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Long-time behavior of a nonautonomous stochastic predator–prey model with jumps

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Abstract The existence and uniqueness of a global positive solution is proven for the system of stochastic differential equations describing a nonautonomous stochastic predator–prey model with a modified version of the Leslie–Gower term and Holling-type II functional response disturbed by white noise, centered and noncentered Poisson noises. Sufficient conditions are obtained for stochastic ultimate boundedness, stochastic permanence, nonpersistence in the mean, weak persistence in the mean and extinction of a solution to the considered system.

Keywords Stochastic predator–prey model, Leslie–Gower and Holling-type II functional response, global solution, stochastic ultimate boundedness, stochastic permanence, extinction, nonpersistence in the mean, weak persistence in the mean

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1 Introduction

The deterministic predator–prey model with modified version of Leslie–Gower term and Holling-type II functional response is studied in [1]. This model has the form

$$\begin{aligned} dx_1(t) &= x_1(t)\left(a - bx_1(t) - \frac{cx_2(t)}{m_1 + x_1(t)}\right)dt, \\ dx_2(t) &= x_2(t)\left(r - \frac{fx_2(t)}{m_2 + x_1(t)}\right)dt, \end{aligned} \tag{1}$$

where $x_1(t)$ and $x_2(t)$ are the prey and predator population densities at time t, respectively. The positive constants a, b, c, r, f, m_1, m_2 are defined as follows: a is the growth rate of prey x_1; b measures the strength of competition among individuals of species x_1; c is the maximum value of the per capita reduction rate of x_1 due to x_2; m_1 and m_2 measure the extent to which the environment provides protection to the prey x_1 and to the predator x_2, respectively; r is the growth rate of the predator x_2, and f has a similar meaning to c. In [1] the authors study boundedness and global stability of the positive equilibrium of the model (1).

In the papers [6, 7, 9] the stochastic version of model (1) is considered in the form

$$\begin{aligned} dx_1(t) &= x_1(t)\left(a - bx_1(t) - \frac{cx_2(t)}{m_1 + x_1(t)}\right)dt + \alpha x_1(t)dw_1(t), \\ dx_2(t) &= x_2(t)\left(r - \frac{fx_2(t)}{m_2 + x_1(t)}\right)dt + \beta x_2(t)dw_2(t), \end{aligned} \tag{2}$$

where $w_1(t)$ and $w_2(t)$ are mutually independent Wiener processes in [6, 7], and processes $w_1(t)$, $w_2(t)$ are correlated in [9]. In [6] the authors proved that there is a unique positive solution to the system (2), obtaining the sufficient conditions for extinction and persistence in the mean of predator and prey. In [7] it is shown that, under appropriate conditions, there is a stationary distribution of the solution to the system (2) which is ergodic. In [9] the authors prove that the densities of the distributions of the solution to the system (2) can converge in L^1 to an invariant density or can converge weakly to a singular measure under appropriate conditions.

Population systems may suffer abrupt environmental perturbations, such as epidemics, fires, earthquakes, etc. So it is natural to introduce Poisson noises into the population model for describing such discontinuous systems.

In this paper, we consider the nonautonomous predator–prey model with modified version of the Leslie–Gower term and Holling-type II functional response, disturbed by white noise and jumps generated by centered and noncentered Poisson measures. So, we take into account not only "small" jumps, corresponding to the centered Poisson measure, but also the "large" jumps, corresponding to the noncentered Poisson measure. This model is driven by the system of stochastic differential equations

$$dx_i(t) = x_i(t)\left[a_i(t) - b_i(t)x_i(t) - \frac{c_i(t)x_2(t)}{m(t) + x_1(t)}\right]dt + \sigma_i(t)x_i(t)dw_i(t)$$

$$+\int_{\mathbb{R}} \gamma_i(t,z)x_i(t)\tilde{\nu}_1(dt,dz) + \int_{\mathbb{R}} \delta_i(t,z)x_i(t)\nu_2(dt,dz),$$

$$x_i(0) = x_{i0} > 0, \ i = 1, 2, \tag{3}$$

where $x_1(t)$ and $x_2(t)$ are the prey and predator population densities at time t, respectively, $b_2(t) \equiv 0$, $w_i(t), i = 1, 2$, are independent standard one-dimensional Wiener processes, $\nu_i(t, A), i = 1, 2$, are independent Poisson measures, which are independent on $w_i(t), i = 1, 2$, $\tilde{\nu}_1(t, A) = \nu_1(t, A) - t\Pi_1(A)$, $E[\nu_i(t, A)] = t\Pi_i(A)$, $i = 1, 2$, $\Pi_i(A), i = 1, 2$, are finite measures on the Borel sets A in $\mathbb{R}$.

To the best of our knowledge, there have been no papers devoted to the dynamical properties of the stochastic predator–prey model (3), even in the case of centered Poisson noise. It is worth noting that the impact of centered and noncentered Poisson noises to the stochastic nonautonomous logistic model and to the stochastic two-species mutualism model is studied in the papers [2–4].

In the following we will use the notations $X(t) = (x_1(t), x_2(t))$, $X_0 = (x_{10}, x_{20})$, $|X(t)| = \sqrt{x_1^2(t) + x_2^2(t)}$, $\mathbb{R}^2_+ = \{X \in \mathbb{R}^2 : \ x_1 > 0, x_2 > 0\}$,

$$\alpha_i(t) = a_i(t) + \int_{\mathbb{R}} \delta_i(t,z)\Pi_2(dz),$$

$$\beta_i(t) = \frac{\sigma_i^2(t)}{2} + \int_{\mathbb{R}} [\gamma_i(t,z) - \ln(1+\gamma_i(t,z))]\Pi_1(dz) - \int_{\mathbb{R}} \ln(1+\delta_i(t,z))\Pi_2(dz),$$

$i = 1, 2$. For bounded, continuous functions $f_i(t), t \in [0, +\infty)$, $i = 1, 2$, let us denote

$$f_{i\,\sup} = \sup_{t\geq 0} f_i(t), \ f_{i\,\inf} = \inf_{t\geq 0} f_i(t), \ i = 1, 2,$$

$$f_{\max} = \max\{f_{1\,\sup}, f_{2\,\sup}\}, \ f_{\min} = \min\{f_{1\,\inf}, f_{2\,\inf}\}.$$

We prove that the system (3) has a unique, positive, global (no explosion in a finite time) solution for any positive initial value, and that this solution is stochastically ultimately bounded. The sufficient conditions for stochastic permanence, nonpersistence in the mean, weak persistence in the mean and extinction of solution are derived.

The rest of this paper is organized as follows. In Section 2, we prove the existence of the unique global positive solution to the system (3) and derive some auxiliary results. In Section 3, we prove the stochastic ultimate boundedness of the solution to the system (3), obtainig conditions under which the solution is stochastically permanent. The sufficient conditions for nonpersistence in the mean, weak persistence in the mean and extinction of the solution are derived.

2 Existence of global solution and some auxiliary lemmas

Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a probability space, $w_i(t), i = 1, 2, t \geq 0$, are independent standard one-dimensional Wiener processes on $(\Omega, \mathcal{F}, \mathrm{P})$, and $\nu_i(t, A), i = 1, 2$, are independent Poisson measures defined on $(\Omega, \mathcal{F}, \mathrm{P})$ independent on $w_i(t), i = 1, 2$.

Here $\mathrm{E}[\nu_i(t, A)] = t\Pi_i(A)$, $i = 1, 2$, $\tilde{\nu}_i(t, A) = \nu_i(t, A) - t\Pi_i(A)$, $i = 1, 2$, $\Pi_i(\cdot)$, $i = 1, 2$, are finite measures on the Borel sets in $\mathbb{R}$. On the probability space $(\Omega, \mathcal{F}, \mathrm{P})$ we consider an increasing, right continuous family of complete sub-σ-algebras $\{\mathcal{F}_t\}_{t\geq 0}$, where $\mathcal{F}_t = \sigma\{w_i(s), \nu_i(s, A), s \leq t, i = 1, 2\}$.

We need the following assumption.

Assumption 1. It is assumed that $a_i(t)$, $b_1(t)$, $c_i(t)$, $\sigma_i(t)$, $\gamma_i(t, z)$, $\delta_i(t, z)$, $i = 1, 2$, $m(t)$ are bounded, continuous on t functions, $a_i(t) > 0$, $i = 1, 2$, $b_{1\,\inf} > 0$, $c_{i\,\inf} > 0$, $i = 1, 2$, $m_{\inf} = \inf_{t\geq 0} m(t) > 0$, and $\ln(1 + \gamma_i(t, z))$, $\ln(1 + \delta_i(t, z))$, $i = 1, 2$, are bounded, $\Pi_i(\mathbb{R}) < \infty$, $i = 1, 2$.

In what follows we will assume that Assumption 1 holds.

