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Asymptotic genealogies for a class of generalized Wright–Fisher models

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Abstract A class of Cannings models is studied, with population size N having a mixed multinomial offspring distribution with random success probabilities $W_1, \dots, W_N$ induced by independent and identically distributed positive random variables $X_1, X_2, \dots$ via $W_i := X_i / S_N$, $i \in \{1, \dots, N\}$, where $S_N := X_1 + \cdots + X_N$. The ancestral lineages are hence based on a sampling with replacement strategy from a random partition of the unit interval into N subintervals of lengths $W_1, \dots, W_N$. Convergence results for the genealogy of these Cannings models are provided under assumptions that the tail distribution of X_1 is regularly varying. In the limit several coalescent processes with multiple and simultaneous multiple collisions occur. The results extend those obtained by Huillet [J. Math. Biol. 68 (2014), 727–761] for the case when X_1 is Pareto distributed and complement those obtained by Schweinsberg [Stoch. Process. Appl. 106 (2003), 107–139] for models where sampling is performed without replacement from a supercritical branching process.

Keywords Cannings model, exchangeable coalescent, regularly varying function, simultaneous multiple collisions, weak convergence
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1 Introduction

Let $X_1, X_2, \ldots$ be independent copies of a random variable X taking values in $(0, \infty)$. For $N \in \mathbb{N} := \{1, 2, \ldots\}$ define $S_N := X_1 + \cdots + X_N$ and $W_i := X_i/S_N$, $i \in \{1, \ldots, N\}$. The weights $W_1, \ldots, W_N$ are exchangeable random variables with $W_1 + \cdots + W_N = 1$. In particular, $\mathbb{E}(W_i) = 1/N$, $i \in \{1, \ldots, N\}$. Consider the Cannings model [6, 7] with population size N and nonoverlapping generations such that, conditional on $W_1, \ldots, W_N$, the offspring sizes $\nu_1, \ldots, \nu_N$ have a multinomial distribution with parameters N and $W_1, \ldots, W_N$. Thus, the offspring distribution is

$$\mathbb{P}(\nu_1 = i_1, \ldots, \nu_N = i_N) = \frac{N!}{i_1! \cdots i_N!} \mathbb{E}(W_1^{i_1} \cdots W_N^{i_N}), \tag{1}$$

$i_1, \ldots, i_N \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$ with $i_1 + \cdots + i_N = N$. For degenerate X, i.e. $\mathbb{P}(X = c) = 1$ for some real constant $c > 0$, this model reduces to the classical Wright–Fisher model with deterministic weights $W_i = 1/N$, $i \in \{1, \ldots, N\}$. It is straightforward to check that the offspring sizes have joint descending factorial moments

$$\mathbb{E}((\nu_1)_{k_1} \cdots (\nu_N)_{k_N}) = (N)_{k_1+\cdots+k_N} \mathbb{E}(W_1^{k_1} \cdots W_N^{k_N}), \quad k_1, \ldots, k_N \in \mathbb{N}_0, \tag{2}$$

where $(x)_0 := 1$ and $(x)_k := x(x-1)\cdots(x-k+1)$ for $x \in \mathbb{R}$ and $k \in \mathbb{N}$. In [15] this model is studied for the case when X is Pareto distributed. If X is gamma distributed with density $x \mapsto x^{r-1}e^{-x}/\Gamma(r)$, $x > 0$, for some $r > 0$, then $(W_1, \ldots, W_N)$ is symmetric Dirichlet distributed with parameter r, leading to the Cannings model with the offspring distribution

$$\mathbb{P}(\nu_1 = i_1, \ldots, \nu_N = i_N) = \frac{N!}{i_1! \cdots i_N!} \frac{[r]_{i_1} \cdots [r]_{i_N}}{[rN]_N},$$

$i_1, \ldots, i_N \in \mathbb{N}_0$ with $i_1 + \cdots + i_N = N$, where $[x]_0 := 1$ and $[x]_i := x(x+1)\cdots(x+i-1)$ for $x \in \mathbb{R}$ and $i \in \mathbb{N}$. This Dirichlet multinomial model has been studied extensively in the literature (see, for example, Griffiths and Spanò [13]). In a series of papers [16, 17, 19] a subclass of Cannings models, called conditional branching process models in the spirit of Karlin and McGregor [23, 24], has been investigated, whose offspring distributions are (by definition) obtained by assuming that $\mathbb{P}(X_1 + \cdots + X_N = N) > 0$ and conditioning on the event that $X_1 + \cdots + X_N = N$. This construction based on conditioning is rather different from the construction based on sampling from a random partition of the unit interval we are dealing with in this article. Note however that several concrete examples (such as the classical Wright–Fisher model and the above mentioned Dirichlet multinomial model) can be constructed in both ways, either by sampling or by conditioning. For example, the Dirichlet multinomial model is obtained by taking N independent and identically distributed negative binomial random variables $X_1, \ldots, X_N$ with parameter $r > 0$ and $p \in (0, 1)$, so with distribution $\mathbb{P}(X_1 = k) = \binom{r+k-1}{k} p^r (1-p)^k$, $k \in \mathbb{N}_0$, and conditioning on the event that $X_1 + \cdots + X_N = N$.

The closely related model studied by Schweinsberg [37] differs from ours, since sampling is performed without replacement from a discrete super-critical Galton–Watson branching process, as explained in [37, Section 1.3]. In that model, X is integer valued and satisfies $\mathbb{E}(X) > 1$. In our model, X does not need to be integer valued and its mean is allowed to be less than 1. Moreover, the sampling in our multinomial model is with replacement, whereas in Schweinsberg's model it is without replacement.

The same multinomial scheme with an additional dormancy mechanism is considered in the recent work by Cordero et al. [8]. A class of Dirichlet models in the domain of attraction of the Kingman coalescent is also studied in two recent works by Boenkost et al. [4, 5] with an emphasis on Haldane's formula [14]. We refer the reader to Athreya [1] for some more information on Haldane's formula.

Fix $n \in \{1, \ldots, N\}$ and sample n individuals from the current generation. For $r \in \mathbb{N}_0$ define a random partition $\Pi_r^{(N,n)}$ of $\{1, \ldots, n\}$ such that $i, j \in \{1, \ldots, n\}$ belong to the same block of $\Pi_r^{(N,n)}$ if and only if the individual i and j share a common parent r generations backward in time. The process $\Pi^{(N,n)} := (\Pi_r^{(N,n)})_{r\in\mathbb{N}_0}$, called the discrete-time n-coalescent, takes values in the space $\mathcal{P}_n$ of partitions of $\{1, \ldots, n\}$. As in [15] we are interested in the limiting behavior of the discrete-time n-coalescent as the total population size N tends to infinity. It is easily seen (and well known) that the discrete-time n-coalescent is a time-homogeneous Markovian process. The transition probabilities $p_{\pi\pi'} := \mathbb{P}(\Pi_{r+1}^{(N,n)} = \pi' \mid \Pi_r^{(N,n)} = \pi)$ are given by

$$p_{\pi\pi'} = (N)_j \mathbb{E}(W_1^{k_1} \cdots W_j^{k_j}) =: \Phi_j^{(N)}(k_1, \ldots, k_j), \qquad \pi, \pi' \in \mathcal{P}_n, \tag{3}$$

if each block of π' is a union of some blocks of π, where $j := |\pi'|$ denotes the number of blocks of π' and $k_1, \ldots, k_j$ are the group sizes of merging blocks of π. Note that $\Phi_j^{(N)}(k_1, \ldots, k_j)$ is defined for all $N, j, k_1, \ldots, k_j \in \mathbb{N}$. Since the random variables $W_1, \ldots, W_N$ are exchangeable and satisfy $W_1 + \cdots + W_N = 1$, it follows for all $N, j, k_1, \ldots, k_j \in \mathbb{N}$ with $j \leq N$ that

$$\begin{aligned}(N-j)\mathbb{E}(W_1^{k_1} \cdots W_j^{k_j} W_{j+1}) &= \mathbb{E}\big(W_1^{k_1} \cdots W_j^{k_j}(W_{j+1} + \cdots + W_N)\big)\\ &= \mathbb{E}\big(W_1^{k_1} \cdots W_j^{k_j}(1 - (W_1 + \cdots + W_j))\big)\\ &= \mathbb{E}(W_1^{k_1} \cdots W_j^{k_j}) - \sum_{i=1}^{j} \mathbb{E}(W_1^{k_1} \cdots W_{i-1}^{k_{i-1}} W_i^{k_i+1} W_{i+1}^{k_{i+1}} \cdots W_j^{k_j}).\end{aligned}$$

Multiplication by $(N)_j$ ($= 0$ for $j > N$) shows that the consistency relation

$$\begin{aligned}&\Phi_j^{(N)}(k_1, \ldots, k_j)\\ &\quad = \Phi_{j+1}^{(N)}(k_1, \ldots, k_j, 1) + \sum_{i=1}^{j} \Phi_j^{(N)}(k_1, \ldots, k_{i-1}, k_i + 1, k_{i+1}, \ldots, k_j)\end{aligned} \tag{4}$$

holds for all $N, j, k_1, \ldots, k_j \in \mathbb{N}$. Moreover, for all $j, l \in \mathbb{N}$ with $j \geq l$ and all $k_1, \ldots, k_j, m_1, \ldots, m_l \in \mathbb{N}$ with $k_1 \geq m_1, \ldots, k_l \geq m_l$, the monotonicity relation

$$\Phi_j^{(N)}(k_1, \ldots, k_j) \leq \Phi_l^{(N)}(m_1, \ldots, m_l) \tag{5}$$

holds. Note that (5) follows from (4) by induction on the difference $d := j - l \in \mathbb{N}_0$. We refer the reader to [30, Definition 2.2] and the remark thereafter for similar statements and proofs for the full class of Cannings models. Choosing $j = 1$ and $k_1 = 2$ in (3) shows that two individuals share a common ancestor one generation backward in time with probability $c_N := \Phi_1^{(N)}(2) = N\mathbb{E}(W_1^2)$, the so-called coalescence probability. We also introduce the effective population size $N_e := 1/c_N$. Note that $c_N = N\mathbb{E}(W_1^2) \geq N(\mathbb{E}(W_1))^2 = 1/N$ or, equivalently, $N_e \leq N$. All Cannings models having an effective population size strictly larger than N (such as the Moran model having effective population size $N_e = N(N-1)/2 > N$ for $N \geq 4$ and most of the extended Moran models studied by Eldon and Wakeley [11] and Huillet and Möhle [18]) therefore do not belong to the class of models we are dealing with in this article.

