1}^0 G_-(t, x_1, y)u_0(y + x_2 - x_1) dy + \int_{L-x_2+x_1}^L G_-(t, x_1, y)u_0(y) dy \\ &- \left. \int_L^{L+x_2-x_1} G_+(t, x_1, y)u_0(y - x_2 + x_1) dy + \int_0^{x_2-x_1} G_+(t, x_1, y)u_0(y) dy \right| \\ &\leq L u_0(\omega) |x_1 - x_2|^{\beta(u_0)} \left(\int_0^{L-x_2+x_1} |G_-(t, x_1, y)| dy + \int_{x_2-x_1}^L |G_+(t, x_1, y)| dy \right) \\ &+ C(\omega) |x_1 - x_2| \sup_{t \in [\delta, L], x_1, y \in [0, L]} |G(t, x, y)| \leq C(\omega) |x_1 - x_2|^{\beta(u_0)}, \end{aligned}$$

where we use the boundedness of the function u_0 due to its Hölder continuity on $[0, L]$.

To prove the Hölder regularity with respect to the time variable we consider $\delta \leq t_1 < t_2 \leq T$. We get

$$\begin{aligned} |G_+(t_1, x, y) - G_+(t_2, x, y)| &= \left| \frac{e^{-t_1}}{\sqrt{4\pi t_1}} \sum_{n=-\infty}^{\infty} e^{-\frac{(y+x-2nL)^2}{4t_1}} \right. \\ &+ \left. \frac{e^{-t_2}}{\sqrt{4\pi t_2}} \sum_{n=-\infty}^{\infty} e^{-\frac{(y+x-2nL)^2}{4t_2}} \right| \\ &\leq \frac{e^{-t_1} - e^{-t_2}}{\sqrt{4\pi t_1}} \sum_{n=-\infty}^{\infty} e^{-\frac{(y+x-2nL)^2}{4t_1}} \end{aligned}$$

$$\begin{aligned}
 &+ \frac{e^{-t_2}}{\sqrt{4\pi t_1}} \sum_{n=-\infty}^{\infty} \left| e^{-\frac{(y+x-2nL)^2}{4t_1}} - e^{-\frac{(y+x-2nL)^2}{4t_2}} \right| \\
 &+ e^{-t_2} \left(\frac{1}{\sqrt{4\pi t_1}} - \frac{1}{\sqrt{4\pi t_2}} \right) \sum_{n=-\infty}^{\infty} e^{-\frac{(y+x-2nL)^2}{4t_2}} \\
 &= J_1 + J_2 + J_3.
 \end{aligned}$$

According to estimates (34), (25) and (27) from Section 4,

$$\begin{aligned}
 J_1 &\leq \frac{|t_1 - t_2|}{\sqrt{4\pi t_1}} \sum_{n=-\infty}^{\infty} e^{-\frac{(y+x-2nL)^2}{4t_1}} \\
 &\leq \frac{|t_1 - t_2|}{\sqrt{4\pi \delta}} \left(\sum_{n \neq 0} e^{-\frac{\min\{(2n)^2, (2n-2)^2\} L^2}{4t_1}} + 1 \right) \\
 &\leq C|t_1 - t_2|.
 \end{aligned}$$

The similar arguments yield

$$\begin{aligned}
 J_2 &\leq \frac{1}{\sqrt{4\pi t_1}} \sum_{n=-\infty}^{\infty} \left| \frac{(y+x-2nL)^2}{4t_1} - \frac{(y+x-2nL)^2}{4t_2} \right| e^{-\frac{(y+x-2nL)^2}{4t_1}} \\
 &\leq C|t_1 - t_2| \sum_{n=-\infty}^{\infty} \frac{(y+x-2nL)^2}{4t_1} e^{-\frac{(y+x-2nL)^2}{4t_1}} \\
 &\leq C|t_1 - t_2|,
 \end{aligned}$$

where we use bound (30) (see Section 4 below).