Theorem 1. *There exists a unique global solution $X(t)$ to the system (3) for any initial value $X(0) = X_0 \in \mathbb{R}^2_+$, and* $\mathrm{P}\{X(t) \in \mathbb{R}^2_+\} = 1$.

Proof. Let us consider the system of stochastic differential equations

$$d\xi_i(t) = \left[a_i(t) - b_i(t)\exp\{\xi_i(t)\} - \frac{c_i(t)\exp\{\xi_2(t)\}}{m(t) + \exp\{\xi_1(t)\}} - \beta_i(t)\right]dt$$
$$+\sigma_i(t)dw_i(t) + \int_{\mathbb{R}} \ln(1 + \gamma_i(t, z))\tilde{\nu}_1(dt, dz) + \int_{\mathbb{R}} \ln(1 + \delta_i(t, z))\tilde{\nu}_2(dt, dz),$$
$$v_i(0) = \ln x_{i0}, \ i = 1, 2. \quad (4)$$

The coefficients of the system (4) are locally Lipschitz continuous. So, for any initial value $(\xi_1(0), \xi_2(0))$ there exists a unique local solution $\Xi(t) = (\xi_1(t), \xi_2(t))$ on $[0, \tau_e)$, where $\sup_{t<\tau_e} |\Xi(t)| = +\infty$ (cf. Theorem 6, p. 246, [5]). Therefore, from the Itô formula we derive that the process $X(t) = (\exp\{\xi_1(t)\}, \exp\{\xi_2(t)\})$ is a unique, positive local solution to the system (3). To show this solution is global, we need to show that $\tau_e = +\infty$ a.s. Let $n_0 \in \mathbb{N}$ be sufficiently large for $x_{i0} \in [1/n_0, n_0]$, $i = 1, 2$. For any $n \geq n_0$ we define the stopping time

$$\tau_n = \inf\left\{t \in [0, \tau_e) : \ X(t) \notin \left(\frac{1}{n}, n\right) \times \left(\frac{1}{n}, n\right)\right\}.$$

It is easy to see that τ_n is increasing as $n \to +\infty$. Denote $\tau_\infty = \lim_{n\to\infty} \tau_n$, whence $\tau_\infty \leq \tau_e$ a.s. If we prove that $\tau_\infty = \infty$ a.s., then $\tau_e = \infty$ a.s. and $X(t) \in \mathbb{R}^2_+$ a.s. for all $t \in [0, +\infty)$. So we need to show that $\tau_\infty = \infty$ a.s. If it is not true, there are constants $T > 0$ and $\varepsilon \in (0, 1)$, such that $\mathrm{P}\{\tau_\infty < T\} > \varepsilon$. Hence, there is $n_1 \geq n_0$ such that

$$\mathrm{P}\{\tau_n < T\} > \varepsilon, \quad \forall n \geq n_1. \quad (5)$$

For the nonnegative function $V(X) = \sum_{i=1}^{2} k_i(x_i - 1 - \ln x_i)$, $X = (x_1, x_2)$, $x_i > 0$, $k_i > 0$, $i = 1, 2$, by the Itô formula we obtain

$$dV(X(t)) = \sum_{i=1}^{2} k_i\left\{(x_i(t) - 1)\left[a_i(t) - b_i(t)x_i(t) - \frac{c_i(t)x_2(t)}{m(t) + x_1(t)}\right]\right.$$

$$+\beta_i(t) + \int_{\mathbb{R}} \delta_i(t,z)x_i(t)\Pi_2(dz)\Big\}dt + \sum_{i=1}^{2} k_i\Big\{(x_i(t)-1)\sigma_i(t)dw_i(t)$$
$$+\int_{\mathbb{R}}[\gamma_i(t,z)x_i(t) - \ln(1+\gamma_i(t,z))]\tilde{\nu}_1(dt,dz)$$
$$+\int_{\mathbb{R}}[\delta_i(t,z)x_i(t) - \ln(1+\delta_i(t,z))]\tilde{\nu}_2(dt,dz)\Big\}. \tag{6}$$

Let us consider the function $f(t, x_1, x_2) = \phi(t, x_1) + \psi(t, x_1, x_2)$, $x_1 > 0$, $x_2 > 0$, where

$$\phi(t,x_1) = -k_1 b_1(t)x_1^2 + k_1\Big(\alpha_1(t) + b_1(t)\Big)x_1 + k_1\beta_1(t) + k_2\beta_2(t)$$
$$-k_1 a_1(t) - k_2 a_2(t),$$
$$\psi(t,x_1,x_2) = (m(t)+x_1)^{-1}\Big[-k_2 c_2(t)x_2^2 + \Big(k_2\alpha_2(t) - k_1 c_1(t)\Big)x_1 x_2$$
$$+\Big(k_2\alpha_2(t)m(t) + k_1 c_1(t) + k_2 c_2(t)\Big)x_2\Big].$$

Under Assumption 1 there is a constant $L_1(k_1, k_2) > 0$, such that

$$\phi(t,x_1) \le k_1\left[-b_{1\,\inf}x_1^2 + \left(\alpha_{1\,\sup} + b_{1\,\sup}\right)x_1\right] + \beta_{\max}(k_1+k_2) \le L_1(k_1,k_2).$$

If $\alpha_{2\,\sup} \le 0$, then for the function $\psi(t, x_1, x_2)$ we have

$$\psi(t,x_1,x_2) \le \frac{-k_2 c_{2\,\inf}x_2^2 + (k_1+k_2)c_{\max}x_2}{m(t)+x_1} \le L_2(k_1,k_2).$$

If $\alpha_{2\,\sup} > 0$, then for $k_2 = k_1\frac{c_{1\,\inf}}{\alpha_{2\,\sup}}$ there is a constant $L_3(k_1, k_2) > 0$, such that

$$\psi(t,x_1,x_2) \le \Big\{-k_2 c_{2\,\inf}x_2^2 + (k_2\alpha_{2\,\sup} - k_1 c_{1\,\inf})x_1x_2 + \Big[k_2\alpha_{2\,\sup}m_{\sup}$$
$$+(k_1+k_2)c_{\max}\Big]x_2\Big\}(m(t)+x_1)^{-1} = \frac{k_1}{m(t)+x_1}\Big\{-\frac{c_{1\,\inf}c_{2\,\inf}}{\alpha_{2\,\sup}}x_2^2$$
$$+\left[c_{1\,\inf}m_{\sup} + \left(1+\frac{c_{1\,\inf}}{\alpha_{2\,\sup}}\right)c_{\max}\right]x_2\Big\} \le L_3(k_1,k_2).$$

Therefore, there is a constant $L(k_1, k_2) > 0$, such that $f(t, x_1, x_2) \le L(k_1, k_2)$. So, from (6) we obtain by integrating

$$V(X(T\wedge\tau_n)) \le V(X_0) + L(k_1,k_2)(T\wedge\tau_n)$$
$$+\sum_{i=1}^{2} k_i\Big\{\int_0^{T\wedge\tau_n}(x_i(t)-1)\sigma_i(t)dw_i(t) + \int_0^{T\wedge\tau_n}\int_{\mathbb{R}}\Big[\gamma_i(t,z)x_i(t) - \ln(1$$

$$+\gamma_i(t,z))\Big]\tilde{\nu}_1(dt,dz) + \int_0^{T\wedge\tau_n}\int_{\mathbb{R}} [\delta_i(t,z)x_i(t) - \ln(1+\delta_i(t,z))]\tilde{\nu}_2(dt,dz)\Big\}. \quad (7)$$

Taking expectations we derive from (7)

$$\mathrm{E}\,[V(X(T\wedge\tau_n))] \le V(X_0) + L(k_1,k_2)T. \quad (8)$$

Set $\Omega_n = \{\tau_n \le T\}$ for $n \ge n_1$. Then by (5), $\mathrm{P}(\Omega_n) = \mathrm{P}\{\tau_n \le T\} > \varepsilon$, $\forall n \ge n_1$. Note that for every $\omega \in \Omega_n$ at least one of $x_1(\tau_n,\omega)$ and $x_2(\tau_n,\omega)$ equals either n or $1/n$. So

$$V(X(\tau_n)) \ge \min\{k_1,k_2\}\min\{n-1-\ln n, \frac{1}{n} - 1 + \ln n\}.$$

From (8) it follows

$$V(X_0) + L(k_1,k_2)T \ge \mathrm{E}[\mathbf{1}_{\Omega_n}V(X(\tau_n))]$$
$$\ge \varepsilon\min\{k_1,k_2\}\min\{n-1-\ln n, \frac{1}{n} - 1 + \ln n\},$$

where $\mathbf{1}_{\Omega_n}$ is the indicator function of Ω_n. Letting $n\to\infty$ leads to the contradiction $\infty > V(X_0) + L(k_1,k_2)T = \infty$. This completes the proof of the theorem. $\square$