General results for Cannings models concerning the convergence of their genealogical tree to an exchangeable coalescent process as the total population size tends to infinity are provided in [32]. For information on the theory of exchangeable coalescent processes we refer the reader to Pitman [33], Sagitov [34] and Schweinsberg [35, 36]. Coalescents with multiple collisions (Λ-coalescents) are Markovian stochastic processes taking values in the set of partitions of $\mathbb{N}$. They are characterized by a finite measure Λ on the unit interval. Important examples are Dirac-coalescents, where $\Lambda = \delta_a$ is the Dirac measure at a given point $a \in [0, 1]$, including the prominent Kingman coalescent (Kingman [26, 25, 27]), where $\Lambda = \delta_0$ is the Dirac measure at 0, and the star-shaped coalescent, where $\Lambda = \delta_1$. Other important examples are beta coalescents, where $\Lambda = \beta(a, b)$ is the beta distribution with parameters $a, b > 0$, including the Bolthausen–Sznitman coalescent, where Λ is the uniform distribution on the unit interval ($a = b = 1$).

The full class of exchangeable coalescent processes (Ξ-coalescents) allowing for simultaneous multiple collisions of ancestral lineages is characterized by a finite measure Ξ on the infinite simplex $\Delta := \{x = (x_1, x_2, \ldots) : x_1 \geq x_2 \geq \cdots \geq 0, \sum_{i=1}^\infty x_i \leq 1\}$. An example is the two-parameter Poisson–Dirichlet coalescent with parameters $\alpha > 0$ and $\theta > -\alpha$, where the characterizing measure $\nu(dx) := \Xi(dx)/\sum_{i=1}^\infty x_i^2$ on Δ is (by definition) the Poisson–Dirichlet distribution $\nu = \text{PD}(\alpha, \theta)$ with parameters $\alpha > 0$ and $\theta > -\alpha$. For more information on the Poisson–Dirichlet coalescent we refer the reader to Section 6 of [31]. In most studies, continuous-time coalescent processes $(\Pi_t)_{t \in T}$ with index set $T = [0, \infty)$ are considered. Note however that all Ξ-coalescents can as well be introduced with discrete time $T = \mathbb{N}_0$. In this case one speaks about a discrete-time Ξ-coalescent $(\Pi_r)_{r \in \mathbb{N}_0}$. The following terminology is taken from [16, Definition 2.1].

Definition 1. (i) A Cannings model is said to be in the domain of attraction of a continuous-time coalescent $\Pi = (\Pi_t)_{t \geq 0}$ if for each sample size $n \in \mathbb{N}$ the time-scaled ancestral process $(\Pi_{\lfloor t/c_N \rfloor}^{(N,n)})_{t \geq 0}$ converges in $D_{\mathcal{P}_n}([0, \infty))$ to $\Pi^{(n)}$ as $N \to \infty$, where $\Pi^{(n)} = (\Pi_t^{(n)})_{t \geq 0}$ denotes the restriction of Π to a sample of size n.

(ii) Analogously, a Cannings model is said to be in the domain of attraction of a discrete-time coalescent $\Pi = (\Pi_r)_{r \in \mathbb{N}_0}$ if for each sample size $n \in \mathbb{N}$ the ancestral process $(\Pi_r^{(N,n)})_{r \in \mathbb{N}_0}$ converges in $D_{\mathcal{P}_n}(\mathbb{N}_0)$ to $\Pi^{(n)}$ as $N \to \infty$, where $\Pi^{(n)} = (\Pi_r^{(n)})_{r \in \mathbb{N}_0}$ denotes the restriction of Π to a sample of size n.

Conditions on the tails of the distribution of X are provided which ensure that the population model with the offspring distribution (1) is in the domain of attraction of some exchangeable coalescent process. The tail condition is of the standard form $\mathbb{P}(X > x) \sim x^{-\alpha}\ell(x)$ as $x \to \infty$, where $\alpha \geq 0$ and ℓ is a function slowly varying at ∞. The results are collected in Theorem 1 in Section 2. It turns out that the three parameter values $\alpha \in \{0, 1, 2\}$ are boundary cases. Consequently, six different regimes ($\alpha > 2, \alpha = 2, \alpha \in (1, 2), \alpha = 1, \alpha \in (0, 1)$ and $\alpha = 0$) are considered leading to different limiting behaviors of the ancestral process. Theorem 1 also provides the asymptotics of the coalescence probability c_N as $N \to \infty$ for all six cases. In Section 3 some illustrating examples are provided including the case studied in [15] when X is Pareto distributed. The proofs are provided in the main Section 4. They are based on general convergence-to-the-coalescent theorems for Cannings models provided in [32] and combine (Abelian and Tauberian) arguments from the theory of regularly varying functions in the spirit of Karamata [20–22] with techniques used by Huillet [15] for the Pareto case and by Schweinsberg [37] for the related model where the sampling is performed without replacement.

2 Results

For most of the results it is assumed that there exist a constant $\alpha \geq 0$ and a function $\ell : (0, \infty) \to (0, \infty)$ slowly varying at ∞ such that

$$\mathbb{P}(X > x) \sim x^{-\alpha}\ell(x), \qquad x \to \infty. \tag{6}$$

Our main result (Theorem 1) clarifies the limiting behavior of the ancestral structure of the Cannings model with the offspring distribution (1) as the total population size N tends to infinity under the assumption (6). It turns out that the parameter values $\alpha \in \{0, 1, 2\}$ are boundary cases. It is hence natural to distinguish six regimes corresponding to the parameter ranges $\alpha > 2$, $\alpha = 2$, $\alpha \in (1, 2)$, $\alpha = 1$, $\alpha \in (0, 1)$ and $\alpha = 0$. In order to state the result it is convenient to introduce the function $\ell^* : (1, \infty) \to (0, \infty)$ via

$$\ell^*(x) := \int_1^x \frac{\ell(t)}{t}\,\mathrm{d}t. \tag{7}$$

Note that ℓ^* is nondecreasing, slowly varying at ∞ and satisfies $\ell(x)/\ell^*(x) \to 0$ as $x \to \infty$, see, for example, Bingham and Doney [3, p. 717 and 718] or Eq. (1.5.8) on p. 26 of Bingham, Goldie and Teugels [2] and the remarks thereafter. More precisely, for every $\lambda > 0$, as $x \to \infty$,

$$\frac{\ell^*(\lambda x) - \ell^*(x)}{\ell(x)} = \frac{1}{\ell(x)}\int_x^{\lambda x} \frac{\ell(t)}{t}\,\mathrm{d}t = \int_1^{\lambda} \frac{\ell(xu)}{\ell(x)}\frac{1}{u}\,\mathrm{d}u \to \int_1^{\lambda} \frac{1}{u}\,\mathrm{d}u = \log\lambda,$$

where the convergence holds by the uniform convergence theorem for slowly varying functions. Thus, ℓ^* is a de Haan function (with ℓ-index 1) and hence slowly varying. For general information on de Haan theory we refer the reader to Chapter 3 of [2].

The main (and only) result of this article is the following.

Theorem 1. *For the Cannings model with the offspring distribution (1) the following assertions hold.*

(i) If $\mathbb{E}(X^2) < \infty$ (in particular if (6) holds with $\alpha > 2$) then the model is in the domain of attraction of the continuous-time Kingman coalescent and the coalescence probability c_N satisfies $c_N \sim \rho/(\mu^2 N)$ as $N \to \infty$, where $\mu := \mathbb{E}(X)$ and $\rho := \mathbb{E}(X^2)$.

(ii) If (6) holds with $\alpha = 2$ then the model is in the domain of attraction of the continuous-time Kingman coalescent and the coalescence probability c_N satisfies $c_N \sim 2\ell^*(N)/(\mu^2 N)$ as $N \to \infty$, where $\mu := \mathbb{E}(X)$ and ℓ^* is defined via (7).

(iii) If (6) holds with $\alpha \in (1,2)$ then the model is in the domain of attraction of the continuous-time Λ-coalescent with $\Lambda := \beta(2-\alpha,\alpha)$ being the beta distribution with parameters $2-\alpha$ and α. Moreover, the coalescence probability c_N satisfies $c_N \sim \alpha\mathrm{B}(2-\alpha,\alpha)\mu^{-\alpha}\ell(N)/N^{\alpha-1} = \Gamma(2-\alpha)\Gamma(\alpha+1)\mu^{-\alpha}\ell(N)/N^{\alpha-1}$ as $N \to \infty$, where $\mu := \mathbb{E}(X)$.

