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Existence and uniqueness of weak solution to a three-dimensional stochastic modified-Leray-alpha model of fluid turbulence

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Abstract In this paper, we study the stochastic three-dimensional modified Leray-alpha model arising from the turbulent flows of fluids. We prove the existence of the probabilistic weak solution under the non-Lipschitz condition for the nonlinear forcing terms. We also discuss its uniqueness.

Keywords Stochastic modified-Leray-alpha model, existence, uniqueness, weak probabilistic solution
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1 Introduction

In this paper, we focus on the study of the probabilistic weak solution to the following three-dimensional modified Leray-alpha model in the periodic box $\mathcal{T} = [0, 2\pi L]^3$,

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$L > 0$:

$$\begin{cases} \partial_t(u^\alpha - \alpha^2 \Delta u^\alpha) - \nu \Delta(u^\alpha - \alpha^2 \Delta u^\alpha) + ((u^\alpha - \alpha^2 \Delta u^\alpha) \cdot \nabla) u^\alpha \\ = -\nabla p^\alpha + f(t, u^\alpha) + g(t, u^\alpha) \dfrac{dW}{dt}, \quad \text{in } \mathcal{T} \times [0, T] \\ \nabla \cdot u^\alpha = 0, \quad \text{in } \mathcal{T} \times [0, T] \\ u^\alpha(x, 0) = u_0(x), \quad \text{in } \mathcal{T} \\ u^\alpha = u^\alpha(x, t) \quad \text{is periodic in } \mathcal{T}, \quad \displaystyle\int_{\mathcal{T}} u^\alpha dx = 0, \end{cases} \tag{1}$$

where the vector $u^\alpha = (u_1^\alpha, u_2^\alpha, u_3^\alpha)$ is the unknown fluid velocity vector, the scalar p^α is the unknown pressure, the constant $\nu > 0$ is the kinematic viscosity and the real number $\alpha > 0$ is a given parameter. The vectors $f(t, u^\alpha)$ and $g(t, u^\alpha)\frac{dW}{dt}$ are external forces, where W is an $\mathbb{R}^m$-valued standard Wiener process. The initial velocity is given by u_0. The time $T > 0$ is the final time.

Alpha models are known to be a good regularisation of the three-dimensional Navier-Stokes equations. For a physical motivations, model derivation, and analytical and numerical aspects of these models, readers can consult [6–8, 21, 24]. Specifically, the deterministic three-dimensional modified Leray-alpha equations were studied in [12], where the global well-posedness was shown, and it was explained that this model can provide a reliable, computationally sound analytical subgrid large eddy simulation model of turbulence. The study of the stochastic incompressible Navier-Stokes equations driven by a white noise dates back to the early 1970s with the pioneering work [3]. For the stochastic versions of the alpha models, we refer the reader to [5, 10, 11]. In [5], the authors proved the existence and uniqueness of the variational solution to the three-dimensional stochastic Navier-Stokes-alpha (NS-α) model equations in a bounded domain, with Lipschitz assumptions on the nonlinear forcing terms. In [10], the authors ameliorated the result of [5] by avoiding the Lipschitz conditions on the nonlinear forcing terms when proving existence. Furthermore, the authors of [11] obtained the existence of a probabilistic weak solution for the stochastic version of the three-dimensional Bardina model arising from the turbulent flows of fluids with non-Lipschitz conditions. A stochastic three-dimensional inviscid simplified Bardina turbulence model was recently investigated, in [17], and the existence of a global strong and pathwise solution was proven in a periodic domain.

In [23], we established the exponential mixing and ergodic theorems for a stochastic damped nonlinear quintic wave equation driven by a localised space-time noise. Our aim in this paper is to show that the deterministic model in the case of a periodic domain introduced in [12] is reasonable, in the sense that, when some stochastic terms are present in the model, we can suggest a stochastic version with an accurate mathematical setting, yielding the existence and uniqueness of weak solutions to the problem. As is known in the fluid mechanics literature, the challenge is to handle the nonlinear term. The main difference with the publications cited above is the structure of the nonlinearity we consider, especially in a stochastic framework. Our nonlinear term generates new difficulties to be overcome and estimates to be established. To the best of our knowledge, this paper is the first work dealing with the existence and uniqueness of solution to a three-dimensional stochastic modified Leray-alpha subgrid scale model of fluid turbulence. To prove the existence of a weak solution in a periodic domain, in Theorem 2.2, we put the system in an abstract form. Then, using the Galerkin method, we construct an approximating solution. Next, we establish a priori estimates on the approximating solution that allow us to prove the compactness properties of the corresponding probability measures. Thereafter, we use Prokhorov's criterion and Skorokhod's theorem to prove the existence of a subsequence that converges strongly as the approximating parameter goes to infinity. For the proof of the existence of the pressure, we use a generalisation of Rham's theorem [15]. Finally, we will prove the pathwise uniqueness of the weak solution to establish the uniqueness of the weak solution in Theorem 2.3. Naturally, the uniqueness property needs hold with the Lipschitz condition of the external nonlinear terms.

The paper is organised as follows. Section 2 is devoted to stating our problem, the functional settings and the main results. The proofs of our main results are included in Section 3. The convergence results of the unique weak solution of the three-dimensional modified Leray-alpha model as the regularising parameter alpha vanishes was addressed in [22].

2 Functional setting and main results

2.1 Functional setting

Before presenting our main results, we introduce some functional settings.

- We set $\mathcal{V} = \Big\{\phi, \phi$ to be a trigonometric polynomial with period $2\pi L$, such that $\nabla \cdot \phi = 0$ and $\int_{\mathcal{T}} \phi(x)dx = 0\Big\}$.

- Let H and V be the closure of the set $\mathcal{V}$ in the space $(L^2(\mathcal{T}))^3$ and $(H^1(\mathcal{T}))^3$, respectively. As mentioned in [9], we denote by $|\cdot|$ and $(\cdot,\cdot)$ the associated norm and the inner product, respectively, of $(L^2(\mathcal{T}))^3$ and by $\|\cdot\|$ and $((\cdot,\cdot)) = (\nabla\cdot, \nabla\cdot)$ the $(H^1(\mathcal{T}))^3$ associated norm and inner product, respectively. We note that H is a Hilbert space equipped with the inner product of $(L^2(\mathcal{T}))^3$ and V is a Hilbert space for the scalar product $((\cdot,\cdot))_V = (\cdot,\cdot) + \alpha^2(\nabla\cdot, \nabla\cdot)$. Its associate norm, denoted by $\|\cdot\|_V$, is equivalent to the usual H^1-norm. Indeed, we can deduce from the definition of $\|\cdot\|_V$ and Poincaré inequality [16, Chapter 5] that
$$(L^{-2} + \alpha^2)^{-1}\|v\|_V \le \|v\| \le \alpha^{-2}\|v\|_V, \quad v \in V. \tag{2}$$

- The orthogonal projection of $(L^2(\mathcal{T}))^3$ onto H (called the Helmholtz-Leray projection) is denoted by $P : (L^2(\mathcal{T}))^3 \to H$.

- Let the operator $A = -P\Delta$ be the Stokes operator with the domain $D(A) = (H^2(\mathcal{T}))^3 \cap V$. In the space-periodic case, for all $u \in D(A)$, we have $Au = -P\Delta u = -\Delta u$. Throughout, we will denote by Y' the dual of any topological space Y and by $\langle\cdot,\cdot\rangle_{Y'}$ the duality between Y and Y'. Moreover, as mentioned in [9, 26], the operator A is an isomorphism from V to V', and it is a self-adjoint, positive, and compact operator on H. Hence, the space H has an orthonormal basis $\{e_j\}_{j\ge1}$ of eigenfunctions of A, such that $Ae_j = \lambda_j e_j$, where $L^{-2} = \lambda_1 \le \lambda_2 \le \cdots \lambda_j \ldots$ are the eigenvalues of A, repeated according to their multiplicities. Furthermore, from [9, 26], we have $D(A^\rho)' = D(A^{-\rho})$, for any $\rho > 0$ and

$$\|v\|_{D(A^\rho)} = |A^\rho v|, \quad v \in V, \quad \rho \in \mathbb{R}. \tag{3}$$

By the Riesz representation, we will identify H with its dual, and we will consider the chain of inclusions

$$D(A) \subset V \subset H \equiv H' \subset V' \subset D(A)', \tag{4}$$

where each space is densely and compactly embedded into the next one. Next, let I denote the identity operator in H. Recall that $(I + \alpha^2 A)^{-1}$ is an isomorphism from H onto $D(A)$, and for all $\phi \in H$, $\varphi \in V$, we have

$$(((I + \alpha^2 A)^{-1}\phi, \varphi))_V = (\phi, \varphi), \tag{5}$$

$$\|(I + \alpha^2 A)^{-1}\phi\|_V \leq |\phi|. \tag{6}$$

- It follows from the Poincaré inequality [16] that

$$|v|^2 \leq \lambda_1^{-1}\|v\|^2, \qquad v \in V. \tag{7}$$

- Hereafter, C denotes some uniform constant that is independent of the parameter α in the equations.

