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Convergence rate of CLT for the drift estimation of sub-fractional Ornstein–Uhlenbeck process of second kind

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Abstract In this paper, we deal with an Ornstein–Uhlenbeck process driven by sub-fractional Brownian motion of the second kind with Hurst index $H \in (\frac{1}{2}, 1)$. We provide a least squares estimator (LSE) of the drift parameter based on continuous-time observations. The strong consistency and the upper bound $O(1/\sqrt{n})$ in Kolmogorov distance for central limit theorem of the LSE are obtained. We use a Malliavin–Stein approach for normal approximations.

Keywords Sub-fractional Ornstein–Uhlenbeck process of second kind, least squares estimator, Berry–Esséen bound, Malliavin–Stein approach for normal approximations
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1 Introduction

Let $S^H := \left\{S_t^H, t \geq 0\right\}$ be a sub-fractional Brownian motion (sub-fBm) with Hurst parameter $H \in (0, 1)$ that is a centered Gaussian process, defined on a complete probability space $(\Omega, \mathcal{F}, P)$, with the covariance function

$$E\left(S_s^H S_t^H\right) = t^{2H} + s^{2H} - \frac{1}{2}\left(|t-s|^{2H} + |t+s|^{2H}\right), \quad s, t \geq 0.$$

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Note that, when $H = \frac{1}{2}$, $S^{\frac{1}{2}}$ is a standard Brownian motion. We only refer to [21] for information about the sub-fBm and additional references.

Consider the sub-fractional Ornstein–Uhlenbeck process (sub-fOU) of the second kind, defined as the unique pathwise solution to

$$\begin{cases} dX_t = -\theta X_t dt + dY_t, \quad t \geq 0, \\ X_0 = 0, \end{cases} \tag{1}$$

where $Y_t := \int_0^t e^{-s} dS^H_{a_s}$ with $a_t = He^{\frac{t}{H}}$, and S^H is a sub-fBm with Hurst parameter $H \in (\frac{1}{2}, 1)$, whereas $\theta > 0$ is considered as unknown parameter. Equivalently, X is the process given explicitly by

$$X_t = e^{-\theta t} \int_0^t e^{\theta s} dY_s, \tag{2}$$

where the integral with respect to Y can be understood in the Skorohod sense. When $H = \frac{1}{2}$, the process $Y_t = \int_0^t e^{-s} dS^{\frac{1}{2}}_{a_s}$ is a standard Brownian motion, by Lévy's characterization theorem. Therefore, the process X given by (1) is a standard Ornstein–Uhlenbeck process. Notice also that the model (1) was originally introduced in [15], where the driving process is a fractional Brownian motion, and its definition is related to the Lamperti transform of the fractional Brownian motion.

Our aim is to estimate the parameter θ based the continuous observations of the process $(X_t)_{t \geq 0}$ given by (1). We will restrict to the case when $\theta > 0$ since the case when $\theta < 0$ has been treated in [1]. Throughout the paper we denote by $\int_0^n u_t dY_t$ the Skorohod integral (or, say, a divergence-type integral) with respect to the Gaussian process Y (see Preliminaries for definition). Let us recall the idea to construct the least squares estimator (LSE) for the drift coefficient θ, introduced in [13]. The LSE is obtained by minimizing

$$\theta \longmapsto \int_0^n |\dot{X}_t + \theta X_t|^2 dt.$$

In this way, we obtain the LSE proposed in (1.4) in the paper [13], which is defined by

$$\hat{\theta}_n = \theta - \frac{\int_0^n X_t dY_t}{\int_0^n X_t^2 dt}. \tag{3}$$

In recent years, several researchers have been interested in studying statistical estimation problems for Gaussian Ornstein–Uhlenbeck processes. We aim to bring a new contribution to the statistical inference for fractional diffusions by estimating the drift parameter of a sub-fOU process of the second kind. Our paper is relevant to the literature on parameter estimation for processes with Gaussian long-memory processes, including [1–5, 8–10, 12, 13, 17, 20]. Estimation of the drift parameters for Ornstein–Uhlenbeck processes driven by fractional noise is a problem that is both well-motivated by practical needs and theoretically challenging. In the finance context, a practical motivation to study this estimation problem is to provide tools to understand volatility modeling in finance. Let us mention some important results in this field where the volatility exhibits long-memory, which means that the volatility today is correlated to past volatility values with a dependence that decays very slowly. Following the approach of [7], the authors of [6] considered the problem of option pricing under a stochastic volatility model that exhibits long-range dependence. More precisely they considered and analyzed the dynamics of the volatility that are described by the equation (1), where the driving process Y is replaced by a standard fractional Brownian motion (fBm) with Hurst parameter H greater than $1/2$.

The study of the asymptotic distribution of an estimator is not very useful in general for practical purposes unless the rate of convergence is known. As far as we know, no result of the Berry–Esséen type is known for the distribution of the LSE $\hat{\theta}_n$ of the drift parameter θ of the sub-fOU of the second kind (1).

In order to describe the asymptotic behavior of the LSE $\hat{\theta}_n$ when $n \to \infty$, we first need the following proposition given in [16, Corollary 1]. This result is proved based on techniques relied on the combination of Malliavin calculus and Stein's method (see, e.g., [18]). More precisely, the authors of [16] provided an upper bound of the Kolmogorov distance for central limit theorem of sequences of the form F_n/G_n, where F_n and G_n are functionals of Gaussian fields.

In the following proposition, $\mathfrak{H}^{\odot 2}$ denote the symmetric tensor product.

Proposition 1 ([16]). *Let $f_n, g_n \in \mathfrak{H}^{\odot 2}$ for all $n \geq 1$, and let b_n be a positive function of n such that $I_2(g_n) + b_n > 0$ almost surely for all $n \geq 1$. Define for all sufficiently large positive n,*

$$\begin{aligned}
\psi_1(n) &:= \frac{1}{b_n^2}\sqrt{\left(b_n^2 - 2\|f_n\|^2_{\mathfrak{H}^{\otimes 2}}\right)^2 + 8\|f_n \otimes_1 f_n\|^2_{\mathfrak{H}^{\otimes 2}}},\\
\psi_2(n) &:= \frac{2}{b_n^2}\sqrt{2\|f_n \otimes_1 g_n\|_{\mathfrak{H}^{\otimes 2}} + \langle f_n, g_n\rangle^2_{\mathfrak{H}^{\otimes 2}}},\\
\psi_3(n) &:= \frac{2}{b_n^2}\sqrt{\|g_n\|^4_{\mathfrak{H}^{\otimes 2}} + 2\|g_n \otimes_1 g_n\|^2_{\mathfrak{H}^{\otimes 2}}}.
\end{aligned}$$

Suppose that $\psi_i(n) \to 0$ for $i = 1, 2, 3$, as $n \to \infty$. Then there exists a positive constant C such that for all sufficiently large positive n,

$$\sup_{z\in\mathbb{R}}\left|\mathbb{P}\left(\frac{I_2(f_n)}{I_2(g_n) + b_n} \leq z\right) - \mathbb{P}(Z \leq z)\right| \leq C \max_{i=1,2,3} \psi_i(n).$$

Let us now describe the results we prove in the present paper. First, in (4) we show that the strong consistency of the LSE $\hat{\theta}_n$ defined by (3), as $n \to \infty$. Then, in (5) we provide, when $H \in (\frac{1}{2}, 1)$, an upper bound of Kolmogorov distance for central limit theorem of the LSE $\hat{\theta}_n$.