(iv) If (6) holds with $\alpha = 1$, then the model is in the domain of attraction of the continuous-time Bolthausen–Sznitman coalescent. If $(a_N)_{N\in\mathbb{N}}$ is a sequence of positive real numbers satisfying $\ell^*(a_N) \sim a_N/N$ as $N \to \infty$, where ℓ^* is defined via (7), then the coalescence probability c_N satisfies $c_N \sim \ell(a_N)/\ell^*(a_N) \sim N\ell(a_N)/a_N$ as $N \to \infty$.

(v) If (6) holds with $\alpha \in (0,1)$, then the model is in the domain of attraction of the discrete-time Ξ-coalescent, where the characterizing measure $\nu(\mathrm{d}x) := \Xi(\mathrm{d}x)/\sum_{i=1}^\infty x_i^2$ is the Poisson–Dirichlet distribution $\nu = \mathrm{PD}(\alpha,0)$ with parameters α and $\theta := 0$. The coalescence probability satisfies $c_N \to 1-\alpha$ as $N \to \infty$.

(vi) If (6) holds with $\alpha = 0$, then the model is in the domain of attraction of the discrete-time star-shaped coalescent and the coalescence probability satisfies $c_N \to 1$ as $N \to \infty$.

In particular, for the first four cases (i)–(iv), $c_N \to 0$ as $N \to \infty$.

The six cases of Theorem 1 are summarized in Table 1. In the table, $\mu := \mathbb{E}(X)$, $\rho := \mathbb{E}(X^2)$, $\ell^*(x) := \int_1^x \ell(t)/t\,\mathrm{d}t$, $x > 1$, and $(a_N)_{N\in\mathbb{N}}$ is a sequence such that $\ell^*(a_N) \sim a_N/N$ as $N \to \infty$.

Remark 1. If $\ell(x) \equiv C$ for some constant $C > 0$, then $\ell^*(x) = C\int_1^x t^{-1}\,\mathrm{d}t = C\log x$ as $x \to \infty$. Assume now in addition that $\alpha = 1$. In this case, in part (iv) of Theorem 1 one can choose $a_1 := 1$ and $a_N := CN\log N$, $N \in \mathbb{N}\setminus\{1\}$. The coalescence probability thus satisfies $c_N \sim CN/a_N \sim 1/\log N$, in agreement with Proposition 6 of Huillet [15] for the Pareto example $\mathbb{P}(X > x) = 1/x$, $x > 1$. The same asymptotics for the coalescence probability holds for the related model considered by Schweinsberg (see [37, Lemma 16]) and, for example, when X is discrete taking the value $k \in \mathbb{N}$ with probability $\mathbb{P}(X = k) = 1/(k(k+1))$.

Table 1. Asymptotics of the ancestry of mixed multinomial Cannings models of the form (1) under the tail condition $\mathbb{P}(X > x) \sim x^{-\alpha}\ell(x)$ as $x \to \infty$

Condition	Limiting coalescent	Coalescence probability
$\mathbb{E}(X^2) < \infty$	Kingman	$\sim \dfrac{\rho}{\mu^2 N}$
$\alpha = 2$	Kingman	$\sim \dfrac{2\ell^*(N)}{\mu^2 N}$
$1 < \alpha < 2$	$\beta(2-\alpha, \alpha)$	$\sim \dfrac{\Gamma(2-\alpha)\Gamma(\alpha+1)\ell(N)}{\mu^\alpha N^{\alpha-1}}$
$\alpha = 1$	Bolthausen–Sznitman	$\sim \dfrac{\ell(a_N)}{\ell^*(a_N)} \sim \dfrac{N\ell(a_N)}{a_N}$
$\alpha \in (0, 1)$	discrete time PD$(\alpha, 0)$	$\sim 1 - \alpha$
$\alpha = 0$	discrete time star-shaped	~ 1

Remark 2. One may doubt that Theorem 1 is valid when X takes values close to 0 with high probability such that $\mathbb{E}(1/S_N) = \infty$ for all $N \in \mathbb{N}$. Typical examples of this form arise when the Laplace transform ψ of X satisfies $\psi(u) \sim L(u)$ as $u \to \infty$ for some function L slowly varying at ∞, or, equivalently (see Feller [12], p. 445, Theorem 2 and p. 446, Theorem 3), if $\mathbb{P}(X \le x) \sim L(1/x)$ as $x \to 0$. A concrete example is $P(X \le x) = 1/(1 - \log x)$, $0 < x \le 1$. In this case, $L(x) = 1/\log x$, $x > 0$, and, hence, $\mathbb{E}(1/S_N) = \int_0^\infty (\psi(u))^N \, du = \infty$ for all $N \in \mathbb{N}$. By Theorem 1 this model is in the domain of attraction of the Kingman coalescent, since $\mathbb{E}(X^2) < \infty$.

The finiteness or infiniteness of $\mathbb{E}(1/S_N)$ turns out to be irrelevant for the statements in Theorem 1, since the convergence results of Theorem 1 solely depend on the limiting behavior of the joint moments of the weights $W_1, \ldots, W_j$ as $N \to \infty$. For example (see Lemma 3), the asymptotics of $\mathbb{E}(W_1^p)$, $p > 0$, as $N \to \infty$ is determined by the values $\psi(u)$ of the Laplace transform ψ for values of u close to 0. For any fixed $\delta > 0$ the values $u > \delta$ do not play any role.

Conjectures and open problems.

Theorem 1 should also hold for Schweinsberg's model [37], since sampling without replacement (instead of sampling with replacement) should neither influence the asymptotics of the coalescence probability nor the limiting processes arising in Theorem 1. Note that in [37] the subclass of models without replacement is studied where the function ℓ in (6) is constant. We leave the analysis of Schweinsberg's model under the more general assumption (6) for the interested reader.

In contrast, conditional branching process models [16, 17, 19] seem to be harder to analyse and behave quite differently in general. Even for the subclass of so-called compound Poisson models, only partial results are available. Theorems 2.2 and 2.3 of [19] clarify that many unbiased compound Poisson models are in the domain of attraction of the Kingman coalescent, and [19, Theorem 2.5] (subcritical case) demonstrates that the limiting behavior of compound Poisson models can differ substantially from all scenarios arising in Theorem 1. To the best of the authors knowledge, the limiting behavior of the ancestral structure of unbiased conditional branching process models as $N \to \infty$ under assumptions of the form (6) has not been fully addressed in the literature. We leave this analysis for future research.

3 Examples

Example 1 (Pareto distribution)**.** Let X be Pareto distributed with parameter $\alpha > 0$ having tail probabilities $\mathbb{P}(X > x) = x^{-\alpha}$, $x > 1$. Clearly, (6) holds with $\ell \equiv 1$, so Theorem 1 is applicable. Note that $\mathbb{E}(X^p) < \infty$ if and only if $p < \alpha$ and in this case $\mathbb{E}(X^p) = \alpha \int_1^\infty x^{p-\alpha-1}\,\mathrm{d}x = \alpha/(\alpha - p)$. In particular $\mu := \mathbb{E}(X) = \alpha/(\alpha-1) < \infty$ for $\alpha > 1$ and $\rho := \mathbb{E}(X^2) = \alpha/(\alpha-2) < \infty$ for $\alpha > 2$. By Theorem 1, for $\alpha \geq 2$ the model is in the domain of attraction of the Kingman coalescent, for $\alpha \in [1,2)$ in the domain of attraction of the $\beta(2-\alpha,\alpha)$-coalescent, and for $\alpha \in (0,1)$ in the domain of attraction of the discrete-time Poisson–Dirichlet coalescent with parameter α.

Note that $\ell^*(x) = \int_1^x 1/t\,\mathrm{d}t = \log x$, $x > 1$. In part (iv) of Theorem 1, we can therefore choose $a_N := N \log N$ and obtain $c_N \sim \ell(a_N)/\ell^*(a_N) = 1/\ell^*(a_N) \sim 1/\log N$ as $N \to \infty$. Thus, by Theorem 1, the coalescence probability c_N satisfies

$$c_N \sim \begin{cases} \frac{\rho}{\mu^2 N} = \frac{(\alpha-1)^2}{\alpha(\alpha-2)N} & \text{if } \alpha > 2, \\ \frac{2\ell^*(N)}{\mu^2 N} = \frac{\log N}{2N} & \text{if } \alpha = 2, \\ \frac{\Gamma(2-\alpha)\Gamma(\alpha+1)}{\mu^\alpha N^{\alpha-1}} & \text{if } \alpha \in (1,2), \\ \frac{1}{\log N} & \text{if } \alpha = 1, \\ 1-\alpha & \text{if } \alpha \in (0,1). \end{cases}$$

For $\alpha > 2$ these results coincide with Proposition 7 of [15] with $\beta = 0$, for $\alpha = 2$ with Proposition 9 of [15], for $\alpha \in (1,2)$ with Lemma 4 and Proposition 5 of [15] with $\beta = 0$, for $\alpha = 1$ with Proposition 6 of [15] with $\beta = 0$, and for $\alpha \in (0,1)$ with Theorem 3 of [15] with $\beta = 0$.

The Pareto example is easily generalized in various ways by replacing $\ell \equiv 1$ by some other slowly varying function. For example, choosing for ℓ (a power of) the logarithm leads to the following example.