- For any $v_1, v_2 \in V$, we define the standard bilinear form associated with the Navier-Stokes equation.

$$B(v_1, v_2) = P((v_1 \cdot \nabla)v_2). \tag{8}$$

In particular, for every $v_1, v_2, v_3 \in V$, the bilinear form $B : V \times V \to V'$ is continuous and satisfies the following estimates [9, 26].

$$|\langle B(v_1, v_2), v_3\rangle_{V'}| \leq c|v_1|^{1/2}\|v_1\|^{1/2}\|v_2\|\|v_3\|. \tag{9}$$

Moreover,

$$\langle B(v_1, v_2), v_3\rangle_{V'} = -\langle B(v_1, v_3), v_2\rangle_{V'}, \tag{10}$$

and in particular,

$$\langle B(v_1, v_2), v_2\rangle_{V'} = 0. \tag{11}$$

For any $v_1 \in H$, $v_2 \in V$ and $v_3 \in D(A)$, we have

$$|\langle B(v_1, v_2), v_3\rangle_{D(A)'}| \leq c|v_1|\|v_2\|\|v_3\|^{1/2}|Av_3|^{1/2}, \tag{12}$$

and therefore

$$\|B(v_1, v_2)\|_{D(A)'} \leq c|v_1|\|v_2\|. \tag{13}$$

Applying the Helmholtz-Leray orthogonal projection P to equation (1), we obtain the following equivalent abstract stochastic evolution equation:

$$\begin{cases} dv^\alpha + [\nu A v^\alpha + B(v^\alpha, u^\alpha)]dt = f(t, u^\alpha)dt + g(t, u^\alpha)dW \\ v^\alpha = u^\alpha + \alpha^2 A u^\alpha \\ u^\alpha(0) = u_0, \end{cases} \tag{14}$$

where $\nabla \cdot u^\alpha = 0$, $\int_{\mathcal{T}} u^\alpha \, dx = 0$ and f and g are two nonlinear operators such that

- $f : (0, T) \times H \to H$ is measurable, almost every (a.e.) t. Moreover, $u \mapsto f(t, u)$ is continuous from H to H, and we have a.e.

$$|f(t, u)| \leq C(1 + |u|). \tag{15}$$

- $g : (0, T) \times H \to (L^2(\mathcal{T}))^{3m}$ is measurable, a.e. t. Moreover, $u \mapsto g(t, u)$ is continuous from H to $(L^2(\mathcal{T}))^{3m}$, and we have a.e.

$$|g(t, u)|_{\left(L^2(\mathcal{T})\right)^{3m}} \leq C(1 + |u|). \tag{16}$$

Furthermore, we assume that $Pf = f$. We always can do so due to the modification of the pressure p in such a way that it includes the gradient part of f. Using the Poincaré inequality and the second equation of (14), we have

$$|v^\alpha| \leq (\lambda_1^{-1} + \alpha^2)|Au^\alpha| = (L^2 + \alpha^2)|Au^\alpha|. \tag{17}$$

Now, we introduce some probabilistic functional spaces.

- Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and $\{\mathcal{F}_t\}_{0\leq t\leq T}$ be an increasing and right-continuous family of sub-σ-algebras of $\mathcal{F}$ such that $\mathcal{F}_0$ contains all the $\mathbb{P}$-null sets of $\mathcal{F}$. Let W be an $\mathbb{R}^m$-valued Wiener process on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t\leq T}, \mathbb{P})$.

- Let X be a Banach space. For any r, $p \geq 1$, we denote by the functional space $L^p(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t\leq T}, \mathbb{P}, L^r((0, T), X))$ the set of functions $u = u(x, t, \omega)$ with values in X defined on $\mathcal{T} \times (0, T) \times \Omega$, such that u is measurable with respect to (t, ω) and for almost all t, u is $\mathcal{F}_t$-measurable. Moreover, we have

$$\|u\|_{L^p(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{0\leq t\leq T},\mathbb{P},L^r((0,T),X))} = \Big[\mathbb{E}\Big(\int_0^T \|u\|_X^r dt\Big)^{p/r}\Big]^{1/r} < \infty,$$

where $\mathbb{E}$ is the mathematical expectation with respect to the probability measure $\mathbb{P}$. The space $L^p(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t\leq T}, \mathbb{P}, L^r((0, T), X))$ is a Banach space. If $r = \infty$, then $L^p(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t\leq T}, \mathbb{P}, L^\infty((0, T), X))$ is endowed with the norm

$$\|u\|_{L^p(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{0\leq t\leq T},\mathbb{P},L^\infty((0,T),X))} = \Big(\mathbb{E} \sup_{0\leq t\leq T} \mathrm{ess}\, \|u\|_X^p\Big)^{1/p}.$$

Finally, we recall Vitali's convergence theorem [20], which is crucial in the proof of our existence result.

Theorem 2.1. *Suppose that $(\varphi_k)_{k\geq 1}$ is a sequence of real integrable functions on Ω. Let φ be a real function on Ω, such that $\varphi_k \longrightarrow \varphi$ a.e.as $k \to \infty$. Then, the following insertions are equivalent*

(i) (φ_k) *is uniformly integrable.*

(ii) $\varphi \in L^1(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{R})$ *and* $\varphi_k \longrightarrow \varphi$ *in* $L^1(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{R})$ *as* $k \to \infty$.

2.2 Main results

The meaning of a weak solution to (1) is understood as follows.

Definition 2.1. A weak solution to (1) is a system $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P}, W, u^\alpha)$, where

(i) $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ is a filtered probability space, where $\{\mathcal{F}_t\}_{0\le t\le T}$ is an increasing and right-continuous family of sub σ-algebra of $\mathcal{F}$.

(ii) W is an m-dimensional $\mathcal{F}_t$-standard Wiener process.

(iii) $u^\alpha(t)$ is $\mathcal{F}_t$-adapted for all $t \in [0, T]$, and for all $1 \le p < \infty$, u^α belongs to $L^p(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P}; L^2(0, T; D(A))) \cap L^p(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P}; L^\infty(0, T; V))$.

(iv) For any $t \in [0, T]$, the following equation holds $\mathbb{P}$-a.s.

$$\begin{aligned} ((u^\alpha(t), \varphi))_V - ((u^\alpha(0), \varphi))_V + \nu \int_0^t (v^\alpha(s), A\varphi) ds \\ + \int_0^t \langle B(v^\alpha(s), u^\alpha(s)), \varphi\rangle_{V'} ds \\ = \int_0^t \langle f(s, u^\alpha(s)), \varphi\rangle_{V'} ds + \Big(\int_0^t g(s, u^\alpha(s)) dW(s), \varphi\Big), \end{aligned} \tag{18}$$

for all $\varphi \in D(A)$.

Our first result is the following existence theorem.

Theorem 2.2. *Let $u_0 \in V$, and suppose that (15) and (16) hold. Then, there exists a weak solution $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\le t\le T}, \mathbb{P}, W, u^\alpha)$ of (1) in the sense of Definition 2.1, and we have $u^\alpha \in L^p(\Omega, \mathcal{F}, \mathbb{P}, C(0, T; V))$.*

Our second result is the following uniqueness theorem.

Theorem 2.3. *If f and g are Lipschitz with respect to the variable u^α, then the solution given by Theorem 2.2 is a unique weak solution of (1) in the sense of Definition 2.1.*

We note that uniqueness is the main target behind regularisations of three-dimensional stochastic Navier-Stokes equations. As stated by Theorem 2.3, in the case of the three-dimensional stochastic modified Leray-alpha model, a pathwise uniqueness holds, in contrast with the case of the original Navier-Stokes equations.

3 Proofs of main results

3.1 Proof of Theorem 2.2

We will perform the proof of Theorem 2.2 in five steps.

Step 1: Construction of an approximating sequence
Let $\{e_j\}_{j\geq 1}$ be an orthonormal basis of H, such that $\{e_j\}$ are eigenfunctions of the operator A. Denote by $H_N = \text{span}\{e_1, \dots, e_N\}$ and by P_N the L^2-orthogonal projection from H onto H_N. We look for a sequence $\{u_N^\alpha\}$ in H_N that is a solution to the following system of stochastic differential equations:

$$\begin{aligned}
&dv_N^\alpha + \nu A v_N^\alpha + P_N B(v_N^\alpha, u_N^\alpha)dt = P_N f(t, u_N^\alpha(t))dt\\
&+ P_N g(t, u_N^\alpha(t))d\bar{W}\\
&v_N^\alpha = u_N^\alpha + \alpha^2 A u_N^\alpha\\
&u_N^\alpha(0) = P_N u_0,
\end{aligned} \tag{19}$$

defined on a fixed stochastic basis $(\bar{\Omega}, \bar{\mathcal{F}}, \{\bar{\mathcal{F}}_t\}_{0\leq t\leq T}, \bar{\mathbb{P}}, \bar{W})$. By classical existence results of solutions to stochastic differential equations, see i.e., [18, Theorem 3.1.1], there exists a local probabilistic weak solution to (19), denoted by $(\Omega_N, \mathcal{F}_N, \{\mathcal{F}_t^N\}_{0\leq t\leq T_N}, \mathbb{P}_N, W^N, u_N^\alpha)$ and defined on $[0, T_N]$, $T_N > 0$. We denote by $\mathbb{E}_N$ the mathematical expectation with respect to $(\Omega_N, \mathcal{F}_N, \mathbb{P}_N)$. The following a priori estimates will allow us to show that $T_N = T$.

Step 2: Estimates for the approximating sequence
Throughout this step, $C, C_i, i = 1, \dots$ denote positive constants that are independent of N and α that may need change from line to line.