Theorem 1. *Assume $H \in (\frac{1}{2}, 1)$ and let $\hat{\theta}_n$ be given by (3). Then, as $n \to \infty$,*

$$\hat{\theta}_n \longrightarrow \theta \quad \textit{almost surely.} \tag{4}$$

Moreover, there exists a constant $0 < C < \infty$, depending only on θ and H, such that for all sufficiently large positive n,

$$\sup_{z\in\mathbb{R}}\left|P\left(\frac{\sqrt{n}}{\sigma_{\theta,H}}\left(\theta - \hat{\theta}_n\right) \leq z\right) - P(Z \leq z)\right| \leq \frac{C}{\sqrt{n}}, \tag{5}$$

where Z denotes a standard normal random variable, and the positive constant $\sigma_{\theta,H}$ is given by

$$\sigma_{\theta,H} \quad := \quad \frac{\theta\sqrt{2\int_{(0,\infty)^3} F(y_1,y_2,y_3)dy_1dy_2dy_3}}{H\left[\beta(H\theta+1-H,2H-1)-k(\theta,H)\right]} < \infty, \tag{6}$$

with β denoting the classical Beta function, $\sigma_{\theta,H} < \infty$ (due to [3] and $0 \le |x-y|^{2H-2} - |x+y|^{2H-2} \le |x-y|^{2H-2}$ for very $x, y \ge 0$), whereas the function F is defined by

$$\begin{aligned} &F(y_1,y_2,y_3)\\ &:= e^{-\theta|y_1-y_3|}e^{-\theta y_2}e^{(1-\frac{1}{H})(y_1+y_2+y_3)}\left(\left|1-e^{-\frac{y_1}{H}}\right|^{2H-2}-\left|1+e^{-\frac{y_1}{H}}\right|^{2H-2}\right)\\ &\qquad\qquad \times\left(\left|e^{-\frac{y_2}{H}}-e^{-\frac{y_3}{H}}\right|^{2H-2}-\left|e^{-\frac{y_2}{H}}+e^{-\frac{y_3}{H}}\right|^{2H-2}\right). \end{aligned} \tag{7}$$

The rest of the paper is structured as follows. Section 2 presents some basic elements of Malliavin calculus which are helpful for some of the arguments we use throughout the paper. Section 3 is devoted to the proof of Theorem 1.

Throughout the paper Z denotes a standard normal random variable, and C denotes a generic positive constant (perhaps depending on θ and H, but not on anything else), which may change from line to line.

2 Preliminaries

In this section, we briefly recall some basic elements of Gaussian analysis, and Malliavin calculus which are helpful for some of the arguments we use throughout the paper. For more details we refer to [18] and [19].

Consider the Gaussian process $Y_t = \int_0^t e^{-s}dS^H_{a_s}, t \ge 0$, with $a_t = He^{\frac{t}{H}}$. Assume that $\frac{1}{2} < H < 1$. Setting $a_u^{-1} = H\log(u/H)$, it follows from [5] that, for every f, g in $\mathcal{C}^1$,

$$\begin{aligned} &E\left(\int_s^t f(r)dY_r \int_u^v g(r)dY_r\right)\\ &= H(2H-1)\int_{a_s}^{a_t}\int_{a_u}^{a_v} f(a_x^{-1})g(a_y^{-1})e^{-a_x^{-1}-a_y^{-1}}\\ &\qquad\qquad\qquad \left[|x-y|^{2H-2}-(x+y)^{2H-2}\right]dxdy \qquad (8)\\ &= \int_s^t\int_u^v f(w)g(z)r_H(w,z)dwdz, \qquad (9) \end{aligned}$$

where $r_H(x,y)$ is a symmetric kernel given by

$$\begin{aligned} r_H(w,z) &= H^{2H-1}(2H-1)(a_wa_z)^{1-H}\Big[\,|a_w-a_z|^{2H-2}-|a_w+a_z|^{2H-2}\Big]\\ &= H^{2H-1}(2H-1)\left(e^{w/H}e^{z/H}\right)^{1-H} \end{aligned}$$

$$\times\Big[\left|e^{w/H}-e^{z/H}\right|^{2H-2}-\left|e^{w/H}+e^{z/H}\right|^{2H-2}\Big].$$

In particular, we obtain the following covariance:

$$E\left((Y_t-Y_s)(Y_v-Y_u)\right) \;=\; \int_s^t\int_u^v r_H(w,z)dwdz.$$

Fix a time interval $[0,T]$. We denote by $\mathfrak{H}$ the canonical Hilbert space associated to the Gaussian process Y. It is the closure of the linear span $\mathcal{E}$ generated by the indicator functions $1_{[0,t]}$, $t\in[0,T]$, with respect to the inner product

$$\langle 1_{[s,t]},1_{[u,v]}\rangle_{\mathfrak{H}}=E\left((Y_t-Y_s)(Y_v-Y_u)\right).$$

The mapping $1_{[0,t]}\mapsto Y_t$ can be extended to a linear isometry between $\mathfrak{H}$ and the Gaussian space $\mathcal{H}_1$ spanned by Y. We denote this isometry by $\varphi\in\mathfrak{H}\mapsto Y(\varphi)$.

For $\frac{1}{2}<H<1$, we introduce $|\mathfrak{H}|$ as the set of measurable functions φ on $[0,T]$ such that

$$\|\varphi\|^2_{|\mathfrak{H}|}:=\int_0^T\int_0^T|\varphi(u)||\varphi(v)|r_H(u,v)dudv<\infty.$$

Note that, if $\varphi,\psi\in|\mathfrak{H}|$,

$$E(Y(\varphi)Y(\psi))=\int_0^T\int_0^T\varphi(u)\psi(v)r_H(u,v)dudv.$$

The space $|\mathfrak{H}|$ is a Banach space with the norm $\|.\|_{|\mathfrak{H}|}$ and it is included in $\mathfrak{H}$. Let $C_b^\infty(\mathbb{R}^n,\mathbb{R})$ be the class of infinitely differentiable functions $f:\mathbb{R}^n\longrightarrow\mathbb{R}$ such that f and all its partial derivatives are bounded. We denote by $\mathcal{S}$ the class of smooth cylindrical random variables G of the form

$$F=f(Y(\varphi_1),\ldots,Y(\varphi_n)),\tag{10}$$

where $n\geq 1$, $f\in C_b^\infty(\mathbb{R}^n,\mathbb{R})$ and $\varphi_1,\ldots,\varphi_n\in\mathfrak{H}$.