Example 2. Fix $\alpha \geq 0$ and assume that X has tail behavior $\mathbb{P}(X > x) \sim x^{-\alpha}\ell(x)$ as $x \to \infty$ with $\ell(x) := c(\log x)^{\beta-1}$, $x > 1$, for some constants $c > 0$ and $\beta > 0$. This example includes the Pareto model ($c = \beta = 1$). Clearly, (6) holds, since ℓ slowly varies at ∞. By Theorem 1, for $\alpha \geq 2$ the model is in the domain of attraction of the Kingman coalescent, for $\alpha \in [1,2)$ in the domain of attraction of the $\beta(2-\alpha,\alpha)$-coalescent, for $\alpha \in (0,1)$ in the domain of attraction of the discrete-time Poisson–Dirichlet coalescent with parameter α, and for $\alpha = 0$ in the domain of attraction of the discrete-time star-shaped coalescent. Note that

$$\ell^*(x) = \int_1^x \frac{\ell(t)}{t}\,\mathrm{d}t = c\int_1^x \frac{(\log t)^{\beta-1}}{t}\,\mathrm{d}t = \frac{c}{\beta}(\log x)^\beta, \qquad x \to \infty.$$

The asymptotics of the coalescence probability c_N as $N \to \infty$ can hence be obtained from the formulas provided in Theorem 1. In particular, for $\alpha > 1$ the asymptotics of c_N depends on the concrete value of $\mu := \mathbb{E}(X)$. For $\alpha = 1$ the asymptotics of c_N is obtained as follows. The sequence $(a_N)_{N\in\mathbb{N}}$, defined via $a_1 := 1$ and $a_N := (c/\beta)N(\log N)^\beta$ for $N \in \mathbb{N}\setminus\{1\}$, satisfies $\ell^*(a_N) \sim (c/\beta)(\log a_N)^\beta \sim (c/\beta)(\log N)^\beta = a_N/N$ as $N \to \infty$. By Theorem 1 (iv), the coalescence probability c_N satisfies $c_N \sim \ell(a_N)/\ell^*(a_N) \sim \beta/\log N$ as $N \to \infty$.

For illustration three examples with discrete X are provided.

Example 3 (Yule–Simon distribution)**.** Let X be Yule–Simon distributed [28, 38] with parameter $\alpha > 0$ having distribution $\mathbb{P}(X = k) = \alpha \mathrm{B}(\alpha + 1, k) = \alpha\Gamma(\alpha + 1)\Gamma(k)/\Gamma(\alpha + 1 + k)$, $k \in \mathbb{N}$, where $\mathrm{B}(.,.)$ and $\Gamma(.)$ denote the beta and the gamma function respectively. It is easily checked that $\mathbb{P}(X > k) = \Gamma(\alpha+1)\Gamma(k+1)/\Gamma(k+\alpha+1)$, $k \in \mathbb{N}_0$. In particular, $\mathbb{P}(X > x) \sim \Gamma(\alpha + 1)x^{-\alpha}$ as $x \to \infty$. Thus, (6) holds with $\ell \equiv \Gamma(\alpha + 1)$. Note that $\mathbb{E}((X)_k) < \infty$ if and only if $k < \alpha$ and in this case $\mathbb{E}((X)_k) = \alpha k!\mathrm{B}(\alpha - k, k)$. In particular, $\mu = \mathbb{E}(X) = \alpha/(\alpha - 1)$ for $\alpha > 1$ and $\mathbb{E}((X)_2) = 2\alpha/((\alpha - 1)(\alpha - 2))$ for $\alpha > 2$, which yields $\rho = \mathbb{E}(X^2) = \alpha^2/((\alpha - 1)(\alpha - 2))$ for $\alpha > 2$. By Theorem 1, for $\alpha \geq 2$ the model is in the domain of attraction of the Kingman coalescent, for $\alpha \in [1, 2)$ in the domain of attraction of the $\beta(2 - \alpha, \alpha)$-coalescent, and for $\alpha \in (0, 1)$ in the domain of attraction of the discrete-time Poisson–Dirichlet coalescent with parameter α. Note that $\ell^*(x) = \Gamma(\alpha + 1) \int_1^x 1/t\, \mathrm{d}t = \Gamma(\alpha + 1) \log x$, $x > 1$. In part (iv) of Theorem 1 we can thus choose $a_N := \Gamma(\alpha + 1)N \log N$ and obtain $c_N \sim \ell(a_N)/\ell^*(a_N) = 1/\log a_N \sim 1/\log N$ as $N \to \infty$. Thus, by Theorem 1, the coalescence probability c_N satisfies

$$c_N \sim \begin{cases} \frac{\rho}{\mu^2 N} = \frac{\alpha-1}{(\alpha-2)N} & \text{if } \alpha > 2, \\ \frac{2\ell^*(N)}{\mu^2 N} = \frac{\log N}{N} & \text{if } \alpha = 2, \\ \frac{\Gamma(2-\alpha)(\Gamma(\alpha+1))^2}{\mu^\alpha N^{\alpha-1}} & \text{if } \alpha \in (1,2), \\ \frac{1}{\log N} & \text{if } \alpha = 1, \\ 1-\alpha & \text{if } \alpha \in (0,1). \end{cases}$$

The Yule–Simon model is a discrete analog of the Pareto model discussed in Example 1. We refer the reader to Kozubowski and Podgórski [28] for some further information on Sibuya and Yule–Simon distributions.

Example 4 (Sibuya distribution)**.** Let X be Sibuya distributed with parameter $\alpha \in (0, 1)$ having probability generating function $f(s) = 1 - (1 - s)^\alpha$, $s \in [0, 1]$. Note that $f(s) = \sum_{k=1}^\infty (-1)^{k-1}\binom{\alpha}{k}s^k$, so X takes the value $k \in \mathbb{N}$ with probability $\mathbb{P}(X = k) = (-1)^{k-1}\binom{\alpha}{k} = \alpha\Gamma(k - \alpha)/(\Gamma(1 - \alpha)k!)$. The Laplace transform ψ of X satisfies $1 - \psi(u) = 1 - f(e^{-u}) = (1 - e^{-u})^\alpha \sim u^\alpha$ as $u \to 0$, i.e. relation (2.1) of Bingham and Doney [3] holds with $n = 0$, $\beta = \alpha \in (0, 1)$ and $L \equiv 1$. By Theorem A of [3] this relation is equivalent (see Eq. (2.3b) of [3]) to $\mathbb{P}(X > x) \sim (\Gamma(1 - \alpha))^{-1}x^{-\alpha}$ as $x \to \infty$, which shows that (6) holds with $\ell \equiv 1/\Gamma(1 - \alpha)$. Part (v) of Theorem 1 ensures that the model is in the domain of attraction of the Poisson–Dirichlet coalescent with parameter α and the coalescence probability c_N satisfies $c_N \to 1 - \alpha$ as $N \to \infty$. The same results are valid when X is α-stable, $\alpha \in (0, 1)$, with Laplace transform $\psi(u) := e^{-u^\alpha}$, $u \geq 0$, since in this case the same asymptotics $1 - \psi(u) \sim u^\alpha$ as $u \to 0$ holds. In this sense the Sibuya example is a discrete version of the α-stable case with $\alpha \in (0, 1)$.

Example 5. Let $\alpha \in (1, 2)$ and $b \in (0, 1/(\alpha - 1)]$. Assume that X has probability generating function $f(s) = (b + 1)s + b((1 - s)^\alpha - 1)$, $s \in [0, 1]$. Note that X is discrete taking values in $\mathbb{N}$ with probabilities $p_k := \mathbb{P}(X = k)$, $k \in \mathbb{N}$, given by $p_1 = b + 1 - b\alpha$ and $p_k = b(-1)^k\binom{\alpha}{k} = b\Gamma(k - \alpha)/(\Gamma(-\alpha)k!)$ for $k \in \{2, 3, \ldots\}$.

From $f'(s) = b + 1 - b\alpha(1-s)^{\alpha-1}$ it follows that $\mu := \mathbb{E}(X) = f'(1) = b + 1$. The Laplace transform ψ of X satisfies $\psi(u) - 1 + (b+1)u \sim bu^\alpha$ as $u \to 0$, i.e. relation (2.1) of Bingham and Doney [3] holds with $n = 1$, $\beta = \alpha - 1 \in (0,1)$, and $L \equiv b$. By Theorem A of [3] this relation is equivalent (see Eq. (2.3b) of [3]) to $\mathbb{P}(X > x) \sim b(-\Gamma(1-\alpha))^{-1}x^{-\alpha}$ as $x \to \infty$, which shows that (6) holds with $\ell(x) \equiv b/(-\Gamma(1-\alpha))$. By Theorem 1 (iii) the model is in the domain of attraction of the $\beta(2-\alpha,\alpha)$-coalescent and $c_N \sim (\alpha-1)\Gamma(\alpha+1)b/(\mu^\alpha N^{\alpha-1})$ as $N \to \infty$.

We close this section with a concrete example belonging to the boundary case (vi) ($\alpha = 0$).

Example 6. Let $\beta > 0$. If $\mathbb{P}(X > x) = 1/(1+\log x)^\beta$, $x \geq 1$, then $\mathbb{P}(X > x) \sim \ell(x)$ as $x \to \infty$ with $\ell(x) := 1/(\log x)^\beta$. By Theorem 1 (vi), the model is in the domain of attraction of the discrete-time star-shaped coalescent and $c_N \to 1$ as $N \to \infty$.

4 Proofs

The following auxiliary result (Lemma 1) is a modified version of Lemma 5 of Schweinsberg [37], adapted to our model. The result may be also viewed as a weak version of Cramér's large deviation theorem (see, for example, [10, Theorem 2.2.3]). Recall that $\mu := \mathbb{E}(X) \in (0,\infty]$.

