Modern Stochastics: Theory and Applications 8 (4) (2021) 465–473
https://doi.org/10.15559/21-VMSTA189

Bounded in the mean solutions of a second-order difference equation

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Received: 9 June 2021, Revised: 9 August 2021, Accepted: 17 August 2021,
Published online: 9 September 2021

Abstract Sufficient conditions are given for the existence of a unique bounded in the mean solution to a second-order difference equation with jumps of operator coefficients in a Banach space. The question of the proximity of this solution to the stationary solution of the corresponding difference equation with constant operator coefficients is studied.

Keywords Difference equation, bounded in the mean solution, stationary solution, proximity of solutions
2010 MSC 60H99, 39A10

1 Introduction

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, X a complex separable Banach space with norm $\|\cdot\|_X$ and zero element 0_X, $\mathcal{L}(X)$ the Banach algebra of bounded linear operators defined on X, and $\mathcal{B}(X)$ the σ-algebra of Borel sets in X.

Definition 1. A sequence of X-valued random elements $\{\xi_n, n \in \mathbb{Z}\}$ defined on $(\Omega, \mathcal{F}, P)$ is called

- bounded in the mean if $\sup\limits_{n\in\mathbb{Z}} E\|\xi_n\|_X < +\infty$;

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– stationary (in the restricted sense) if
$\forall\, m \in \mathbb{N}\, \forall\, n_1, n_2, \ldots, n_m \in \mathbb{Z}\, \forall\, Q_1, Q_2, \ldots, Q_m \in \mathcal{B}(X)$:
$P\{\xi_{n_k+1} \in Q_k, 1 \leqslant k \leqslant m\} = P\{\xi_{n_k} \in Q_k, 1 \leqslant k \leqslant m\}$.

Consider the difference equation

$$\begin{cases} \xi_{n+1} - 2\xi_n + \xi_{n-1} = A\xi_n + \eta_n,\ n \geq 1, \\ \xi_{n+1} - 2\xi_n + \xi_{n-1} = B\xi_n + \eta_n,\ n \leq 0, \end{cases} \tag{1}$$

where A, B are fixed operators belonging to $\mathcal{L}(X)$, $\{\eta_n,\ n \in \mathbb{Z}\}$ is the given bounded in the mean sequence of X-valued random elements.

Definition 2. A sequence of X-valued random elements $\{\xi_n,\ n \in \mathbb{Z}\}$ is called a bounded in the mean solution of equation (1) corresponding to a bounded in the mean sequence $\{\eta_n,\ n \in \mathbb{Z}\}$ if the sequence $\{\xi_n,\ n \in \mathbb{Z}\}$ is bounded in the mean and equality (1) holds with probability 1 for all $n \in \mathbb{Z}$.

The purpose of this article is to obtain sufficient conditions for the operators A, B under which the difference equation (1) has a unique bounded in the mean solution $\{\xi_n,\ n \in \mathbb{Z}\}$ for each bounded in the mean sequence $\{\eta_n,\ n \in \mathbb{Z}\}$ and also to prove that $E\|\xi_n - \zeta_n\|_X \to 0$, as $n \to \infty$, where $\{\zeta_n,\ n \in \mathbb{Z}\}$ is the unique bounded in the mean solution of the difference equation with a constant operator coefficient A

$$\zeta_{n+1} - 2\zeta_n + \zeta_{n-1} = A\zeta_n + \eta_n,\ n \in \mathbb{Z}. \tag{2}$$

Bounded solutions of second-order deterministic difference equations with constant operator coefficients are studied in [3, 8], stationary solutions of the second-order equation (2) in [3, 2], bounded in the mean solutions of a first-order difference equation with a jump of the operator coefficient in [5], and bounded solutions of a deterministic analogue of equation (1) in [6]. Some applications of difference equations with operator coefficients in the deterministic case are given in [3, 7, 10, 1], and in the stochastic case in [3, 2, 9] and in references therein.