The derivative operator D of a smooth cylindrical random variable G of the form (10) is defined as the $\mathfrak{H}$-valued random variable

$$DG=\sum_{i=1}^n\frac{\partial f}{\partial x_i}(Y(\varphi_1),\ldots,Y(\varphi_n))\varphi_i.$$

In this way the derivative DG is an element of $L^2(\Omega;\mathfrak{H})$. We denote by $D^{1,2}$ the closure of $\mathcal{S}$ with respect to the norm defined by

$$\|G\|^2_{1,2}=E(G^2)+E(\|DG\|^2_{\mathfrak{H}}).$$

The divergence operator δ is the adjoint of the derivative operator D. Concretely, a random variable $u\in L^2(\Omega;\mathfrak{H})$ belongs to the domain of the divergence operator $Dom\delta$ if

$$E\left|\langle DG,u\rangle_{\mathfrak{H}}\right|\leq c_u\|G\|_{L^2(\Omega)}$$

for every $G \in \mathcal{S}$, where c_u is a constant which depends only on u. In this case $\delta(u)$ is given by the duality relation

$$E(G\delta(u)) = E\langle DG, u\rangle_{\mathfrak{H}}$$

for any $F \in D^{1,2}$. We will make use of the notation

$$\delta(u) = \int_0^T u_s dY_s, \quad u \in Dom\delta.$$

In particular, for $h \in \mathfrak{H}$, $Y(h) = \delta(h) = \int_0^T h_s dY_s$.

For every $n \geq 1$, let $\mathfrak{H}_n$ be the nth Wiener chaos of B, that is, the closed linear subspace of $L^2(\Omega)$ generated by the random variables $\{H_n(Y(h)), h \in \mathfrak{H}, \|h\|_{\mathfrak{H}} = 1\}$ where H_n is the nth Hermite polynomial. The mapping $I_n(h^{\otimes n}) = n! H_n(Y(h))$ provides a linear isometry between the symmetric tensor product $\mathfrak{H}^{\odot n}$ (equipped with the modified norm $\|.\|_{\mathfrak{H}^{\odot n}} = \sqrt{n!}\|.\|_{\mathfrak{H}^{\otimes n}}$) and $\mathfrak{H}_n$. For every $f, g \in \mathfrak{H}^{\odot n}$ the following product formula holds

$$E\left(I_n(f)I_n(g)\right) = n!\langle f, g\rangle_{\mathfrak{H}^{\otimes n}}.$$

Notice that for every nonrandom Hölder continuous function φ of order $\alpha \in (1 - H, 1)$, we have

$$\int_0^t \varphi_s dY_s = \int_0^t \varphi_s dY_s = Y(\varphi). \tag{11}$$

For a smooth and cylindrical random variable $F = (Y(\varphi_1), \ldots, Y(\varphi_n))$, with $\varphi_i \in \mathfrak{H}$, $i = 1, \ldots, n$, and $f \in \mathcal{C}_b^\infty(\mathbb{R}^n)$ (f and all of its partial derivatives are bounded), we define its Malliavin derivative as the $\mathfrak{H}$-valued random variable given by

$$DF = \sum_{i=1}^n \frac{\partial f}{\partial x_i}\left(Y(\varphi_1), \ldots, Y(\varphi_n)\right)\varphi_i.$$

For every $q \geq 1$, $\mathcal{H}_q$ denotes the qth Wiener chaos of Y, defined as the closed linear subspace of $L^2(\Omega)$ generated by the random variables $\{H_q(Y(h)), h \in \mathfrak{H}, \|h\|_{\mathfrak{H}} = 1\}$ where H_q is the qth Hermite polynomial. Wiener chaoses of different orders are orthogonal in $L^2(\Omega)$.

The mapping $I_q(h^{\otimes q}) := q! H_q(Y(h))$ is a linear isometry between the symmetric tensor product $\mathfrak{H}^{\odot q}$ (equipped with the modified norm $\|.\|_{\mathfrak{H}^{\odot q}} = \sqrt{q!}\|.\|_{\mathfrak{H}^{\otimes q}}$) and $\mathcal{H}_q$. For every $f, g \in \mathfrak{H}^{\odot q}$ the following extended isometry property holds

$$E\left(I_q(f)I_q(g)\right) = q!\langle f, g\rangle_{\mathfrak{H}^{\otimes q}}.$$

We will only need to know the product formula for $q = 1$ (see [18, Section 2.7.3]), which is

$$I_1(f)I_1(g) = I_2(f \otimes g) + \langle f, g\rangle_{\mathfrak{H}}. \tag{12}$$

Let $\{e_k, k \geq 1\}$ be a complete orthonormal system in the Hilbert space $\mathfrak{H}$. Given $f \in \mathfrak{H}^{\odot n}$, $g \in \mathfrak{H}^{\odot m}$, and $p = 1, \ldots, n \wedge m$, the pth contraction between f and g is the element of $\mathfrak{H}^{\otimes(m+n-2p)}$ defined by

$$f \otimes_p g = \sum_{i_1,\ldots,i_p=1}^{\infty} \langle f, e_{i_1} \otimes \cdots \otimes e_{i_p} \rangle_{\mathfrak{H}^{\otimes p}} \otimes \langle g, e_{i_1} \otimes \cdots \otimes e_{i_p} \rangle_{\mathfrak{H}^{\otimes p}}.$$

Let us also recall the hypercontractivity property in Wiener chaos. For $h \in \mathfrak{H}^{\otimes q}$, the multiple Wiener integrals $I_q(h)$, which exhaust the set $\mathfrak{H}_q$, satisfy a hypercontractivity property (equivalence in $\mathfrak{H}_q$ of all L^p norms for all $p \geq 2$), which implies that for any $G \in \oplus_{l=1}^{q} \mathfrak{H}_l$ (i.e. in a fixed sum of Wiener chaoses), we have

$$\big(E\big[|G|^p\big]\big)^{1/p} \leqslant c_{p,q} \left(E\big[|G|^2\big]\right)^{1/2} \quad \text{for any } p \geq 2. \tag{13}$$

It should be noted that the constants $c_{p,q}$ above are known with some precision when G is a single chaos term: indeed, by Corollary 2.8.14 in [18], $c_{p,q} = (p-1)^{q/2}$.

The following result is a direct consequence of the Borel–Cantelli Lemma (the proof is elementary; see, e.g., [14, Lemma 2.1]). It is convenient for establishing almost sure convergences from L^p convergences.

Lemma 1. *Let $\gamma > 0$. Let $(Z_n)_{n\geq 1}$ be a sequence of random variables. If for every $p \geq 1$ there exists a constant $c_p > 0$ such that for all $n \geq 1$,*

$$\|Z_n\|_{L^p(\Omega)} \leqslant c_p \cdot n^{-\gamma},$$

then for all $\varepsilon > 0$ there exists a random variable α_ε which is almost surely finite such that

$$|Z_n| \leqslant \alpha_\varepsilon \cdot n^{-\gamma+\varepsilon} \quad \textit{almost surely}$$

for all $n \geq 1$. Moreover, $E|\alpha_\varepsilon|^p < \infty$ for all $p \geq 1$.