Lemma 1. *For every $a \in (0,\mu)$ there exists $q \in (0,1)$ such that $\mathbb{P}(S_N \leq aN) \leq q^N$ for all $N \in \mathbb{N}$.*

Proof. Let f denote the moment generating function of $Y := X/a$, i.e. $f(x) := \mathbb{E}(x^Y)$, $x \in [0,1]$. From $\mathbb{E}(x^{S_N/a}) \geq \int_{\{S_N \leq aN\}} x^{S_N/a}\,\mathrm{d}\mathbb{P} \geq x^N\mathbb{P}(S_N \leq aN)$ it follows that $\mathbb{P}(S_N \leq aN) \leq x^{-N}\mathbb{E}(x^{S_N/a}) = (x^{-1}f(x))^N$ for all $x \in (0,1]$. Since $f(1) = 1$ and $f'(1) = \mathbb{E}(Y) = \mu/a > 1$, there exists $x_0 \in (0,1)$ such that $f(x_0) < x_0$. The result follows with $q := x_0^{-1}f(x_0)$. □

We now prove part (i) of Theorem 1.

Proof of Theorem 1 (i). We first verify that $Nc_N \to \rho/\mu^2$ as $N \to \infty$. We have

$$Nc_N = N^2\mathbb{E}(W_1^2) = N^2\int_{(0,\infty)} \mathbb{E}\bigg(\bigg(\frac{x}{x+S_{N-1}}\bigg)^2\bigg)\,\mathbb{P}_X(\mathrm{d}x) = \int_{(0,\infty)} f_N(x)\,\mathbb{P}_X(\mathrm{d}x),$$

where $f_N(x) := \mathbb{E}((x/(x/N+S_{N-1}/N))^2)$. By the law of large numbers, $(x/(x/N+S_{N-1}/N))^2 \to (x/\mu)^2$ almost surely and, hence, also in distribution as $N \to \infty$. For any $r > 0$ the map $x \mapsto x \wedge r$ is bounded and continuous on $[0,\infty)$. Thus,

$$\liminf_{N\to\infty} f_N(x) \geq \liminf_{N\to\infty} \mathbb{E}\bigg(\bigg(\frac{x}{\frac{x}{N}+\frac{S_{N-1}}{N}}\bigg)^2 \wedge r\bigg) = (x/\mu)^2 \wedge r.$$

Letting $r \to \infty$ yields $\liminf_{N\to\infty} f_N(x) \geq (x/\mu)^2$. Therefore, by Fatou's lemma,

$$\liminf_{N\to\infty} Nc_N = \liminf_{N\to\infty} \int_{(0,\infty)} f_N(x)\,\mathbb{P}_X(\mathrm{d}x) \geq \int_{(0,\infty)} (x/\mu)^2\,\mathbb{P}_X(\mathrm{d}x) = \frac{\rho}{\mu^2}.$$

In order to see that $\limsup_{N\to\infty} Nc_N \le \rho/\mu^2$ fix $a \in (0,\mu)$. By Lemma 1 there exists $q \in (0,1)$ such that $\mathbb{P}(S_N \le aN) \le q^N$ for all $N \in \mathbb{N}$. Therefore,

$$\begin{aligned} Nc_N &= N^2\mathbb{E}(W_1^2) = N^2\mathbb{E}(W_1^2 1_{\{S_N\le aN\}}) + N^2\mathbb{E}((X_1/S_N)^2 1_{\{S_N>aN\}}) \\ &\le N^2\mathbb{P}(S_N \le aN) + N^2\mathbb{E}(((X_1/(aN))^2) \le N^2q^N + \frac{\rho}{a^2} \to \frac{\rho}{a^2} \end{aligned}$$

as $N \to \infty$. Thus, $\limsup_{N\to\infty} Nc_N \le \rho/a^2$. Letting $a \uparrow \mu$ shows that $\limsup_{N\to\infty} Nc_N \le \rho/\mu^2$ and $Nc_N \to \rho/\mu^2$ is established.

It is well known (see [29, Section 4]) that any sequence of Cannings models with population sizes N is in the domain of attraction of the Kingman coalescent if and only if $\Phi_1^{(N)}(3)/c_N \to 0$ as $N \to \infty$. Thus, we have to verify that $\mathbb{E}(W_1^3)/\mathbb{E}(W_1^2) \to 0$ as $N \to \infty$. Since $\mathbb{E}(W_1^2) \ge (\mathbb{E}(W_1))^2 = 1/N^2$ it suffices to verify that $N^2\mathbb{E}(W_1^3) \to 0$ as $N \to \infty$. Fix again $a \in (0,\mu)$ and choose $q \in (0,1)$ as above. We have

$$N^2\mathbb{E}(W_1^3) = N^2\mathbb{E}(W_1^3 1_{\{S_N\le aN\}}) + N^2\mathbb{E}(W_1^3 1_{\{S_N>aN\}}).$$

Since $N^2\mathbb{E}(W_1^3 1_{\{S_N\le aN\}}) \le N^2\mathbb{P}(S_N \le aN) \le N^2q^N \to 0$ as $N \to \infty$ it remains to verify that $N^2\mathbb{E}(W_1^3 1_{\{S_N>aN\}}) \to 0$ as $N \to \infty$. We have

$$\begin{aligned} N^2\mathbb{E}(W_1^3 1_{\{S_N>aN\}}) &= N^2\mathbb{E}(X_1^3 S_N^{-3} 1_{\{S_N>aN, X_1\le aN\}}) \\ &\qquad + N^2\mathbb{E}(W_1^3 1_{\{S_n>aN, X_1>aN\}}) \\ &\le \frac{1}{a^3N}\mathbb{E}(X^3 1_{\{X\le aN\}}) + N^2\mathbb{P}(X > aN). \end{aligned}$$

Clearly, $N^2\mathbb{P}(X > aN) \le a^{-2}\mathbb{E}(X^2 1_{\{X>aN\}}) \to 0$ as $N \to \infty$, since $\rho := \mathbb{E}(X^2) < \infty$. It hence remains to verify that $N^{-1}\mathbb{E}(X^3 1_{\{X\le aN\}}) \to 0$ as $N \to \infty$. Let $\varepsilon > 0$. Choose L sufficiently large such that $\mathbb{E}(X^2 1_{\{X>L\}}) \le \varepsilon/(2a)$. Then, for all $N \in \mathbb{N}$ with $N \ge 2\rho L/\varepsilon$,

$$\begin{aligned} N^{-1}\mathbb{E}(X^3 1_{\{X\le aN\}}) &= N^{-1}\mathbb{E}(X^3 1_{\{X\le aN, X\le L\}}) + N^{-1}\mathbb{E}(X^3 1_{\{L<X\le aN\}}) \\ &\le N^{-1}L\rho + a\mathbb{E}(X^2 1_{\{X>L\}}) \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

which shows that $N^{-1}\mathbb{E}(X^3 1_{\{X\le aN\}}) \to 0$ as $N \to \infty$. □

We now prepare the proofs of the parts (ii) and (iii) of Theorem 1. We need the following two auxiliary results.

Lemma 2. *If (6) holds for some $\alpha \ge 0$ then for all $p > \alpha$,*

$$\mathbb{E}\left(\left(\frac{X}{X+x}\right)^p\right) \sim \frac{\Gamma(\alpha+1)\Gamma(p-\alpha)}{\Gamma(p)} x^{-\alpha}\ell(x), \qquad x \to \infty,$$

and

$$\mathbb{E}\left(\left(\frac{X}{X\vee x}\right)^p\right) \sim \frac{p}{p-\alpha} x^{-\alpha}\ell(x), \qquad x \to \infty.$$

Proof. Let T be a nonnegative random variable and $f : [0,\infty) \to \mathbb{R}$ be a continuous and piecewise continuously differentiable function such that $f(T)$ is integrable. Then,

$$\begin{aligned}
&\mathbb{E}(f(T)) - f(0)\\
&\quad = \int_{[0,\infty)} \big(f(x) - f(0)\big)\,\mathbb{P}_T(\mathrm{d}x) \;=\; \int_{[0,\infty)}\int_{[0,x)} f'(t)\,\lambda(\mathrm{d}t)\,\mathbb{P}_T(\mathrm{d}x)\\
&\quad = \int_{[0,\infty)} f'(t)\int_{(t,\infty)} \mathbb{P}_T(\mathrm{d}x)\,\lambda(\mathrm{d}t) \;=\; \int_0^\infty f'(t)\,\mathbb{P}(T>t)\,\mathrm{d}t. \qquad (8)
\end{aligned}$$

Let $x>0$. Applying (8) to $T := X/x$ and $f(t) := (t/(t+1))^p$ shows that

$$\mathbb{E}\bigg(\Big(\frac{X}{X+x}\Big)^p\bigg) \;=\; \int_0^\infty \frac{pt^{p-1}}{(t+1)^{p+1}}\mathbb{P}(X>xt)\,\mathrm{d}t.$$

By Theorem 3 of Karamata [22], applied to the function $\varphi(x) := \mathbb{P}(X>x)$, which is regularly varying at ∞ with index $\gamma := -\alpha$, it follows that, as $x\to\infty$,

$$\begin{aligned}
\mathbb{E}\bigg(\Big(\frac{X}{X+x}\Big)^p\bigg) \;&\sim\; \mathbb{P}(X>x)\int_0^\infty \frac{pt^{p-1}}{(t+1)^{p+1}}t^{-\alpha}\,\mathrm{d}t\\
&=\; \mathbb{P}(X>x)\frac{\Gamma(\alpha+1)\Gamma(p-\alpha)}{\Gamma(p)}.
\end{aligned}$$