Finally,

$$J_3 \leq \frac{C}{\sqrt{4\pi t_1 t_2}} (\sqrt{t_2} - \sqrt{t_1}) \sum_{n=-\infty}^{\infty} e^{-\frac{(y+x-2nL)^2}{4t_2}} \leq \frac{C}{\delta \sqrt{4\pi}} \sqrt{t_2 - t_1} \leq C \sqrt{t_2 - t_1}.$$

In consequence,

$$|G_+(t_1, x, y) - G_+(t_2, x, y)| \leq C \sqrt{t_2 - t_1}.$$

Likewise, we get the same estimate for $|G_+(t_1, x, y) - G_+(t_2, x, y)|$. Hence,

$$\left| \int_0^L G(t_1, x, y) u_0(y) dy - \int_0^L G(t_2, x, y) u_0(y) dy \right| \leq C(\omega) \sqrt{t_2 - t_1}.$$

The rest of the proof runs as respective part of the proof of Theorem 5.1 in [25] with the use of Steps 1 and 2 instead of [25, Theorems 3.1 and 4.1]. \square

4 Averaging principle

In this section we consider the random functions u_ε and \bar{u} given by equations (3) and (4).

Theorem 2. Assume that Assumptions A1–A3 hold. Then there exist versions of u_ε and \bar{u} such that for any $\gamma < \frac{1}{2}\left(1 - \frac{1}{2\beta(\sigma)}\right)$, we have

$$\sup_{\varepsilon \in (0, T], t \in [0, T], x \in [0, L]} \varepsilon^{-\gamma} |u_\varepsilon(t, x) - \bar{u}(t, x)| < +\infty \quad a. s.$$

Proof. By Theorem 1, if Assumptions A1–A2 hold, functions (3) and (4) have continuous in (t, x) versions. We consider these versions.

Let $\varepsilon > 0, t \in [0, T], x \in [0, L]$ be fixed. Put

$$g(t, x, y) = \int_0^t G(t - s, x, y)[\sigma(s/\varepsilon, y) - \bar{\sigma}(y)] ds, \tag{22}$$

where the fundamental solution $G(t, x, y)$ is given by (2).

Then

$$|u_\varepsilon(t, x) - \bar{u}(t, x)| = \left| \int_{[0, L]} g(t, x, y) d\mu(y) \right|.$$

With the aim of using inequality (7), we estimate $|g(t, x, y)|, y \in [0, L]$, and $|g(t, x, y + h) - g(t, x, y)|, h \in [0, L - y]$, in terms of some positive powers of h and ε .

Denote

$$D_{n\pm} = \frac{(y \pm x - 2nL)^2}{4}$$

and

$$\Sigma_\varepsilon(r) = \int_0^r [\sigma(t/\varepsilon - \tau, y) - \bar{\sigma}(y)] d\tau, \quad 0 \leq r \leq t/\varepsilon.$$

Then $|\Sigma_\varepsilon(r)| \leq C_\Sigma$, where C_Σ does not depend on ε , while $\int_0^t [\sigma(s, y) - \bar{\sigma}(y)] ds$ is bounded by A3.

In the integral from (22) we substitute $\tau = (t - s)/\varepsilon$ and obtain

$$\begin{aligned} |g(t, x, y)| &= \varepsilon \left| \int_0^{t/\varepsilon} G(\tau\varepsilon, x, y)[\sigma(t/\varepsilon - \tau, y) - \bar{\sigma}(y)] d\tau \right| \\ &= \sqrt{\varepsilon} \left| \int_0^{t/\varepsilon} \frac{e^{-\tau\varepsilon}}{\sqrt{4\pi\tau}} \sum_{n=-\infty}^{\infty} \left(e^{-\frac{(y-x-2nL)^2}{4\tau\varepsilon}} + e^{-\frac{(y+x-2nL)^2}{4\tau\varepsilon}} \right) \right. \\ &\quad \left. \times [\sigma(t/\varepsilon - \tau, y) - \bar{\sigma}(y)] d\tau \right| \\ &\leq \frac{\sqrt{\varepsilon}}{\sqrt{4\pi}} \left| \sum_{n=-\infty}^{\infty} \int_0^{t/\varepsilon} \frac{e^{-\frac{D_{n-}}{\tau\varepsilon}}}{\sqrt{\tau}} [\sigma(t/\varepsilon - \tau, y) - \bar{\sigma}(y)] d\tau \right. \\ &\quad \left. + \sum_{n=-\infty}^{\infty} \int_0^{t/\varepsilon} \frac{e^{-\frac{D_{n+}}{\tau\varepsilon}}}{\sqrt{\tau}} [\sigma(t/\varepsilon - \tau, y) - \bar{\sigma}(y)] d\tau \right|. \end{aligned}$$

