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Statistical inference for nonergodic weighted fractional Vasicek models

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Abstract A problem of drift parameter estimation is studied for a nonergodic weighted fractional Vasicek model defined as $dX_t = \theta(\mu + X_t)dt + dB_t^{a,b}$, $t \geq 0$, with unknown parameters $\theta > 0$, $\mu \in \mathbb{R}$ and $\alpha := \theta\mu$, whereas $B^{a,b} := \{B_t^{a,b}, t \geq 0\}$ is a weighted fractional Brownian motion with parameters $a > -1$, $|b| < 1$, $|b| < a + 1$. Least square-type estimators $(\widetilde{\theta}_T, \widetilde{\mu}_T)$ and $(\widetilde{\theta}_T, \widetilde{\alpha}_T)$ are provided, respectively, for (θ, μ) and (θ, α) based on a continuous-time observation of $\{X_t,\ t \in [0, T]\}$ as $T \to \infty$. The strong consistency and the joint asymptotic distribution of $(\widetilde{\theta}_T, \widetilde{\mu}_T)$ and $(\widetilde{\theta}_T, \widetilde{\alpha}_T)$ are studied. Moreover, it is obtained that the limit distribution of $\widetilde{\theta}_T$ is a Cauchy-type distribution, and $\widetilde{\mu}_T$ and $\widetilde{\alpha}_T$ are asymptotically normal.

Keywords Weighted fractional Vasicek model, parameter estimation, strong consistency, joint asymptotic distribution, Young integral
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1 Introduction

The statistical inference of non-ergodic Itô-type diffusions has a long history. For motivation and further references, we refer the reader to Basawa and Scott [2], Dietz and Kutoyants [9], Jacod [17] and Shimizu [24]. On the one hand, the statistical analysis of equations driven by fractional Gaussian processes is obviously more recent.

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The development of stochastic calculus with respect to the fractional Gaussian processes allowed to study such models. On the other hand, the long range dependence property makes the fractional Gaussian processes important driving noises in modeling several phenomena arising from finance, economic, telecommunication networks and physics.

Let $B^{a,b} := \{B_t^{a,b}, t \geq 0\}$ be a weighted fractional Brownian motion (wfBm) with parameters (a, b) such that $a > -1$, $|b| < 1$ and $|b| < a + 1$, that is, $B^{a,b}$ is defined as a centered Gaussian process starting from zero with covariance

$$R^{a,b}(t,s) = E\left(B_t^{a,b} B_s^{a,b}\right) = \int_0^{s\wedge t} u^a \left[(t-u)^b + (s-u)^b\right] du, \quad s,t \geq 0. \qquad (1)$$

For $a = 0$, $-1 < b < 1$, the wfBm is a fractional Brownian motion (fBm), up to a multiplicative constant $\frac{2}{b+1}$, with the Hurst parameter $\frac{b+1}{2}$. The process $B^{a,b}$ was introduced in [5] as an extension of fBm. Moreover, it shares several properties with fBm, such as self-similarity, path continuity, behavior of increments, long-range dependence, nonsemimartingale, and others. But, unlike fBm, the wfBm does not have stationary increments for $a \neq 0$. For more details about the subject, we refer the reader to [5].

The purpose of this paper is to estimate jointly the drift parameters of the weighted fractional Vasicek (also called weighted fractional mean-reverting Ornstein–Uhlenbeck) process $X := \{X_t, t \geq 0\}$ that is defined as the unique (pathwise) solution to

$$X_0 = 0, \qquad dX_t = \theta\,(\mu + X_t)\,dt + dB_t^{a,b}, \quad t \geq 0, \qquad (2)$$

where $\theta > 0$ and $\mu \in \mathbb{R}$ are considered as unknown parameters. When $a = b = 0$, $B^{a,b}$ is a standard Brownian motion, and in this case the model (2) with $\mu = 0$ was originally proposed by Ornstein and Uhlenbeck and then it was generalized by Vasicek, see [23].

In recent years, several researchers have been interested in studying statistical estimation problems for Gaussian Ornstein–Uhlenbeck processes. Estimation of the drift parameters in fractional-noise-driven Ornstein–Uhlenbeck processes is a problem that is both well-motivated by practical needs and theoretically challenging. In the finance context, our practical motivation to study this estimation problem is to provide tools to understand volatility modeling in finance. Indeed, any mean-reverting model in discrete or continuous time can be taken as a model for stochastic volatility. Let us mention some important results in this field where the volatility exhibits long-memory, which means that the volatility today is correlated to past volatility values with a dependence that decays very slowly. The authors of [6, 7] considered the problem of option pricing under a stochastic volatility model that exhibits long-range dependence. More precisely they assumed that the dynamics of the volatility are described by the equation (2), where the driving process $B^{a,b}$ is a standard fractional Brownian motion (corresponding to $a = 0$) with the Hurst parameter $H = \frac{b+1}{2}$ greater than $1/2$. On the other hand, the paper [16] on rough volatility contends that the short-time behavior indicates that the Hurst parameter H in the volatility is less than $1/2$.

An example of interesting problem related to (2) is the statistical estimation of μ and θ when one observes the whole trajectory of X. In order to estimate the unknown parameters θ and μ when the whole trajectory of X defined in (2) is observed, we will consider the following least squares estimators (LSEs) proposed in [14]:

$$\widetilde{\theta}_T = \frac{\frac{1}{2}TX_T^2 - X_T\int_0^T X_s ds}{T\int_0^T X_s^2 ds - \left(\int_0^T X_s ds\right)^2} \tag{3}$$

and

$$\widetilde{\alpha}_T = \frac{X_T\int_0^T X_s^2 ds - \frac{1}{2}X_T^2\int_0^T X_s ds}{T\int_0^T X_s^2 ds - \left(\int_0^T X_s ds\right)^2}, \tag{4}$$

as statistics to estimate θ and $\alpha := \mu\theta$, respectively. Furthermore, we can obtain a least squares-type estimator $\widetilde{\mu}_T$ for μ, that is, the statistic

$$\widetilde{\mu}_T = \widetilde{\alpha}_T/\widetilde{\theta}_T = \frac{\int_0^T X_s^2 ds - \frac{1}{2}X_T\int_0^T X_s ds}{\frac{1}{2}TX_T - \int_0^T X_s ds}. \tag{5}$$

Let us mention that similar drift statistical problems for nonergodic Vasicek models were recently studied. The work [14] studied the case when the process $B^{a,b}$ in (2) is replaced by a Gaussian process, and provided sufficient conditions on the driving Gaussian process in order to ensure the strong consistency and the asymptotic distribution of the estimators given by (3), (4) and (5). Note that we cannot apply directly this result here, because $B^{a,b}$ verifies properties different of those given in [14].