The same steps, but applied to $f(t) := (t/(t\vee 1))^p$, show that

$$\begin{aligned}
\mathbb{E}\bigg(\Big(\frac{X}{X\vee x}\Big)^p\bigg) \;&=\; \int_0^\infty f'(t)\mathbb{P}(X>xt)\,\mathrm{d}t \;\sim\; \mathbb{P}(X>x)\int_0^\infty f'(t)t^{-\alpha}\,\mathrm{d}t\\
&=\; \mathbb{P}(X>x)\int_0^1 pt^{p-\alpha-1}\,\mathrm{d}t \;=\; \mathbb{P}(X>x)\frac{p}{p-\alpha}.
\end{aligned}$$

□

Lemma 3. *For all $j\in\{1,\ldots,N\}$ and $p_1,\ldots,p_j>0$,*

$$\mathbb{E}(W_1^{p_1}\cdots W_j^{p_j}) \;=\; \frac{1}{\Gamma(p)}\int_0^\infty u^{p-1}\mathbb{E}(e^{-uS_{N-j}})\prod_{i=1}^j \mathbb{E}(X^{p_i}e^{-uX})\,\mathrm{d}u, \qquad (9)$$

where $p := p_1+\cdots+p_j$ and $S_0 := 0$. Moreover, for any fixed $j\in\mathbb{N}$ the asymptotics of the latter integral as $N\to\infty$ is determined by the values of u close to 0, i.e. for any fixed $j\in\mathbb{N}$ and $\delta>0$, as $N\to\infty$,

$$\mathbb{E}(W_1^{p_1}\cdots W_j^{p_j}) \;\sim\; \frac{1}{\Gamma(p)}\int_0^\delta u^{p-1}\mathbb{E}(e^{-uS_{N-j}})\prod_{i=1}^j \mathbb{E}(X^{p_i}e^{-uX})\,\mathrm{d}u. \qquad (10)$$

In particular, for any fixed $\delta>0$,

$$\frac{1}{N} \;=\; \mathbb{E}(W_1) \;\sim\; \int_0^\delta \mathbb{E}(Xe^{-uX})\mathbb{E}(e^{-uS_{N-1}})\,\mathrm{d}u, \qquad N\to\infty. \qquad (11)$$

Remark 3. The fundamental relation (9) is well known from several references (see, for example, Cortines [9, Proposition 4.4] or Huillet [15]).

Proof. Let $j \in \{1,\ldots,N\}$ and $p_1,\ldots,p_j > 0$. From the representation $S_N^{-p} = (\Gamma(p))^{-1}\int_0^\infty u^{p-1}e^{-uS_N}\,\mathrm{d}u$ it follows that

$$\begin{aligned}\mathbb{E}(W_1^{p_1}\cdots W_j^{p_j}) &= \mathbb{E}(X_1^{p_1}\cdots X_j^{p_j}S_N^{-p})\\ &= \frac{1}{\Gamma(p)}\int_0^\infty u^{p-1}\mathbb{E}(X_1^{p_1}\cdots X_j^{p_j}e^{-uS_N})\,\mathrm{d}u\\ &= \frac{1}{\Gamma(p)}\int_0^\infty u^{p-1}\mathbb{E}(e^{-uS_{N-j}})\prod_{i=1}^{j}\mathbb{E}(X^{p_i}e^{-uX})\,\mathrm{d}u,\end{aligned}$$

which is (9). To check (10) fix $j \in \mathbb{N}$ and $\delta > 0$ and let ψ denote the Laplace transform of X. Decompose $\mathbb{E}(W_1^{p_1}\cdots W_j^{p_j}) = A_N + B_N$ with

$$A_N := \frac{1}{\Gamma(p)}\int_0^\delta u^{p-1}\mathbb{E}(e^{-uS_{N-j}})\prod_{i=1}^{j}\mathbb{E}(X^{p_i}e^{-uX})\,\mathrm{d}u$$

and

$$B_N := \frac{1}{\Gamma(p)}\int_\delta^\infty u^{p-1}\mathbb{E}(e^{-uS_{N-j}})\prod_{i=1}^{j}\mathbb{E}(X^{p_i}e^{-uX})\,\mathrm{d}u.$$

The map $u \mapsto \mathbb{E}(e^{-uS_{N-1}})$ is nonincreasing on $[0,\infty)$. Thus,

$$B_N \le \mathbb{E}(e^{-\delta S_{N-j}})\frac{1}{\Gamma(p)}\int_\delta^\infty u^{p-1}\prod_{i=1}^{j}\mathbb{E}(X^{p_i}e^{-uX})\,\mathrm{d}u = c_1(\psi(\delta))^{N-j}$$

and

$$A_N \ge \frac{1}{\Gamma(p)}\int_0^{\delta/2} u^{p-1}\mathbb{E}(e^{-uS_{N-j}})\prod_{i=1}^{j}\mathbb{E}(X^{p_i}e^{-uX})\,\mathrm{d}u \ge c_2(\psi(\delta/2))^{N-j}$$

with constants

$$c_1 := c_1(p_1,\ldots,p_j,\delta) := \frac{1}{\Gamma(p)}\int_\delta^\infty u^{p-1}\prod_{i=1}^{j}\mathbb{E}(X^{p_i}e^{-uX})\,\mathrm{d}u$$

and

$$c_2 := c_2(p_1,\ldots,p_j,\delta) := \frac{1}{\Gamma(p)}\int_0^{\delta/2} u^{p-1}\prod_{i=1}^{j}\mathbb{E}(X^{p_i}e^{-uX})\,\mathrm{d}u.$$

Note that $0 < c_1, c_2 < \infty$ and that c_1 and c_2 do not depend on N. Thus,

$$\begin{aligned}1 \le \frac{\mathbb{E}(W_1^{p_1}\cdots W_j^{p_j})}{A_N} &= 1+\frac{B_N}{A_N} \le 1+\frac{c_1(\psi(\delta))^{N-j}}{c_2(\psi(\delta/2))^{N-j}}\\ &= 1+\frac{c_1}{c_2}\bigg(\frac{\psi(\delta)}{\psi(\delta/2)}\bigg)^{N-j} \to 1\end{aligned}$$

as $N \to \infty$, since $\psi(\delta/2) > \psi(\delta)$. Eq. (11) follows by choosing $j := 1$ and $p_1 := 1$ in (10). □

We now turn to the proofs of the parts (ii) and (iii) of Theorem 1. We first consider part (iii) ($1 < \alpha < 2$). The boundary case $\alpha = 2$ (part (ii) of Theorem 1) will be studied afterwards.

Proof of Theorem 1 (iii). The idea of the proof is to apply the general convergence result [32, Theorem 2.1]. Having (3) in mind the main task is to derive the asymptotics of the moments of W_1 or, more generally, the asymptotics of the joint moments of the random variables $W_1, \ldots, W_j$ as $N \to \infty$. The following proof is based on Schweinsberg's [37] method. We first verify that

$$\lim_{N\to\infty} \frac{(\mu N)^\alpha}{\ell(N)} \mathbb{E}(W_1^k) \;=\; \alpha \mathrm{B}(k-\alpha,\alpha), \qquad k \in \mathbb{N}\setminus\{1\}. \tag{12}$$

For all $\lambda > \mu := \mathbb{E}(X)$, by the law of large numbers, $\mathbb{P}(S_{N-1} \le \lambda N) \to 1$ as $N \to \infty$. Thus,

$$\begin{aligned}
\mathbb{E}(W_1^k) &\ge \mathbb{E}(W_1^k 1_{\{X_2+\cdots+X_N \le \lambda N\}})\\
&\ge \mathbb{E}\bigg(\Big(\frac{X_1}{X_1+\lambda N}\Big)^k\bigg) \mathbb{P}(X_2+\cdots+X_N \le \lambda N)\\
&\sim \mathbb{E}\bigg(\Big(\frac{X}{X+\lambda N}\Big)^k\bigg) \sim \alpha \mathrm{B}(k-\alpha,\alpha)\frac{\ell(N)}{(\lambda N)^\alpha}, \qquad N \to \infty,
\end{aligned}$$

where the last asymptotics holds by Lemma 2, since ℓ is slowly varying at ∞. Multiplication with $N^\alpha/\ell(N)$ and taking lim inf shows that

$$\liminf_{N\to\infty} \frac{N^\alpha}{\ell(N)} \mathbb{E}(W_1^k) \;\ge\; \alpha \mathrm{B}(k-\alpha,\alpha)/\lambda^\alpha.$$

Letting $\lambda \downarrow \mu$ it follows that $\liminf_{N\to\infty} N^\alpha/\ell(N)\mathbb{E}(W_1^k) \ge \alpha \mathrm{B}(k-\alpha,\alpha)/\mu^\alpha$.