Lemma 1. *The density of the prey population $x_1(t)$ obeys*

$$\limsup_{t\to\infty}\frac{\ln(m+x_1(t))}{t} \le 0,\ \forall m>0, \qquad a.s. \quad (9)$$

Proof. By the Itô formula for the process $e^t\ln(m+x_1(t))$ we have

$$e^t\ln(m+x_1(t)) - \ln(m+x_{10}) = \int_0^t e^s\Big\{\ln(m+x_1(s))$$
$$+\frac{x_1(s)}{m+x_1(s)}\Big[a_1(s) - b_1(s)x_1(s) - \frac{c_1(s)x_2(s)}{m(s)+x_1(s)}\Big] - \frac{\sigma_1^2(s)x_1^2(s)}{2(m+x_1(s))^2}$$
$$+\int_{\mathbb{R}}\Big[\ln\Big(1+\frac{\gamma_1(s,z)x_1(s)}{m+x_1(s)}\Big) - \frac{\gamma_1(s,z)x_1(s)}{m+x_1(s)}\Big]\Pi_1(dz)\Big\}ds$$
$$+\int_0^t e^s\frac{\sigma_1(s)x_1(s)}{m+x_1(s)}dw_1(s) + \int_0^t\int_{\mathbb{R}} e^s\ln\Big(1+\frac{\gamma_1(s,z)x_1(s)}{m+x_1(s)}\Big)\tilde{\nu}_1(ds,dz)$$
$$+\int_0^t\int_{\mathbb{R}} e^s\ln\Big(1+\frac{\delta_1(s,z)x_1(s)}{m+x_1(s)}\Big)\nu_2(ds,dz). \quad (10)$$

For $0<\kappa\le 1$, let us denote the process

$$\zeta_\kappa(t) = \int_0^t e^s\frac{\sigma_1(s)x_1(s)}{m+x_1(s)}dw_1(s) + \int_0^t\int_{\mathbb{R}} e^s\ln\Big(1+\frac{\gamma_1(s,z)x_1(s)}{m+x_1(s)}\Big)\tilde{\nu}_1(ds,dz)$$

$$+\int_0^t\int_{\mathbb{R}} e^s \ln\left(1+\frac{\delta_1(s,z)x_1(s)}{m+x_1(s)}\right)\nu_2(ds,dz)-\frac{\kappa}{2}\int_0^t e^{2s}\sigma_1^2(s)\left(\frac{x_1(s)}{m+x_1(s)}\right)^2 ds$$

$$-\frac{1}{\kappa}\int_0^t\int_{\mathbb{R}}\left[\left(1+\frac{\gamma_1(s,z)x_1(s)}{m+x_1(s)}\right)^{\kappa e^s}-1-\kappa e^s\ln\left(1+\frac{\gamma_1(s,z)x_1(s)}{m+x_1(s)}\right)\right]\Pi_1(dz)ds$$

$$-\frac{1}{\kappa}\int_0^t\int_{\mathbb{R}}\left[\left(1+\frac{\delta_1(s,z)x_1(s)}{m+x_1(s)}\right)^{\kappa e^s}-1\right]\Pi_2(dz)ds.$$

By virtue of the exponential inequality ([3], Lemma 2.2) for any $T>0$, $0<\kappa\le 1$, $\beta>0$, we have

$$\mathrm{P}\{\sup_{0\le t\le T}\zeta_\kappa(t)>\beta\}\le e^{-\kappa\beta}. \tag{11}$$

Choosing $T=k\tau$, $k\in\mathbb{N}$, $\tau>0$, $\kappa=\varepsilon e^{-k\tau}$, $\beta=\theta e^{k\tau}\varepsilon^{-1}\ln k$, $0<\varepsilon<1$, $\theta>1$, we get

$$\mathrm{P}\{\sup_{0\le t\le T}\zeta_\kappa(t)>\theta e^{k\tau}\varepsilon^{-1}\ln k\}\le\frac{1}{k^\theta}.$$

By the Borel–Cantelli lemma, for almost all $\omega\in\Omega$, there is a random integer $k_0(\omega)$, such that, for $\forall k\ge k_0(\omega)$ and $0\le t\le k\tau$, we have

$$\int_0^t e^s\frac{\sigma_1(s)x_1(s)}{m+x_1(s)}dw_1(s)+\int_0^t\int_{\mathbb{R}} e^s\ln\left(1+\frac{\gamma_1(s,z)x_1(s)}{m+x_1(s)}\right)\tilde{\nu}_1(ds,dz)$$

$$+\int_0^t\int_{\mathbb{R}} e^s\ln\left(1+\frac{\delta_1(s,z)x_1(s)}{m+x_1(s)}\right)\nu_2(ds,dz)\le$$

$$\frac{\varepsilon}{2e^{k\tau}}\int_0^t e^{2s}\left(\frac{\sigma_1(s)x_1(s)}{m+x_1(s)}\right)^2 ds+\frac{e^{k\tau}}{\varepsilon}\int_0^t\int_{\mathbb{R}}\left[\left(1+\frac{\gamma_1(s,z)x_1(s)}{m+x_1(s)}\right)^{\varepsilon e^{s-k\tau}}\right.$$

$$\left.-1-\varepsilon e^{s-k\tau}\ln\left(1+\frac{\gamma_1(s,z)x_1(s)}{m+x_1(s)}\right)\right]\Pi_1(dz)ds$$

$$+\frac{e^{k\tau}}{\varepsilon}\int_0^t\int_{\mathbb{R}}\left[\left(1+\frac{\delta_1(s,z)x_1(s)}{m+x_1(s)}\right)^{\varepsilon e^{s-k\tau}}-1\right]\Pi_2(dz)ds+\frac{\theta e^{k\tau}\ln k}{\varepsilon}. \tag{12}$$

By using the inequality $x^r\le 1+r(x-1)$, $\forall x\ge 0$, $0\le r\le 1$, with $x=1+\frac{\gamma_1(s,z)x_1(s)}{m+x_1(s)}$, $r=\varepsilon e^{s-k\tau}$, and then with $x=1+\frac{\delta_1(s,z)x_1(s)}{m+x_1(s)}$, $r=\varepsilon e^{s-k\tau}$, we derive the estimates

$$\frac{e^{k\tau}}{\varepsilon}\int_0^t\int_{\mathbb{R}}\left[\left(1+\frac{\gamma_1(s,z)x_1(s)}{m+x_1(s)}\right)^{\varepsilon e^{s-k\tau}}-1\right.$$

$$-\varepsilon e^{s-k\tau}\ln\left(1+\frac{\gamma_1(s,z)x_1(s)}{m+x_1(s)}\right)\Bigg]\Pi_1(dz)ds$$

$$\leq\int_0^t\int_{\mathbb{R}} e^s\left[\frac{\gamma_1(s,z)x_1(s)}{m+x_1(s)}-\ln\left(1+\frac{\gamma_1(s,z)x_1(s)}{m+x_1(s)}\right)\right]\Pi_1(dz)ds, \tag{13}$$

$$\frac{e^{k\tau}}{\varepsilon}\int_0^t\int_{\mathbb{R}}\left[\left(1+\frac{\delta_1(s,z)x_1(s)}{m+x_1(s)}\right)^{\varepsilon e^{s-k\tau}}-1\right]\Pi_2(dz)ds$$

$$\leq\int_0^t\int_{\mathbb{R}} e^s\frac{\delta_1(s,z)x_1(s)}{m+x_1(s)}\Pi_2(dz)ds. \tag{14}$$

From (10), by using (12)–(14) we get

$$e^t\ln(m+x_1(t))\leq\ln(m+x_{10})+\int_0^t e^s\Bigg\{\ln(m+x_1(s))$$

$$+\frac{x_1(s)}{m+x_1(s)}\left[a_1(s)-b_1(s)x_1(s)-\frac{c_1(s)x_2(s)}{m(s)+x_1(s)}\right]-\frac{\sigma_1^2(s)x_1^2(s)}{2(m+x_1(s))^2}$$

$$\times\left(1-\varepsilon e^{s-k\tau}\right)+\int_{\mathbb{R}}\frac{\delta_1(s,z)x_1(s)}{m+x_1(s)}\Pi_2(dz)\Bigg\}ds+\frac{\theta e^{k\tau}\ln k}{\varepsilon},\ \text{a.s.} \tag{15}$$

It is easy to see that, under Assumption 1, for any $x>0$ there exists a constant $L>0$ independent on k, s and x, such that

$$\ln(m+x)-\frac{x^2b_1(s)}{m+x}+\frac{x\alpha_1(s)}{m+x}\leq L.$$

So, from (15) for any $(k-1)\tau\leq t\leq k\tau$ we have (a.s.)

$$\frac{\ln(m+x_1(t))}{\ln t}\leq e^{-t}\frac{\ln(m+x_{10})}{\ln t}+\frac{L}{\ln t}(1-e^{-t})+\frac{\theta e^{k\tau}\ln k}{\varepsilon e^{(k-1)\tau}\ln(k-1)\tau}.$$