2 Auxiliary statements

Put $X^2 = \left\{ \begin{pmatrix} x^{(1)} \\ x^{(2)} \end{pmatrix} \mid x^{(1)}, x^{(2)} \in X \right\}$. Then X^2 will be a complex separable Banach space with coordinatewise addition and multiplication by a scalar and with norm $||\overline{x}||_{X^2} = ||x^{(1)}||_X + ||x^{(2)}||_X$, $\overline{x} = \begin{pmatrix} x^{(1)} \\ x^{(2)} \end{pmatrix} \in X^2$. If operators E, F, G, H belong to $\mathcal{L}(X)$, then, as in the case of numerical matrices $T = \begin{pmatrix} E & F \\ G & H \end{pmatrix}$ defines an operator belonging to $\mathcal{L}(X^2)$ by the rule $T\,\overline{x} = \begin{pmatrix} Ex^{(1)} + Fx^{(2)} \\ Gx^{(1)} + Hx^{(2)} \end{pmatrix}$, $\overline{x} = \begin{pmatrix} x^{(1)} \\ x^{(2)} \end{pmatrix} \in X^2$.

Consider an operator $T_A = \begin{pmatrix} A + 2I & -I \\ I & O \end{pmatrix}$, where I and O are the identity and zero operators in X, respectively. Denote by $\sigma(T_A)$, $\rho(T_A)$, $r(T_A)$ the spectrum, resolvent set and spectral radius of the operator T_A, respectively. In what follows, we will use the following statements.

Lemma 1. *The number* $\lambda \neq 0$ *belongs to* $\rho(T_A)$ *if and only if* $\lambda + \frac{1}{\lambda} - 2$ *belongs to* $\rho(A)$.

Proof. Sufficiency. Since $(\lambda + \frac{1}{\lambda} - 2) \in \rho(A)$, the operator $\Delta_\lambda = \lambda^2 I - (A+2I)\lambda + I$ has a continuous inverse operator Δ_λ^{-1}. Let J be the identity operator in X^2. It is easy to verify that the operator

$$(T_A - \lambda J)^{-1} = \begin{pmatrix} -\lambda\Delta_\lambda^{-1} & \Delta_\lambda^{-1} \\ -\Delta_\lambda^{-1} & (A + 2I - \lambda I)\Delta_\lambda^{-1} \end{pmatrix}$$

is a continuous inverse operator to $T_A - \lambda J$. Therefore, $\lambda \in \rho(T_A)$.

Necessity. Let us fix $\lambda \in \rho(T_A)$, $\lambda \neq 0$. It suffices to prove that the operator Δ_λ has a continuous inverse operator.

From the Banach theorem on the inverse operator, it follows that if Δ_λ^{-1} does not exist, then one of the following conditions is satisfied:

$(a1)$ there exists $u \neq 0_X$ such that $\Delta_\lambda u = 0_X$;

$(a2)$ there exists $v \in X$ such that the operator equation $\Delta_\lambda x = v$ has no solutions.

If condition $(a1)$ is satisfied then $(T_A - \lambda J)\begin{pmatrix} \lambda u \\ u \end{pmatrix} = \begin{pmatrix} 0_X \\ 0_X \end{pmatrix}$. This contradicts inclusion $\lambda \in \rho(T_A)$.

Since $\lambda \in \rho(T_A)$, the equation

$$\begin{pmatrix} A + 2I - \lambda I & -I \\ I & -\lambda I \end{pmatrix}\begin{pmatrix} x^{(1)} \\ x^{(2)} \end{pmatrix} = \begin{pmatrix} v \\ 0_X \end{pmatrix} \tag{3}$$

has a solution. Writing the equation 3 coordinatewise, we successively obtain the equalities $x^{(1)} = \lambda x^{(2)}$, $(A+2I-\lambda I)\lambda x^{(2)} - x^{(2)} = v$. Hence, the equation $\Delta_\lambda x = v$ has a solution $x = -x^{(2)}$. Thus, condition $(a2)$ is also not satisfied. □

Let $S = \{z \in \mathbb{C} \mid |z| = 1\}$ be the unit circle on the complex plane $\mathbb{C}$.