To handle the lim sup, fix $a \in (0,\mu)$ and decompose

$$\mathbb{E}(W_1^k) \;=\; \mathbb{E}(W_1^k 1_{\{X_2+\cdots+X_N \le aN\}}) + \mathbb{E}(W_1^k 1_{\{X_2+\cdots+X_N > aN\}}).$$

From Lemma 1 it follows that there exists $N_0 \in \mathbb{N}$ and $q \in (0,1)$ such that $\mathbb{P}(S_{N-1} \le aN) \le q^N$ for all $N > N_0$. Thus, $\mathbb{E}(W_1^k 1_{\{X_2+\cdots+X_N \le aN\}}) \le \mathbb{P}(X_2+\cdots+X_N \le aN) = \mathbb{P}(S_{N-1} \le aN) \le q^N$ for all $N \in \mathbb{N}$ with $N > N_0$. It hence suffices to verify that

$$\limsup_{N\to\infty} \frac{(\mu N)^\alpha}{\ell(N)} \mathbb{E}(W_1^k 1_{\{X_2+\cdots+X_N > aN\}}) \;=\; \alpha \mathrm{B}(k-\alpha,\alpha). \tag{13}$$

In order to see this, let $\lambda \in (a,\mu)$ and decompose

$$\begin{aligned}
&\mathbb{E}(W_1^k 1_{\{X_2+\cdots+X_N > aN\}})\\
&\quad = \mathbb{E}(W_1^k 1_{\{aN < X_2+\cdots+X_N \le \lambda N\}}) + \mathbb{E}(W_1^k 1_{\{X_2+\cdots+X_N > \lambda N\}})\\
&\quad \le \mathbb{E}\bigg(\Big(\frac{X_1}{X_1+aN}\Big)^k\bigg) \mathbb{P}(S_{N-1} \le \lambda N)
\end{aligned}$$

$$+\mathbb{E}\bigg(\Big(\frac{X_1}{X_1+\lambda N}\Big)^k\bigg)\mathbb{P}(S_{N-1}>\lambda N).$$

The two expectations on the right hand side are both $O(\ell(N)/N^\alpha)$ by Lemma 2. Moreover, $\mathbb{P}(S_{N-1}\le\lambda N)\to 0$ and $\mathbb{P}(S_{N-1}>\lambda N)\to 1$ as $N\to\infty$. Therefore, only the last term contributes to the lim sup, and we obtain

$$\begin{aligned}
&\limsup_{N\to\infty}\frac{N^\alpha}{\ell(N)}\mathbb{E}(W_1^k 1_{\{X_2+\cdots+X_N>aN\}})\\
&\qquad\le \limsup_{N\to\infty}\frac{N^\alpha}{\ell(N)}\mathbb{E}\bigg(\Big(\frac{X_1}{X_1+\lambda N}\Big)^k\bigg)\mathbb{P}(S_{N-1}>\lambda N)\\
&\qquad\sim \frac{N^\alpha}{\ell(N)}\alpha\mathrm{B}(k-\alpha,\alpha)\frac{\ell(\lambda N)}{(\lambda N)^\alpha}\;=\;\alpha\mathrm{B}(k-\alpha,\alpha)/\lambda^\alpha.
\end{aligned}$$

Letting $\lambda\uparrow\mu$ shows that (13) holds. Thus, (12) is established.

Choosing $k=2$ in (12) yields the asymptotic formula for the coalescence probability $c_N=N\mathbb{E}(W_1^2)$ stated in Theorem 1 (iii). In particular, $c_N=O(\ell(N)/N^{\alpha-1})$. In summary we conclude that

$$\frac{\Phi_1^{(N)}(k)}{c_N}=\frac{\mathbb{E}(W_1^k)}{\mathbb{E}(W_1^2)}\to\frac{\Gamma(k-\alpha)}{\Gamma(k)\Gamma(2-\alpha)}=\int_{(0,1)}x^{k-2}\Lambda(\mathrm{d}x),\qquad N\to\infty,$$

where $\Lambda:=\beta(2-\alpha,\alpha)$ denotes the beta distribution with parameters $2-\alpha$ and α. Moreover,

$$\begin{aligned}
\mathbb{E}(W_1^2W_2^2 1_{\{S_N>aN\}}) &\le \mathbb{E}\bigg(\frac{X_1^2X_2^2}{(X_1\vee aN)^2(X_2\vee aN)^2}\bigg)\\
&= \bigg(\mathbb{E}\Big(\frac{X^2}{(X\vee aN)^2}\Big)\bigg)^2\sim\Big(\frac{2}{2-\alpha}\frac{\ell(aN)}{(aN)^\alpha}\Big)^2\\
&= O\Big(\frac{(\ell(N))^2}{N^{2\alpha}}\Big).
\end{aligned}$$

Since $c_N\ge K\ell(N)/N^{\alpha-1}$ for some $K>0$ it follows that $\Phi_2^{(N)}(2,2)/c_N=O(\ell(N)/N^{\alpha-1})=O(c_N)\to 0$ as $N\to\infty$. Thus, for all $j,k_1,\ldots,k_j\in\mathbb{N}\setminus\{1\}$, $\Phi_j^{(N)}(k_1,\ldots,k_j)/c_N\le\Phi_2^{(N)}(2,2)/c_N\to 0$ as $N\to\infty$. By [32, Theorem 2.1], the model is in the domain of attraction of the $\beta(2-\alpha,\alpha)$-coalescent. □

We now turn to the boundary case $\alpha=2$, so we prove part (ii) of Theorem 1.

Proof of Theorem 1 (ii). For all $x>0$,

$$\begin{aligned}
\mathbb{E}(X^2 1_{\{X\le x\}}) &= \int_0^\infty \mathbb{P}(X^2 1_{\{X\le x\}}>y)\,\mathrm{d}y\\
&= \int_0^\infty 2t\mathbb{P}(X^2 1_{\{X\le x\}}>t^2)\,\mathrm{d}t = \int_0^x 2t\mathbb{P}(t<X\le x)\,\mathrm{d}t\\
&= \int_0^x 2t\big(\mathbb{P}(X>t)-\mathbb{P}(X>x)\big)\,\mathrm{d}t = \int_0^x 2t\mathbb{P}(X>t)\,\mathrm{d}t-x^2\mathbb{P}(X>x).
\end{aligned}$$

Since $\int_0^x t\mathbb{P}(X > t)\,\mathrm{d}t \sim \int_1^x \ell(t)/t\,\mathrm{d}t = \ell^*(x)$, $x^2\mathbb{P}(X > x) \sim \ell(x)$, and $\ell(x)/\ell^*(x) \to 0$ as $x \to \infty$, it follows that $\mathbb{E}(X^2 1_{\{X\le x\}}) \sim 2\ell^*(x)$ as $x \to \infty$. Thus, relation (2.3c) of Bingham and Doney [3] holds with $n = 1$ and $L := \ell^*$. This relation is equivalent (see (2.4) in Theorem A of [3]) to $\psi''(u) \sim 2\ell^*(1/u)$ as $u \to 0$.

Recall that $c_N = N\mathbb{E}(W_1^2)$. We now verify the asymptotic relation $c_N \sim 2\mu^{-2}\ell^*(N)/N$ as $N \to \infty$ or, equivalently, that

$$\lim_{N\to\infty} \frac{N^2}{\ell^*(N)}\mathbb{E}(W_1^2) = \frac{2}{\mu^2}. \tag{14}$$

We have

$$\mathbb{E}(W_1^2) = \int_0^\infty u\psi''(u)\mathbb{E}(e^{-uS_{N-1}})\,\mathrm{d}u = \frac{1}{N^2}\int_0^\infty t\psi''(t/N)\mathbb{E}(e^{-tS_{N-1}/N})\,\mathrm{d}t.$$

Multiplication by $N^2/\ell^*(N)$ and Fatou's lemma yield

$$\begin{aligned}\liminf_{N\to\infty}\frac{N^2}{\ell^*(N)}\mathbb{E}(W_1^2) &\ge \int_0^\infty t\liminf_{N\to\infty}\frac{\psi''(t/N)}{\ell^*(N)}\mathbb{E}(e^{-tS_{N-1}/N})\,\mathrm{d}t\\ &= \int_0^\infty 2te^{-\mu t}\,\mathrm{d}t = \frac{2}{\mu^2},\end{aligned}$$

since $\psi''(t/N) \sim 2\ell^*(N/t) \sim 2\ell^*(N)$ and $\mathbb{E}(e^{-tS_{N-1}/N}) \to e^{-\mu t}$ as $N \to \infty$. To see that $\limsup_{N\to\infty} \frac{N^2}{\ell^*(N)}\mathbb{E}(W_1^2) \le 2/\mu^2$, fix $a \in (0,\mu)$. By Lemma 1 there exists $N_0 \in \mathbb{N}$ and $q \in (0,1)$ such that $\mathbb{P}(S_{N-1} \le aN) \le q^N$ for all $N \in \mathbb{N}$ with $N > N_0$. Noting that $\mathbb{E}(W_1^2 1_{\{X_2+\cdots+X_N\le aN\}}) \le \mathbb{P}(S_{N-1} \le aN) \le q^N$, it suffices to verify that

$$\limsup_{N\to\infty}\frac{N^2}{\ell^*(N)}\mathbb{E}(W_1^2 1_{\{X_2+\cdots+X_N>aN\}}) \le \frac{2}{\mu^2}. \tag{15}$$