We have

$$\begin{aligned} \left| \sum_{n=-\infty}^{\infty} \int_0^1 \frac{e^{-\frac{D_{n\pm}}{\tau\varepsilon}}}{\sqrt{\tau}} [\sigma(t/\varepsilon - \tau, y) - \bar{\sigma}(y)] d\tau \right| &\leq 2C_\sigma \int_0^1 \sum_{n=-\infty}^{\infty} \frac{e^{-\frac{D_{n\pm}}{\tau\varepsilon}}}{\sqrt{\tau}} d\tau \\ &\leq 4C_\sigma \sup_{\tau \leq 1} \sum_{n=-\infty}^{\infty} e^{-\frac{D_{n\pm}}{\tau\varepsilon}} = 4C_\sigma \sum_{n=-\infty}^{\infty} e^{-\frac{D_{n\pm}}{\varepsilon}} \end{aligned} \tag{23}$$

and

$$\begin{aligned} \left| \sum_{n=-\infty}^{\infty} \int_1^{t/\varepsilon} \frac{e^{-\frac{D_{n\pm}}{\tau\varepsilon}}}{\sqrt{\tau}} [\sigma(t/\varepsilon - \tau, y) - \bar{\sigma}(y)] d\tau \right| &= \left| \sum_{n=-\infty}^{\infty} \int_1^{t/\varepsilon} \frac{e^{-\frac{D_{n\pm}}{\tau\varepsilon}}}{\sqrt{\tau}} d\Sigma_\varepsilon(\tau) \right| \\ &= \left| \sum_{n=-\infty}^{\infty} \frac{e^{-\frac{D_{n\pm}}{\tau\varepsilon}}}{\sqrt{\tau}} \Sigma_\varepsilon(\tau) \Big|_1^{t/\varepsilon} - \sum_{n=-\infty}^{\infty} \int_1^{t/\varepsilon} \Sigma_\varepsilon(\tau) e^{-\frac{D_{n\pm}}{\tau\varepsilon}} \left(\frac{D_{n\pm}}{\varepsilon\sqrt{\tau^5}} - \frac{1}{2\sqrt{\tau^3}} \right) d\tau \right| \\ &\leq 2C_\Sigma \sup_{1 \leq \tau \leq t/\varepsilon} \sum_{n=-\infty}^{\infty} e^{-\frac{D_{n\pm}}{\tau\varepsilon}} + C_\Sigma \sum_{n=-\infty}^{\infty} \int_1^{t/\varepsilon} \frac{D_{n\pm} e^{-\frac{D_{n\pm}}{\tau\varepsilon}}}{\varepsilon\sqrt{\tau^5}} d\tau \\ &\leq 2C_\Sigma \sum_{n=-\infty}^{\infty} e^{-\frac{D_{n\pm}}{T}} + C_\Sigma \int_1^{t/\varepsilon} \frac{\sum_{n=-\infty}^{\infty} D_{n\pm} e^{-\frac{D_{n\pm}}{\tau\varepsilon}}}{\varepsilon\sqrt{\tau^5}} d\tau. \end{aligned} \tag{24}$$

Now we estimate $e^{-\frac{D_{n\pm}}{s}}$, $s \in [0, T]$. For $n = 0$,

$$e^{-\frac{D_{n\pm}}{s}} = e^{-\frac{(y\pm x)^2}{4s}} \leq 1.$$

Consider $n \neq 0$. In this case

$$e^{-\frac{D_{n-}}{s}} \leq e^{-\frac{(2|n|-1)^2 L^2}{4s}} \quad \text{and} \quad e^{-\frac{D_{n+}}{s}} \leq e^{-\frac{\min\{(2n)^2, (2n-2)^2\} L^2}{4s}}. \tag{25}$$