Lemma 2. $\sigma(T_A) \cap S = \varnothing$ *if and only if* $\sigma(A) \cap [-4; 0] = \varnothing$.

Since $\{\lambda + \frac{1}{\lambda} - 2 \mid \lambda \in S\} = [-4; 0]$, Lemma 2 is a direct consequence of Lemma 1.

Lemma 3. *The difference equation (1) has a unique bounded in the mean solution* $\{\xi_n, n \in \mathbb{Z}\}$ *for each bounded in the mean sequence* $\{\eta_n, n \in \mathbb{Z}\}$ *if and only if the difference equation*

$$\begin{cases} \overline{\xi}_{n+1} = T_A\overline{\xi}_n + \overline{\eta}_n, n \geq 1, \\ \overline{\xi}_{n+1} = T_B\overline{\xi}_n + \overline{\eta}_n, n \leq 0, \end{cases} \tag{4}$$

has a unique bounded in the mean solution $\{\overline{\xi}_n, n \in \mathbb{Z}\}$ *for each bounded in the mean sequence of* X^2*-valued random elements* $\{\overline{\eta}_n, n \in \mathbb{Z}\}$ *defined on* $(\Omega, \mathcal{F}, P)$.

The proof of Lemma 3 is standard and is omitted here.

Remark 1. If $\left\{\begin{pmatrix}\xi_n^{(1)}\\ \xi_n^{(2)}\end{pmatrix} n \in \mathbb{Z}\right\}$ is a bounded in the mean solution of equation (4) corresponding to the bounded in the mean sequence $\left\{\begin{pmatrix}\eta_n\\ 0_X\end{pmatrix}, n \in \mathbb{Z}\right\}$, then $\xi_n^{(2)} = \xi_{n-1}^{(1)}$ with probability 1 for all $n \in \mathbb{Z}$ and therefore $\{\xi_n^{(1)}, n \in \mathbb{Z}\}$ is a bounded in the mean solution of equation (1) corresponding to the sequence $\{\eta_n, n \in \mathbb{Z}\}$.

Denote by Y the Banach space $\mathcal{L}_1(\Omega, X)$ of all equivalence classes of random elements $\xi : \Omega \to X$ such that $\|\xi\|_Y = E\|\xi\|_X < +\infty$. Each operator G belonging to $\mathcal{L}(X)$ induces an operator $\widetilde{G}$ belonging to $\mathcal{L}(Y)$ and defined by the rule

$$\forall\, \xi \in Y : (\widetilde{G}\xi)(\omega) = G\xi(\omega), \omega \in \Omega. \tag{5}$$

The following lemma is a direct consequence of Definitions 1 and 2.

Lemma 4. *The difference equation (4) has a unique bounded in the mean solution $\{\overline{\xi}_n, n \in \mathbb{Z}\}$ for each bounded in the mean sequence $\{\overline{\eta}_n, n \in \mathbb{Z}\}$ if and only if the deterministic difference equation*

$$\begin{cases} \overline{\xi}_{n+1} = \widetilde{T}_A\overline{\xi}_n + \overline{\eta}_n, n \geq 1, \\ \overline{\xi}_{n+1} = \widetilde{T}_B\overline{\xi}_n + \overline{\eta}_n, n \leq 0, \end{cases} \tag{6}$$

has a unique bounded solution $\{\overline{\xi}_n, n \in \mathbb{Z}\}$ for each sequence $\{\overline{\eta}_n, n \in \mathbb{Z}\}$ bounded in Y^2.