Let us also describe what is known about the parameter estimation for the model (2) when $B^{a,b}$ is a fBm, i.e., $a = 0$, with the Hurst parameter $H = \frac{b+1}{2}$. Let $B^H := \left\{B_t^H, t \geq 0\right\}$ denote a fBm with the Hurst parameter $H \in (0, 1)$. Consider the following fractional Vasicek model driven by B^H,

$$dX_t \quad = \quad \theta\left(\mu + X_t\right)dt + dB_t^H, \quad X_0 = 0, \tag{6}$$

where $\theta,\ \mu \in \mathbb{R}$ are unknown parameters. Notice that the process (6) is *ergodic* if $\theta < 0$, $\mu = 0$ and $X_0 = \int_{-\infty}^0 e^{-\theta s}dB_s^H$. Otherwise, the process (6) is *nonergodic* if $\theta > 0$.

Now we recall several approaches to estimate the parameters of (6). For the maximum likelihood estimation approach, in general the techniques used to construct maximum likelihood estimators (MLEs) for the drift parameters of (6) are based on Girsanov transforms for fBm and depend on the properties of the deterministic fractional operators (determined by the Hurst parameter) related to the fBm. In general, the MLE is not easily computable. In particular, it relies on being able to constitute a discretization of an MLE. For a more recent comprehensive discussion via this method, we refer to [18].

A least squares approach has been also considered by several researchers to study statistical estimation problems for (6). Let us mention some works in this direction: in the case when $\theta < 0$, the statistical estimation for the parameters μ and θ based on continuous-time observations of $\{X_t,\ t \in [0, T]\}$ as $T \to \infty$, has been studied by several papers, for instance [8, 3] and the references therein. When $\mu = 0$ in (6), the estimation of θ has been investigated by using least squares method as follows: the case of ergodic-type fractional Ornstein–Uhlenbeck processes, corresponding to $\theta < 0$, has been considered in [11, 15], and the case nonergodic fractional Ornstein–Uhlenbeck processes has been studied in [10, 12].

The paper is organized as follows. In Section 2 we analyze some pathwise properties of the Vasicek model (2). In Section 3 we prove the strong consistency of the estimators $\widetilde{\theta}_T$, $\widetilde{\mu}_T$ and $\widetilde{\alpha}_T$ as $T \to \infty$. Section 4 is devoted to analyze the joint asymptotic distribution of the LSEs $(\widetilde{\theta}_T, \widetilde{\mu}_T)$ and $(\widetilde{\theta}_T, \widetilde{\alpha}_T)$ as $T \to \infty$.

Throughout the paper, we shall use notation C for various constants whose value is not important and may change from line to line and even in the same line.

2 Notations and auxiliary results

This section is devoted to study pathwise properties of the nonergodic weighted fractional Vasicek model (2). These properties will be needed in order to analyze the asymptotic behavior of the LSEs $(\widetilde{\theta}_T, \widetilde{\mu}_T)$ and $(\widetilde{\theta}_T, \widetilde{\alpha}_T)$.

Because (2) is linear, it is immediate to solve it explicitly; one then gets the following formula

$$X_t = \mu\left(e^{\theta t} - 1\right) + e^{\theta t}\int_0^t e^{-\theta s} dB_s^{a,b}, \quad t \geq 0, \tag{7}$$

where the integral with respect to $B^{a,b}$ is understood in the Young sense (see Appendix).

Let us introduce the following processes, for every $t \geq 0$:

$$\zeta_t := \int_0^t e^{-\theta s} dB_s^{a,b}, \quad Z_t := \int_0^t e^{-\theta s} B_s^{a,b} ds, \quad \Sigma_t := \int_0^t X_s ds. \tag{8}$$

Thus, using (7), we can write

$$X_t = \mu\left(e^{\theta t} - 1\right) + e^{\theta t}\zeta_t. \tag{9}$$

Furthermore, by (2),

$$X_t = \mu\theta t + \theta\Sigma_t + B_t^{a,b}. \tag{10}$$

Moreover, applying the formula (49), we have

$$\zeta_t = e^{-\theta t} B_t^{a,b} + \theta\int_0^t e^{-\theta s} B_s^{a,b} ds = e^{-\theta t} B_t^{a,b} + \theta Z_t. \tag{11}$$

Lemma 1 ([1]). *Suppose that $a > -1$, $|b| < 1$ and $|b| < a+1$. Then, we can rewrite the covariance $R^{a,b}(t,s)$ of $B^{a,b}$, given in (1), as follows:*

$$R^{a,b}(t,s) \;=\; \beta\,(a+1, b+1)\left[t^{a+b+1} + s^{a+b+1}\right] - m(t,s), \tag{12}$$

where $\beta(c,d) = \int_0^1 x^{c-1}(1-x)^{d-1}dx$, $c > 0$, $d > 0$, denotes the usual Beta function, and the function $m(t,s)$ is defined by

$$m(t,s) \;:=\; \int_{s\wedge t}^{s\vee t} u^a (t\vee s - u)^b du.$$

There exists a constant C depending only on a, b, such that for every $s, t \geq 0$,

$$E\left[\left(B_t^{a,b} - B_s^{a,b}\right)^2\right] \leq C(s \vee t)^a |t-s|^{b+1}. \tag{13}$$

Since the process $B^{a,b}$ has $((a+b+1) \wedge (b+1) - \varepsilon)$-Hölder continuous paths for all $\varepsilon \in (0, (a+b+1) \wedge (b+1))$, and $E\left[(B_t^{a,b})^2\right] = 2\beta\,(a+1, b+1)\, t^{a+b+1}$, we can deduce from [10, Lemma 2.1] and [10, Lemma 2.2] the following result.