Therefore,

$$\limsup_{t\to\infty}\frac{\ln(m+x_1(t))}{\ln t}\leq\frac{\theta e^{\tau}}{\varepsilon},\ \forall\theta>1,\tau>0,\varepsilon\in(0,1),\quad \text{a.s.}$$

If $\theta\downarrow 1$, $\tau\downarrow 0$, $\varepsilon\uparrow 1$, we obtain

$$\limsup_{t\to\infty}\frac{\ln(m+x_1(t))}{\ln t}\leq 1,\quad \text{a.s.}$$

So,

$$\limsup_{t\to\infty}\frac{\ln(m+x_1(t))}{t}\leq 0,\quad \text{a.s.}$$

□

Corollary 1. *The density of the prey population $x_1(t)$ obeys*

$$\limsup_{t\to\infty} \frac{\ln x_1(t)}{t} \le 0, \quad a.s.$$

Lemma 2. *The density of the predator population $x_2(t)$ has the property that*

$$\limsup_{t\to\infty} \frac{\ln x_2(t)}{t} \le 0, \quad a.s.$$

Proof. Making use of the Itô formula we get

$$e^t \ln x_2(t) - \ln x_{20} = \int_0^t e^s \left\{ \ln x_2(s) + a_2(s) - \frac{c_2(s)x_2(s)}{m(s)+x_1(s)} - \frac{\sigma_2^2(s)}{2} + \int_{\mathbb{R}} \left[\ln\left(1+\gamma_2(s,z)\right) - \gamma_2(s,z) \right] \Pi_1(dz) \right\} ds + \psi(t), \tag{16}$$

where

$$\psi(t) = \int_0^t e^s \sigma_2(s) dw_2(s) + \int_0^t \int_{\mathbb{R}} e^s \ln\left(1+\gamma_2(s,z)\right) \tilde{\nu}_1(ds,dz) + \int_0^t \int_{\mathbb{R}} e^s \ln\left(1+\delta_2(s,z)\right) \nu_2(ds,dz).$$

By virtue of the exponential inequality (11) we have

$$\mathrm{P}\{\sup_{0\le t\le T} \zeta_\kappa(t) > \beta\} \le e^{-\kappa\beta}, \forall 0 < \kappa \le 1, \beta > 0,$$

where

$$\zeta_\kappa(t) = \psi(t) - \frac{\kappa}{2} \int_0^t e^{2s} \sigma_2^2(s) ds - \frac{1}{\kappa} \int_0^t \int_{\mathbb{R}} \Big[(1+\gamma_2(s,z))^{\kappa e^s} - 1 - \kappa e^s \ln(1+\gamma_2(s,z)) \Big] \Pi_1(dz) ds - \frac{1}{\kappa} \int_0^t \int_{\mathbb{R}} \left[(1+\delta_2(s,z))^{\kappa e^s} - 1 \right] \Pi_2(dz) ds.$$

Choosing $T = k\tau$, $k \in \mathbb{N}$, $\tau > 0$, $\kappa = e^{-k\tau}$, $\beta = \theta e^{k\tau} \ln k$, $\theta > 1$, we get

$$\mathrm{P}\{\sup_{0\le t\le T} \zeta_\kappa(t) > \theta e^{k\tau} \ln k\} \le \frac{1}{k^\theta}.$$

By the same arguments as in the proof of Lemma 1, using the Borel–Cantelli lemma, we derive from (16)

$$e^t \ln x_2(t) \le \ln x_{20} + \int_0^t e^s \left\{ \ln x_2(s) + a_2(s) - \frac{c_2(s)x_2(s)}{m(s)+x_1(s)} \right.$$

$$-\frac{\sigma_2^2(s)}{2}\left(1-e^{s-k\tau}\right)+\int_{\mathbb{R}}\delta_2(s,z)\Pi_2(dz)\Bigg\}ds+\theta e^{k\tau}\ln k,\quad \text{a.s.,} \tag{17}$$

for all sufficiently large $k \geq k_0(\omega)$ and $0 \leq t \leq k\tau$.

Using inequality $\ln x - cx \leq -\ln c - 1$, $\forall x \geq 0$, $c > 0$, with $x = x_2(s)$, $c = \frac{c_2(s)}{m(s)+x_1(s)}$, we derive from (17) the estimate

$$e^t \ln x_2(t) \leq \ln x_{20} + \int_0^t e^s \ln\Big(m_{\sup} + x_1(s)\Big)ds + L(e^t-1) + \theta e^{k\tau}\ln k,$$

for some constant $L > 0$.

So, for $(k-1)\tau \leq t \leq k\tau$, $k \geq k_0(\omega)$, we have

$$\limsup_{t\to\infty}\frac{\ln x_2(t)}{t} \leq \limsup_{t\to\infty}\frac{1}{t}\int_0^t e^{s-t}\ln\Big(m_{\sup}+x_1(s)\Big)ds \leq 0,$$

by virtue of Lemma 1. □

Lemma 3. *Let $p > 0$. Then for any initial value $x_{10} > 0$, the pth-moment of the prey population density $x_1(t)$ obeys*

$$\limsup_{t\to\infty} \mathrm{E}\left[x_1^p(t)\right] \leq K_1(p), \tag{18}$$

where $K_1(p) > 0$ is independent of x_{10}.

For any initial value $x_{20} > 0$, the expectation of the predator population density $x_2(t)$ obeys

$$\limsup_{t\to\infty} \mathrm{E}\left[x_2(t)\right] \leq K_2, \tag{19}$$

where $K_2 > 0$ is independent of x_{20}.

Proof. Let τ_n be the stopping time defined in Theorem 1. Applying the Itô formula to the process $V(t, x_1(t)) = e^t x_1^p(t)$, $p > 0$, we obtain

$$V(t\wedge\tau_n, x_1(t\wedge\tau_n)) = x_{10}^p + \int_0^{t\wedge\tau_n} e^s x_1^p(s)\Bigg\{1 + p\Bigg[a_1(s) - b_1(s)x_1(s)$$

$$-\frac{c_1(s)x_2(s)}{m(s)+x_1(s)}\Bigg] + \frac{p(p-1)\sigma_1^2(s)}{2} + \int_{\mathbb{R}}\left[(1+\gamma_1(s,z))^p - 1 - p\gamma_1(s,z)\right]\Pi_1(dz)$$

$$+\int_{\mathbb{R}}\left[(1+\delta_1(s,z))^p - 1\right]\Pi_2(dz)\Bigg\}ds + \int_0^{t\wedge\tau_n} pe^s x_1^p(s)\sigma_1(s)dw_1(s)$$

$$+\int_0^{t\wedge\tau_n}\int_{\mathbb{R}} e^s x_1^p(s)\left[(1+\gamma_1(s,z))^p-1\right]\tilde{\nu}_1(ds,dz)$$
$$+\int_0^{t\wedge\tau_n}\int_{\mathbb{R}} e^s x_1^p(s)\left[(1+\delta_1(s,z))^p-1\right]\tilde{\nu}_2(ds,dz). \tag{20}$$

Under Assumption 1 there is a constant $K_1(p)>0$, such that

$$e^s x_1^p\Bigg\{1+p\left[a_1(s)-b_1(s)x_1-\frac{c_1(s)x_2}{m(s)+x_1}\right]+\frac{p(p-1)\sigma_1^2(s)}{2}+$$
$$+\int_{\mathbb{R}}\left[(1+\gamma_1(s,z))^p-1-p\gamma_1(s,z)\right]\Pi_1(dz)+\int_{\mathbb{R}}\left[(1+\delta_1(s,z))^p-1\right]\Pi_2(dz)\Bigg\}$$
$$\le K_1(p)e^s. \tag{21}$$

From (20) and (21), taking expectations, we obtain

$$\mathrm{E}[V(t\wedge\tau_n,x_1(t\wedge\tau_n))]\le x_{10}^p+K_1(p)e^t.$$

Letting $n\to\infty$ leads to the estimate

$$e^t\mathrm{E}[x_1^p(t)]\le x_{10}^p+e^tK_1(p). \tag{22}$$

So from (22) we derive (18).