3 Proof of Theorem 1

From (3) we can write

$$\theta - \hat{\theta}_n = \frac{\int_0^n X_t dY_t}{\int_0^n X_t^2 dt}. \tag{14}$$

It follows from (2) that

$$\frac{1}{\sqrt{n}} \int_0^n X_t dY_t \quad = \quad I_2(h_n),$$

with

$$h_n(s,t) := \frac{1}{2\sqrt{n}} e^{-\theta|t-s|} 1_{[0,n]^2}(s,t). \tag{15}$$

On the other hand, using the product formula (12),

$$\begin{aligned} X_t^2 &= \left(I_1\left(e^{-\theta(t-.)}1_{[0,t]}(.)\right)\right)^2 \\ &= I_2\left(e^{-2\theta t}e^{\theta u}e^{\theta v}1_{[0,t]^2}(u,v)\right) + \left\|e^{-\theta(t-.)}1_{[0,t]}(.)\right\|_{\mathfrak{H}}^2. \end{aligned}$$

Let us introduce the positive constant

$$\rho_{\theta,H} := \frac{H^{2H}(2H-1)}{\theta}\Big[\beta(H\theta+1-H, 2H-1) - k(\theta,H)\Big], \tag{16}$$

with

$$k(\theta,H) := \int_0^1 u^{H\theta-H}(1+u)^{2H-2}du.$$

Thus

$$\begin{aligned} \frac{1}{n\rho_{\theta,H}}\int_0^n X_t^2 dt &= I_2\left(\frac{1}{n\rho_{\theta,H}}\int_0^n e^{-2\theta t}e^{\theta u}e^{\theta v}1_{[0,t]^2}(u,v)dt\right) \\ &\quad + \frac{1}{n\rho_{\theta,H}}\int_0^n e^{-2\theta t}\left\|e^{\theta u}1_{[0,t]}(u)\right\|_{\mathfrak{H}}^2 dt \\ &=: I_2(g_n) + b_n, \end{aligned} \tag{17}$$

where

$$b_n := \frac{1}{n\rho_{\theta,H}}\int_0^n e^{-2\theta t}\left\|e^{\theta u}1_{[0,t]}(u)\right\|_{\mathfrak{H}}^2 dt, \tag{18}$$

and

$$\begin{aligned} g_n(u,v) &:= \frac{1}{n\rho_{\theta,H}}e^{\theta u}e^{\theta v}\frac{e^{-2\theta(u\vee v)} - e^{-2\theta n}}{2\theta}1_{[0,n]^2}(u,v) \\ &= \frac{1}{2\theta\rho_{\theta,H}n}\left(e^{-\theta|u-v|} - e^{-2\theta n}e^{\theta u}e^{\theta v}\right)1_{[0,n]^2}(u,v) \\ &= \frac{1}{\theta\rho_{\theta,H}\sqrt{n}}h_n(u,v) - l_n(u,v), \end{aligned} \tag{19}$$

with h_n given by (15), and

$$l_n(u,v) := \frac{1}{2n\theta\rho_{\theta,H}}e^{-2\theta n}e^{\theta u}e^{\theta v}1_{[0,n]^2}(u,v).$$

Therefore, combining (14), (15) and (17), we get

$$\frac{\sqrt{n}}{\sigma_{\theta,H}}\left(\theta - \hat{\theta}_n\right) = \frac{I_2(f_n)}{I_2(g_n) + b_n}, \tag{20}$$

where $\sigma_{\theta,H}$ is given by (6), and

$$f_n := \frac{1}{\rho_{\theta,H}\sigma_{\theta,H}}h_n. \tag{21}$$

In order to prove our main result we make use of the following technical lemmas.

Lemma 2. *Let $H \in (\frac{1}{2}, 1)$, and let b_n and f_n be the functions given by (18) and (21), respectively. Then, for all $n \geq 1$,*

$$|b_n - 1| \leq \frac{C}{n}, \tag{22}$$

$$\left|1 - 2\| f_n \|^2_{\mathfrak{H}^{\otimes 2}}\right| \leq \frac{C}{n}. \tag{23}$$

Consequently, for all $n \geq 1$,

$$\left|b_n^2 - 2\| f_n \|^2_{\mathfrak{H}^{\otimes 2}}\right| \leq \frac{C}{n}.$$

Proof. Using (8) and making the change of variables $u = x/y$, we get

$$\begin{aligned}
&\left\| e^{\theta u} 1_{[0,t]}(u) \right\|^2_{\mathfrak{H}} \\
&= H(2H-1) \int_{a_0}^{a_t} \int_{a_0}^{a_t} (x/H)^{H\theta - H} (y/H)^{H\theta - H} \\
&\qquad \left[|x-y|^{2H-2} - (x+y)^{2H-2}\right] dx dy \\
&= 2H^{2H(1-\theta)+1}(2H-1) \int_{a_0}^{a_t} y^{2H\theta - 1} \int_{a_0/y}^{1} u^{H\theta - H} \\
&\qquad \left[|1-u|^{2H-2} - (1+u)^{2H-2}\right] du dy \\
&= 2H^{2H(1-\theta)+1}(2H-1) \int_{a_0}^{a_t} y^{2H\theta - 1} \int_{0}^{1} u^{H\theta - H} \\
&\qquad \left[|1-u|^{2H-2} - (1+u)^{2H-2}\right] du dy \\
&\quad - 2H^{2H(1-\theta)+1}(2H-1) \int_{a_0}^{a_t} y^{2H\theta - 1} \int_{0}^{a_0/y} u^{H\theta - H} \\
&\qquad \left[|1-u|^{2H-2} - (1+u)^{2H-2}\right] du dy \\
&:= A_t - B_t,
\end{aligned} \tag{24}$$

where

$$\begin{aligned}
A_t &= 2H^{2H(1-\theta)+1}(2H-1) \int_{a_0}^{a_t} y^{2H\theta-1} dy \\
&\qquad \times \left(\int_0^1 u^{H\theta - H} \left[|1-u|^{2H-2} - (1+u)^{2H-2}\right] du \right) \\
&= 2H^{2H(1-\theta)+1}(2H-1) \Big[\beta(H\theta + 1 - H, 2H-1) - k(\theta, H)\Big] \\
&\qquad \times \int_{a_0}^{a_t} y^{2H\theta - 1} dy \\
&= \frac{H^{2H}(2H-1)}{\theta} \Big[\beta(H\theta + 1 - H, 2H-1) - k(\theta, H)\Big](e^{2\theta t} - 1).
\end{aligned}$$

Moreover,

$$\frac{1}{n}\int_0^n e^{-2\theta t}A_t dt = \frac{H^{2H}(2H-1)}{\theta}\Big[\beta(H\theta+1-H,2H-1)-k(\theta,H)\Big] \times\left(1+\frac{e^{-2\theta n}-1}{2\theta n}\right).$$