Lemma 2. *Assume that $a > -1$, $|b| < 1$ and $|b| < a+1$. Let Z and ζ be given by (8). Then for all $\varepsilon \in (0, (a+b+1) \wedge (b+1))$ the process ζ admits a modification with $((a+b+1) \wedge (b+1) - \varepsilon)$-Hölder continuous paths, still denoted ζ in the sequel.*

Moreover,

$$Z_T \longrightarrow Z_\infty := \int_0^\infty e^{-\theta s} B_s^{a,b} ds, \qquad \zeta_T \longrightarrow \zeta_\infty := \theta Z_\infty \tag{14}$$

almost surely and in $L^2(\Omega)$ as $T \to \infty$.

Also,

$$\lim_{T\to\infty} e^{-2\theta T} \int_0^T e^{2\theta s} \zeta_s^2 ds = \frac{1}{2\theta} \zeta_\infty^2 \text{ almost surely.} \tag{15}$$

We will make use of the following two technical lemmas.

Lemma 3. *Suppose that $a > -1$, $|b| < 1$ and $|b| < a+1$. Then, almost surely, as $T \to \infty$,*

$$\frac{B_T^{a,b}}{T^\delta} \longrightarrow 0 \quad \text{for all } \delta > \frac{a+b+1}{2}, \tag{16}$$

$$e^{-\theta T} X_T \longrightarrow \mu + \zeta_\infty, \tag{17}$$

$$e^{-\theta T} \int_0^T X_s ds \longrightarrow \frac{1}{\theta}(\mu + \zeta_\infty), \tag{18}$$

$$\frac{e^{-\theta T}}{T} \int_0^T s X_s ds \longrightarrow \frac{1}{\theta}(\mu + \zeta_\infty), \tag{19}$$

$$e^{-2\theta T} \int_0^T X_s^2 ds \longrightarrow \frac{1}{2\theta}(\mu + \zeta_\infty)^2, \tag{20}$$

$$\frac{e^{-\theta T}}{T^\delta} \int_0^T |X_s| ds \longrightarrow 0 \quad \text{for any } \delta > 0, \tag{21}$$

$$\frac{e^{-\theta T}}{T} \int_0^T |B_t^{a,b} X_t| dt \longrightarrow 0 \quad \text{if } a+b < 1, \tag{22}$$

where ζ_∞ is defined in Lemma 2.

Proof. Let us prove (16). Let $\delta > \frac{a+b+1}{2}$. By the Borel–Cantelli lemma, it is sufficient to prove that, for any $\varepsilon > 0$,

$$\sum_{n\geq 0} P\left(\sup_{n\leq T\leq n+1} \left|\frac{B_T^{a,b}}{T^\delta}\right| > \varepsilon\right) < \infty.$$

Let $q > 0$ such that $q(\delta - \frac{a+b+1}{2}) > 1$ and $\frac{q(b+1)}{2} > 1$. Applying Markov's inequality, we obtain

$$\begin{aligned} P\left(\sup_{n\leq T\leq n+1}\left|\frac{B_T^{a,b}}{T^\delta}\right| > \varepsilon\right) &\leq \frac{1}{\varepsilon^q} E\left[\sup_{n\leq T\leq n+1}\left|\frac{B_T^{a,b}}{T^\delta}\right|^q\right] \\ &\leq \frac{1}{\varepsilon^q n^{q\delta}} E\left[\sup_{n\leq T\leq n+1}\left|B_T^{a,b}\right|^q\right]. \end{aligned}$$

Further, applying the Garsia–Rodemich–Rumsey Lemma (see [21, Lemma A.3.1]) for $\psi(x) = x^q$, $p(x) = x^{\frac{m+2}{q}}$, with $0 < m < \frac{q(b+1)}{2} - 1$, we get for every $n \leq s, t \leq n+1$,

$$\left|B_t^{a,b} - B_s^{a,b}\right|^q \leq C\,|t-s|^m \int_n^{n+1}\int_n^{n+1} \frac{\left|B_u^{a,b} - B_v^{a,b}\right|^q}{|u-v|^{m+2}} du dv.$$

This together with (13) implies

$$\begin{aligned} E\left[\sup_{n\leq s,t\leq n+1}\left|B_t^{a,b} - B_s^{a,b}\right|^q\right] &\leq C\int_n^{n+1}\int_n^{n+1} (u\vee v)^{\frac{qa}{2}} \frac{|u-v|^{\frac{q(b+1)}{2}}}{|u-v|^{m+2}} du dv \\ &\leq C n^{\frac{qa}{2}} \int_0^1\int_0^1 |x-y|^{\frac{q(b+1)}{2}-m-2} dx dy \\ &\leq C n^{\frac{qa}{2}}. \end{aligned}$$

Hence,

$$\begin{aligned} E\left[\sup_{n\leq T\leq n+1}\left|B_T^{a,b}\right|^q\right] &\leq C\left(E\left[\sup_{n\leq T\leq n+1}\left|B_T^{a,b} - B_n^{a,b}\right|^q\right] + E\left|B_n^{a,b}\right|^q\right) \\ &\leq C\left(E\left[\sup_{n\leq s,t\leq n+1}\left|B_t^{a,b} - B_s^{a,b}\right|^q\right] + n^{\frac{q(a+b+1)}{2}}\right) \\ &\leq C\left(n^{\frac{qa}{2}} + n^{\frac{q(a+b+1)}{2}}\right) \\ &\leq C n^{\frac{q(a+b+1)}{2}}. \end{aligned}$$

Then,

$$P\left(\sup_{n\leq T\leq n+1}\left|\frac{B_T^{a,b}}{T^\delta}\right| > \varepsilon\right) \leq \frac{C}{\varepsilon^q n^{q(\delta - \frac{a+b+1}{2})}}.$$

As a consequence, since $q(\delta - \frac{a+b+1}{2}) > 1$, the above series converges, which proves (16).

Notice that the convergence (17) is a direct consequence of (9) and (14).

On the other hand, using (49), (14) and (16) we have almost surely as $T \to \infty$,

$$e^{-\theta T}\int_0^T e^{\theta s} Z_s ds = \frac{Z_T}{\theta} - \frac{e^{-\theta T}}{\theta}\int_0^T B_s^{a,b} ds \longrightarrow \frac{\zeta_\infty}{\theta^2}. \tag{23}$$

Combining (9), (11) and (23), we get almost surely as $T \to \infty$,

$$e^{-\theta T}\int_0^T X_s ds = e^{-\theta T}\int_0^T \left(\mu(e^{\theta s}-1) + B_s^{a,b} + \theta e^{\theta s} Z_s\right) ds \longrightarrow \frac{1}{\theta}\left(\mu + \zeta_\infty\right),$$

which proves (18). Similarly, (49), (14), (16) and (23) imply that almost surely as $T \to \infty$,

$$\frac{e^{-\theta T}}{T}\int_0^T s e^{\theta s} Z_s ds = \frac{Z_T}{\theta} - \frac{e^{-\theta T}}{\theta T}\int_0^T s B_s^{a,b} ds - \frac{e^{-\theta T}}{\theta T}\int_0^T e^{\theta s} Z_s ds \longrightarrow \frac{\zeta_\infty}{\theta^2}.$$

Combined with (9) and (11), a straightforward calculation as above, leads to (19).