Lemma 3.1. *The approximating sequence u_N^α satisfies the a priori estimates:*

a. $\mathbb{E}_N \sup\limits_{0\leq t\leq T} \|u_N^\alpha(t)\|_V + 4\nu\alpha^2 \mathbb{E}_N \int_0^T \|u_N^\alpha(t)\|^2_{D(A)}dt \leq C_1,$

b. $\mathbb{E}_N \sup\limits_{0\leq t\leq T} \|u_N^\alpha(t)\|_V^p \leq C_{2,p},\ p \in [1, \infty),$

c. $\mathbb{E}_N\Big(\int_0^T \|u_N^\alpha(t)\|^2_{D(A)}dt\Big)^p \leq \dfrac{C_{3,p}}{\alpha^{2p}},\ p \in [1, \infty).$

Proof. Applying $(I + \alpha^2 A)^{-1}$ to the first equation of (19), it follows that

$$\begin{aligned}
du_N^\alpha &+ \Big[\nu A u_N^\alpha + (I + \alpha^2 A)^{-1} P_N B(v_N^\alpha, u_N^\alpha)\Big]dt\\
&= (I + \alpha^2 A)^{-1} P_N f(t, u_N^\alpha)dt + (I + \alpha^2 A)^{-1} P_N g(t, u_N^\alpha)dW^N.
\end{aligned} \tag{20}$$

By Itô's formula for $\|u_N^\alpha\|_V^2$, we have

$$\begin{aligned}
d\|u_N^\alpha\|_V^2 &+ 2\Big[\nu((Au_N^\alpha, u_N^\alpha))_V + (((I + \alpha^2 A)^{-1} P_N B(v_N^\alpha, u_N^\alpha), u_N^\alpha))_V\Big]dt\\
&= 2(((I + \alpha^2 A)^{-1} P_N f(t, u_N^\alpha), u_N^\alpha))_V dt + \|(I + \alpha^2 A)^{-1} P_N g(t, u_N^\alpha)\|_V^2 dt\\
&\qquad + 2(((I + \alpha^2 A)^{-1} P_N g(t, u_N^\alpha(t)), u_N^\alpha))_V dW^N.
\end{aligned}$$

From (5) and (11), we obtain that

$$((Au_N^\alpha, u_N^\alpha))_V = (Au_N^\alpha, u_N^\alpha) + \alpha^2(Au_N^\alpha, Au_N^\alpha) = \|u_N^\alpha\|^2 + \alpha^2|Au_N^\alpha|^2,$$

$$
\begin{aligned}
(((I+\alpha^2 A)^{-1}P_N B(v_N^\alpha,u_N^\alpha),u_N^\alpha))_V &= (P_N B(v_N^\alpha,u_N^\alpha),u_N^\alpha)=0,\\
(((I+\alpha^2 A)^{-1}P_N f(t,u_N^\alpha),u_N^\alpha))_V &= (P_N f(t,u_N^\alpha),u_N^\alpha)=(f(t,u_N^\alpha),u_N^\alpha),\\
(((I+\alpha^2 A)^{-1}P_N g(t,u_N^\alpha),u_N^\alpha))_V &= (P_N g(t,u_N^\alpha),u_N^\alpha)=(g(t,u_N^\alpha),u_N^\alpha).
\end{aligned}
$$

Thus, we infer that

$$
\begin{aligned}
d\|u_N^\alpha\|_V^2 + 2\nu(\|u_N^\alpha\|^2+\alpha^2|Au_N^\alpha|^2)dt &= 2(f(t,u_N^\alpha),u_N^\alpha)dt\\
&+\|(I+\alpha^2 A)^{-1}g(t,u_N^\alpha)\|_V^2 dt + 2(g(t,u_N^\alpha),u_N^\alpha)dW^N.
\end{aligned}
\tag{21}
$$

By (6), we have $\|(I+\alpha^2 A)^{-1}g(t,u_N^\alpha)\|_V^2 \le |g(t,u_N^\alpha)|^2$. It follows that

$$
\begin{aligned}
d\|u_N^\alpha\|_V^2 + 2\nu(\|u_N^\alpha\|^2+\alpha^2|Au_N^\alpha|^2)dt &\le 2(f(t,u_N^\alpha),u_N^\alpha)dt\\
&+|g(t,u_N^\alpha)|^2 dt + 2(g(t,u_N^\alpha),u_N^\alpha)dW^N.
\end{aligned}
\tag{22}
$$

Since $|\cdot| \le \|\cdot\|_V$, inequalities (15) and (16) imply that

$$
(f(t,u_N^\alpha),u_N^\alpha) \le C(1+\|u_N^\alpha\|_V^2) \text{ and } (g(t,u_N^\alpha),u_N^\alpha) \le C(1+\|u_N^\alpha\|_V^2). \tag{23}
$$

Hence, (22) and (23) yield

$$
\begin{aligned}
&d\|u_N^\alpha\|_V^2 + 2\nu(\|u_N^\alpha\|_V^2+\alpha^2|Au_N^\alpha|^2)dt\\
&\qquad \le C(1+\|u_N^\alpha\|_V^2)dt + 2(g(t,u_N^\alpha(t)),u_N^\alpha)dW^N.
\end{aligned}
\tag{24}
$$

Dropping the term $2\nu\|u_N^\alpha\|_V^2$ and integrating with respect to time, we obtain

$$
\begin{aligned}
&\|u_N^\alpha(t)\|_V^2 + 2\nu\alpha^2\int_0^t \|u_N^\alpha(s)\|_{D(A)}^2 dt \le \|u_N^\alpha(0)\|_V^2\\
&+CT + C\int_0^t \|u_N^\alpha(s)\|_V^2 dt + 2\int_0^t (g(t,u_N^\alpha(s)),u_N^\alpha(s))dW^N(s).
\end{aligned}
\tag{25}
$$

For each integer $n>0$, we consider the $\mathcal{F}_t^N$-stopping time τ_N^n defined by

$$
\tau_N^n = \inf\left\{t\in[0,T], \|u_N^\alpha(t)\|_V^2 + 2\nu\alpha^2\int_0^t \|u_N^\alpha(s)\|_{D(A)}^2 dt \ge n^2\right\}\wedge T,
$$

where $a\wedge b=\min(a,b)$. This stopping time will be useful later on to apply Burkhölder-Davis-Gundy's inequality. Moreover, the sequence $(\tau_N^n)_n$ increases to T when n goes to ∞. Taking the supremum and the mathematical expectation of (25), we obtain that

$$
\begin{aligned}
&\mathbb{E}_N \sup_{0\le s\le t\wedge\tau_N^n} \|u_N^\alpha(s)\|_V^2 + 2\nu\alpha^2\mathbb{E}_N\int_0^{t\wedge\tau_N^n}\|u_N^\alpha(s)\|_{D(A)}^2 ds\\
&\qquad \le \|u_N^\alpha(0)\|_V^2 + CT + C\mathbb{E}_N\int_0^{t\wedge\tau_N^n}\|u_N^\alpha(s)\|_V^2 ds\\
&\qquad +2\mathbb{E}_N \sup_{0\le s\le t\wedge\tau_N^n}\int_0^s (g(r,u_N^\alpha(r)),u_N^\alpha(r))dW^N(r),
\end{aligned}
\tag{26}
$$

for all $t \in [0, T]$ and all $n, N \geq 1$. Let us estimate the last term of the right-hand side of (26). Using Burkhölder-Davis Gundy's inequality [13, Chapter 3, Theorem 3.28], we have

$$\begin{aligned} &2\mathbb{E}_N \sup_{0\leq s\leq t\wedge\tau_N^n} \Big| \int_0^s (g(r, u_N^\alpha(r)), u_N^\alpha(r))dW^N(r)\Big| \\ &\leq 6C\mathbb{E}_N\Big(\int_0^{t\wedge\tau_N^n} (g(t, u_N^\alpha(s)), u_N^\alpha(s))^2 ds\Big)^{1/2}. \end{aligned}$$

Using Young's inequality and (23), we deduce that

$$\begin{aligned} &2\mathbb{E}_N \sup_{0\leq s\leq t\wedge\tau_N^n} \Big| \int_0^s (g(r, u_N^\alpha(r)), u_N^\alpha(r))W^N(r)\Big| \\ &\leq C_1\mathbb{E}_N\Big(\int_0^{t\wedge\tau_N^n} (1 + \|u_N^\alpha(s)\|_V^2)\|u_N^\alpha(s)\|_V^2 ds\Big)^{1/2} \\ &\leq \frac{1}{2}\mathbb{E}_N \sup_{0\leq s\leq t\wedge\tau_N^n} \|u_N^\alpha(s)\|_V^2 + C_2T + C_2\mathbb{E}_N\int_0^{t\wedge\tau_N^n} \|u_N^\alpha(s)\|_V^2 ds. \end{aligned}$$

We obtain from (26) that

$$\begin{aligned} &\mathbb{E}_N \sup_{0\leq s\leq t\wedge\tau_N^n} \|u_N^\alpha(s)\|_V^2 + 4\nu\alpha^2\mathbb{E}_N\int_0^{t\wedge\tau_N^n} \|u_N^\alpha(s)\|_{D(A)}^2 ds \\ &\leq 2\|u_N^\alpha(0)\|_V^2 + C_3T + C_3\mathbb{E}_N\int_0^{t\wedge\tau_N^n} \|u_N^\alpha(s)\|_V^2 ds. \end{aligned} \tag{27}$$

Dropping the viscous term from the left-hand side of (27), we obtain

$$\mathbb{E}_N \sup_{0\leq s\leq t\wedge\tau_N^n} \|u_N^\alpha(s)\|_V^2 \leq 2\|u_N^\alpha(0)\|_V^2 + C_3T + C_3\mathbb{E}_N\int_0^{t\wedge\tau_N^n} \|u_N^\alpha(s)\|_V^2 ds. \tag{28}$$

Since τ_N^n increases to T as n goes to ∞, Gronwall's lemma implies that

$$\mathbb{E}_N \sup_{0\leq s\leq T} \|u_N^\alpha(s)\|_V^2 \leq C.$$

Thus, it follows from (27) that

$$\mathbb{E}_N \sup_{0\leq t\leq T} \|u_N^\alpha(s)\|_V^2 + 4\nu\alpha^2\mathbb{E}_N\int_0^T \|u_N^\alpha(s)\|_{D(A)}^2 ds \leq C. \tag{29}$$