Thus

$$\left|\frac{1}{n\rho_{\theta,H}}\int_0^n e^{-2\theta t}A_t dt-1\right| = \frac{1-e^{-2\theta n}}{2\theta n}\le\frac{1}{2\theta n}. \quad (25)$$

On the other hand,

$$\begin{aligned}|B_t| &\le 2H^{2H(1-\theta)+1}(2H-1)\int_{a_0}^{a_t}y^{2H\theta-1}\\ &\qquad\times\int_0^{a_0/y}u^{H\theta-H}\Big||1-u|^{2H-2}-(1+u)^{2H-2}\Big|dudy\\ &\le 2H^{2H(1-\theta)+1}(2H-1)\int_{a_0}^{a_t}y^{2H\theta-1}\int_0^{a_0/y}u^{H\theta-H}|1-u|^{2H-2}dudy\\ &\le 2H^{2H(1-\theta)+1}(2H-1)\int_{a_0}^{a_t}y^{2H\theta-1}(a_0/y)^{H\theta}\int_0^{a_0/y}u^{-H}\\ &\qquad(1-u)^{2H-2}dudy\\ &\le 2H^{2H-\theta H+1}(2H-1)\int_{a_0}^{a_t}y^{H\theta-1}dy\int_0^1u^{-H}(1-u)^{2H-2}du\\ &= 2H^{2H+1}(2H-1)\beta(1-H,2H-1)\frac{e^{\theta t}-1}{H\theta}\\ &\le Ce^{\theta t}.\end{aligned}$$

Hence,

$$\frac{1}{n\rho_{\theta,H}}\int_0^n e^{-2\theta t}|B_t|dt \le \frac{C}{n}\int_0^n e^{-\theta t}dt\le\frac{C}{n}. \quad (26)$$

Therefore, combining (24), (25) and (26), we deduce (22).

Now let us prove (23). First we decompose the integral $\int_{[0,n]^4}$ into

$$\int_{[0,n]^4}=\int_{\cup_{i=1}^5 A_{i,n}}=\sum_{i=1}^5\int_{A_{i,n}}, \quad (27)$$

where

$$A_{1,n}=\cup_{i=1}^8 D_{i,n},\ A_{2,n}=\cup_{i=9}^{12}D_{i,n},\ A_{3,n}=\cup_{i=13}^{16}D_{i,n},$$
$$A_{4,n}=\cup_{i=17}^{20}D_{i,n},\ A_{5,n}=\cup_{i=21}^{24}D_{i,n},$$

with

$$\begin{aligned}
D_{1,n} &:= \{0 < x_1 < x_2 < x_3 < x_4 < n\} \\
D_{2,n} &:= \{0 < x_1 < x_2 < x_4 < x_3 < n\} \\
D_{3,n} &:= \{0 < x_2 < x_1 < x_3 < x_4 < n\} \\
D_{4,n} &:= \{0 < x_2 < x_1 < x_4 < x_3 < n\} \\
D_{5,n} &:= \{0 < x_3 < x_4 < x_1 < x_2 < n\} \\
D_{6,n} &:= \{0 < x_3 < x_4 < x_2 < x_1 < n\} \\
D_{7,n} &:= \{0 < x_4 < x_3 < x_1 < x_2 < n\} \\
D_{8,n} &:= \{0 < x_4 < x_3 < x_2 < x_1 < n\},
\end{aligned}$$

$$\begin{aligned}
D_{9,n} &:= \{0 < x_1 < x_3 < x_2 < x_4 < n\} \\
D_{10,n} &:= \{0 < x_3 < x_1 < x_4 < x_2 < n\} \\
D_{11,n} &:= \{0 < x_2 < x_4 < x_1 < x_3 < n\} \\
D_{12,n} &:= \{0 < x_4 < x_2 < x_3 < x_1 < n\},
\end{aligned}$$

$$\begin{aligned}
D_{13,n} &:= \{0 < x_1 < x_3 < x_4 < x_2 < n\} \\
D_{14,n} &:= \{0 < x_3 < x_1 < x_2 < x_4 < n\} \\
D_{15,n} &:= \{0 < x_2 < x_4 < x_3 < x_1 < n\} \\
D_{16,n} &:= \{0 < x_4 < x_2 < x_1 < x_3 < n\},
\end{aligned}$$

$$\begin{aligned}
D_{17,n} &:= \{0 < x_1 < x_4 < x_2 < x_3 < n\} \\
D_{18,n} &:= \{0 < x_4 < x_1 < x_3 < x_2 < n\} \\
D_{19,n} &:= \{0 < x_2 < x_3 < x_1 < x_4 < n\} \\
D_{20,n} &:= \{0 < x_3 < x_2 < x_4 < x_1 < n\},
\end{aligned}$$

$$\begin{aligned}
D_{21,n} &:= \{0 < x_1 < x_4 < x_3 < x_2 < n\} \\
D_{22,n} &:= \{0 < x_4 < x_1 < x_2 < x_3 < n\} \\
D_{23,n} &:= \{0 < x_3 < x_2 < x_1 < x_4 < n\} \\
D_{24,n} &:= \{0 < x_2 < x_3 < x_4 < x_1 < n\}.
\end{aligned}$$

Therefore, using (9), (27), and setting

$$m_H(x_1, x_2, x_3, x_4) := e^{-\theta|x_1-x_3|} e^{-\theta|x_2-x_4|} r_H(x_1, x_2) r_H(x_3, x_4),$$

we have

$$\begin{aligned}
\|h_n\|^2_{\mathfrak{H}^{\otimes 2}} &= \frac{1}{4n} \int_{[0,n]^4} m_H(x_1, x_2, x_3, x_4) dx_1 \dots dx_4 \\
&= \frac{1}{4n} \left(\int_{A_{1,n}} + \int_{A_{2,n}} + \int_{A_{3,n}} + \int_{A_{4,n}} + \int_{A_{5,n}} \right) \\
&\qquad \times m_H(x_1, x_2, x_3, x_4) dx_1 \dots dx_4
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4n}\left(8\int_{D_{1,n}} +4\int_{D_{9,n}} +4\int_{D_{13,n}} +4\int_{D_{17,n}} +4\int_{D_{21,n}}\right) \\
&\qquad\qquad \times m_H(x_1,x_2,x_3,x_4)dx_1\dots dx_4 \\
&=: 2I_{1,n}+I_{2,n}+I_{3,n}+I_{4,n}+I_{5,n}, \qquad (28)
\end{aligned}
$$

where we used the fact that

$$
\begin{aligned}
&\int_{D_{1,n}} m_H(x_1,x_2,x_3,x_4)dx_1dx_2dx_3dx_4 \\
&\qquad = \dots = \int_{D_{8,n}} m_H(x_1,x_2,x_3,x_4)dx_1dx_2dx_3dx_4, \\
&\int_{D_{9,n}} m_H(x_1,x_2,x_3,x_4)dx_1dx_2dx_3dx_4 \\
&\qquad = \dots = \int_{D_{12,n}} m_H(x_1,x_2,x_3,x_4)dx_1dx_2dx_3dx_4, \\
&\int_{D_{13,n}} m_H(x_1,x_2,x_3,x_4)dx_1dx_2dx_3dx_4 \\
&\qquad = \dots = \int_{D_{16,n}} m_H(x_1,x_2,x_3,x_4)dx_1dx_2dx_3dx_4, \\
&\int_{D_{17,n}} m_H(x_1,x_2,x_3,x_4)ddx_1dx_2dx_3dx_4 \\
&\qquad = \dots = \int_{D_{20,n}} m_H(x_1,x_2,x_3,x_4)dx_1dx_2dx_3dx_4, \\
&\int_{D_{21,n}} m_H(x_1,x_2,x_3,x_4)dx_1dx_2dx_3dx_4 \\
&\qquad = \dots = \int_{D_{24,n}} m_H(x_1,x_2,x_3,x_4)dx_1dx_2dx_3dx_4.
\end{aligned}
$$