Let W be a complex Banach space. Suppose that the spectrum $\sigma(U)$ of the operator $U \in \mathcal{L}(W)$ satisfies the condition $\sigma(U) \cap S = \varnothing$. Let $\sigma_-(U)$ be the part of the spectrum $\sigma(U)$ lying inside the circle S and $\sigma_+(U) = \sigma(U)\backslash\sigma_-(U)$. In what follows, we will consider the case when $\sigma_-(U) \neq \varnothing$, $\sigma_+(U) \neq \varnothing$. Note that all the results obtained below are also true in the case when one of the sets $\sigma_-(U)$, $\sigma_+(U)$ is empty, with obvious changes in the formulas obtained.

From the theorem on the spectral decomposition of an operator in a Banach space (see, for example, [3, p. 8]) it follows that the space W is represented as a direct sum $W = W_-(U)\dot{+}W_+(U)$ of subspaces $W_-(U)$, $W_+(U)$, for which the following conditions are satisfied:

the subspaces $W_-(U)$, $W_+(U)$ are invariant under the operator U;

the restrictions U_-, U_+ of the operator U to the subspaces $W_-(U)$, $W_+(U)$ have the spectra $\sigma_-(U)$, $\sigma_+(U)$, respectively;

the spectral radii of the operators U_-, U_+^{-1} satisfy the inequalities

$$r(U_-) < 1, r\left(U_+^{-1}\right) < 1. \tag{7}$$

3 The bounded in the mean solutions of the difference equation (1)

The following theorem is one of the main results of this article.

Theorem 1. *Let the operators A, B satisfy the following conditions:*

($i1$) $\sigma(A) \cap [-4; 0] = \varnothing, \sigma(B) \cap [-4; 0] = \varnothing$;

($i2$) $X^2 = X^2_-(T_A) \dot{+} X^2_+(T_B)$.

Then the difference equation (1) has a unique bounded in the mean solution $\{\xi_n, n \in \mathbb{Z}\}$ *for each bounded in the mean* X*-valued sequence* $\{\eta_n, n \in \mathbb{Z}\}$.

Proof. Condition ($i1$) and Lemma 2 imply that $\sigma(T_A) \cap S = \varnothing, \sigma(T_B) \cap S = \varnothing$. Also, using condition ($i2$) and Theorem 2 from [5], we conclude that the difference equation (4) has a unique bounded in the mean solution $\{\overline{\xi}_n, n \in \mathbb{Z}\}$ for every bounded in the mean sequence $\{\overline{\eta}_n, n \in \mathbb{Z}\}$. Therefore the assertion of the theorem holds by Lemma 3. □

Remark 2. In paper [6] it was established that if, in addition, the space X is finite-dimensional and the matrices of the operators A, B have the Jordan normal form in the same basis, then condition ($i1$) implies condition ($i2$).

Example 1. In the complex Euclidean space $X = \mathbb{C}^2$, consider the operators $A = \begin{pmatrix} 1/2 & 0 \\ 0 & 4/3 \end{pmatrix}$, $B = \begin{pmatrix} 1/2 & 0 \\ -5/6 & 4/3 \end{pmatrix}$. It is easy to verify that $\sigma(A) = \sigma(B) = \{1/2, 4/3\}$, $\sigma(T_A) = \sigma(T_B) = \{1/2, 2, 1/3, 3\}$. It follows from the proof of Lemma 1 that if $\lambda \neq 0$, then $Au = (\lambda + \frac{1}{\lambda} - 2)u$ if and only if $T_A \begin{pmatrix} \lambda u \\ u \end{pmatrix} = \lambda \begin{pmatrix} \lambda u \\ u \end{pmatrix}$. Consequently, $X^2_-(T_A)$, $X^2_+(T_B)$ are, respectively, the linear spans of the eigenvectors $\begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 2 \\ 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 3 \\ 0 \\ 1 \end{pmatrix}$ of the operators T_A, T_B. These four vectors are linearly independent. Therefore, for the operators A, B, conditions ($i1$) and ($i2$) of Theorem 1 are satisfied.