Since

$$\sum_{n \geq 0} e^{-n^2 a} \leq \sum_{n \geq 0} e^{-na} = (1 - e^{-a})^{-1} \leq (1 - e^{-b})^{-1}, \quad a \geq b > 0, \tag{26}$$

and

$$\sum_{n \geq 1} e^{-n^2 a} \leq e^{-a} (1 - e^{-a})^{-1} \leq e^{-b} (1 - e^{-b})^{-1}, \quad a \geq b > 0,$$

we conclude that, for $s \in [0, T]$,

$$\begin{aligned} \sum_{n \neq 0, n = -\infty}^{\infty} e^{-\frac{D_{n-}}{s}} &\leq \sum_{n \neq 0, n = -\infty}^{\infty} e^{-\frac{(2|n|-1)^2 L^2}{4s}} \leq 2e^{-\frac{L^2}{4s}} \sum_{n \geq 0} e^{-\frac{n^2 L^2}{s}} \\ &\leq 2e^{-\frac{L^2}{4s}} \left(1 - e^{-\frac{L^2}{s}} \right)^{-1} \leq 2 \left(1 - e^{-\frac{L^2}{T}} \right)^{-1}. \end{aligned}$$

Analogously,

$$\begin{aligned} \sum_{n \neq 0, n = -\infty}^{\infty} e^{-\frac{D_{n+}}{s}} &\leq \sum_{n \neq 0, n = -\infty}^{\infty} e^{-\frac{\min\{(2n)^2, (2n-2)^2\}L^2}{4s}} \\ &= \sum_{n \geq 1} e^{-\frac{(2n)^2L^2}{4s}} + \sum_{n \geq 1} e^{-\frac{(2n-2)^2L^2}{4s}} \leq 2 \left(1 - e^{-\frac{L^2}{T}}\right)^{-1}. \end{aligned} \tag{27}$$

Substitute obtained bounds to (23) and (24). We thus get

$$\left| \sum_{n=-\infty}^{\infty} \int_0^1 \frac{e^{-\frac{D_{n\pm}}{\tau\varepsilon}}}{\sqrt{\tau}} [\sigma(t/\varepsilon - \tau, y) - \bar{\sigma}(y)] d\tau \right| \leq 4C_{\sigma} \left(1 + 2 \left(1 - e^{-\frac{L^2}{T}}\right)^{-1}\right)$$

and

$$\begin{aligned} &\left| \sum_{n=-\infty}^{\infty} \int_1^{t/\varepsilon} \frac{e^{-\frac{D_{n\pm}}{\tau\varepsilon}}}{\sqrt{\tau}} [\sigma(t/\varepsilon - \tau, y) - \bar{\sigma}(y)] d\tau \right| \\ &\leq 2C_{\Sigma} \left(1 + 2 \left(1 - e^{-\frac{L^2}{T}}\right)^{-1}\right) + C_{\Sigma} \int_1^{t/\varepsilon} \frac{\sum_{n=-\infty}^{\infty} D_{n\pm} e^{-\frac{D_{n\pm}}{\tau\varepsilon}}}{\varepsilon\sqrt{\tau^5}} d\tau. \end{aligned}$$

Hence,

$$\begin{aligned} &|g(t, x, y)| \\ &\leq C\sqrt{\varepsilon} \left[C + \left| \int_1^{t/\varepsilon} \frac{\sum_{n=-\infty}^{\infty} D_{n-} e^{-\frac{D_{n-}}{\tau\varepsilon}}}{\varepsilon\sqrt{\tau^5}} d\tau \right| + \left| \int_1^{t/\varepsilon} \frac{\sum_{n=-\infty}^{\infty} D_{n+} e^{-\frac{D_{n+}}{\tau\varepsilon}}}{\varepsilon\sqrt{\tau^5}} d\tau \right| \right]. \end{aligned}$$

Next we estimate the integrals from the last relation. For $|y| \leq L$ and $|x| \leq L$ it is clear that

$$D_{n\pm} \leq (1 + |n|)^2 L^2. \tag{28}$$

Therefore,

$$\begin{aligned} \sum_{n=-\infty}^{\infty} D_{n-} e^{-\frac{D_{n-}}{\tau\varepsilon}} &= D_{0-} e^{-\frac{D_{0-}}{\tau\varepsilon}} + \sum_{n \neq 0, n = -\infty}^{\infty} D_{n-} e^{-\frac{D_{n-}}{\tau\varepsilon}} \\ &\leq D_{0-} e^{-\frac{D_{0-}}{\tau\varepsilon}} + 2L^2 e^{-\frac{L^2}{4\tau\varepsilon}} \sum_{n \geq 0} (n+2)^2 e^{-\frac{nL^2}{\tau\varepsilon}} \\ &\leq D_{0-} e^{-\frac{D_{0-}}{\tau\varepsilon}} + CL^2 e^{-\frac{L^2}{4\tau\varepsilon}} \left(1 - e^{-\frac{L^2}{T}}\right)^{-1}. \end{aligned}$$