Let us prove the estimate (19). Applying the Itô formula to the process $U(t,X(t))=e^t[k_1x_1(t)+k_2x_2(t)]$, $k_i>0$, $i=1,2$, we obtain

$$dU(t,X(t))=e^t\Bigg\{k_1x_1(t)+k_2x_2(t)+k_1\Bigg[a_1(t)x_1(t)-b_1(t)x_1^2(t)$$
$$-\frac{c_1(t)x_1(t)x_2(t)}{m(t)+x_1(t)}\Bigg]+k_2\left[a_2(t)x_2(t)-\frac{c_2(t)x_2^2(t)}{m(t)+x_1(t)}\right]$$
$$+\sum_{i=1}^2 k_i\int_{\mathbb{R}} x_i(t)\delta_i(t,z)\Pi_2(dz)\Bigg\}dt+e^t\Bigg\{\sum_{i=1}^2 k_i\Bigg[x_i(t)\sigma_i(t)dw_i(t)$$
$$+\int_{\mathbb{R}} x_i(t)\gamma_i(t,z)\tilde{\nu}_1(dt,dz)+\int_{\mathbb{R}} x_i(t)\delta_i(t,z)\tilde{\nu}_2(dt,dz)\Bigg]\Bigg\}. \tag{23}$$

For the function

$$f(t,x_1,x_2)=\frac{1}{m(t)+x_1}\Bigg\{k_1\Big[-b_1(t)x_1^3+\Big(1+a_1(t)+\bar{\delta}_1(t)-b_1(t)m(t)\Big)x_1^2$$

$$+m(t)\Big(1+a_1(t)+\bar{\delta}_1(t)\Big)x_1\Big]+\Big[k_2\Big(1+a_2(t)+\bar{\delta}_2(t)\Big)-k_1c_1(t)\Big]x_1x_2$$
$$+k_2\Big[-c_2(t)x_2^2+m(t)\Big(1+a_2(t)+\bar{\delta}_2(t)\Big)x_2\Big]\Big\},$$
$$\text{where } \bar{\delta}_i(t)=\int_{\mathbb{R}}\delta_i(t,z)\Pi_2(dz),\ i=1,2,$$

we have

$$f(t,x_1,x_2)\le\frac{\phi_1(x_1,x_2)+\phi_2(x_2)}{m(t)+x_1},$$

where

$$\phi_1(x_1,x_2)=k_1\Big[-b_{1\,\inf}x_1^3+\Big(d_1-b_{1\,\inf}m_{\inf}\Big)x_1^2+m_{\sup}d_1x_1\Big]$$
$$+\Big[k_2d_2-k_1c_{1\,\inf}\Big]x_1x_2,$$
$$\phi_2(x_2)=k_2\Big[-c_{2\,\inf}x_2^2+m_{\sup}d_2x_2\Big],\ d_i=1+a_{i\,\sup}+|\bar{\delta}_i|_{\sup},\ i=1,2.$$

For $k_2=k_1c_{1\,\inf}/d_2$ there is a constant $L'>0$, such that $\phi_1(x_1,x_2)\le L'k_1$ and $\phi_2(x_2)\le L'k_1$. So, there is a constant $L>0$, such that

$$f(t,x_1,x_2)\le Lk_1. \tag{24}$$

From (23) and (24) by integrating and taking expectation, we derive

$$\mathrm{E}[U(t\wedge\tau_n,X(t\wedge\tau_n))]\le k_1\left[x_{10}+\frac{c_{1\,\inf}}{d_2}x_{20}+Le^t\right].$$

Letting $n\to\infty$ leads to the estimate

$$e^t\mathrm{E}\left[x_1(t)+\frac{c_{1\,\inf}}{d_2}x_2(t)\right]\le x_{10}+\frac{c_{1\,\inf}}{d_2}x_{20}+Le^t.$$

So,

$$\mathrm{E}[x_2(t)]\le\left(\frac{d_2}{c_{1\,\inf}}x_{10}+x_{20}\right)e^{-t}+\frac{d_2}{c_{1\,\inf}}L. \tag{25}$$

From (25) we have (19). □

Lemma 4. *If $p_{2\,\inf}>0$, where $p_2(t)=a_2(t)-\beta_2(t)$, then for any initial value $x_{20}>0$, the predator population density $x_2(t)$ satisfies*

$$\limsup_{t\to\infty}\mathrm{E}\left[\left(\frac{1}{x_2(t)}\right)^{\theta}\right]\le K(\theta),\ 0<\theta<1, \tag{26}$$

Proof. For the process $U(t) = 1/x_2(t)$ by the Itô formula we derive

$$U(t) = U(0) + \int_0^t U(s)\Big[\frac{c_2(s)x_2(s)}{m(s)+x_1(s)} - a_2(s) + \sigma_2^2(s) + \int_{\mathbb{R}} \frac{\gamma_2^2(s,z)}{1+\gamma_2(s,z)}\Pi_1(dz)\Big]ds - \int_0^t U(s)\sigma_2(s)dw_2(s)$$
$$- \int_0^t\int_{\mathbb{R}} U(s)\frac{\gamma_2(s,z)}{1+\gamma_2(s,z)}\tilde{\nu}_1(ds,dz) - \int_0^t\int_{\mathbb{R}} U(s)\frac{\delta_2(s,z)}{1+\delta_2(s,z)}\nu_2(ds,dz).$$

Then, by applying the Itô formula, we derive, for $0 < \theta < 1$,

$$(1+U(t))^\theta = (1+U(0))^\theta + \int_0^t \theta(1+U(s))^{\theta-2}\Big\{(1+U(s))U(s)$$
$$\times\Big[\frac{c_2(s)x_2(s)}{m(s)+x_1(s)} - a_2(s) + \sigma_2^2(s) + \int_{\mathbb{R}} \frac{\gamma_2^2(s,z)}{1+\gamma_2(s,z)}\Pi_1(dz)\Big]$$
$$+\frac{\theta-1}{2}U^2(s)\sigma_2^2(s)$$
$$+\frac{1}{\theta}\int_{\mathbb{R}}\Big[(1+U(s))^2\Big(\Big(\frac{1+U(s)+\gamma_2(s,z)}{(1+\gamma_2(s,z))(1+U(s))}\Big)^\theta - 1\Big)$$
$$+\theta(1+U(s))\frac{U(s)\gamma_2(s,z)}{1+\gamma_2(s,z)}\Big]\Pi_1(dz)$$
$$+\frac{1}{\theta}\int_{\mathbb{R}}(1+U(s))^2\Big[\Big(\frac{1+U(s)+\delta_2(s,z)}{(1+\delta_2(s,z))(1+U(s))}\Big)^\theta - 1\Big]\Pi_2(dz)\Big\}ds$$
$$-\int_0^t \theta(1+U(s))^{\theta-1}U(s)\sigma_2(s)dw_2(s)$$
$$+\int_0^t\int_{\mathbb{R}}\Big[\Big(1+\frac{U(s)}{1+\gamma_2(s,z)}\Big)^\theta - (1+U(s))^\theta\Big]\tilde{\nu}_1(ds,dz)$$
$$+\int_0^t\int_{\mathbb{R}}\Big[\Big(1+\frac{U(s)}{1+\delta_2(s,z)}\Big)^\theta - (1+U(s))^\theta\Big]\tilde{\nu}_2(ds,dz)$$
$$= (1+U(0))^\theta + \int_0^t \theta(1+U(s))^{\theta-2}J(s)ds$$
$$-I_{1,stoch}(t) + I_{2,stoch}(t) + I_{3,stoch}(t), \qquad (27)$$

where $I_{j,stoch}(t)$, $j=\overline{1,3}$, are the corresponding stochastic integrals in (27). Under Assumption 1 there exist constants $|K_1(\theta)|<\infty$, $|K_2(\theta)|<\infty$, such that for the process $J(t)$ we have the estimate

$$\begin{aligned}
J(t) \le (1+U(t))U(t)\Big[-a_2(t)+\frac{c_{2\sup}U^{-1}(t)}{m_{inf}}+\sigma_2^2(t) \\
+\int_{\mathbb{R}}\frac{\gamma_2^2(s,z)}{1+\gamma_2(s,z)}\Pi_1(dz)\Big]+\frac{\theta-1}{2}U^2(s)\sigma_2^2(s) \\
+\frac{1}{\theta}\int_{\mathbb{R}}\Big[(1+U(s))^2\Big(\Big(\frac{1}{1+\gamma_2(s,z)}+\frac{1}{1+U(s)}\Big)^\theta-1\Big) \\
+\theta(1+U(s))\frac{U(s)\gamma_2(s,z)}{1+\gamma_2(s,z)}\Big]\Pi_1(dz) \\
+\frac{1}{\theta}\int_{\mathbb{R}}(1+U(s))^2\Big[\Big(\frac{1}{1+\delta_2(s,z)}+\frac{1}{1+U(s)}\Big)^\theta-1\Big]\Pi_2(dz) \\
\le U^2(t)\Big[-a_2(t)+\frac{\sigma_2^2(t)}{2}+\int_{\mathbb{R}}\gamma_2(t,z)\Pi_1(dz)+\frac{\theta}{2}\sigma_2^2(t) \\
+\frac{1}{\theta}\int_{\mathbb{R}}[(1+\gamma_2(t,z))^{-\theta}-1]\Pi_1(dz)+\frac{1}{\theta}\int_{\mathbb{R}}[(1+\delta_2(t,z))^{-\theta}-1]\Pi_2(dz)\Big] \\
+K_1(\theta)U(t)+K_2(\theta)=-K_0(t,\theta)U^2(t)+K_1(\theta)U(t)+K_2(\theta),
\end{aligned}$$