By Itô's formula and (21), it follows, for any $p \geq 4$, that

$$\begin{aligned} d\|u_N^\alpha\|_V^{\frac{p}{2}} &= \frac{p}{2}\|u_N^\alpha\|_V^{\frac{p}{2}-2}\Big(- \nu(\|u_N^\alpha\|_V^2 + \alpha^2|Au_N^\alpha|^2)dt + 2(f(t, u_N^\alpha), u_N^\alpha) \\ &\quad + \frac{p-4}{4}\frac{(g(t, u_N^\alpha), u_N^\alpha)^2}{\|u_N^\alpha\|_V^2}\Big)dt + \frac{p}{2}\|u_N^\alpha\|_V^{\frac{p}{2}-2}(g(t, u_N^\alpha), u_N^\alpha)W^N. \end{aligned} \tag{30}$$

Young's inequality and (23), yield

$$\|u_N^\alpha\|_V^{\frac{p}{2}-2}(f(t, u_N^\alpha), u_N^\alpha) \leq C(1 + \|u_N^\alpha\|_V^{\frac{p}{2}}). \tag{31}$$

Thanks to Cauchy-Schwarz's and Young's inequalities and (23), we obtain that

$$\frac{(g(t,u_N^\alpha),u_N^\alpha)^2}{\|u_N^\alpha\|_V^2}\le C(1+\|u_N^\alpha\|_V^2). \tag{32}$$

Next, we drop $-\nu(\|u_N^\alpha\|^2+\alpha^2|Au_N^\alpha|^2)$ from the right-hand side of (30). Using (31) and (32) and integrating with respect to $t\in[0,T]$, we have

$$\begin{aligned}\|u_N^\alpha(t)\|_V^{\frac{p}{2}} \le \|u_N^\alpha(0)\|_V^{\frac{p}{2}} + C\int_0^t (1+\|u_N^\alpha(s)\|_V^{\frac{p}{2}})ds \\ + \frac{p}{2}\int_0^t \|u_N^\alpha\|_V^{\frac{p}{2}-1}(g(t,u_N^\alpha),u_N^\alpha)W^N. \end{aligned}\tag{33}$$

For each integer $n>0$, we consider the $\mathcal{F}_t^N$-stopping time τ_N^n defined by

$$\tilde{\tau}_N^n=\inf\left\{t\in[0,T],\int_0^t\|u_N^\alpha(s)\|_V^p dt\ge n^2\right\}\wedge T.$$

Taking the supremum and squaring both sides of (33), Young's inequality leads to

$$\begin{aligned}\Big(\sup_{0\le s\le t\wedge\tilde{\tau}_N^n}\|u_N^\alpha(s)\|_V^{\frac{p}{2}}\Big)^2 \le 2\Big(\|u_N^\alpha(0)\|_V^{\frac{p}{2}}+C\int_0^{t\wedge\tilde{\tau}_N^n}(1+\|u_N^\alpha(s)\|_V^{\frac{p}{2}})ds\Big)^2 \\ +\frac{p^2}{2}\sup_{0\le s\le t\wedge\tilde{\tau}_N^n}\Big|\int_0^s\|u_N^\alpha\|_V^{\frac{p}{2}-1}(g(r,u_N^\alpha),u_N^\alpha)W^N\Big|^2.\end{aligned}\tag{34}$$

Using Cauchy-Schwarz's and Young's inequalities, we deduce that

$$\begin{aligned}&\Big(\|u_N^\alpha(0)\|_V^{\frac{p}{2}}+C\int_0^{t\wedge\tilde{\tau}_N^n}(1+\|u_N^\alpha(s)\|_V^{\frac{p}{2}})ds\Big)^2\\ &\qquad\le 2\|u_N^\alpha(0)\|_V^p+2C(T^2+T\int_0^{t\wedge\tilde{\tau}_N^n}\|u_N^\alpha(s)\|_V^p ds).\end{aligned}$$

Taking the mathematical expectation in (34), we obtain

$$\begin{aligned}\mathbb{E}_N\sup_{0\le s\le t\wedge\tilde{\tau}_N^n}\|u_N^\alpha(s)\|_V^p\le C\Big(\|u_{0,N}^\alpha\|_V^p+T+\mathbb{E}_N\int_0^{t\wedge\tilde{\tau}_N^n}\|u_N^\alpha(s)\|_V^p ds\Big)\\ +\frac{p^2}{2}\mathbb{E}_N\sup_{0\le s\le t\wedge\tilde{\tau}_N^n}\Big|\int_0^s\|u_N^\alpha(r)\|_V^{\frac{p}{2}-1}g(r,u_N^\alpha(r)),u_N^\alpha(r))W^N(r)\Big|^2.\end{aligned}$$

The Burkhölder-Davis-Gundy inequality implies that

$$\begin{aligned}&\mathbb{E}_N\sup_{0\le s\le t\wedge\tilde{\tau}_N^n}\Big|\int_0^s\|u_N^\alpha(r)\|_V^{\frac{p}{2}-1}(g(r,u_N^\alpha(r)),u_N^\alpha(r))W^N(r)\Big|^2\\ &\qquad\le C_p\mathbb{E}_N\int_0^{t\wedge\tilde{\tau}_N^n}\|u_N^\alpha(s)\|_V^{p-2}(g(s,u_N^\alpha(s)),u_N^\alpha(s))ds\end{aligned}$$

$$\leq C_p \mathbb{E}_N \int_0^{t\wedge\tilde{\tau}_N^n} \|u_N^\alpha(s)\|_V^{p-2}(1+\|u_N^\alpha(s)\|_V^2)ds$$
$$\leq C_p + \mathbb{E}_N \int_0^{t\wedge\tilde{\tau}_N^n} \|u_N^\alpha(s)\|_V^p ds,$$

where we use Young's inequality in the last step. Thus, it follows that

$$\mathbb{E}_N \sup_{0\leq t\leq t\wedge\tilde{\tau}_N^n} \|u_N^\alpha(s)\|_V^p \leq C_p + C\mathbb{E}_N \int_0^{t\wedge\tilde{\tau}_N^n} \|u_N^\alpha(s)\|_V^p ds.$$

Since τ_N^n increases to T, as n goes to ∞, Gronwall's inequality implies that

$$\mathbb{E}_N \sup_{0\leq t\leq T} \|u_N^\alpha(t)\|_V^p \leq C_p, \quad p\geq 4. \tag{35}$$

As (35) is proven for any $p \geq 4$, it is consequently true for any $p \in [1,\infty)$. From (26), we obtain

$$(2\nu\alpha^2)^p\Big(\int_0^T \|u_N^\alpha(t)\|_{D(A)}^2 dt\Big)^p \leq C_p\Big(\|u_N^\alpha(0)\|_V^{2p} + T^p$$
$$+\Big(\int_0^T \|u_N^\alpha(t)\|_V^2 dt\Big)^p\Big) + c_p \sup_{0\leq t\leq T} \Big|\int_0^t (g(s,u_N^\alpha(s)),u_N^\alpha(s))W^N(s)\Big|^p. \tag{36}$$

To handle the last term on the right-hand side of inequality (36), we take the mathematical expectation. Then, by the stopping time techniques used in the proof of (29), we apply Burkhölder-Davis-Gundy's inequality. Finally, we obtain

$$\mathbb{E}_N\Big(\int_0^T \|u_N^\alpha(t)\|_{D(A)}^2 dt\Big)^p \leq \frac{C_p}{\alpha^{2p}}, \quad p\in[1,\infty). \tag{37}$$

□

Estimate (38) given in the lemma below is crucial in the proof of the tightness of the law to the Galerkin solution u_N^α.

Lemma 3.2. *There exists a positive constant $C(\alpha)$ such that*

$$\mathbb{E}_N \sup_{0\leq|\theta|\leq\eta} \int_0^T \|u_N^\alpha(t+\theta)-u_N^\alpha(t)\|_{D(A)'}^2 dt \leq C(\alpha)\eta, \tag{38}$$

where $0<\eta\leq 1$ is a fixed constant.

Proof. For $\theta>0$, we obtain from (19)

$$\|v_N^\alpha(t+\theta)-v_N^\alpha(t)\|_{D(A)'}^2 \leq 2\Big\|\int_t^{t+\theta} P_N\Big(\nu A v_N^\alpha(s)+B(v_N^\alpha(s),u_N^\alpha(s))$$
$$-f(s,u_N^\alpha(s))\Big)ds\Big\|_{D(A)'}^2 + 2\Big|\int_t^{t+\theta} g(s,u_N^\alpha(s))W^N(s)\Big|^2.$$

This implies that

$$\|v_N^\alpha(t+\theta)-v_N^\alpha(t)\|_{D(A)'}^2 \le 6\theta\Big[\nu^2\int_t^{t+\theta}\|Av_N^\alpha(s)\|_{D(A)'}^2 ds + \int_t^{t+\theta}\|P_N B(v_N^\alpha(s),u_N^\alpha(s))\|_{D(A)'}^2 ds + \int_t^{t+\theta}|f(s,u_N^\alpha(s))|^2 ds\Big] + 2\Big|\int_t^{t+\theta} g(s,u_N^\alpha(s))W^N(s)\Big|^2.$$

Since

$$\|Av_N^\alpha\|_{D(A)'} = |v_N| \le |u_N| + \alpha^2|Au_N|$$

and

$$\|P_N B(v_N^\alpha(s),u_N^\alpha(s))\|_{D(A)'}^2 \le c|v_N^\alpha(s)|^2\|u_N^\alpha(t)\|^2,$$