where we use the relations

$$\begin{aligned} \sum_{n \geq 1} n^2 e^{-na} &= (e^{-a} + e^{-2a})(1 - e^{-a})^{-3} \leq 2e^{-a}(1 - e^{-a})^{-3} \leq 2(1 - e^{-a})^{-1}, \\ \sum_{n \geq 1} n e^{-na} &\leq \sum_{n \geq 1} n^2 e^{-na} \leq 2(1 - e^{-a})^{-1}, \quad a > 0, \end{aligned}$$

and estimate (26). Then, due to the fact that $\max_{D>0} De^{-\frac{D}{\tau\varepsilon}} = \tau\varepsilon e^{-1}$, we have

$$\sum_{n=-\infty}^{\infty} D_{n-} e^{-\frac{D_{n-}}{\tau\varepsilon}} \leq C\tau\varepsilon, \quad C = C(L, T). \tag{29}$$

Consequently,

$$\int_1^{t/\varepsilon} \frac{\sum_{n=-\infty}^{\infty} D_{n-} e^{-\frac{D_{n-}}{\tau\varepsilon}}}{\varepsilon\sqrt{\tau^5}} d\tau \leq C \int_1^{t/\varepsilon} \frac{1}{\sqrt{\tau^3}} d\tau \leq 2C.$$

In the same way we obtain

$$\begin{aligned} \sum_{n=-\infty}^{\infty} D_{n+} e^{-\frac{D_{n+}}{\tau\varepsilon}} &= D_{0+} e^{-\frac{D_{0+}}{\tau\varepsilon}} + D_{1+} e^{-\frac{D_{1+}}{\tau\varepsilon}} + \sum_{n \neq 0, 1, n=-\infty}^{\infty} D_{n+} e^{-\frac{D_{n+}}{\tau\varepsilon}} \\ &\leq 2\tau\varepsilon e^{-1} + L^2 \sum_{n \geq 1} (n+1)^2 e^{-\frac{(2n)^2 L^2}{4\tau\varepsilon}} + L^2 \sum_{n \geq 2} (n+1)^2 e^{-\frac{(2n-2)^2 L^2}{4\tau\varepsilon}} \\ &\leq C\tau\varepsilon (1 - e^{-\frac{L^2}{T}})^{-1} \leq C\tau\varepsilon, \end{aligned} \tag{30}$$

which yields

$$\int_1^{t/\varepsilon} \frac{\sum_{n=-\infty}^{\infty} D_{n+} e^{-\frac{D_{n+}}{\tau\varepsilon}}}{\varepsilon\sqrt{\tau^5}} d\tau \leq 2C.$$

Therefore, we get

$$|g(t, x, y)| \leq C(\sigma, L, T)\sqrt{\varepsilon} = C\sqrt{\varepsilon}$$

and so

$$\begin{aligned} |g(t, x, y+h) - g(t, x, y)| &\leq C\sqrt{\varepsilon}, \quad \|g(t, x, \cdot)\|_{L_2[0, L]} \leq C\sqrt{\varepsilon}, \\ w_{2, [0, L]}^2(g, r) &\leq C\varepsilon. \end{aligned} \tag{31}$$

Now we estimate $|g(t, x, y+h) - g(t, x, y)|$ and then $w_{2, [0, L]}^2(g, r)$ in terms of some positive power of h . Thus,

$$\begin{aligned} &|g(t, x, y+h) - g(t, x, y)| \\ &= \left| \int_0^t (G(t-s, x, y+h) - G(t-s, x, y)) [\sigma(s/\varepsilon, y+h) - \bar{\sigma}(y+h)] ds \right. \\ &\quad \left. + \int_0^t G(t-s, x, y) ([\sigma(s/\varepsilon, y+h) - \sigma(s/\varepsilon, y)] - [\bar{\sigma}(y+h) - \bar{\sigma}(y)]) ds \right| \\ &= |I_1 + I_2|, \end{aligned}$$

and then

$$\int_0^{L-h} |g(t, x, y + h) - g(t, x, y)|^2 dy \leq 2 \int_0^{L-h} I_1^2 dy + 2 \int_0^{L-h} I_2^2 dy. \quad (32)$$