Let us now estimate $I_{1,n}$. Making the change of variables $y_3 = x_4 - x_1$, $y_2 = x_4 - x_2$, $y_1 = x_4 - x_3$, and $y_4 = x_4$, we obtain that $\frac{1}{H^{4H-2}(2H-1)^2} I_{1,n}$ is equal to

$$
\begin{aligned}
&\frac{1}{n}\int_0^n \int_{0<x_1<x_2<x_3<x_4} e^{-\theta|x_1-x_3|}e^{-\theta|x_2-x_4|}e^{(1/H-1)(x_1+x_2+x_3+x_4)} \\
&\times\left(\left|e^{x_1/H}-e^{x_2/H}\right|^{2H-2}-\left|e^{x_1/H}+e^{x_2/H}\right|^{2H-2}\right) \\
&\qquad \times\left(\left|e^{x_3/H}-e^{x_4/H}\right|^{2H-2}-\left|e^{x_3/H}+e^{x_4/H}\right|^{2H-2}\right)dx_1dx_2dx_3dx_4 \\
= &\frac{1}{n}\int_0^n \int_{0<y_1<y_2<y_3<y_4} F(y_1,y_2,y_3)dy_1dy_2dy_3dy_4 \\
= &\frac{1}{n}\left[\int_0^n \int_{0<y_1<y_2<y_3<\infty} - \int_0^n\int_{y_4}^{\infty}\int_0^{y_3}\int_0^{y_2}\right] F(y_1,y_2,y_3)dy_1dy_2dy_3dy_4
\end{aligned}
$$

$$= \int_{0<y_1<y_2<y_3<\infty} F(y_1,y_2,y_3)dy_1dy_2dy_3 \\ -\frac{1}{n}\int_0^n\int_{y_4}^{\infty}\int_0^{y_3}\int_0^{y_2} F(y_1,y_2,y_3)dy_1dy_2dy_3dy_4, \tag{29}$$

where the function F is given by (7). Moreover, $\frac{1}{n}\int_0^n\int_{y_4}^{\infty}\int_0^{y_3}\int_0^{y_2} F(y_1,y_2,y_3)dy_1dy_2dy_3dy_4$ is equal to

$$\begin{aligned}
&\frac{1}{n}\int_0^n\int_{y_4}^{\infty}\int_0^{y_3}\int_0^{y_2} e^{-\theta|y_1-y_3|}e^{-\theta y_2}e^{(1-\frac{1}{H})(y_1+y_2+y_3)}\\
&\qquad\left[\left|e^{-\frac{y_2}{H}}-e^{-\frac{y_3}{H}}\right|^{2H-2}-\left|e^{-\frac{y_2}{H}}+e^{-\frac{y_3}{H}}\right|^{2H-2}\right]\\
&\qquad\qquad\times\left[\left|1-e^{-\frac{y_1}{H}}\right|^{2H-2}-\left|1+e^{-\frac{y_1}{H}}\right|^{2H-2}\right]dy_1dy_2dy_3dy_4\\
&=\frac{1}{n}\int_0^n\int_{y_4}^{\infty}\int_0^{y_3}\int_0^{y_2} e^{-\theta|y_1-y_3|}e^{-\theta y_2}e^{(1-\frac{1}{H})(y_1+y_2+y_3)}\\
&\qquad\Bigg(\left|e^{-\frac{y_2}{H}}-e^{-\frac{y_3}{H}}\right|^{2H-2}\left|1-e^{-\frac{y_1}{H}}\right|^{2H-2}\\
&\qquad-\left|e^{-\frac{y_2}{H}}-e^{-\frac{y_3}{H}}\right|^{2H-2}\left|1+e^{-\frac{y_1}{H}}\right|^{2H-2}\\
&\qquad-\left|e^{-\frac{y_2}{H}}+e^{-\frac{y_3}{H}}\right|^{2H-2}\left|1-e^{-\frac{y_1}{H}}\right|^{2H-2}\\
&\qquad+\left|e^{-\frac{y_2}{H}}+e^{-\frac{y_3}{H}}\right|^{2H-2}\left|1+e^{-\frac{y_1}{H}}\right|^{2H-2}\Bigg)dy_1dy_2dy_3dy_4\\
&:= A_n^{(1)}-A_n^{(2)}-A_n^{(3)}+A_n^{(4)},
\end{aligned}\tag{30}$$

where

$$\begin{aligned}
A_n^{(1)} &= \frac{1}{n}\int_0^n\int_{y_4}^{\infty}\int_0^{y_3}\int_0^{y_2} e^{-\theta|y_1-y_3|}e^{-\theta y_2}e^{(1-\frac{1}{H})(y_1+y_2+y_3)}\\
&\qquad\times\left|e^{-\frac{y_2}{H}}-e^{-\frac{y_3}{H}}\right|^{2H-2}\left|1-e^{-\frac{y_1}{H}}\right|^{2H-2}dy_1dy_2dy_3dy_4\\
&\le \frac{1}{n}\int_0^n\int_{y_4}^{\infty}\int_0^{y_3}\int_0^{y_2} e^{-\theta y_3}e^{(1-\frac{1}{H})(y_1+y_2+y_3)}\\
&\qquad\times\left|e^{-\frac{y_2}{H}}-e^{\frac{y_3}{H}}\right|^{2H-2}\left|1-e^{-\frac{y_1}{H}}\right|^{2H-2}dy_1dy_2dy_3dy_4\\
&\le \frac{H\beta(1-H,2H-1)}{n}\int_0^n\int_{y_4}^{\infty}\int_0^{y_3} e^{-\theta y_3}e^{(1-\frac{1}{H})(y_2+y_3)}\\
&\qquad\times\left|e^{-\frac{y_2}{H}}-e^{-\frac{y_3}{H}}\right|^{2H-2}dy_2dy_3dy_4\\
&= \frac{H\beta(1-H,2H-1)}{n}\int_0^n\int_{y_4}^{\infty}\int_0^{y_3} e^{-\theta y_3}e^{(1-\frac{1}{H})(y_3-y_2)}\\
&\qquad\times\left|1-e^{-(y_3-y_2)/H}\right|^{2H-2}dy_2dy_3dy_4
\end{aligned}$$

$$
\begin{aligned}
&= \frac{H\beta(1-H,2H-1)}{n}\int_0^n\int_{y_4}^{\infty}\int_0^{y_3} e^{-\theta y_3}e^{(1-\frac{1}{H})x_2} \\
&\qquad\qquad \times\left|1-e^{-\frac{x_2}{H}}\right|^{2H-2}dx_2dy_3dy_4 \\
&\le \frac{(H\beta(1-H,2H-1))^2}{n}\int_0^n\int_{y_4}^{\infty} e^{-\theta y_3}dy_3dy_4 \\
&\le \frac{(H\beta(1-H,2H-1))^2}{\theta^2 n}.
\end{aligned}
$$