Lemma 3.1 implies that

$$\mathbb{E}_N \sup_{0\le\theta\le\eta}\int_0^T\int_t^{t+\theta}\|Av_N^\alpha(s)\|_{D(A)'}^2 dsdt \le \mathbb{E}_N\int_0^T\int_t^{t+\eta}|v_N^\alpha(s)|dsdt \le \mathbb{E}_N\int_0^T\int_t^{t+\eta}\sup_{0\le t\le T}\|u_N^\alpha(t)\|_V^2 dsdt + \alpha^4\mathbb{E}_N\int_0^T\int_t^{t+\eta}\|u_N^\alpha(s)\|_{D(A)}^2 dsdt \le CT\eta + \alpha^2 C.$$

Moreover, we have

$$\begin{aligned}\mathbb{E}_N \sup_{0\le\theta\le\eta}\int_0^T\int_t^{t+\theta}\|P_N B(v_N^\alpha(s),u_N^\alpha(s))\|_{D(A)'}^2 dsdt &\le c\mathbb{E}_N\int_0^T\int_t^{t+\eta}|v_N^\alpha(s)|^2\|u_N^\alpha(t)\|^2 dsdt\\ &\le C\alpha^{-4}\eta\mathbb{E}_N\sup_{0\le t\le T}\|u_N^\alpha(t)\|_V^4 + C\eta\mathbb{E}_N\Big(\int_0^T\big(|u_N|^2+\alpha^4|Au_N|^2\big)dt\Big)^2 dt\\ &\le C\alpha^{-4}\eta + C\eta\mathbb{E}_N\sup_{0\le t\le T}\|u_N^\alpha(t)\|_V^4 + C\alpha^4\eta\mathbb{E}_N\Big(\alpha^2\int_0^T\|u_N^\alpha(s)\|_{D(A)}dt\Big)^2 dt\\ &\le C\alpha^{-4}\eta + C\eta + C\alpha^4\eta.\end{aligned}$$

It follows from (15) that

$$\begin{aligned}\mathbb{E}_N \sup_{0\le\theta\le\eta}\int_0^T\int_t^{t+\theta}|f(s,u_N^\alpha(s))|^2 ds &\le C\mathbb{E}_N\int_0^T\int_t^{t+\eta}(1+\|u_N^\alpha(t)\|_V^2)dsdt\\ &\le C\eta + \mathbb{E}_N\int_0^T\int_t^{t+\eta}\sup_{0\le t\le T}\|u_N^\alpha(t)\|_V^2\\ &\le C\eta.\end{aligned}$$

As previously, using Burkhölder-Davis-Gundy's inequality and (16), we obtain

$$\mathbb{E}_N \int_0^T \sup_{0\le\theta\le\eta} \Big| \int_t^{t+\theta} g(s,u_N^\alpha(s))W^N(s)\Big|^2 dt \le \mathbb{E}_N \int_0^T \int_t^{t+\eta} \Big| g(s,u_N^\alpha(s))\Big|^2 ds dt$$
$$\le C\eta.$$

Collecting the estimates above, we obtain that

$$\mathbb{E}_N \sup_{0\le\theta\le\eta} \int_0^T \|v_N^\alpha(t+\theta) - v_N^\alpha(t)\|^2_{D(A)'} dt \le C(\alpha^2\eta + 1 + \alpha^{-4} + \alpha^4)\eta^2$$
$$\le C(\alpha)\eta. \tag{39}$$

In fact, as $0 \le \eta \le 1$, we have $\eta^2 \le \eta$. Therefore, from (4) and the fact that

$$\alpha^4 |u_N^\alpha(t+\theta) - u_N^\alpha(t)|^2 \le |A^{-1}(v_N^\alpha(t+\theta) - v_N^\alpha(t))|^2$$
$$\le \|v_N^\alpha(t+\theta) - v_N^\alpha(t)\|^2_{D(A)'},$$

estimate (39) implies that

$$\mathbb{E}_N \sup_{0\le\theta\le\eta} \int_0^T \|u_N^\alpha(t+\theta) - u_N^\alpha(t)\|^2_{D(A)'} dt \le C(\alpha)\eta.$$

To achieve the proof of (38), we can use similar discussions to prove a similar estimate for the case $\theta < 0$. □

Step 3: Tightness Property

First, we introduce the concept of tightness of probability measure (see, e.g., [13]). Let X be a separable Banach space, equipped with the Borel σ-algebra $\mathcal{B}(X)$.

Definition 3.3. A set Γ of probability measures defined on $(X, \mathcal{B}(X))$ is called tight if, for each $\varepsilon > 0$, there is a compact set A such that $P(A) > 1 - \varepsilon$, for all $P \in \Gamma$.

Next, we will prove the tightness property of the Galerkin solution. Similar to the proof in [2, Section 4.3.3], we have the following lemma.

Lemma 3.4. *For any sequences of positive real numbers μ_n, ν_n that tend to 0 as $n \to \infty$, the injection of*

$$\chi_{\mu_n,\nu_n} = \Big\{ z \in L^2(0,T; D(A) \cap L^\infty(0,T;V),$$
$$\sup_n \frac{1}{\nu_n} \sup_{|\tau|\le\mu_n} \Big(\int_0^T \|z(t+\tau) - z(t)\|^2_{D(A)'} dt\Big)^{1/2} < \infty \Big\},$$

in $L^2(0,T;V)$ is compact.

The space χ_{μ_n,ν_n} is a Banach space with the norm

$$\|z\|_{\chi_{\mu_n,\nu_n}} = \sup_{0\le t\le T} \|z(t)\|_V + \Big(\int_0^T \|z(t)\|^2_{D(A)} dt\Big)^{1/2}$$

$$+\sup_n \frac{1}{\nu_n} \sup_{|\tau|\le\mu_n} \Big(\int_0^T \|z(t+\tau)-z(t)\|^2_{D(A)'}dt\Big)^{1/2}.$$

We consider a Banach space $\mathcal{B}_{p,\mu_n,\nu_n}$ $(1 \le p \le \infty)$ of random variables z defined on some probability space. If we denote the expectation on $\mathcal{B}_{p,\mu_n,\nu_n}$ by $\hat{\mathbb{E}}$, then we have

$$\hat{\mathbb{E}} \sup_{0\le t\le T} \|z(t)\|_V^p < \infty, \qquad \hat{\mathbb{E}}\Big(\int_0^T \|z(t)\|^2_{D(A)}dt\Big)^{p/2} < \infty$$

and

$$\hat{\mathbb{E}} \sup_n \frac{1}{\nu_n} \sup_{|\tau|\le\mu_n} \int_0^T \|z(t+\tau)-z(t)\|^2_{D(A)'}dt < \infty.$$

The space $\mathcal{B}_{p,\mu,\nu}$ is endowed with the norm

$$\|z\|_{\mathcal{B}_{p,\mu_n,\nu_n}} = \Big(\hat{\mathbb{E}} \sup_{0\le t\le T} \|z(t)\|_V^p\Big)^{1/p} + \Big(\hat{\mathbb{E}}^\alpha\Big(\int_0^T \|z(t)\|^2_{D(A)}dt\Big)^{p/2}\Big)^{2/p}$$
$$+\hat{\mathbb{E}} \sup_n \frac{1}{\nu_n}\Big(\sup_{|\tau|\le\mu_n} \int_0^T \|z(t+\tau)-z(t)\|^2_{D(A)'}dt\Big)^{1/2}.$$

According to a priori estimates in Lemmas 3.1 and 3.2, we deduce that for every μ_n, ν_n such that the series $\sum_{n=1}^{\infty} \frac{\sqrt{\mu_n}}{\nu_n}$ converges, the approximate solutions $\{u_N^\alpha\}_{N\in\mathbb{N}}$ remain bounded in $\mathcal{B}_{p,\mu_n,\nu_n}$, for any $1 \le p < \infty$.

Now, we consider the set $S = C((0,T),\mathbb{R}^m) \times L^2((0,T),V)$, equipped with its Borel σ-algebra $\mathcal{B}(S)$. We denote by ψ the measurable S-valued map defined on $(\Omega_N, \mathcal{F}_N, \mathbb{P}_N)$ by $\psi(\omega) = W^N(\omega), u_N^\alpha(\omega))$. For each N, we introduce a probability measure Γ_N defined on $(S,\mathcal{B}(S))$ by $\Gamma_N(\Lambda) = \mathbb{P}_N(\psi^{-1}(\Lambda))$, for $\Lambda \in \mathcal{B}(S)$.