where we used the inequality $(x+y)^\theta\le x^\theta+\theta x^{\theta-1}y$, $0<\theta<1$, $x,y>0$. Due to

$$\begin{aligned}
\lim_{\theta\to 0+}\Big[\frac{\theta}{2}\sigma_2^2(t)+\frac{1}{\theta}\int_{\mathbb{R}}[(1+\gamma_2(t,z))^{-\theta}-1]\Pi_1(dz) \\
+\frac{1}{\theta}\int_{\mathbb{R}}[(1+\delta_2(t,z))^{-\theta}-1]\Pi_2(dz)+\int_{\mathbb{R}}\ln(1+\gamma_2(t,z))\Pi_1(dz) \\
+\int_{\mathbb{R}}\ln(1+\delta_2(t,z))\Pi_2(dz)\Big]=\lim_{\theta\to 0+}\Delta(\theta)=0,
\end{aligned}$$

and the condition $p_{2\inf}>0$ we can choose a sufficiently small $0<\theta<1$ so that

$$K_0(\theta)=\inf_{t\ge 0}K_0(t,\theta)=\inf_{t\ge 0}[p_2(t)-\Delta(\theta)]=p_{2\inf}-\Delta(\theta)>0$$

is satisfied. So, from (27) and the estimate for $J(t)$ we derive

$$d\big[(1+U(t))^\theta\big]\le\theta(1+U(t))^{\theta-2}[-K_0(\theta)U^2(t)+K_1(\theta)U(t)+K_2(\theta)]dt$$

$$-\theta(1+U(t))^{\theta-1}U(t)\sigma_2(t)dw_2(t)+\int\limits_{\mathbb{R}}\Bigg[\bigg(1+\frac{U(t)}{1+\gamma_2(t,z)}\bigg)^{\theta}$$

$$-(1+U(t))^{\theta}\Bigg]\tilde{\nu}_1(dt,dz)+\int\limits_{\mathbb{R}}\Bigg[\bigg(1+\frac{U(t)}{1+\delta_2(t,z)}\bigg)^{\theta}-(1+U(t))^{\theta}\Bigg]\tilde{\nu}_2(dt,dz). \quad (28)$$

By the Itô formula and (28) we have

$$\begin{aligned} d\left[e^{\lambda t}(1+U(t))^{\theta}\right] &= \lambda e^{\lambda t}(1+U(t))^{\theta}dt+e^{\lambda t}d\left[(1+U(t))^{\theta}\right] \\ \le e^{\lambda t}\theta(1+U(t))^{\theta-2}\Bigg[&-\bigg(K_0(\theta)-\frac{\lambda}{\theta}\bigg)U^2(t)+\bigg(K_1(\theta)+\frac{2\lambda}{\theta}\bigg)U(t) \\ &+K_2(\theta)+\frac{\lambda}{\theta}\Bigg]dt-\theta e^{\lambda t}(1+U(t))^{\theta-1}U(t)\sigma_2(t)dw_2(t) \\ &+e^{\lambda t}\int\limits_{\mathbb{R}}\Bigg[\bigg(1+\frac{U(t)}{1+\gamma_2(t,z)}\bigg)^{\theta}-(1+U(t))^{\theta}\Bigg]\tilde{\nu}_1(dt,dz) \\ &+e^{\lambda t}\int\limits_{\mathbb{R}}\Bigg[\bigg(1+\frac{U(t)}{1+\delta_2(t,z)}\bigg)^{\theta}-(1+U(t))^{\theta}\Bigg]\tilde{\nu}_2(dt,dz). \end{aligned} \quad (29)$$

Let us choose $\lambda=\lambda(\theta)>0$, such that $K_0(\theta)-\lambda/\theta>0$. Then there is a constant $K>0$, such that

$$\begin{aligned} &(1+U(t))^{\theta-2}\Bigg[-\bigg(K_0(\theta)-\frac{\lambda}{\theta}\bigg)U^2(t) \\ &+\bigg(K_1(\theta)+\frac{2\lambda}{\theta}\bigg)U(t)+K_2(\theta)+\frac{\lambda}{\theta}\Bigg]\le K. \end{aligned} \quad (30)$$

Let τ_n be the stopping time defined in Theorem 1. Then by integrating (29), using (30) and taking the expectation we obtain

$$\mathrm{E}\left[e^{\lambda(t\wedge\tau_n)}(1+U(t\wedge\tau_n))^{\theta}\right]\le\bigg(1+\frac{1}{x_{20}}\bigg)^{\theta}+\frac{\theta}{\lambda}K\left(e^{\lambda t}-1\right).$$

Letting $n\to\infty$ leads to the estimate

$$e^{t}\mathrm{E}\left[(1+U(t))^{\theta}\right]\le\bigg(1+\frac{1}{x_{20}}\bigg)^{\theta}+\frac{\theta}{\lambda}K\left(e^{\lambda t}-1\right). \quad (31)$$

From (31) we obtain

$$\begin{aligned} \limsup_{t\to\infty}\mathrm{E}\left[\bigg(\frac{1}{x_2(t)}\bigg)^{\theta}\right] &= \limsup_{t\to\infty}\mathrm{E}\left[U^{\theta}(t)\right] \\ &\le \limsup_{t\to\infty}\mathrm{E}\left[(1+U(t))^{\theta}\right]\le\frac{\theta}{\lambda(\theta)}K, \end{aligned}$$

this implies (26). □

3 The long-time behaviour

Definition 1 ([8])**.** The solution $X(t)$ to the system (3) is said to be stochastically ultimately bounded, if for any $\varepsilon \in (0, 1)$, there is a positive constant $\chi = \chi(\varepsilon) > 0$, such that for any initial value $X_0 \in \mathbb{R}^2_+$, the solution to the system (3) has the property that

$$\limsup_{t\to\infty} \mathrm{P}\{|X(t)| > \chi\} < \varepsilon.$$

In what follows in this section we will assume that Assumption 1 holds.

Theorem 2. *The solution $X(t)$ to the system* (3) *with the initial value $X_0 \in \mathbb{R}^2_+$ is stochastically ultimately bounded.*

Proof. From Lemma 3 we have the estimate

$$\limsup_{t\to\infty} E[x_i(t)] \le K_i, \; i = 1, 2. \tag{32}$$

For $X = (x_1, x_2) \in \mathbb{R}^2_+$ we have $|X| \le x_1 + x_2$, therefore, from (32) $\limsup_{t\to\infty} E[|X(t)|] \le L = K_1 + K_2$. Let $\chi > L/\varepsilon$, $\forall\varepsilon \in (0, 1)$. Then applying the Chebyshev inequality yields

$$\limsup_{t\to\infty} \mathrm{P}\{|X(t)| > \chi\} \le \frac{1}{\chi} \limsup_{t\to\infty} E[|X(t)|] \le \frac{L}{\chi} < \varepsilon.$$

□

The property of stochastic permanence is important since it means the long-time survival in a population dynamics.

Definition 2. The population density $x(t)$ is said to be stochastically permanent if for any $\varepsilon > 0$, there are positive constants $H = H(\varepsilon)$, $h = h(\varepsilon)$ such that

$$\liminf_{t\to\infty} \mathrm{P}\{x(t) \le H\} \ge 1 - \varepsilon, \quad \liminf_{t\to\infty} \mathrm{P}\{x(t) \ge h\} \ge 1 - \varepsilon,$$

for any inial value $x_0 > 0$.

Theorem 3. *If $p_{2\,\mathrm{inf}} > 0$, where $p_2(t) = a_2(t) - \beta_2(t)$, then for any initial value $x_{20} > 0$, the predator population density $x_2(t)$ is stochastically permanent.*

Proof. From Lemma 3 we have estimate

$$\limsup_{t\to\infty} E[x_2(t)] \le K.$$

Thus for any given $\varepsilon > 0$, let $H = K/\varepsilon$, by virtue of Chebyshev's inequality, we can derive that

$$\limsup_{t\to\infty} \mathrm{P}\{x_2(t) \ge H\} \le \frac{1}{H} \limsup_{t\to\infty} E[x_2(t)] \le \varepsilon.$$

Consequently, $\liminf_{t\to\infty} \mathrm{P}\{x_2(t) \le H\} \ge 1 - \varepsilon$.