Using similar arguments as in (23), we deduce that almost surely as $T \to \infty$,

$$e^{-2\theta T}\int_0^T e^{2\theta s} Z_s ds \longrightarrow \frac{\zeta_\infty}{2\theta^2}.$$

Combining this latter convergence together with (9), (14), (15) and (16), we can deduce (20).

Using (17), we have almost surely as $T \to \infty$,

$$\frac{e^{-\theta T}}{T^\delta}\int_0^T |X_s| ds \le \sup_{t\ge 0}\left|\frac{X_t}{e^{\theta t}}\right| \frac{e^{-\theta T}}{T^\delta}\int_0^T e^{\theta s} ds \longrightarrow 0,$$

which implies (21).

Now we prove (22). Let $\frac{a+b+1}{2} < \delta < 1$,

$$\begin{aligned}
\frac{e^{-\theta T}}{T}\int_0^T |B_t^{a,b} X_t| dt &\le \sup_{t\ge 0}\left|\frac{B_t^{a,b} X_t}{t^\delta e^{\theta t}}\right| \frac{e^{-\theta T}}{T}\int_0^T t^\delta e^{\theta t} dt \\
&\le \sup_{t\ge 0}\left|\frac{B_t^{a,b} X_t}{t^\delta e^{\theta t}}\right| \frac{e^{-\theta T}}{T^{1-\delta}}\int_0^T e^{\theta t} dt \\
&\le \sup_{t\ge 0}\left|\frac{B_t^{a,b} X_t}{t^\delta e^{\theta t}}\right| \frac{1}{\theta T^{1-\delta}} \\
&\longrightarrow 0
\end{aligned}$$

almost surely as $T \to \infty$, where we used that $\sup_{t\ge 0}\left|\frac{B_t^{a,b} X_t}{t^\delta e^{\theta t}}\right| < \infty$ almost surely, thanks to (16) and (17). Thus the proof of (22) is done. □

3 Strong consistency

In this section we will prove the strong consistency of the estimators $\widetilde{\theta}_T$, $\widetilde{\mu}_T$ and $\widetilde{\alpha}_T$.

Theorem 1. *Suppose that $a > -1$, $|b| < 1$ and $|b| < a+1$. Let $\widetilde{\theta}_T$, $\widetilde{\alpha}_T$ and $\widetilde{\mu}_T$ be given by (3), (4) and (5), respectively. Then, almost surely, as $T \to \infty$,*

$$\widetilde{\theta}_T \longrightarrow \theta. \tag{24}$$

Moreover, if $a + b < 1$,

$$\widetilde{\mu}_T \longrightarrow \mu, \tag{25}$$

and consequently,

$$\widetilde{\alpha}_T = \widetilde{\mu}_T \widetilde{\theta}_T \longrightarrow \alpha = \mu\theta$$

almost surely, as $T \to \infty$.

Proof. Combining (3) and the convergences (17), (20) and (21), we obtain

$$\begin{aligned} \widetilde{\theta}_T &= \frac{\frac{1}{2}\left(e^{-\theta T} X_T\right)^2 - e^{-\theta T} X_T \frac{e^{-\theta T}}{T} \int_0^T X_s ds}{e^{-2\theta T} \int_0^T X_s^2 ds - \left(\frac{e^{-\theta T}}{\sqrt{T}} \int_0^T X_s ds\right)^2} \\ &\longrightarrow \theta, \quad \text{almost surely, as } T \to \infty. \end{aligned}$$

Thus the convergence (24) is obtained.

Now we prove (25). It follows from (5) that $\widetilde{\mu}_T$ can be written as follows:

$$\widetilde{\mu}_T = \frac{e^{-\theta T}}{T}\left[\int_0^T X_s^2 ds - \frac{X_T}{2}\int_0^T X_s ds\right] \times \frac{1}{\frac{1}{2} e^{-\theta T} X_T - \frac{e^{-\theta T}}{T} \int_0^T X_s ds}.$$

According to the convergences (17) and (21) we have, almost surely, as $T \to \infty$,

$$\frac{1}{\frac{1}{2} e^{-\theta T} X_T - \frac{e^{-\theta T}}{T} \int_0^T X_s ds} \longrightarrow \frac{2}{\mu + \zeta_\infty}.$$

Therefore, it remains to prove

$$\frac{e^{-\theta T}}{T}\left[\int_0^T X_s^2 ds - \frac{X_T}{2}\int_0^T X_s ds\right] \longrightarrow \frac{\mu}{2}\left(\mu + \zeta_\infty\right) \tag{26}$$

almost surely, as $T \to \infty$.

Using the formula (49) and the equation (2), we have

$$\begin{aligned} &\int_0^T X_s^2 ds - \frac{X_T}{2}\int_0^T X_s ds \\ &= \int_0^T X_s d\Sigma_s - \frac{1}{2}\left(\mu\theta T + \theta\Sigma_T + B_T^{a,b}\right)\Sigma_T \\ &= \int_0^T \left(\mu\theta s + \theta\Sigma_s + B_s^{a,b}\right) d\Sigma_s - \frac{\mu\theta}{2} T\Sigma_T - \frac{\theta}{2}\Sigma_T^2 - \frac{1}{2} B_T^{a,b}\Sigma_T \\ &= \mu\theta\int_0^T sX_s ds + \frac{\theta}{2}\Sigma_T^2 + \int_0^T B_s^{a,b} d\Sigma_s - \frac{\mu\theta}{2} T\Sigma_T - \frac{\theta}{2}\Sigma_T^2 - \frac{1}{2} B_T^{a,b}\Sigma_T \\ &= \left(\mu\theta\int_0^T sX_s ds - \frac{\mu\theta}{2} T\Sigma_T\right) + \left(\int_0^T B_s^{a,b} d\Sigma_s - \frac{1}{2} B_T^{a,b}\Sigma_T\right) \\ &=: I_T + J_T. \end{aligned}$$

Moreover, by L'Hôpital's rule and (17) we have

$$
\begin{aligned}
\frac{e^{-\theta T}}{T} I_T &= \frac{e^{-\theta T}}{T}\left(\mu\theta\int_0^T sX_s ds - \frac{\mu\theta}{2} T\Sigma_T\right) \\
&\longrightarrow \frac{\mu}{2}\left(\mu+\zeta_\infty\right)
\end{aligned}
$$

almost surely, as $T \to \infty$.