Example 2. Let A be the operator from Example 1 and

$$B = \frac{1}{21} \begin{pmatrix} 14 \cdot 17 + 15 \cdot 50 & -64 \cdot 15 \\ 64 \cdot 17 & -(14 \cdot 15 + 17 \cdot 50) \end{pmatrix}.$$

Then $\sigma(B) = \{4/3, -100/21\}, \sigma(T_B) = \{1/3, 3, -3/7, -7/3\}$ and also $X^2_-(T_A)$, $X^2_+(T_B)$ are the linear spans of the eigenvectors $\begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 3 \\ 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} -7 \cdot 15 \\ -7 \cdot 17 \\ 3 \cdot 15 \\ 3 \cdot 17 \end{pmatrix}$ of the operators T_A, T_B, respectively. Since these four vectors are linearly dependent, condition ($i2$) of Theorem 1 is not satisfied.

4 Proximity of components of the bounded in the mean solutions of the difference equations (1) and (2) for $n \to \infty$

First, consider the deterministic analogs of equations (1) and (2). Let U, V be fixed operators belonging to $\mathcal{L}(W)$. In what follows, we need the following statements.

Theorem 2 (See Theorem 1 in [3, p. 9]). *The difference equation*

$$u_{n+1} = U u_n + y_n, n \in \mathbb{Z}, \tag{8}$$

has a unique bounded solution $\{u_n, n \in \mathbb{Z}\}$ *for each sequence* $\{y_n, n \in \mathbb{Z}\}$ *bounded in* W *if and only if* $\sigma(U) \cap S = \varnothing$.

Remark 3. It follows from the proof of Theorem 2 that if $\sigma(U) \cap S = \varnothing$ then the unique bounded solution of equation (8) corresponding to the bounded sequence $\{y_n, n \in \mathbb{Z}\}$ has the form

$$\left\{ u_n = \sum_{j=0}^{\infty} U_-^j P_-^U y_{n-1-j} - \sum_{j=-\infty}^{-1} U_+^j P_+^U y_{n-1-j}, n \in \mathbb{Z} \right\}, \tag{9}$$

where P_-^U, P_+^U are the projectors in W onto the subspaces $W_-(U)$ and $W_+(U)$, respectively. Due to inequalities (7), the series in (9) converge.

Theorem 3 (See Theorem 1 in [4]). *Assume that the following conditions are fulfilled:*

$(j1)$ $\sigma(U) \cap S = \varnothing, \sigma(V) \cap S = \varnothing$;

$(j2)$ $W = W_-(U) \dot{+} W_+(V)$.

Then the difference equation

$$\begin{cases} x_{n+1} = U x_n + y_n, n \geq 1, \\ x_{n+1} = V x_n + y_n, n \leq 0, \end{cases} \tag{10}$$

has a unique bounded solution $\{x_n, n \in \mathbb{Z}\}$ *for each sequence* $\{y_n, n \in \mathbb{Z}\}$ *bounded in* W.

Remark 4. It was also shown in [4] that for equation (10) under conditions (j1), (j2) for each $n \geq 1$ the element x_n of the unique bounded solution $\{x_n, n \in \mathbb{Z}\}$ corresponding to a bounded sequence $\{y_n, n \in \mathbb{Z}\}$ can be obtained as follows. Let P_-^0, P_+^0 be projectors in W onto the subspaces $W_-(U)$, $W_+(V)$, respectively, corresponding to the representation $W = W_-(U) \dot{+} W_+(V)$. Put

$$\forall n \geq 1 : P_+^n = U^n P_+^0 U_+^{-n} P_+^U, \; P_-^n = I_W - P_+^n, \tag{11}$$

where I_W is the identity operator in W. Then

$$\begin{aligned} \forall n \geq 1 : x_n &= P_-^{n-1} y_{n-1} + U_- P_-^{n-2} y_{n-2} + \cdots + U_-^{n-2} P_-^1 y_1 \\ &+ \sum_{j=-\infty}^{0} U_-^{n-1} P_-^0 V_-^{|j|} P_-^V y_j - \sum_{j=n}^{\infty} P_+^{n-1} U_+^{n-1-j} P_+^U y_j. \end{aligned} \tag{12}$$

Conditions (j1), (j2) ensure the existence of the projectors $P_\pm^U$, $P_\pm^V$, $P_\pm^0$, and also, taking into account inequalities (7), the convergence in the norm in W of the series from (12) and the boundedness of the sequence $\{x_n, n \in \mathbb{Z}\}$.