By condition A3 and finiteness of $\sum_{n=-\infty}^{\infty} e^{-\frac{D_{n\pm}}{s}}$, we obtain

$$\begin{aligned} |I_2| &\leq C2L_\sigma h^{\beta(\sigma)} \int_0^t \frac{e^{-(t-s)}}{\sqrt{4\pi(t-s)}} ds \leq C2L_\sigma h^{\beta(\sigma)} \sqrt{t} \leq Ch^{\beta(\sigma)}, \\ \int_0^{L-h} I_2^2 dy &\leq Ch^{2\beta(\sigma)}. \end{aligned} \quad (33)$$

Denote

$$D_{n\pm}^h = \frac{(y + h \pm x - 2nL)^2}{4}.$$

Recall that here $h \in [0, L]$ and we consider the integrals in (32) such that $y + h \leq L$. So

$$\begin{aligned} |I_1| &= \left| \int_0^t \frac{e^{-(t-s)}}{\sqrt{4\pi(t-s)}} \sum_{n=-\infty}^{\infty} \left(e^{-\frac{D_{n-}^h}{t-s}} - e^{-\frac{D_{n-}}{t-s}} \right) [\sigma(s/\varepsilon, y + h) - \bar{\sigma}(y + h)] ds \right. \\ &\quad \left. + \int_0^t \frac{e^{-(t-s)}}{\sqrt{4\pi(t-s)}} \sum_{n=-\infty}^{\infty} \left(e^{-\frac{D_{n+}^h}{t-s}} - e^{-\frac{D_{n+}}{t-s}} \right) [\sigma(s/\varepsilon, y + h) - \bar{\sigma}(y + h)] ds \right| \\ &= |I_{1-} + I_{1+}|. \end{aligned}$$

Since $e^{-(t-s)} \leq 1$, for $t - s \geq 0$, we conclude by Assumption A2 that

$$\begin{aligned} |I_{1-}| &\leq 2C_\sigma \left| \int_0^t \frac{1}{\sqrt{4\pi(t-s)}} \sum_{n=-\infty}^{\infty} \left(e^{-\frac{D_{n-}^h}{t-s}} - e^{-\frac{D_{n-}}{t-s}} \right) ds \right| \\ &= 2C_\sigma \left| \sum_{n=0, \pm 1} \int_0^t \frac{1}{\sqrt{4\pi(t-s)}} \left(e^{-\frac{D_{n-}^h}{t-s}} - e^{-\frac{D_{n-}}{t-s}} \right) ds \right. \\ &\quad \left. + \int_0^t \frac{1}{\sqrt{4\pi(t-s)}} \sum_{|n| \geq 2} \left(e^{-\frac{D_{n-}^h}{t-s}} - e^{-\frac{D_{n-}}{t-s}} \right) ds \right|. \end{aligned}$$

For $n = 0, \pm 1$, the values of D_{n-}^h and D_{n-} are bounded by $4L^2$. Thus, the same reasoning as that used in obtaining the bound of I_2 in [22, inequality (13)] proves

$$\left| \int_0^t \frac{1}{\sqrt{4\pi(t-s)}} \left(e^{-\frac{D_{n-}^h}{t-s}} - e^{-\frac{D_{n-}}{t-s}} \right) ds \right| \leq C|h \ln h| \leq Ch^{\beta(\sigma)}.$$