On the other hand, taking $\frac{a+b+1}{2} < \delta < 1$,

$$
\begin{aligned}
\frac{e^{-\theta T}}{T}|J_T| &= \frac{e^{-\theta T}}{T}\left|\int_0^T B_s^{a,b} d\Sigma_s - \frac{1}{2}B_T^{a,b}\Sigma_T\right| \\
&= \frac{e^{-\theta T}}{T}\left|\int_0^T B_s^{a,b} X_s ds - \frac{1}{2}B_T^{a,b}\Sigma_T\right| \\
&\leq \frac{3}{2}\left(\sup_{s\geq 0}\left|\frac{B_s^{a,b}}{s^\delta}\right|\right)\frac{e^{-\theta T}}{T^{1-\delta}}\int_0^T |X_s| ds \\
&\longrightarrow 0
\end{aligned}
$$

almost surely, as $T \to \infty$, where we used (16) and (21).

Consequently, the convergence (26) is proved. Thus the desired results are obtained. ☐

4 Joint asymptotic distribution

In this section we analyze the joint asymptotic distribution of the LSEs $(\widetilde{\theta}_T, \widetilde{\mu}_T)$ and $(\widetilde{\theta}_T, \widetilde{\alpha}_T)$.

Lemma 4. *Suppose that $a > -1$, $|b| < 1$ and $|b| < a+1$. Then*

1) *The limit of the variance of $T^{-\frac{a}{2}}e^{-\theta T}\int_0^T e^{\theta s} dB_s^{a,b}$ exists as $T \to \infty$. More precisely,*

$$
\lim_{T\to\infty} E\left[\left(T^{-\frac{a}{2}}e^{-\theta T}\int_0^T e^{\theta s} dB_s^{a,b}\right)^2\right] \longrightarrow \frac{\Gamma(b+1)}{\theta^{b+1}}. \tag{27}
$$

2) *For all fixed $s \geq 0$,*

$$
\lim_{T\to\infty} E\left(B_s^{a,b} T^{-\frac{a}{2}} e^{-\theta T}\int_0^T e^{\theta r} dB_r^{a,b}\right) = 0. \tag{28}
$$

3) *For all fixed $s \geq 0$, as $T \longrightarrow \infty$*

$$
\frac{E\left[B_s^{a,b}B_T^{a,b}\right]}{T^{\frac{a+b+1}{2}}} \longrightarrow 0, \quad E\left[\frac{B_T^{a,b}}{T^{\frac{a+b+1}{2}}} T^{-\frac{a}{2}} e^{-\theta T}\int_0^T e^{\theta r} dB_r^{a,b}\right] \longrightarrow 0. \tag{29}
$$

Proof. Both convergences (27) and (28) are proved in [1]. Let us now prove the convergences given in (29). Fix $s \geq 0$. Using (12), and $b-a-1<0$ and change of variables $x=u/T$, we get

$$
\begin{aligned}
&\lim_{T\to\infty} \frac{E\left[B_s^{a,b} B_T^{a,b}\right]}{T^{\frac{a+b+1}{2}}} \\
&= \lim_{T\to\infty} T^{\frac{-a-b-1}{2}} \left[\beta\left(a+1,b+1\right)\left[s^{a+b+1}+T^{a+b+1}\right] - m(s,T)\right] \\
&= \lim_{T\to\infty} T^{\frac{-a-b-1}{2}} \left[\beta\left(a+1,b+1\right) T^{a+b+1} - \int_s^T u^a (T-u)^b du\right] \\
&= \lim_{T\to\infty} T^{\frac{-a-b-1}{2}} \left[-\int_0^T u^a (T-u)^b du + \int_0^s u^a (T-u)^b du\right] \\
&= \lim_{T\to\infty} T^{\frac{-a-b-1}{2}} \left[-T^{a+b+1}\int_0^1 x^a (1-x)^b dx + \int_0^s u^a (T-u)^b du\right] \\
&= \lim_{T\to\infty} T^{\frac{-a-b-1}{2}} \int_0^s u^a (T-u)^b du.
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
T^{\frac{-a-b-1}{2}} \int_0^s u^a (T-u)^b du &= T^{\frac{b-a-1}{2}} \int_0^s u^a \left(1-\frac{u}{T}\right)^b du \\
&\leq \begin{cases} T^{\frac{b-a-1}{2}} \int_0^s u^a du & \text{if } b \geq 0, \\ T^{\frac{b-a-1}{2}} \int_0^s u^a \left(1-\frac{s}{T}\right)^b du & \text{if } b < 0 \end{cases} \\
&\longrightarrow 0.
\end{aligned}
$$

Thus, we deduce that, for all fixed $s \geq 0$,

$$
\lim_{T\to\infty} \frac{E\left[B_s^{a,b} B_T^{a,b}\right]}{T^{\frac{a+b+1}{2}}} = 0.
$$

In order to complete the proof, we show that

$$
\lim_{T\to\infty} E\left[\frac{B_T^{a,b}}{T^{\frac{a+b+1}{2}}} T^{-\frac{a}{2}} e^{-\theta T} \int_0^T e^{\theta r} dB_r^{a,b}\right] = 0. \tag{30}
$$