The next theorem shows how close the solutions of equations (8) and (10) are, as $n \to \infty$.

Theorem 4. *Let conditions (j1), (j2) of Theorem 3 be satisfied. Then there exist constants $\rho \in (0; 1)$, $C > 0$, $n_0 \in \mathbb{N}$ depending only on the operators U, V and such that for each sequence $\{y_n, n \in \mathbb{Z}\}$ bounded in W, for bounded solutions $\{u_n, n \in \mathbb{Z}\}$ and $\{x_n, n \in \mathbb{Z}\}$ of equations (8) and (10) corresponding to the sequence $\{y_n, n \in \mathbb{Z}\}$, the following estimate holds:*

$$\forall\, n \geq n_0 : \|x_n - u_n\|_W \leq C\rho^n \sup_{n\in\mathbb{Z}} \|y_n\|_W. \tag{13}$$

Proof. From (7) it follows that the spectral radii of the operators U_-, U_+^{-1}, V_- are less than one. Therefore, there exist constants $\rho \in (0, 1)$, $m_0 \in \mathbb{N}$ such that

$$\forall\, m \geq m_0 : \max\bigl(\|U_-^m\|, \|U_+^{-m}\|, \|V_-^m\|\bigr) \leq \rho^m. \tag{14}$$

Fix a bounded sequence $\{y_n, n \in \mathbb{Z}\}$ and, for $n \geq m_0 + 2$, estimate $\|u_n - x_n\|_W$ using (9), (12). Since P_-^0 is a projector onto $W_-(U)$, then if we also use (11), we get

$$\begin{aligned}
&\forall\, 0 \leq k \leq n-2 : \|U_-^k P_-^U y_{n-1-k} - U_-^k P_-^{n-1-k} y_{n-1-k}\|_W \\
&= \|U_-^k\bigl(P_-^U - I_W + P_+^{n-1-k}\bigr)y_{n-1-k}\|_W = \|U_-^k\bigl(P_+^{n-1-k} - P_+^U\bigr)y_{n-1-k}\|_W \\
&= \|U_-^k\bigl(U^{n-1-k}P_+^0U_+^{-(n-1-k)} - U^{n-1-k}U_+^{-(n-1-k)}\bigr)P_+^U y_{n-1-k}\|_W \\
&= \| - U_-^k U^{n-1-k} P_-^0 U_+^{-(n-1-k)} P_+^U y_{n-1-k}\|_W \\
&= \|U_-^{n-1} P_-^0 U_+^{-(n-1-k)} P_+^U y_{n-1-k}\|_W.
\end{aligned}$$

Therefore denoting by C_1 the maximum of the squared norms of the operators $P_\pm^0$, $P_\pm^U$, $P_\pm^V$ we obtain

$$\begin{aligned}
&\forall\, 0 \leq k \leq n-1-m_0 : \|U_-^k P_-^U y_{n-1-k} - U_-^k P_-^{n-1-k} y_{n-1-k}\|_W \\
&\leq \rho^{n-1}\rho^{n-1-k}C_1\|y\|_\infty, && (15)\\
&\forall\, n-m_0 \leq k \leq n-2 : \|U_-^k P_-^U y_{n-1-k} - U_-^k P_-^{n-1-k} y_{n-1-k}\|_W \\
&\leq \rho^{n-1}C_1 \max_{1\leq j\leq m_0-1} \|U_+^{-j}\| \cdot \|y\|_\infty. && (16)
\end{aligned}$$

Here $\|y\|_\infty = \sup\limits_{n\in\mathbb{Z}} \|y_n\|_W$.