Proposition 3.5. *The family of probability measures* $\{\Gamma_N;\, N \in \mathbb{N}\}$ *is tight.*

Proof. For any $\varepsilon > 0$, we shall find two compact subsets

$$\Theta_\varepsilon \subset C(0,T;\mathbb{R}^m), \qquad Z_\varepsilon \subset L^2(0,T;V),$$

such that

$$\mathbb{P}_N(\omega : W^N(\omega,.) \notin \Theta_\varepsilon) \le \frac{\varepsilon}{2} \tag{40}$$
$$\mathbb{P}_N(\omega : u_N^\alpha(\omega,.) \notin Z_\varepsilon) \le \frac{\varepsilon}{2}. \tag{41}$$

For Θ_ε, we use the following classical results about the Wiener process

$$\mathbb{E}_N\Big|W^N(t_2) - W^N(t_1)\Big|^{2j} = (2j-1)!(t_2-t_1)^j, \quad j \in \mathbb{N}.$$

For a constant ℓ_ε to be chosen depending on ε, we consider the set

$$\Theta_\varepsilon = \Big\{W^N(.) \in C(0,T;\mathbb{R}^m) : \sup_{t_1,t_2\in[0,T],|t_2-t_1|\le 1/n^6} n|W^N(t_2) - W^N(t_1)| \le \ell_\varepsilon\Big\}.$$

Using an extended version for monotonically increasing functions of Markov's inequality [4, Chapter 1] (for a nonnegative random variable ζ, $\beta > 0$ and $k \in \mathbb{N}$, we have $\mathbb{P}(|\zeta| \geq \beta) \leq \frac{\mathbb{E}(|\zeta|^k)}{\beta^k}$), we obtain that

$$\mathbb{P}_N(\omega : W^N(\omega, .) \notin \Theta_\varepsilon) \leq c \sum_{n=1}^{\infty} \left(\frac{n}{\ell_\varepsilon}\right)^4 (Tn^{-6})^2 n^6 = \frac{c}{\ell_\varepsilon^4} \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

To obtain (40), we take $\ell_\varepsilon^4 = 2\frac{C_\varepsilon}{\varepsilon} \sum_{n=1}^{\infty} \frac{1}{n^2}$. Next, we choose Z_ε as a ball in χ_{μ_n, ν_n} of radius M_ε centred at zero, where μ_n and ν_n are independent of ε, the series $\sum_{n=1}^{\infty} \frac{\sqrt{\mu_n}}{\nu_n}$ converges and μ_n, $\nu_n \longrightarrow 0$ as $n \longrightarrow \infty$. From Lemma 3.4, we know that Z_ε is a compact subset of $L^2(0, T; V)$, and

$$\begin{aligned}\mathbb{P}_N\big(\omega : u_N^\alpha(\omega, .) \notin Z_\varepsilon\big) &\leq \mathbb{P}_N\big(\omega : \|u_N^\alpha\|_{\chi_{\mu_n,\nu_n}} > M_\varepsilon\big) \\ &\leq \frac{1}{M_\varepsilon} \mathbb{E}_N \|u_N^\alpha\|_{\chi_{\mu_n,\nu_n}} \leq \frac{c}{M_\varepsilon}.\end{aligned}$$

Choosing $M_\varepsilon = \dfrac{2c}{\varepsilon}$, (41) holds. Finally, using (40) and (41), we obtain that

$$\mathbb{P}_N\big(\omega : W^N(\omega, .) \in \Theta_\varepsilon, u_N^\alpha(\omega, .) \in Z_\varepsilon\big) \geq 1 - \varepsilon.$$

This completes the proof of Proposition 3.5. □

Step 4: Applications of Prokhorov's and Skorokhod's Theorems

In this step, we recall two lemmas due to Prokhorov [19] and Skorokhod [25], which will be crucial in the following.

Lemma 3.6 (Prokhorov's theorem [19]). *If a set Γ of probability measures on $(X, \mathcal{B}(X))$ is tight, then for each sequence $\Gamma_n \subset \Gamma$, there exists a subsequence $\{\Gamma_{n_k}\}$ that converges weakly to a probability measure Γ:*

$$\int \phi \, d\Gamma_{n_k} \longrightarrow \int \phi \, d\Gamma, \ \textit{as} \ n_k \longrightarrow \infty,$$

for all bounded continuous integrands ϕ.

Lemma 3.7 (Skorokhod's theorem [25]). *For an arbitrary sequence of probability measures $\{\Gamma_k\}$ on $(X, \mathcal{B}(X))$ weakly convergent to a probability measure Γ, there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and random variables ξ_n, $n \in \mathbb{N}$, and ξ with values in X, such that the probability law of ξ_n is Γ_n, the probability law of ξ is Γ and $\lim_{n\to\infty} \xi_n = \xi$, $\mathbb{P}$-a.s.*

Using the tightness property of $\{\Gamma_N\}_N$ and Prokhorov's theorem, we obtain the existence of a subsequence $\{\Gamma_{N_k}\}$ and a probability measure Γ such that Γ_{N_k} converges weakly to Γ. Hence, by Skorokhod's theorem, there exist a probability space

$(\Omega, \mathcal{F}, \mathbb{P})$ and random variables $(W_{N_k}, u^\alpha_{N_k})$, (W, u^α) with values in S, such that the law of $(W_{N_k}, u^\alpha_{N_k})$ is Γ_{N_k} and the law of (W, u^α) is Γ. Moreover, we have

$$(W_{N_k}, u^\alpha_{N_k}) \longrightarrow (W, u^\alpha) \text{ strongly, in } S \ \ \mathbb{P}-a.s., \tag{42}$$

where (W_{N_k}) is a sequence of an m-dimensional standard Wiener process. Let $\mathcal{F}_t = \sigma\{(W(s), u^\alpha(s)), 0 \le s \le t\}$. As in [1], we can show that $W(t)$ is an m-dimensional $\mathcal{F}_t$ standard Wiener process. Using the same arguments as in [2], we can prove that the pair $(W_{N_k}, u^\alpha_{N_k})$ satisfies the following equation, $d\mathbb{P} \otimes dt$-almost everywhere:

$$\begin{aligned} & v^\alpha_{N_k}(t) + \int_0^t P_{N_k}(\nu A v^\alpha_{N_k}(s) + B(v^\alpha_{N_k}(s), u^\alpha_{N_k}(s)) - f(s, u^\alpha_{N_k}(s)))ds \\ & = v^\alpha_{N_k}(0) + \int_0^t P_{N_k} g(s, u^\alpha_{N_k}(s)) dW_{N_k}(s), \end{aligned} \tag{43}$$

where $v^\alpha_{N_k}(t) = u^\alpha_{N_k}(t) + \alpha^2 A u^\alpha_{N_k}(t)$.

Step 5: Convergence
By (43), $u^\alpha_{N_k}$ satisfies the same estimates of u^α_N, that is

$$\mathbb{E}_N \sup_{0\le t\le T} \|u^\alpha_{N_k}(t)\|^p_V \le C_p, \quad p \in [1, \infty) \tag{44}$$

$$\mathbb{E}_N \Big(\int_0^T \|u^\alpha_{N_k}(t)\|^2_{D(A)} dt \Big)^p \le C_p, \quad p \in [1, \infty) \tag{45}$$

$$\mathbb{E}_N \sup_{0\le|\theta|\le\eta} \int_0^T \|u^\alpha_{N_k}(t+\theta) - u^\alpha_{N_k}(t)\|^2_{D(A)'} dt \le C(\alpha)\eta, \tag{46}$$

where $0 < \eta \le 1$. Thus, we can extract a subsequence, denoted again by $u^\alpha_{N_k}$, such that

$$u^\alpha_{N_k} \rightharpoonup u^\alpha \text{ weakly*, in } L^p(\Omega, \mathcal{F}, \mathbb{P}; L^\infty(0, T; V)) \tag{47}$$

$$u^\alpha_{N_k} \rightharpoonup u^\alpha \text{ weakly, in } L^p(\Omega, \mathcal{F}, \mathbb{P}; L^2(0, T; D(A))). \tag{48}$$

Using (42), it follows that

$$u^\alpha_{N_k} \longrightarrow u^\alpha \text{ strongly, in } L^2(0, T; V), \mathbb{P}-a.s.$$

Then, estimates (44)–(46) and Vitali's theorem yield

$$u^\alpha_{N_k} \longrightarrow u^\alpha \text{ strongly, in } L^2(\Omega, \mathcal{F}, \mathbb{P}; L^2(0, T; V)). \tag{49}$$

Thus, we can extract from $(u^\alpha_{N_k})$ a subsequence denoted again by $u^\alpha_{N_k}$, such that

$$u^\alpha_{N_k} \longrightarrow u^\alpha \text{ in } V, \tag{50}$$

for almost all (ω, t) with respect to the measure $d\mathbb{P} \otimes dt$.

Using (48), we obtain, for $p = 2$, that

$$u^\alpha_{N_k} \rightharpoonup u^\alpha \text{ weakly, in } L^2(\Omega, \mathcal{F}, \mathbb{P}, L^2(0, T; D(A))).$$

Hence, we deduce that

$$v^\alpha_{N_k} \rightharpoonup v^\alpha \text{ weakly, in } L^2(\Omega, \mathcal{F}, \mathbb{P}, L^2(0, T; H)), \tag{51}$$

for almost all (ω, t) with respect to the measure $d\mathbb{P} \otimes dt$. Since A is a linear bounded operator, it follows that

$$\int_0^t P_{N_k} A v^\alpha_{N_k}(s)ds \rightharpoonup \int_0^t Av^\alpha(s)ds \text{ weakly, in } L^2(\Omega, \mathcal{F}, \mathbb{P}, L^2(0, T; D(A)')).$$

The strong convergence (49) and the weak convergence (51) allow us to pass to the limit in the nonlinear terms of (43).

Thanks to (50), the continuity of f and g and Vitali's theorem, we obtain

$$P_{N_k} f(s, u^\alpha_{N_k}(s)) \longrightarrow f(s, u^\alpha(s)) \text{ strongly, in } L^2(\Omega, \mathcal{F}, \mathbb{P}, L^2(0, T; H)) \tag{52}$$

$$P_{N_k} g(s, u^\alpha_{N_k}(s)) \longrightarrow g(s, u^\alpha(s)) \text{ strongly, in } L^2(\Omega, \mathcal{F}, \mathbb{P}, L^2(0, T; H)). \tag{53}$$

In view of (53), we use the same arguments, as in [2] to prove that

$$\int_0^t P_{N_k} g(s, u^\alpha_{N_k}(s))dW_{N_k}(s) \rightharpoonup \int_0^t g(s, u^\alpha(s))dW(s)$$

weakly in $L^2(\Omega, \mathcal{F}, \mathbb{P}, L^2(0, T; H))$.