From Lemma 4 we have the estimate

$$\limsup_{t\to\infty} \mathrm{E}\left[\left(\frac{1}{x_2(t)}\right)^{\theta}\right] \le K(\theta),\ \ 0<\theta<1.$$

For any given $\varepsilon > 0$, let $h = (\varepsilon/K(\theta))^{1/\theta}$, then by Chebyshev's inequality, we have

$$\begin{aligned}\limsup_{t\to\infty} \mathrm{P}\{x_2(t) < h\} &\le \limsup_{t\to\infty} \mathrm{P}\left\{\left(\frac{1}{x_2(t)}\right)^{\theta} > h^{-\theta}\right\}\\ &\le h^{\theta}\limsup_{t\to\infty} \mathrm{E}\left[\left(\frac{1}{x_2(t)}\right)^{\theta}\right] \le \varepsilon.\end{aligned}$$

Consequently, $\liminf_{t\to\infty} \mathrm{P}\{x_2(t) \ge h\} \ge 1-\varepsilon$. □

Theorem 4. *If the predator is absent, i.e.* $x_2(t) = 0$ *a.s., and* $p_{1\,\mathrm{inf}} > 0$*, where* $p_1(t) = a_1(t) - \beta_1(t)$*, then for any initial value* $x_{10} > 0$*, the prey population density* $x_1(t)$ *is stochastically permanent.*

Proof. From Lemma 3 we have the estimate

$$\limsup_{t\to\infty} E[x_1(t)] \le K.$$

Thus for any given $\varepsilon > 0$, let $H = K/\varepsilon$, by virtue of Chebyshev's inequality, we can derive that

$$\limsup_{t\to\infty} \mathrm{P}\{x_1(t) \ge H\} \le \frac{1}{H}\limsup_{t\to\infty} E[x_1(t)] \le \varepsilon.$$

Consequently, $\liminf_{t\to\infty} \mathrm{P}\{x_1(t) \le H\} \ge 1-\varepsilon$.

For the process $U(t) = 1/x_1(t)$, by the Itô formula we have

$$\begin{aligned}U(t) = U(0) &+ \int_0^t U(s)\Big[b_1(s)x_1(s) - a_1(s) + \sigma_1^2(s)\\ &+ \int_{\mathbb{R}} \frac{\gamma_1^2(s,z)}{1+\gamma_1(s,z)}\Pi_1(dz)\Big]ds - \int_0^t U(s)\sigma_1(s)dw_1(s)\\ &- \int_0^t\!\!\int_{\mathbb{R}} U(s)\frac{\gamma_1(s,z)}{1+\gamma_1(s,z)}\tilde{\nu}_1(ds,dz) - \int_0^t\!\!\int_{\mathbb{R}} U(s)\frac{\delta_1(s,z)}{1+\delta_1(s,z)}\nu_2(ds,dz).\end{aligned}$$

Then, using the same arguments as in the proof of Lemma 4 we can derive the estimate

$$\limsup_{t\to\infty} \mathrm{E}\left[\left(\frac{1}{x_1(t)}\right)^{\theta}\right] \le K(\theta),\ \ 0<\theta<1.$$

For any given $\varepsilon > 0$, let $h = (\varepsilon/K(\theta))^{1/\theta}$. Then by Chebyshev's inequality, we have

$$\limsup_{t\to\infty} \mathrm{P}\{x_1(t) < h\} = \limsup_{t\to\infty} \mathrm{P}\left\{\left(\frac{1}{x_1(t)}\right)^{\theta} > h^{-\theta}\right\}$$
$$\leq h^{\theta} \limsup_{t\to\infty} \mathrm{E}\left[\left(\frac{1}{x_1(t)}\right)^{\theta}\right] \leq \varepsilon.$$

Consequently, $\liminf_{t\to\infty} \mathrm{P}\{x_1(t) \geq h\} \geq 1 - \varepsilon$. □

Remark 1. If the predator is absent, i.e. $x_2(t) = 0$ a.s., then the equation for the prey $x_1(t)$ has the logistic form. So, Theorem 4 gives us the sufficient conditions for the stochastic permanence of the solution to the stochastic nonautonomous logistic equation disturbed by white noise, centered and noncentered Poisson noises.

Definition 3. The solution $X(t) = (x_1(t), x_2(t))$, $t \geq 0$, to equation (3) will be said extinct if for every initial data $X_0 \in \mathbb{R}^2_+$, we have $\lim_{t\to\infty} x_i(t) = 0$ almost surely (a.s.), $i = 1, 2$.

Theorem 5. *If*

$$\bar{p}_i^* = \limsup_{t\to\infty} \frac{1}{t}\int_0^t p_i(s)ds < 0, \text{ where } p_i(t) = a_i(t) - \beta_i(t),\ i = 1, 2,$$

then the solution $X(t)$ to equation (3) with the initial condition $X_0 \in \mathbb{R}^2_+$ will be extinct.

Proof. By the Itô formula, we have

$$d\ln x_i(t) = \left[a_i(t) - b_i(t)x_i(t) - \frac{c_i(t)x_2(t)}{m(t) + x_1(t)} - \beta_i(t)\right]dt + dM_i(t)$$
$$\leq [a_i(t) - \beta_i(t)]dt + dM_i(t),\ i = 1, 2, \tag{33}$$

where the martingale

$$M_i(t) = \int_0^t \sigma_i(s)dw_i(s) + \int_0^t\int_{\mathbb{R}} \ln(1 + \gamma_i(s, z))\tilde{\nu}_1(ds, dz)$$
$$+ \int_0^t\int_{\mathbb{R}} \ln(1 + \delta_i(s, z))\tilde{\nu}_2(ds, dz),\ i = 1, 2, \tag{34}$$

has quadratic variation

$$\langle M_i, M_i\rangle(t) = \int_0^t \sigma_i^2(s)ds + \int_0^t\int_{\mathbb{R}} \ln^2(1 + \gamma_i(s, z))\Pi_1(dz)ds$$

$$+\int_0^t\int_{\mathbb{R}} \ln^2(1+\delta_i(s,z))\Pi_2(dz)ds \le Kt,\ i=1,2.$$

Then the strong law of large numbers for local martingales ([10]) yields $\lim_{t\to\infty} M_i(t)/t = 0$, $i=1,2$, a.s. Therefore, from (33) we obtain

$$\limsup_{t\to\infty} \frac{\ln x_i(t)}{t} \le \limsup_{t\to\infty} \frac{1}{t}\int_0^t p_i(s)ds < 0, \quad \text{a.s.}$$

So, $\lim_{t\to\infty} x_i(t) = 0$, $i=1,2$, a.s. □

Definition 4 ([11])**.** The population density $x(t)$ will be said nonpersistent in the mean if

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t x(s)ds = 0 \text{ a.s.}$$

Theorem 6. *If $\bar{p}_1^* = 0$, then the prey population density $x_1(t)$ with the initial condition $x_{10} > 0$ will be nonpersistent in the mean.*

Proof. From the first equality in (33) for $i=1$ we have

$$\ln x_1(t) \le \ln x_{10} + \int_0^t p_1(s)ds - b_{1\inf}\int_0^t x_1(s)ds + M_1(t), \tag{35}$$

where the martingale $M_1(t)$ is defined in (34). From the definition of $\bar{p}_1^*$ and the strong law of large numbers for $M_1(t)$ it follows, that $\forall\varepsilon > 0$, $\exists t_0 \ge 0$, and $\exists\Omega_\varepsilon \subset \Omega$, with $\mathrm{P}(\Omega_\varepsilon) \ge 1-\varepsilon$, such that

$$\frac{1}{t}\int_0^t p_1(s)ds \le \bar{p}_1^* + \frac{\varepsilon}{2},\ \frac{M_1(t)}{t} \le \frac{\varepsilon}{2},\ \forall t \ge t_0,\ \omega\in\Omega_\varepsilon.$$

So, from (35) we derive

$$\begin{aligned}\ln x_1(t) - \ln x_{10} &\le t(\bar{p}_1^* + \varepsilon) - b_{1\inf}\int_0^t x_1(s)ds \\ &= t\varepsilon - b_{1\inf}\int_0^t x_1(s)ds,\ \forall t\ge t_0,\ \omega\in\Omega_\varepsilon.\end{aligned} \tag{36}$$

Let $y_1(t) = \int_0^t x_1(s)ds$, then from (36) we have

$$\ln\left(\frac{dy_1(t)}{dt}\right) \le \varepsilon t - b_{1\inf}y_1(t) + \ln x_{10}$$

$$\Rightarrow e^{b_{1\inf} y_1(t)} \frac{dy_1(t)}{dt} \le x_{10} e^{\varepsilon t}, \forall t \ge t_0,\ \omega \in \Omega_\varepsilon.$$

By integrating the last inequality from t_0 to t we obtain

$$e^{b_{1\inf} y_1(t)} \le \frac{b_{1\inf} x_{10}}{\varepsilon} \left(e^{\varepsilon t} - e^{\varepsilon t_0}\right) + e^{b_{1\inf} y_1(t_0)},\ \forall t \ge t_0,\ \omega \in \Omega_\varepsilon.$$