Since

$$
\left|e^{-\frac{y_2}{H}}+e^{-\frac{y_3}{H}}\right|^{2H-2} \le \left|e^{-\frac{y_2}{H}}-e^{-\frac{y_3}{H}}\right|^{2H-2} \quad \text{and} \quad \left|1+e^{-\frac{y_1}{H}}\right|^{2H-2} \le \left|1-e^{-\frac{y_1}{H}}\right|^{2H-2},
$$

we have

$$
|A_n^{(2)}| \le |A_n^{(1)}| \le \frac{C}{n}, \quad |A_n^{(3)}| \le |A_n^{(1)}| \le \frac{C}{n} \quad \text{and} \quad |A_n^{(4)}| \le |A_n^{(1)}| \le \frac{C}{n}.
$$

Consequently,

$$
\frac{1}{n}\int_0^n\int_{y_4}^{\infty}\int_0^{y_3}\int_0^{y_2} F(y_1,y_2,y_3)dy_1dy_2dy_3dy_4 \le \frac{C}{n}. \tag{31}
$$

Combining (29) and (31) we deduce

$$
\left|I_{1,n} - H^{4H-2}(2H-1)^2\int_{0<y_1<y_2<y_3<\infty} F(y_1,y_2,y_3)\right|dy_1dy_2dy_3 \le \frac{C}{n}. \tag{32}
$$

Moreover,

$$
\left|I_{1,n} - H^{4H-2}(2H-1)^2\int_{0<y_1<y_3<y_2<\infty} F(y_1,y_2,y_3)dy_1dy_2dy_3\right| \le \frac{C}{n}, \tag{33}
$$

since

$$
\begin{aligned}
&\int_{0<y_1<y_3<y_2<\infty} F(y_1,y_2,y_3)dy_1dy_3dy_2 \\
&\qquad\qquad = \int_{0<y_1<y_2<y_3<\infty} F(y_1,y_2,y_3)dy_1dy_2dy_3.
\end{aligned}
$$

Using similar arguments as above, we can conclude

$$
\left|I_{2,n} - H^{4H-2}(2H-1)^2\int_{0<y_2<y_1<y_3<\infty} F(y_1,y_2,y_3)dy_2dy_1dy_3\right| \le \frac{C}{n}, \tag{34}
$$

$$
\left|I_{3,n} - H^{4H-2}(2H-1)^2\int_{0<y_2<y_3<y_1<\infty} F(y_1,y_2,y_3)dy_2dy_3dy_1\right| \le \frac{C}{n}, \tag{35}
$$

$$
\left|I_{4,n} - H^{4H-2}(2H-1)^2\int_{0<y_3<y_1<y_2<\infty} F(y_1,y_2,y_3)dy_3dy_1dy_2\right| \le \frac{C}{n}, \tag{36}
$$

$$\left| I_{5,n} - H^{4H-2}(2H-1)^2 \int_{0<y_3<y_2<y_1<\infty} F(y_1, y_2, y_3) dy_3 dy_2 dy_1 \right| \leq \frac{C}{n}. \tag{37}$$

Combining (28), (32)–(37) and the fact that

$$(0,\infty)^3 = \bigcup_{\sigma \in \mathfrak{S}} \{0 < y_{\sigma(1)} < y_{\sigma(2)} < y_{\sigma(3)} < \infty\},$$

where $\mathfrak{S}$ is a set of permutations on a set $\{1, 2, 3\}$, we deduce that

$$\left| \|h_n\|^2_{\mathfrak{H}^{\otimes 2}} - H^{4H-2}(2H-1)^2 \int_{(0,\infty)^3} F(y_1, y_2, y_3) dy_1 dy_2 dy_3 \right| \leq \frac{C}{n}, \tag{38}$$

which proves (23). □

Lemma 3. *Suppose $H \in (\frac{1}{2}, 1)$. Let g_n and f_n be the functions given by (19) and (21), respectively. Then, for all $n \geq 1$,*

$$\|f_n \otimes_1 f_n\|_{\mathfrak{H}^{\otimes 2}} \leq \frac{C}{\sqrt{n}}, \tag{39}$$

$$\|g_n\|_{\mathfrak{H}^{\otimes 2}} \leq \frac{C}{\sqrt{n}}, \tag{40}$$

$$\|g_n \otimes_1 g_n\|_{\mathfrak{H}^{\otimes 2}} \leq \frac{C}{n^{3/2}}, \tag{41}$$

$$\|f_n \otimes_1 g_n\|_{\mathfrak{H}^{\otimes 2}} \leq \frac{C}{n}, \tag{42}$$

$$\left| \langle f_n, g_n \rangle_{\mathfrak{H}^{\otimes 2}} \right| \leq \frac{C}{\sqrt{n}}. \tag{43}$$

Proof. Taking into account that

$$8\|f_n \otimes_1 f_n\|^2_{\mathfrak{H}^{\otimes 2}} = Var\left(\frac{1}{2}\|DF_n\|^2_{\mathfrak{H}^{\otimes 2}}\right), \qquad \text{for} \quad F_n := I_2(f_n),$$

in order to obtain (39), it is sufficient to show that

$$Var\left(\|DF_n\|^2_{\mathfrak{H}^{\otimes 2}}\right) \leq \frac{C}{n}. \tag{44}$$

Now using (9), we can write

$$\|DF_n\|^2_{\mathfrak{H}^{\otimes 2}} = 4\int_0^n \int_0^n I_1(f_n(u,.)) I_1(f_n(v,.)) r_H(u,v) du dv$$

Using the multiplicative formula (12), we see that

$$\begin{aligned} I_1(f_n(u,.)) I_1(f_n(v,.)) &= \langle f_n(u,.), f_n(v,.)\rangle_{\mathfrak{H}^{\otimes 2}} + I_2\Big(f_n(u,.)\tilde{\otimes} f_n(v,.)\Big) \\ &=: A_1(u,v) + A_2(u,v). \end{aligned}$$

Here A_1 is deterministic and A_2 has expectation zero. Thus, we obtain that

$$\|DF_n\|^2_{\mathfrak{H}^{\otimes 2}} - E\Big(\|DF_n\|^2_{\mathfrak{H}^{\otimes 2}}\Big) = 4\int_0^n\int_0^n A_2(u,v)r_H(u,v)dudv$$