Consider the second integral. We have

$$\sum_{|n| \geq 2} \left| e^{-\frac{D_{n-}^h}{t-s}} - e^{-\frac{D_{n-}}{t-s}} \right| \leq \sum_{|n| \geq 2} \frac{2h|y + \frac{h}{2} - x - 2nL|}{4(t-s)} e^{-\frac{\min(D_{n-}^h, D_{n-})}{t-s}},$$

where we use the following relation

$$|e^{-a} - e^{-b}| \leq e^{-\min\{a,b\}}|a - b|, \quad a, b \geq 0. \tag{34}$$

According to (28),

$$\left|y + \frac{h}{2} - x - 2nL\right| \leq 2(1 + |n|)^2L,$$

and analogously to (25) we obtain

$$e^{-\frac{\min\{D_{n-}^h, D_{n-}\}}{t-s}} \leq e^{-\frac{(2|n|-1)^2L^2}{4(t-s)}}, \quad |n| \geq 2.$$

Hence,

$$\begin{aligned} & \left| \int_0^t \frac{1}{\sqrt{4\pi(t-s)}} \sum_{|n| \geq 2} \left(e^{-\frac{D_{n-}^h}{t-s}} - e^{-\frac{D_{n-}}{t-s}} \right) ds \right| \\ & \leq Ch \int_0^t \frac{1}{4(t-s)^{3/2}} \sum_{n \geq 2} (1+n)^2 L^2 e^{-\frac{(2n-1)^2L^2}{4(t-s)}} ds. \end{aligned} \tag{35}$$

Then we use the same reasoning as in getting (29) and have

$$\sum_{n \geq 2} (1+n)^2 L^2 e^{-\frac{(2n-1)^2L^2}{4(t-s)}} \leq C4(t-s).$$

It follows that

$$\left| \int_0^t \frac{1}{\sqrt{4\pi(t-s)}} \sum_{|n| \geq 2} \left(e^{-\frac{D_{n-}^h}{t-s}} - e^{-\frac{D_{n-}}{t-s}} \right) ds \right| \leq Ch\sqrt{t} \leq Ch,$$

and finally that

$$|I_{1-}| \leq Ch^{\beta(\sigma)}.$$

The same estimate holds for term $|I_{1+}| \leq Ch^{\beta(\sigma)}$. Hence,

$$\int_0^{L-h} I_1^2 dy \leq Ch^{2\beta(\sigma)}.$$

Taking into account (33), we deduce that

$$\begin{aligned} \int_0^{L-h} |g(t, x, y+h) - g(t, x, y)|^2 dy & \leq Ch^{2\beta(\sigma)}, \\ w_{2,[0,L]}^2(g, r) & \leq Cr^{2\beta(\sigma)}. \end{aligned}$$

Raise the latter inequality to the power θ and multiply by inequality (31) raised to the power $1 - \theta$, for an arbitrary $\theta \in (0, 1)$. We have

$$w_{2,[0,L]}^2(g, r) \leq Cr^{2\theta\beta(\sigma)} \varepsilon^{1-\theta}.$$

Thus, for any $\gamma < \frac{1}{2} \left(1 - \frac{1}{2\beta(\sigma)}\right)$, there exists $\alpha < \theta\beta(\sigma)$ such that

$$\|g\|_{B_{22}^\alpha([b,c])} \leq C\sqrt{\varepsilon} + C\varepsilon^{(1-\theta)/2} \left(\int_0^t r^{2\theta\beta(\sigma)-2\alpha-1} dr\right)^{\frac{1}{2}} \leq C\varepsilon^\gamma.$$

Consequently, by relation (7),

$$\begin{aligned} |u_\varepsilon(t, x) - \bar{u}(t, x)| &= \left| \int_{[0,L]} g(t, x, y) d\mu(y) \right| \\ &\leq |g(0, z)\mu([0, L])| + C \|g(\cdot, z)\|_{B_{22}^\alpha([0,L])} \left\{ \sum_{n \geq 1} 2^{-n\varepsilon} \sum_{1 \leq k \leq 2^n} \left| \mu(\Delta_{kn}^{(L)}) \right|^2 \right\}^{\frac{1}{2}} \\ &\leq C\varepsilon^\gamma \left(|\mu([0, L])| + \left\{ \sum_{n \geq 1} 2^{-n\varepsilon} \sum_{1 \leq k \leq 2^n} \left| \mu(\Delta_{kn}^{(L)}) \right|^2 \right\}^{\frac{1}{2}} \right) \leq C(\omega)\varepsilon^\gamma, \end{aligned}$$

where the sum with the stochastic measure is finite in view of [21, Lemma 3.1], and the constant $C(\omega)$ does not depend on t, x and ε . □

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