Now, it remains to prove that

$$\int_0^t P_{N_k} B(v^\alpha_{N_k}(s), u^\alpha_{N_k}(s))ds \rightharpoonup \int_0^t B(v^\alpha(s), u^\alpha(s))ds$$

weakly in $L^2(\Omega, \mathcal{F}, \mathbb{P}, L^2(0, T; D(A)'))$. Then, we introduce $\mathcal{M}^\infty_{\mathcal{F}_t}(0, T, D(A))$ the space of all processes $w \in L^\infty(\Omega, (0, T), dP \otimes dt, D(A))$ that are $\mathcal{F}_t$-progressively measurable. The space $\mathcal{M}^\infty_{\mathcal{F}_t}(0, T, D(A))$ is a Banach subspace of $L^\infty(\Omega, (0, T), dP \otimes dt, D(A))$. For any $w \in \mathcal{M}^\infty_{\mathcal{F}_t}(0, T, D(A))$, we have

$$\left|\mathbb{E}\int_0^t \langle P_{N_k} B(v^\alpha_{N_k}(s), u^\alpha_{N_k}(s)) - B(v^\alpha(s), u^\alpha(s)), w\rangle_{D(A)'} ds\right| \leq I^{(1)}_{N_k} + I^{(2)}_{N_k} + I^{(3)}_{N_k},$$

where

$$I^{(1)}_{N_k} = \left|\mathbb{E}\int_0^t \langle B(v^\alpha_{N_k}(s), u^\alpha_{N_k}(s)), P_{N_k} w - w\rangle_{D(A)'} ds\right|$$

$$I^{(2)}_{N_k} = \left|\mathbb{E}\int_0^t \langle B(v^\alpha_{N_k}(s) - v^\alpha(s), u^\alpha_{N_k}(s)), w\rangle_{D(A)'} ds\right|$$

$$I^{(3)}_{N_k} = \left|\mathbb{E}\int_0^t \langle B(v, u^\alpha_{N_k}(s) - u^\alpha(s)), w\rangle_{D(A)'} ds\right|.$$

Recall that for any $\varphi \in H$, we have $|P_{N_k}\varphi - \varphi| \longrightarrow 0$, as $N_k \to \infty$. Additionally, we obtain from (4) that

$$\|P_{N_k}\varphi - \varphi\|_{D(A)} \longrightarrow 0, \text{ as } N_k \to \infty. \tag{54}$$

Using (12), (4) and Young's inequality, we have

$$\begin{aligned} I_{N_k}^{(1)} &\leq c\mathbb{E}\int_0^t |v_{N_k}^\alpha|\|u_{N_k}^\alpha\|\|P_{N_k}w - w\|^{1/2}|A(P_{N_k}w - w)|^{1/2}ds \\ &\leq C\mathbb{E}\int_0^t |A(P_{N_k}w - w)||v_{N_k}^\alpha|\|u_{N_k}^\alpha\|ds \\ &\leq C\|P_{N_k}w - w\|_{\mathcal{M}_{\mathcal{F}_t}^\infty}\Big[\mathbb{E}\sup_{0\leq t\leq T}\|u_{N_k}\|_V^2 + \mathbb{E}\int_0^T |v_{N_k}^\alpha|^2 ds\Big]. \end{aligned}$$

Since

$$\mathbb{E}\int_0^T |v_{N_k}^\alpha|^2 ds \leq \alpha^{-2}\mathbb{E}\sup_{0\leq t\leq T}\|u_{N_k}\|_V^2 + \alpha^4\mathbb{E}\int_0^T |Au_{N_k}^\alpha|^2 ds < \infty, \tag{55}$$

(54), (44) and (45) imply that $I_{N_k}^{(1)} \longrightarrow 0$, as $N_k \to \infty$.

Next, we investigate $I_{N_k}^{(2)}$. For any $\chi \in L^2(\Omega, \mathcal{F}, \mathbb{P}, L^2(0, T; H))$, we define the function $\mathcal{L}(\chi) = \mathbb{E}\int_0^t \langle B(\chi(s), u_{N_k}^\alpha(s)), w\rangle_{D(A)'}ds$. By (12) and (4), we have

$$\begin{aligned} |\mathcal{L}(\chi)| &\leq c\mathbb{E}\int_0^t |\chi|\|u_{N_k}^\alpha\|\|w\|^{1/2}|Aw|^{1/2}ds \\ &\leq C\mathbb{E}\int_0^t |Aw||\chi|\|u_{N_k}^\alpha\|ds. \end{aligned}$$

Using (2) and Cauchy-Schwarz's inequality, we infer that

$$|\mathcal{L}(\chi)| \leq C\|w\|_{\mathcal{M}_{\mathcal{F}_t}^\infty}\Big(\mathbb{E}\int_0^T \|u_{N_k}\|_V^2 ds\Big)^{1/2}\Big(\mathbb{E}\int_0^T |\chi|^2 ds\Big)^{1/2}. \tag{56}$$

Therefore, (44) and (56) imply that $\mathcal{L}$ is a linear continuous form. By (51), we deduce that $\mathcal{L}(v_{N_k}^\alpha) \longrightarrow \mathcal{L}(v^\alpha)$, as $N_k \to \infty$. As $I_{N_k}^{(2)} = \mathcal{L}(v_{N_k}^\alpha - v^\alpha)$, we obtain that $I_{N_k}^{(2)} \longrightarrow 0$, as $N_k \to \infty$.

Finally, we turn to $I_{N_k}^{(3)}$. Using (12), (4) and Cauchy-Schwarz's inequality, it follows that

$$\begin{aligned} I_{N_k}^{(3)} &\leq c\mathbb{E}\int_0^t |v_{N_k}^\alpha|\|u_{N_k}^\alpha - u^\alpha\|\|w\|^{1/2}|Aw|^{1/2}ds \\ &\leq C\|w\|_{\mathcal{M}_{\mathcal{F}_t}^\infty}\Big(\mathbb{E}\int_0^T |v_{N_k}^\alpha|^2 ds\Big)^{1/2}\Big(\mathbb{E}\int_0^T \|u_{N_k}^\alpha - u^\alpha\|^2 ds\Big)^{1/2}. \end{aligned}$$

As a consequence of (2), (49) and (55), we deduce that $I_{N_k}^{(3)} \longrightarrow 0$, as $N_k \to \infty$.

Gathering all convergence results, we obtain that

$$v^\alpha(t) + \int_0^t Av^\alpha(s)ds + \int_0^t B(v^\alpha(s), u^\alpha(s))ds$$
$$= v^\alpha(0) + \int_0^t f(s, u^\alpha(s))ds + \int_0^t g(s, u^\alpha(s))dW(s), \tag{57}$$

$\mathbb{P}$ -a. s. for a. e. t. Moreover, we note that $Av^\alpha \in L^2(\Omega, \mathcal{F}, \mathbb{P}; L^2(0, T; D(A)'))$, $B(v^\alpha, u^\alpha) \in L^2(\Omega, \mathcal{F}, \mathbb{P}; L^2(0, T; V'))$, $f(t, u^\alpha) \in L^2(\Omega, \mathcal{F}, \mathbb{P}; L^2(0, T; H))$ and $g(t, u^\alpha) \in L^2(\Omega, \mathcal{F}, \mathbb{P}; L^2(0, T; (L^2(\mathcal{T}))^{3m}))$.

To prove the continuity of u^α with respect to time, we note that (57) allows us to apply [14, Theorem 3.2, Chapter I]. By this theorem, we deduce that there exists $\tilde{\Omega} \in \mathcal{F}$, such that $\mathbb{P}(\tilde{\Omega}) = 1$ and v^α is continuous in H, $\mathbb{P}$-a.s. with respect to t, for any $\omega \in \tilde{\Omega}$. Therefore, u^α is $\mathbb{P}$-a. s. continuous with values in V. This concludes the proof of Theorem 2.2.

Remark 3.8. To prove the existence and the uniqueness of the pressure, we may use a generalisation of Rham's theorem for processes [15, Theorem 4.1, Remark 4.3].

3.2 *Proof of Theorem 2.3*

In the following, we will prove the pathwise uniqueness to (1). Let u_1^α and u_2^α be two weak solutions of (1) that have in $D(A)$ almost surely continuous trajectories with the same initial data u_0 and are defined on the same probability space with the same standard Wiener process. Since f and g are Lipschitz with respect to u^α, there exist $L_f > 0$ and $L_g > 0$, such that

$$|f(t, u_1^\alpha) - f(t, u_2^\alpha)| \leq L_f |u_1^\alpha - u_2^\alpha| \tag{58}$$

and

$$|g(t, u_1^\alpha) - g(t, u_2^\alpha)|_{\left(L^2(\mathcal{T})\right)^{3m}} \leq L_g |u_1^\alpha - u_2^\alpha|. \tag{59}$$

We denote by $v_1^\alpha = u_1^\alpha + \alpha^2 A u_1^\alpha$ and $v_2^\alpha = u_2^\alpha + \alpha^2 A u_2^\alpha$. Let $\delta u^\alpha = u_1^\alpha - u_2^\alpha$ and $\delta v^\alpha = v_1^\alpha - v_2^\alpha$. It follows from (14) that

$$d\delta v^\alpha + [\nu A \delta v^\alpha + B(v_1^\alpha, u_1^\alpha) - B(v_2^\alpha, u_2^\alpha)]dt$$
$$= (f(t, u_1^\alpha) - f(t, u_2^\alpha))dt + (g(t, u_1^\alpha) - g(t, u_2^\alpha))dW.$$