Applying twice (49), we can write

$$
\begin{aligned}
&E\left[\frac{B_T^{a,b}}{T^{\frac{a+b+1}{2}}} T^{-\frac{a}{2}} e^{-\theta T} \int_0^T e^{\theta r} dB_r^{a,b}\right] \qquad (31)\\
&= T^{-a-\frac{b+1}{2}} \left[R^{a,b}(T,T) - \theta e^{-\theta T} \int_0^T e^{\theta r} R^{a,b}(T,r) dr\right] \\
&= T^{-a-\frac{b+1}{2}} e^{-\theta T} \int_0^T e^{\theta r} \beta(a+1,b+1)(a+b+1) r^{a+b} dr
\end{aligned}
$$

$$
\begin{aligned}
& -T^{-a-\frac{b+1}{2}} e^{-\theta T} \int_0^T e^{\theta r} \frac{\partial m(T,r)}{\partial r} dr \\
= & \quad (a+b+1)\beta(a+1,b+1) T^{-a-\frac{b+1}{2}} e^{-\theta T} \int_0^T e^{\theta r} r^{a+b} dr \\
& + T^{-a-\frac{b+1}{2}} e^{-\theta T} \int_0^T e^{\theta r} r^a (T-r)^b dr. \qquad (32)
\end{aligned}
$$

Moreover, applying L'Hôspital's rule, we get

$$
\begin{aligned}
& \lim_{T\to\infty} T^{-a-\frac{b+1}{2}} e^{-\theta T} \int_0^T e^{\theta r} r^{a+b} dr \qquad (33) \\
= & \lim_{T\to\infty} \frac{e^{\theta T} T^{a+b}}{e^{\theta T}\left[(a+\frac{b+1}{2}) T^{a+\frac{b-1}{2}} + \theta T^{a+\frac{b+1}{2}}\right]} \\
= & \lim_{T\to\infty} \frac{1}{(a+\frac{b+1}{2}) T^{\frac{-b-1}{2}} + \theta T^{\frac{1-b}{2}}} \\
= & \ 0, \qquad (34)
\end{aligned}
$$

since $|b| < 1$. Also, using change of variables $x = r/T$ and $y = T - r$, we have

$$
\begin{aligned}
& T^{-a-\frac{b+1}{2}} e^{-\theta T} \int_0^T e^{\theta r} r^a (T-r)^b dr \\
= & \quad T^{-a-\frac{b+1}{2}} e^{-\theta T} \left[\int_0^{T/2} e^{\theta r} r^a (T-r)^b dr + \int_{T/2}^T e^{\theta r} r^a (T-r)^b dr\right] \\
\leq & \quad T^{-a-\frac{b+1}{2}} e^{-\theta T/2} \int_0^{T/2} r^a (T-r)^b dr + C T^{-\frac{b+1}{2}} e^{-\theta T} \int_{T/2}^T e^{\theta r} (T-r)^b dr \\
= & \quad T^{\frac{b+1}{2}} e^{-\theta T/2} \int_0^{1/2} x^a (1-x)^b dr + C T^{-\frac{b+1}{2}} \int_0^{T/2} e^{-\theta y} y^b dy \\
\leq & \quad T^{\frac{b+1}{2}} e^{-\theta T/2} \beta(a+1,b+1) + C T^{-\frac{b+1}{2}} \Gamma(b+1) \\
\longrightarrow & \quad 0 \qquad (35)
\end{aligned}
$$

as $T \to \infty$, thanks to $a + 1 > 0$ and $b + 1 > 0$.

Consequently, combining (32), (34) and (35), we obtain (30). □

In order to investigate the asymptotic behavior in distribution of the estimators $(\widetilde{\theta}_T, \widetilde{\mu}_T)$ and $(\widetilde{\theta}_T, \widetilde{\alpha}_T)$, as $T \to \infty$, we will also need the following lemmas.

Lemma 5. *Suppose that $a > -1$, $|b| < 1$ and $|b| < a + 1$. Let X be the process given by (2). Then we have for every $T > 0$,*

$$
\begin{aligned}
\frac{1}{2} X_T^2 - \frac{X_T}{T} \int_0^T X_t dt \quad = \quad & \theta \left(\int_0^T X_t^2 dt - \frac{1}{T} \left(\int_0^T X_t dt \right)^2 \right) \\
& + (\mu + \theta Z_T) \int_0^T e^{\theta t} dB_t^{a,b} + R_T, \qquad (36)
\end{aligned}
$$

where Z_T is given in (8), and the process R_T is defined by

$$\begin{aligned} R_T \quad := \quad & \frac{1}{2}(\mu\theta T)^2 + \frac{1}{2}(B_T^{a,b})^2 - \mu B_T^{a,b} - \frac{(\mu\theta)^2 T^2}{2} - \frac{B_T^{a,b}}{T}\int_0^T X_t dt \\ & -\theta\int_0^T (B_t^{a,b})^2 dt + \theta^2 \int_0^T e^{-\theta t} B_t^{a,b} \int_0^t e^{\theta s} B_s^{a,b} ds dt. \end{aligned}$$

Moreover, as $T \longrightarrow \infty$,

$$T^{-\frac{a}{2}} e^{-\theta T} R_T \longrightarrow 0 \tag{37}$$

almost surely.

Proof. Using similar arguments as in [14, Lemma 3.1] we obtain (36). On the other hand, the convergence (37) is a direct consequence of (16), (18) and $|b| < 1$. □

Lemma 6. *Suppose that $a > -1$, $|b| < 1$ and $|b| < a + 1$. Let F be any $\sigma\{B_t^{a,b}, t \geq 0\}$-measurable random variable such that $P(F < \infty) = 1$. Then as $T \to \infty$,*

$$\left(F, T^{-\frac{a}{2}} e^{-\theta T} \int_0^T e^{\theta t} dB_t^{a,b}\right) \xrightarrow{Law} \left(F, \sqrt{\frac{\Gamma(b+1)}{\theta^{b+1}}} N_2\right), \tag{38}$$

and

$$\begin{aligned} & \left(\frac{B_T^{a,b}}{T^{\frac{a+b+1}{2}}}, F, T^{-\frac{a}{2}} e^{-\theta T} \int_0^T e^{\theta t} dB_t^{a,b}\right) \\ & \xrightarrow{Law} \left(\sqrt{2\beta(a+1, b+1)} N_1, F, \sqrt{\frac{\Gamma(b+1)}{\theta^{b+1}}} N_2\right), \end{aligned} \tag{39}$$

where $N_1 \sim \mathcal{N}(0,1)$, $N_2 \sim \mathcal{N}(0,1)$ and $B^{a,b}$ are independent.