Hence, in order to have (44), we need to show that

$$E\left[\left(\int_0^n\int_0^n A_2(u,v)r_H(u,v)dudv\right)^2\right] \le \frac{C}{n}.$$

We have

$$\begin{aligned}
&E\left[\left(\int_0^n\int_0^n A_2(u,v)r_H(u,v)dudv\right)^2\right]\\
&=\int_{[0,n]^4} E\left[A_2(u_1,v_1)A_2(u_2,v_2)\right] r_H(u_1,v_1)r_H(u_2,v_2)du_1dv_1du_2dv_2\\
&=\int_{[0,n]^4}\Big(\int_{[0,n]^4}\big(f_n(u_1,.)\tilde{\otimes} f_n(v_1,.)(x_1,y_1)\big)\\
&\qquad\times\big(f_n(u_2,.)\tilde{\otimes} f_n(v_2,.)\big)(x_2,y_2)r_H(x_1,y_1)r_H(x_2,y_2)dx_1dy_1dx_2dy_2\Big)\\
&\qquad\qquad\times r_H(u_1,v_1)r_H(u_2,v_2)du_1dv_1du_2dv_2\\
&=\int_{[0,n]^8}\big(f_n(u_1,.)\tilde{\otimes} f_n(v_1,.)(x_1,y_1)\big)\big(f_n(u_2,.)\tilde{\otimes} f_n(v_2,.)\big)(x_2,y_2)\\
&\qquad\times r_H(x_1,y_1)r_H(x_2,y_2)r_H(u_1,v_1)\\
&\qquad\qquad\times r_H(u_2,v_2)dx_1dy_1dx_2dy_2du_1dv_1du_2dv_2.
\end{aligned}$$

Note that for every $0 \le x, y \le T$

$$r_H(x,y) \le H^{2H-1}(2H-1)e^{(\frac{1}{H}-1)(x+y)}\left|e^{\frac{x}{H}} - e^{\frac{y}{H}}\right|^{2H-2} := K(x,y).$$

Thus,

$$\begin{aligned}
&E\left[\left(\int_0^n\int_0^n A_2(u,v)r_H(u,v)dudv\right)^2\right]\\
&\le\int_{[0,n]^8}\big(f_n(u_1,.)\tilde{\otimes} f_n(v_1,.)(x_1,y_1)\big)\big(f_n(u_2,.)\tilde{\otimes} f_n(v_2,.)\big)(x_2,y_2)\\
&\quad\times K(x_1,y_1)K(x_2,y_2)K(u_1,v_1)K(u_2,v_2)dx_1dy_1dx_2dy_2du_1dv_1du_2dv_2\\
&\le\frac{C}{n},
\end{aligned}$$

where the last inequality comes from [3, Lemma 5.1]. Hence the inequality (39) is obtained.

Since for every $(u, v) \in [0, n]^2$, $g_n(u, v) \geq 0$, $h_n(u, v) \geq 0$ and $l_n(u, v) \geq 0$, using (19), we get

$$\|g_n\|_{\mathfrak{H}^{\otimes 2}} \leq \frac{1}{\theta\rho_{\theta,H}\sqrt{n}}\|h_n\|_{\mathfrak{H}^{\otimes 2}}.$$

Combining this with (38), we obtain (40). Similarly, using (19), (21) and (39), we have

$$\begin{aligned}
\|g_n \otimes_1 g_n\|^2_{\mathfrak{H}^{\otimes 2}} &\leq \frac{C}{n^2}\|h_n \otimes_1 h_n\|^2_{\mathfrak{H}^{\otimes 2}} \\
&\leq \frac{C}{n^2}\|f_n \otimes_1 f_n\|^2_{\mathfrak{H}^{\otimes 2}} \\
&\leq \frac{C}{n^3},
\end{aligned}$$

which implies (41).

It is well known that

$$\|f_n \otimes_1 g_n\|^2_{\mathfrak{H}^{\otimes 2}} = \langle f_n \otimes_1 f_n, g_n \otimes_1 g_n\rangle_{\mathfrak{H}^{\otimes 2}},$$

due to a straightforward application of the definition of contractions and the Fubini theorem.

Thus, from (39) and (41), we obtain

$$\begin{aligned}
\|f_n \otimes_1 g_n\|^2_{\mathfrak{H}^{\otimes 2}} &\leq \|f_n \otimes_1 f_n\|_{\mathfrak{H}^{\otimes 2}}\|g_n \otimes_1 g_n\|_{\mathfrak{H}^{\otimes 2}} \\
&\leq \frac{C}{n^3},
\end{aligned}$$

which leads to (42).

Finally, the inequality (43) is a direct consequence of (23) and (40). The proof of the lemma is thus complete. □

Proof of Theorem 1. First we prove the almost sure convergence (4). From (20) we can write

$$\theta - \hat{\theta}_n = \frac{\frac{\sigma_{\theta,H}}{\sqrt{n}} I_2(f_n)}{I_2(g_n) + b_n}.$$

Furthermore, using (23), and (40), we have

$$E\left[\left(\frac{1}{\sqrt{n}} I_2(f_n)\right)^2\right] = \frac{2}{n}\|f_n\|^2_{\mathfrak{H}^{\otimes 2}} \leq \frac{C}{n}, \text{ and } \quad E\left[(I_2(g_n))^2\right] = 2\|g_n\|^2_{\mathfrak{H}^{\otimes 2}} \leq \frac{C}{n}.$$

Combining this with (13) and Lemma 1, we obtain that, as $n \to \infty$,

$$\frac{1}{\sqrt{n}} I_2(f_n) \longrightarrow 0, \text{ and } \quad I_2(g_n) \longrightarrow 0$$

almost surely. Moreover, it follows from (22) that $b_n \longrightarrow 0$, as $n \to \infty$. Therefore (4) is obtained.

Let us now prove the convergence in law (5). It follows from (20), Proposition 1, Lemma 2 and Lemma 3 that for every n large,

$$\begin{aligned}
&\sup_{z\in\mathbb{R}}\left|P\left(\frac{\sqrt{n}}{\sigma_{\theta,H}}\left(\theta-\hat{\theta}_n\right)\le z\right)-P\left(Z\le z\right)\right| \\
&\quad\le C\times\max\Big\{\left|b_n^2-2\|f_n\|^2_{\mathfrak{H}^{\otimes 2}}\right|,\ \|f_n\otimes_1 f_n\|_{\mathfrak{H}^{\otimes 2}},\ \left|\langle f_n,g_n\rangle_{\mathfrak{H}^{\otimes 2}}\right|, \\
&\qquad\qquad \|f_n\otimes_1 g_n\|^{\frac{1}{2}}_{\mathfrak{H}^{\otimes 2}},\ \|g_n\otimes_1 g_n\|_{\mathfrak{H}^{\otimes 2}},\ \|g_n\|^2_{\mathfrak{H}^{\otimes 2}}\Big\} \\
&\quad\le\frac{C}{\sqrt{n}},
\end{aligned}$$

which implies the desired conclusion. □

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