So,

$$y_1(t) \le \frac{1}{b_{1\inf}} \ln \left[e^{b_{1\inf} y_1(t_0)} + \frac{b_{1\inf} x_{10}}{\varepsilon} \left(e^{\varepsilon t} - e^{\varepsilon t_0}\right)\right],\ \forall t \ge t_0,\ \omega \in \Omega_\varepsilon,$$

and therefore

$$\limsup_{t\to\infty} \frac{1}{t} \int_0^t x_1(s) ds \le \frac{\varepsilon}{b_{1\inf}},\ \forall \omega \in \Omega_\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary and $x_1(t) > 0$ a.s., we have

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t x_1(s) ds = 0\ a.s.$$

□

Theorem 7. *If $\bar{p}_2^* = 0$ and $\bar{p}_1^* < 0$, then the predator population density $x_2(t)$ with the initial condition $x_{20} > 0$ will be nonpersistent in the mean.*

Proof. From the first equality in (33) with $i = 2$ we have, for $c = c_{2\inf}/m_{\sup}$,

$$\ln x_2(t) \le \ln x_{20} + \int_0^t p_2(s) ds - c_{2\inf} \int_0^t \frac{x_2(s)}{m(s) + x_1(s)} ds + M_2(t)$$

$$= \ln x_{20} + \int_0^t p_2(s) ds - c_{2\inf} \int_0^t \frac{1}{m(s)} \left[x_2(s) - \frac{x_1(s) x_2(s)}{m(s) + x_1(s)} \right] ds + M_2(t)$$

$$\le \ln x_{20} + \int_0^t p_2(s) ds - c \int_0^t x_2(s) ds + c \int_0^t \frac{x_1(s) x_2(s)}{m_{\sup} + x_1(s)} ds + M_2(t), \qquad (37)$$

where the martingale $M_2(t)$ is defined in (34). From Theorem 5, the definition of $\bar{p}_2^*$ and the strong law of large numbers for $M_2(t)$ it follows, that $\forall \varepsilon > 0$, $\exists t_0 \ge 0$, and $\exists \Omega_\varepsilon \subset \Omega$ with $\mathrm{P}(\Omega_\varepsilon) \ge 1 - \varepsilon$, such that

$$\frac{1}{t} \int_0^t p_2(s) ds \le \bar{p}_2^* + \frac{\varepsilon}{2},\ \frac{M_2(t)}{t} \le \frac{\varepsilon}{2},\ \frac{x_1(t)}{m_{\sup} + x_1(t)} \le \varepsilon,\ \forall t \ge t_0,\ \omega \in \Omega_\varepsilon.$$

So, from (37) we derive

$$\ln x_2(t) - \ln x_{20} \le t(\bar{p}_2^* + \varepsilon) - c(1-\varepsilon) \int\limits_{t_0}^{t} x_2(s)ds$$
$$= t\varepsilon - c(1-\varepsilon) \int\limits_{t_0}^{t} x_2(s)ds, \forall t \ge t_0,\ \omega \in \Omega_\varepsilon. \tag{38}$$

Let $y_2(t) = \int_{t_0}^{t} x_2(s)ds$. Then from (38) we have

$$\ln\left(\frac{dy_2(t)}{dt}\right) \le \varepsilon t - c(1-\varepsilon)y_2(t) + \ln x_{20}$$
$$\Rightarrow e^{c(1-\varepsilon)y_2(t)} \frac{dy_2(t)}{dt} \le x_{20}e^{\varepsilon t}, \forall t \ge t_0,\ \omega \in \Omega_\varepsilon.$$

By integrating the last inequality from t_0 to t we obtain

$$e^{c(1-\varepsilon)y_2(t)} \le \frac{c(1-\varepsilon)x_{20}}{\varepsilon}\left(e^{\varepsilon t} - e^{\varepsilon t_0}\right) + 1,\ \forall t \ge t_0,\ \omega \in \Omega_\varepsilon.$$

So,

$$y_2(t) \le \frac{1}{c(1-\varepsilon)} \ln\left[1 + \frac{c(1-\varepsilon)x_{20}}{\varepsilon}\left(e^{\varepsilon t} - e^{\varepsilon t_0}\right)\right],\ \forall t \ge t_0,\ \omega \in \Omega_\varepsilon,$$

and therefore

$$\limsup_{t\to\infty} \frac{1}{t} \int\limits_0^t x_2(s)ds \le \frac{\varepsilon}{c(1-\varepsilon)},\ \forall \omega \in \Omega_\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary and $x_2(t) > 0$ a.s., we have

$$\lim_{t\to\infty} \frac{1}{t} \int\limits_0^t x_2(s)ds = 0\ a.s.$$

□

Definition 5 ([11])**.** The population density $x(t)$ will be said weakly persistent in the mean if

$$\bar{x}^* = \limsup_{t\to\infty} \frac{1}{t} \int_0^t x(s)ds > 0 \text{ a.s.}$$

Theorem 8. *If $\bar{p}_2^* > 0$, then the predator population density $x_2(t)$ with the initial condition $x_{20} > 0$ will be weakly persistent in the mean.*

Proof. If the assertion of theorem is not true, then $\mathrm{P}\{\bar{x}_2^* = 0\} > 0$. From the first equality in (33) we get

$$\frac{1}{t}(\ln x_2(t) - \ln x_{20}) = \frac{1}{t}\int_0^t p_2(s)ds - \frac{1}{t}\int_0^t \frac{c_2(s)x_2(s)}{m(s)+x_1(s)}ds + \frac{M_2(t)}{t}$$
$$\geq \frac{1}{t}\int_0^t p_2(s)ds - \frac{c_{2\,\sup}}{m_{\inf}t}\int_0^t x_2(s)ds + \frac{M_2(t)}{t},$$

where the martingale $M_2(t)$ is defined in (34). For $\forall\omega \in \{\omega \in \Omega|\ \bar{x}_2^* = 0\}$ in virtue of the strong law of large numbers for the martingale $M_2(t)$ we have

$$\limsup_{t\to\infty} \frac{\ln x_2(t)}{t} \geq \bar{p}_2^* > 0.$$

Therefore,

$$\mathrm{P}\left\{\omega \in \Omega|\ \limsup_{t\to\infty} \frac{\ln x_2(t)}{t} > 0\right\} > 0.$$

But from Lemma 2 we have

$$\mathrm{P}\left\{\omega \in \Omega|\ \limsup_{t\to\infty} \frac{\ln x_2(t)}{t} \leq 0\right\} = 1.$$

This is a contradiction. □

Theorem 9. *If $\bar{p}_1^* > 0$ and $\bar{p}_2^* < 0$, then the prey population density $x_1(t)$ with the initial condition $x_{10} > 0$ will be weakly persistent in the mean.*

Proof. Let $\mathrm{P}\{\bar{x}_1^* = 0\} > 0$. From the first equality in (33) with $i = 1$ we get

$$\frac{1}{t}(\ln x_1(t) - \ln x_{10}) = \frac{1}{t}\int_0^t p_1(s)ds - \frac{1}{t}\int_0^t b_1(s)x_1(s)ds$$
$$-\frac{1}{t}\int_0^t \frac{c_1(s)x_2(s)}{m(s)+x_1(s)}ds + \frac{M_1(t)}{t}$$
$$\geq \frac{1}{t}\int_0^t p_1(s)ds - \frac{b_{1\,\sup}}{t}\int_0^t x_1(s)ds - \frac{c_{1\,\sup}}{m_{\inf}t}\int_0^t x_2(s)ds + \frac{M_1(t)}{t} \quad (39)$$

where the martingale $M_1(t)$ is defined in (34). From the definition of $\bar{p}_1^*$, the strong law of large numbers for the martingale $M_1(t)$ and Theorem 2 for $x_2(t)$, we have $\forall\varepsilon > 0, \exists t_0 \geq 0, \exists\Omega_\varepsilon \subset \Omega$ with $\mathrm{P}(\Omega_\varepsilon) \geq 1 - \varepsilon$, such that

$$\frac{1}{t}\int_0^t p_1(s)ds \geq \bar{p}_1^* - \frac{\varepsilon}{3},\ \frac{M_1(t)}{t} \geq -\frac{\varepsilon}{3},\ \frac{1}{t}\int_0^t x_2(s)ds \leq \frac{\varepsilon m_{\inf}}{3c_{1\,\sup}}, \forall t \geq t_0, \omega \in \Omega_\varepsilon.$$

So, from (39) we get for $\omega \in \{\omega \in \Omega|\bar{x}_1^* = 0\} \cap \Omega_\varepsilon$

$$\limsup_{t\to\infty} \frac{\ln x_1(t)}{t} \geq \bar{p}_1^* - \varepsilon > 0$$

for a sufficiently small $\varepsilon > 0$. Therefore,

$$\mathrm{P}\left\{\omega \in \Omega \mid \limsup_{t\to\infty} \frac{\ln x_1(t)}{t} > 0\right\} > 0.$$

But from Corollary 1

$$\mathrm{P}\left\{\omega \in \Omega \mid \limsup_{t\to\infty} \frac{\ln x_1(t)}{t} \le 0\right\} = 1.$$

Therefore we have a contradiction. □

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