Proof. The convergence (38) is proved in [10, Lemma 2.4]. Now we prove (39). Using similar arguments as in the proof of [13, Lemma 7] it suffices to prove that for every positive integer d, and for all fixed $s_1, \ldots, s_d \geq 0$, $\left(\frac{B_T^{a,b}}{T^{\frac{a+b+1}{2}}}, B_{s_1}^{a,b}, \ldots, B_{s_d}^{a,b},\right.$ $\left. T^{-\frac{a}{2}} e^{-\theta T} \int_0^T e^{\theta t} dB_t^{a,b}\right)$ converges in distribution to

$$\left(\sqrt{2\beta(a+1, b+1)} N_1, B_{s_1}^{a,b}, \ldots, B_{s_d}^{a,b}, \sqrt{\frac{\Gamma(b+1)}{\theta^{b+1}}} N_2\right)$$

as $T \to \infty$. Moreover, since the left-hand side in this latter convergence is a Gaussian vector, it is sufficient to establish the convergence of its covariance matrix. Combining this with Lemma 4, the desired result is obtained. □

Recall that if $X \sim \mathcal{N}(m_1, \sigma_1)$ and $Y \sim \mathcal{N}(m_2, \sigma_2)$ are two independent random variables, then X/Y follows a Cauchy-type distribution. For a motivation and further references, we refer the reader to [22], as well as [19]. Notice also that if $N \sim \mathcal{N}(0,1)$ is independent of $B^{a,b}$, then N is independent of ζ_∞, since $\zeta_\infty = \theta \int_0^\infty e^{-\theta s} B_s^{a,b} ds$ is a functional of $B^{a,b}$.

Theorem 2. *Suppose that $a > -1$, $|b| < 1$ and $|b| < a+1$. Suppose that $N_1 \sim \mathcal{N}(0,1)$, $N_2 \sim \mathcal{N}(0,1)$ and $B^{a,b}$ are independent. Then as $T \to \infty$,*

$$e^{\theta T}\left(\widetilde{\theta}_T - \theta\right) \xrightarrow{Law} \frac{2\sqrt{\frac{\Gamma(b+1)}{\theta^{b-1}}}N_2}{\mu + \zeta_\infty}. \tag{40}$$

Moreover, if $a + b < 1$, then as $T \to \infty$,

$$T^{\frac{1-a-b}{2}}\left(\widetilde{\mu}_T - \mu\right) \xrightarrow{Law} \frac{\sqrt{2\beta(a+1,b+1)}}{\theta}N_1, \tag{41}$$

$$T^{\frac{1-a-b}{2}}\left(\widetilde{\alpha}_T - \alpha\right) \xrightarrow{Law} \sqrt{2\beta(a+1,b+1)}N_1. \tag{42}$$

Also, as $T \to \infty$,

$$\left(e^{\theta T}\left(\widetilde{\theta}_T - \theta\right), T^{\frac{1-a-b}{2}}\left(\widetilde{\mu}_T - \mu\right)\right) \xrightarrow{Law} \left(\frac{2\sqrt{\frac{\Gamma(b+1)}{\theta^{b-1}}}N_2}{\mu + \zeta_\infty}, \frac{\sqrt{2\beta(a+1,b+1)}}{\theta}N_1\right), \tag{43}$$

$$\left(e^{\theta T}\left(\widetilde{\theta}_T - \theta\right), T^{\frac{1-a-b}{2}}\left(\widetilde{\alpha}_T - \alpha\right)\right) \xrightarrow{Law} \left(\frac{2\sqrt{\frac{\Gamma(b+1)}{\theta^{b-1}}}N_2}{\mu + \zeta_\infty}, \sqrt{2\beta(a+1,b+1)}N_1\right). \tag{44}$$

Proof. First we prove (40). From (3) and (36) we can write

$$\begin{aligned}
& T^{-\frac{a}{2}}e^{\theta T}\left(\widetilde{\theta}_T - \theta\right) \\
= & \frac{(\mu + \theta Z_T)\, T^{-\frac{a}{2}}e^{-\theta T}\int_0^T e^{\theta t}dB_t^{a,b} + T^{-\frac{a}{2}}e^{-\theta T}R_T}{e^{-2\theta T}\left(\int_0^T X_t^2 dt - \frac{1}{T}\left(\int_0^T X_t dt\right)^2\right)} \\
= & \frac{T^{-\frac{a}{2}}e^{-\theta T}\int_0^T e^{\theta t}dB_t^{a,b}}{(\mu + \zeta_\infty)} \times \frac{(\mu + \zeta_\infty)(\mu + \theta Z_T)}{e^{-2\theta T}\left(\int_0^T X_t^2 dt - \frac{1}{T}\left(\int_0^T X_t dt\right)^2\right)} \\
& + \frac{T^{-\frac{a}{2}}e^{-\theta T}R_T}{e^{-2\theta T}\left(\int_0^T X_t^2 dt - \frac{1}{T}\left(\int_0^T X_t dt\right)^2\right)} \\
=: & \; a_T \times b_T + c_T.
\end{aligned} \tag{45}$$

Lemma 6 yields, as $T \to \infty$,

$$a_T \xrightarrow{Law} \frac{\sqrt{\frac{\Gamma(b+1)}{\theta^{b+1}}}N_2}{\mu + \zeta_\infty},$$

whereas (20), (21) and (14) imply that $b_T \longrightarrow 2\theta$ almost surely as $T \to \infty$. On the other hand, by (20), (21) and (37), we obtain that $c_T \longrightarrow 0$ almost surely as $T \to \infty$.

Combining all these facts together with (38), we get (40).

Using (3) and (5), a straightforward calculation shows that $\widetilde{\theta}_T$ and $\widetilde{\mu}_T$ verify

$$\widetilde{\theta}_T \widetilde{\mu}_T T \quad = \quad \widetilde{\theta}_T \frac{\widetilde{\mu}_T}{X_T} X_T T = X_T - \widetilde{\theta}_T \int_0^T X_t dt. \tag{46}$$

Combining (46) with (2), we obtain

$$\begin{aligned} & T^{\frac{1-a-b}{2}} \left(\widetilde{\mu}_T - \mu \right) \qquad (47) \\ = \;& \frac{1}{\widetilde{\theta}_T} \left[-e^{\theta T} \left(\widetilde{\theta}_T - \theta \right) \frac{e^{-\theta T}}{T^{\frac{a+b+1}{2}}} \int_0^T X_t dt - \mu \frac{T^{\frac{1-a-b}{2}}}{e^{\theta T}} e^{\theta T} \left(\widetilde{\theta}_T - \theta \right) + \frac{B_T^{a,b}}{T^{\frac{a+b+1}{2}}} \right] \\ =: \;& \frac{1}{\widetilde{\theta}_T} \left[D_T + \frac{B_T^{a,b}}{T^{\frac{a+b+1}{2}}} \right]. \qquad (48) \end{aligned}$$

Using (21), (24), (40) and Slutsky's theorem, we obtain $D_T \longrightarrow 0$ in probability as $T \to \infty$. This together with (24) implies (41).