From (11) and the properties of the projectors it follows that

$$\begin{aligned}
&\forall\, k \geq 0 : \|U_+^{-1-k} P_+^U y_{n+k} - P_+^{n-1} U_+^{-1-k} P_+^U y_{n+k}\|_W \\
&= \|\bigl(U^{n-1}U_+^{-n+1}P_+^U - U^{n-1}P_+^0U^{-n+1}P_+^U\bigr)U_+^{-1-k}P_+^U y_{n+k}\|_W \\
&= \|U_-^{n-1}P_-^0U_+^{-n-k}P_+^U y_{n+k}\|_W \leq \rho^{n-1}\rho^{n+k}C_1\|y\|_\infty. && (17)
\end{aligned}$$

Also

$$\left\| \sum_{j=n-1}^{\infty} U_-^j P_-^U y_{n-1-j}\|_W \leq C_1\|y \right\|_\infty \frac{\rho^{n-1}}{1-\rho}, \tag{18}$$

$$\left\| \sum_{j=-\infty}^{0} U_-^{n-1} P_-^0 V^{|j|} P_-^V y_j \right\|_W$$
$$\leq \rho^{n-1} C_1 \|y\|_\infty \left(m_0 \max_{0 \leq k \leq m_0 - 1} \|V_-^k\| + \frac{\rho^{m_0}}{1-\rho} \right). \tag{19}$$

Note that the constants in (15)–(19) depend only on the operators U and V.

It follows from representations (9), (12) and inequalities (15)–(19) that estimate (13) is true. □

From Theorem 1 with $A = B$ it follows that when $\sigma(A) \cap [-4; 0] = \varnothing$ holds, the difference equation (2) has a unique bounded in the mean solution $\{\zeta_n, n \in \mathbb{Z}\}$ for each bounded in the mean sequence $\{\eta_n, n \in \mathbb{Z}\}$. It also follows from the results established in [4] that if $\sigma(T_A) \cap S = \varnothing$, $\sigma(T_B) \cap S = \varnothing$, $X^2 = X_-^2(T_A) \dot{+} X_+^2(T_B)$, then $\sigma(\widetilde{T}_A) = \sigma(T_A)$, $\sigma(\widetilde{T}_B) = \sigma(T_B)$, $Y^2 = Y_-^2(\widetilde{T}_A) \dot{+} Y_+^2(\widetilde{T}_B)$, where the operators $\widetilde{T}_A$, $\widetilde{T}_B$ are defined according to (5). Therefore, applying Theorem 4 to the difference equation (6) and then using Lemmas 3, 4, Theorem 1 and Remark 1, we conclude that the following theorem holds.

Theorem 5. *Let the conditions of Theorem 1 be satisfied. Then there exist constants $\rho \in (0, 1)$, $C > 0$, $n_0 \in \mathbb{N}$ depending only on the operators A and B and such that for each bounded in the mean sequence of X-valued random elements $\{\eta_n, n \in \mathbb{Z}\}$ for bounded in the mean solutions $\{\xi_n, n \in \mathbb{Z}\}$ and $\{\zeta_n, n \in \mathbb{Z}\}$ of equations (1) and (2) the following estimate holds:*

$$\forall n \geq n_0 : E\|\xi_n - \zeta_n\|_X \leq C\rho^n \sup_{n \in \mathbb{Z}} E\|\eta_n\|_X. \tag{20}$$

Note that when the sequence $\{\eta_n, n \in \mathbb{Z}\}$ is, in addition, stationary, then the corresponding solution $\{\zeta_n, n \in \mathbb{Z}\}$ of equation (2) is also stationary. According to (20), in this case, the elements of the solution to equation (1) are close to the stationary sequence $\{\zeta_n, n \in \mathbb{Z}\}$ when $n \to \infty$, despite the jump in the operator coefficient in (1).

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