Applying $(I + \alpha^2 A)^{-1}$ and using Itô's formula, we obtain that

$$d\|\delta u^\alpha(t)\|_V^2 + 2\nu((A\delta u^\alpha, \delta u^\alpha))_V + 2(((I + \alpha^2 A)^{-1}\big(B(v_1^\alpha, u_1^\alpha)$$
$$-B(v_2^\alpha, u_2^\alpha)\big), \delta u^\alpha))_V = 2(((I + \alpha^2 A)^{-1}\big(f(s, u_1^\alpha) - f(s, u_2^\alpha)\big), \delta u^\alpha))_V$$
$$+\|(I + \alpha^2 A)^{-1}(g(s, u_1^\alpha) - g(s, u_2^\alpha))\|_V^2$$
$$+2(((I + \alpha^2 A)^{-1}\big(g(s, u_1^\alpha) - g(s, u_2^\alpha)\big), \delta u^\alpha))_V dW,$$

for any $t \in [0, T]$. Integrating with respect to t and using (5), we obtain

$$\|\delta u^\alpha(t)\|_V^2 + 2\nu \int_0^t ((A\delta u^\alpha(s), \delta u^\alpha(s)))_V ds$$

$$+2\int_0^t \langle B(v_1^\alpha(s), u_1^\alpha(s)) - B(v_2^\alpha(s), u_2^\alpha(s)), \delta u^\alpha(s)\rangle_{D(A)'} ds$$
$$= 2\int_0^t \Big(f(s, u_1^\alpha(s)) - f(s, u_2^\alpha(s)), \delta u^\alpha(s)\Big) ds$$
$$+ \int_0^t \|(I + \alpha^2 A)^{-1}(g(s, u_1^\alpha(s)) - g(s, u_2^\alpha(s)))\|_V^2 ds$$
$$+2\int_0^t \Big(g(s, u_1^\alpha(s)) - g(s, u_2^\alpha(s)), \delta u^\alpha(s)\Big) dW(s). \tag{60}$$

Next, we define $\vartheta^\alpha(t) = exp\Big(-\int_0^t \gamma(s)\|u_1^\alpha(s)\|^2 ds\Big)$, $0 \le t \le T$, where γ is a real-valued function that will be fixed below. Applying Itô's formula for the real process $\vartheta^\alpha(t)\|\delta u^\alpha(t)\|_V^2$, we deduce from (60) that

$$\vartheta^\alpha(t)\|\delta u^\alpha(t)\|_V^2 + 2\nu\int_0^t \vartheta^\alpha(s)\big(\|\delta u^\alpha(s)\| + \alpha^2|A\delta u^\alpha(s)|\big) ds$$
$$\le 2\int_0^t \vartheta^\alpha(s)|\langle B(v_1^\alpha(s), u_1^\alpha(s)) - B(v_2^\alpha(s), u_2^\alpha(s)), \delta u^\alpha(s)\rangle_{D(A)'}| ds$$
$$+2\int_0^t \vartheta^\alpha(s)|(f(s, u_1^\alpha(s)) - f(s, u_2^\alpha(s)), \delta u^\alpha(s))| ds$$
$$+ \int_0^t \vartheta^\alpha(s)\Big|g(s, u_1^\alpha(s)) - g(s, u_2^\alpha(s))\Big|^2_{\left(L^2(\mathcal{T})\right)^{3m}} ds$$
$$+2\int_0^t \vartheta^\alpha(s)|(g(s, u_1^\alpha(s)) - g(s, u_2^\alpha(s)), \delta u^\alpha(s))| dW(s)$$
$$- \int_0^t \vartheta^\alpha(s)|\gamma(s)|\|u_1^\alpha(s)\|_V^2\|\delta u^\alpha(s)\|_V^2 ds. \tag{61}$$

From (11) and (4), we have

$$\begin{aligned}&|\langle B(v_1^\alpha, u_1^\alpha) - B(v_2^\alpha, u_2^\alpha), \delta u^\alpha\rangle_{D(A)'}|\\ &\qquad = |\langle B(\delta v^\alpha, u_1^\alpha), \delta u^\alpha\rangle_{D(A)'} + \langle B(v_2^\alpha, \delta u^\alpha), \delta u^\alpha\rangle_{D(A)'}|\\ &\qquad = |\langle B(\delta v^\alpha, u_1^\alpha), \delta u^\alpha\rangle_{D(A)'}|.\end{aligned}$$

Using (12), (17), Young's inequality, (7) and (2), we infer that

$$\begin{aligned}|\langle B(\delta v^\alpha, u_1^\alpha), \delta u^\alpha\rangle_{D(A)'}| &\le c|\delta v^\alpha|\|u_1^\alpha\|\|\delta u^\alpha\|^{1/2}|A\delta u^\alpha(s)|^{1/2}\\ &\le c(\lambda_1^{-2} + \alpha^2)|A\delta u^\alpha|^{3/2}\|u_1^\alpha\|\|\delta u^\alpha\|^{1/2}\\ &\le \frac{c(\lambda_1^{-2} + \alpha^2)}{2}|\delta u^\alpha|\|u_1^\alpha\|^2 + \frac{c}{2}|A\delta u^\alpha|^3\\ &\le \frac{c}{2}\Big(\frac{(\lambda_1^{-2} + \alpha^2)}{\alpha^4} + |A\delta u^\alpha|^3\Big)\|u_1^\alpha\|^2\|\delta u^\alpha\|_V^2.\end{aligned}$$

Since $|\cdot| \le \|\cdot\|_V$, (58) and (59) imply that

$$|(f(s, u_1^\alpha(s)) - f(s, u_2^\alpha(s)), \delta u^\alpha(s))| \le L_f|\delta u^\alpha(s)|^2 \le \frac{c^2\lambda_1^{-2}}{\alpha^4}\|\delta u^\alpha(s)\|_V^2$$

and

$$|g(s,u_1^\alpha(s))-g(s,u_2^\alpha(s))|_{(L^2(\mathcal{T}))^{3m}}\le L_g|\delta u^\alpha(s)|\le\frac{c\lambda_1^{-1}}{\alpha^2}\|\delta u^\alpha(s)\|_V^2.$$

It follows from (61) that

$$\begin{aligned}\vartheta^\alpha(t)\|\delta u^\alpha(s)\|_V^2+2\nu\int_0^t\vartheta^\alpha(s)\big(\|\delta u^\alpha(s)\|+\alpha^2|A\delta u^\alpha(s)|\big)ds\\ \le c\int_0^t\vartheta^\alpha(s)\Big(\frac{(\lambda_1^{-1}+\alpha^2)}{\alpha^4}+|A\delta u^\alpha(s)|^3\Big)\|u_1^\alpha(s)\|^2\|\delta u^\alpha(s)\|_V^2ds\\ +\frac{c^2\lambda_1^{-2}}{\alpha^4}(2L_f+L_g^2)\int_0^t\vartheta^\alpha(s)\|\delta u^\alpha(s)\|_V^2ds\\ +2\int_0^t\vartheta^\alpha(s)|(g(s,u_1^\alpha(s))-g(s,u_2^\alpha(s)),\delta u^\alpha(s))|dW(s)\\ -\int_0^t\gamma(s)\vartheta^\alpha(s)\|u_1^\alpha(s)\|^2\|\delta u^\alpha(s)\|_V^2ds,\end{aligned}\tag{62}$$

for any $t\in[0,T]$. Taking $\gamma(s)=c\Big(\frac{(\lambda_1^{-1}+\alpha^2)}{\alpha^4}+|A\delta u^\alpha(s)|^3\Big)$ in (62), we obtain that

$$\begin{aligned}\vartheta^\alpha(t)\|\delta u^\alpha(s)\|_V^2+2\nu\int_0^t\vartheta^\alpha(s)\big(\|\delta u^\alpha(s)\|+\alpha^2|A\delta u^\alpha(s)|\big)ds\\ \le\frac{c^2\lambda_1^{-2}}{\alpha^4}(2L_f+L_g^2)\int_0^t\vartheta^\alpha(s)\|\delta u^\alpha(s)\|_V^2ds\\ +2\int_0^t\vartheta^\alpha(s)|(g(s,u_1^\alpha(s))-g(s,u_2^\alpha(s)),\delta u^\alpha(s))|dW(s).\end{aligned}\tag{63}$$

Since $0<\vartheta^\alpha(t)\le1$, $t\in[0,T]$, we have

$$\begin{aligned}\mathbb{E}\int_0^t\vartheta^\alpha(s)|(g(s,u_1^\alpha(s))-g(s,u_2^\alpha(s)),\delta u^\alpha(s))|dW(s)\\ \le\mathbb{E}\int_0^t|(g(s,u_1^\alpha(s))-g(s,u_2^\alpha(s)),\delta u^\alpha(s))|dW(s).\end{aligned}$$

Then, using the property of the stochastic integral, the expectation of the stochastic integral in (63) vanishes. Dropping the viscous term from the left-hand side of (63), we deduce that for all $t\in[0,T]$

$$\mathbb{E}(\vartheta^\alpha(t)\|\delta u^\alpha(s)\|_V^2)\le\frac{c^2\lambda_1^{-2}}{\alpha^4}(2L_f+L_g^2)\mathbb{E}\Big(\int_0^t\vartheta^\alpha(s)\|\delta u^\alpha(s)\|_V^2ds\Big).$$

By Gronwall's lemma, we obtain that $\|\delta u^\alpha(s)\|_V=0$, $\mathbb{P}$-a.s. for all $t\in[0,T]$. Hence, we deduce that $u_1^\alpha(t)=u_2^\alpha(t)$ $\mathbb{P}$-a.s. for all $t\in[0,T]$. This concludes the proof of Theorem 2.3.

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