Further, according to (46) and (2) we can write

$$T^{\frac{1-a-b}{2}} \left(\widetilde{\alpha}_T - \alpha \right) \quad = \quad -e^{\theta T} \left(\widetilde{\theta}_T - \theta \right) \frac{e^{-\theta T}}{T^{\frac{a+b+1}{2}}} \int_0^T X_t dt + \frac{B_T^{a,b}}{T^{\frac{a+b+1}{2}}},$$

which proves (42), by using (21), (24), (40), and Slutsky's theorem.

Let us now prove (43). By (45) and (48) we have

$$\begin{aligned} & \left(e^{\theta T} (\widetilde{\theta}_T - \theta), T^{\frac{1-a-b}{2}} \left(\widetilde{\mu}_T - \mu \right) \right) \\ = \;& \left(a_T \times b_T + c_T, \frac{1}{\widetilde{\theta}_T} \left[D_T + \frac{B_T^{a,b}}{T^{\frac{a+b+1}{2}}} \right] \right) \\ = \;& \left(a_T \times b_T, \frac{1}{\widetilde{\theta}_T} \frac{B_T^{a,b}}{T^{\frac{a+b+1}{2}}} \right) + \left(c_T, \frac{1}{\widetilde{\theta}_T} D_T \right) \\ = \;& \frac{1}{\widetilde{\theta}_T} \left(2\theta^2 a_T, \frac{B_T^{a,b}}{T^{\frac{a+b+1}{2}}} \right) + \frac{1}{\widetilde{\theta}_T} \left(a_T \times (b_T \times \widetilde{\theta}_T - 2\theta^2), 0 \right) \\ & + \left(c_T, \frac{1}{\widetilde{\theta}_T} D_T \right). \end{aligned}$$

By the above convergences and Slutsky's theorem we deduce, as $T \to \infty$,

$$\frac{1}{\widetilde{\theta}_T} \left(a_T \times (b_T \times \widetilde{\theta}_T - 2\theta^2), 0 \right) \longrightarrow 0, \; \left(c_T, \frac{1}{\widetilde{\theta}_T} D_T \right) \longrightarrow 0 \text{ in probability.}$$

Therefore, using (39), we obtain as $T \to \infty$,

$$\frac{1}{\widetilde{\theta}_T} \left(2\theta^2 a_T, \frac{B_T^{a,b}}{T^{\frac{a+b+1}{2}}} \right) \quad = \quad \frac{1}{\widetilde{\theta}_T} \left(2\theta^2 \frac{e^{-\theta T} \int_0^T e^{\theta t} dB_t^{a,b}}{(\mu + \zeta_\infty)}, \frac{B_T^{a,b}}{T^{\frac{a+b+1}{2}}} \right)$$

$$\xrightarrow{Law} \left(\frac{2\sqrt{\frac{\Gamma(b+1)}{\theta^{b-1}}}N_2}{\mu+\zeta_\infty}, \frac{\sqrt{2\beta(a+1,b+1)}}{\theta}N_1 \right),$$

which completes the proof of (43). Finally, following the same arguments as in the proof of (43) we obtain (44). □

A Appendix: Young integral

In this section, we briefly recall some basic elements of the Young integral (see [25]), which are helpful for some of the arguments we use. For any $\alpha \in [0, 1]$, we denote by $\mathcal{H}^\alpha([0, T])$ the set of α-Hölder continuous functions, that is, the set of functions $f : [0, T] \to \mathbb{R}$ such that

$$|f|_\alpha := \sup_{0\le s<t\le T} \frac{|f(t) - f(s)|}{(t-s)^\alpha} < \infty.$$

We also set $|f|_\infty = \sup_{t\in[0,T]} |f(t)|$, and we equip $\mathcal{H}^\alpha([0, T])$ with the norm

$$\|f\|_\alpha := |f|_\alpha + |f|_\infty.$$

Let $f \in \mathcal{H}^\alpha([0, T])$, and consider the operator $T_f : \mathcal{C}^1([0, T]) \to \mathcal{C}^0([0, T])$ defined as

$$T_f(g)(t) = \int_0^t f(u)g'(u)du, \quad t \in [0, T].$$

It can be shown (see, e.g., [20, Section 3.1]) that, for any $\beta \in (1-\alpha, 1)$, there exists a constant $C_{\alpha,\beta,T} > 0$ depending only on α, β and T such that, for any $g \in \mathcal{C}^1([0, T])$,

$$\left\| \int_0^\cdot f(u)g'(u)du \right\|_\beta \le C_{\alpha,\beta,T}\|f\|_\alpha\|g\|_\beta.$$

We deduce that, for any $\alpha \in (0, 1)$, any $f \in \mathcal{H}^\alpha([0, T])$ and any $\beta \in (1-\alpha, 1)$, the linear operator $T_f : \mathcal{C}^1([0, T]) \subset \mathcal{H}^\beta([0, T]) \to \mathcal{H}^\beta([0, T])$, defined as $T_f(g) = \int_0^\cdot f(u)g'(u)du$, is continuous. By density, it extends (in a unique way) to an operator defined on $\mathcal{H}^\beta$. As a consequence, if $f \in \mathcal{H}^\alpha([0, T])$, if $g \in \mathcal{H}^\beta([0, T])$ and if $\alpha + \beta > 1$, then the (so-called) Young integral $\int_0^\cdot f(u)dg(u)$ is (well) defined as being $T_f(g)$.

The Young integral obeys the following formula. Let $f \in \mathcal{H}^\alpha([0, T])$ with $\alpha \in (0, 1)$, and $g \in \mathcal{H}^\beta([0, T])$ with $\beta \in (0, 1)$. If $\alpha+\beta > 1$, then $\int_0^\cdot g_u df_u$ and $\int_0^\cdot f_u dg_u$ are well-defined as Young integrals, and for all $t \in [0, T]$,

$$f_t g_t = f_0 g_0 + \int_0^t g_u df_u + \int_0^t f_u dg_u. \tag{49}$$

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