In order to see this, let $\lambda \in (a,\mu)$ and decompose $\mathbb{E}(W_1^2 1_{\{X_2+\cdots+X_N>aN\}}) = \mathbb{E}(W_1^2 1_{A_N}) + \mathbb{E}(W_1^2 1_{B_N})$, where $A_N := \{aN < X_2+\cdots+X_N \le \lambda N\}$ and $B_N := \{X_2+\cdots+X_N > \lambda N\}$. We have

$$\begin{aligned}\frac{N^2}{\ell^*(N)}\mathbb{E}(W_1^2 1_{A_N}) &= \frac{N^2}{\ell^*(N)}\int_0^\infty u\psi''(u)\mathbb{E}(e^{-uS_{N-1}}1_{\{aN<S_{N-1}\le\lambda N\}})\,\mathrm{d}u\\ &\le \mathbb{P}(S_{N-1}\le\lambda N)\frac{N^2}{\ell^*(N)}\int_0^\infty u\psi''(u)e^{-uaN}\,\mathrm{d}u\\ &= \mathbb{P}(S_{N-1}\le\lambda N)\frac{1}{\ell^*(N)}\int_0^\infty t\psi''(t/N)e^{-at}\,\mathrm{d}t\\ &\sim \mathbb{P}(S_{N-1}\le\lambda N)\frac{1}{\ell^*(N)}\psi''(1/N)\int_0^\infty te^{-at}\,\mathrm{d}t\\ &\sim \mathbb{P}(S_{N-1}\le\lambda N)\frac{2}{a^2} \to 0, \qquad N\to\infty,\end{aligned}$$

where the second last asymptotics holds by Theorem 3 of Karamata [22], applied with $f(t) := te^{-at}$ and $\varphi := \psi''$, which is slowly varying at 0. For the second part we obtain

$$\begin{aligned}\frac{N^2}{\ell^*(N)}\mathbb{E}(W_1^2 1_{B_N}) &= \frac{N^2}{\ell^*(N)}\int_0^\infty u\psi''(u)\mathbb{E}(e^{-uS_{N-1}}1_{\{S_{N-1}>\lambda N\}})\,\mathrm{d}u\\ &\le \frac{N^2}{\ell^*(N)}\int_0^\infty u\psi''(u)e^{-u\lambda N}\,\mathrm{d}u\\ &= \frac{1}{\ell^*(N)}\int_0^\infty t\psi''(t/N)e^{-\lambda t}\,\mathrm{d}t\\ &\sim \frac{1}{\ell^*(N)}\psi''(1/N)\int_0^\infty te^{-\lambda t}\,\mathrm{d}t \sim \frac{2}{\lambda^2},\end{aligned}$$

where the second last asymptotics holds again by Theorem 3 of Karamata [22], now applied with $f(t) := te^{-\lambda t}$ and $\varphi := \psi''$. Therefore,

$$\begin{aligned}&\limsup_{N\to\infty}\frac{N^2}{\ell^*(N)}\mathbb{E}(W_1^2 1_{\{X_2+\cdots+X_N>aN\}})\\ &\le \limsup_{N\to\infty}\frac{N^2}{\ell^*(N)}\mathbb{E}(W_1^2 1_{A_N}) + \limsup_{N\to\infty}\frac{N^2}{\ell^*(N)}\mathbb{E}(W_1^2 1_{B_N})\\ &\le 0+\frac{2}{\lambda^2} = \frac{2}{\lambda^2}.\end{aligned}$$

Letting $\lambda \uparrow \mu$ shows that (15) holds. Thus, (14) is established. The rest of the proof now works as follows. By the monotone density theorem (Lemma 4), applied with $\rho = 0$,

$$\frac{-u\psi'''(u)}{\psi''(u)} \sim \frac{-u\psi'''(u)}{2\ell^*(1/u)} \to 0, \qquad u \to 0.$$

Thus, for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $-u\psi'''(u) \le \varepsilon\psi''(u)$ for all $u \in (0,\delta)$. Therefore, together with Lemma 3, as $N \to \infty$,

$$\begin{aligned}\mathbb{E}(W_1^3) &\sim \frac{1}{2}\int_0^\delta u^2(-\psi'''(u))(\psi(u))^{N-1}\,\mathrm{d}u\\ &\le \frac{\varepsilon}{2}\int_0^\delta u\psi''(u)(\psi(u))^{N-1}\,\mathrm{d}u \sim \frac{\varepsilon}{2}\mathbb{E}(W_1^2).\end{aligned}$$

Thus, $\limsup_{N\to\infty}\mathbb{E}(W_1^3)/\mathbb{E}(W_1^2) \le \varepsilon/2$. Since ε can be chosen arbitrarily small, it follows that $\lim_{N\to\infty}\Phi_1^{(N)}(3)/c_N = \lim_{N\to\infty}\mathbb{E}(W_1^3)/\mathbb{E}(W_1^2) = 0$, which is equivalent (see, for example, [29, Section 4]) to the property that the model is in the domain of attraction of the Kingman coalescent. □

We now turn to the proofs of the three remaining parts (iv)–(vi) of Theorem 1. We first consider the case $0 < \alpha < 1$ corresponding to part (v) of Theorem 1. The boundary cases (iv) ($\alpha = 1$) and (vi) ($\alpha = 0$) will be considered afterwards. Assume that $0 < \alpha < 1$. Then (6) is exactly Eq. (2.3b) of Bingham and Doney [3] with $n = 0$, $\beta = \alpha \in (0,1)$ and $L(x) := \Gamma(1-\alpha)\ell(x)$. By [3, Theorem A], (6) is hence equivalent (see [3, Eq. (2.1)]) to $1 - \psi(u) \sim u^\alpha L(1/u) = \Gamma(1-\alpha)u^\alpha\ell(1/u)$ as $u \to 0$.

Proof of Theorem 1 (v). For $k \in \mathbb{N}_0$ and $x > 0$ define $h_k(x) := x^k \mathbb{P}(X > x)$. By (6), $h_k(x) \sim x^{k-\alpha}\ell(x)$ as $x \to \infty$. Karamata's Tauberian theorem [2, Theorem 1.7.6], applied with $U := h_k$, $\rho := k - \alpha$ and $c := \Gamma(\rho + 1)$, yields for all $k \in \mathbb{N}_0$ that $\widehat{h}_k(u) := u \int_0^\infty e^{-ux} x^k \mathbb{P}(X > x)\,\mathrm{d}x \sim \Gamma(k - \alpha + 1)u^{\alpha-k}\ell(1/u)$ as $u \to 0$. Thus, by (8), for all $k \in \mathbb{N}$,

$$\begin{aligned}
\varphi_k(u) \;&:=\; \mathbb{E}(X^k e^{-uX}) \;=\; \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}x}(x^k e^{-ux})\mathbb{P}(X > x)\,\mathrm{d}x \\
&=\; \int_0^\infty (kx^{k-1}e^{-ux} - ux^k e^{-ux})\mathbb{P}(X > x)\,\mathrm{d}x \;=\; \frac{k}{u}\widehat{h}_{k-1}(u) - \widehat{h}_k(u) \\
&\sim\; \frac{k}{u}\Gamma(k-\alpha)u^{\alpha-(k-1)}\ell(1/u) - \Gamma(k-\alpha+1)u^{\alpha-k}\ell(1/u) \\
&=\; \alpha\Gamma(k-\alpha)u^{\alpha-k}\ell(1/u), \qquad u \to 0.
\end{aligned} \tag{16}$$

We now turn to the joint moments of $W_1, \ldots, W_j$. Let $a_1, a_2, \ldots$ be positive real numbers satisfying $L(a_N) \sim a_N^\alpha/N$ as $N \to \infty$. Moreover, fix some $\delta \in (0, \infty)$. The exact value of δ is irrelevant but it is important that δ is finite. Let $j, k_1, \ldots, k_j \in \mathbb{N}$. Define $k := k_1 + \cdots + k_j$. By Lemma 3, as $N \to \infty$,

$$\begin{aligned}
\Phi_j^{(N)}(k_1, \ldots, k_j) \;&=\; (N)_j \mathbb{E}(W_1^{k_1} \cdots W_j^{k_j}) \\
&\sim\; \frac{N^j}{\Gamma(k)} \int_0^\delta u^{k-1}\mathbb{E}(e^{-uS_{N-j}}) \prod_{i=1}^j \varphi_{k_i}(u)\,\mathrm{d}u \\
&=\; \frac{N^j}{\Gamma(k)a_N^k} \int_0^{\delta a_N} t^{k-1}\mathbb{E}(e^{-tS_{N-j}/a_N}) \prod_{i=1}^j \varphi_{k_i}(t/a_N)\,\mathrm{d}t.
\end{aligned}$$

Corollary 1, an Abelian result á la Karamata provided in the appendix for convenience, applied to $x_N := 1/a_N$, $f_N(t) := t^{k-1}\mathbb{E}(e^{-tS_{N-j}/a_N})1_{(0,\delta a_N)}(t)$ and $\varphi := \prod_{i=1}^j \varphi_{k_i}$, which is regularly varying at 0 with index $\sum_{i=1}^j (\alpha - k_i) = j\alpha - k$, yields, as $N \to \infty$,

$$\Phi_j^{(N)}(k_1 \ldots, k_j) \;\sim\; \frac{N^j \prod_{i=1}^j \varphi_{k_i}(1/a_N)}{\Gamma(k)a_N^k} \int_0^{\delta a_N} t^{j\alpha-1}\mathbb{E}(e^{-tS_{N-j}/a_N})\,\mathrm{d}t. \tag{17}$$

In the following the asymptotic relation (17) is used to verify by induction on $j \in \mathbb{N}$ that, for all $k_1, \ldots, k_j \in \mathbb{N}$,

$$\lim_{N\to\infty} \Phi_j^{(N)}(k_1 \ldots, k_j) \;=\; \alpha^{j-1}\frac{\Gamma(j)}{\Gamma(k)} \prod_{i=1}^j \frac{\Gamma(k_i - \alpha)}{\Gamma(1-\alpha)}. \tag{18}$$

Since $\Phi_1^{(N)}(1) = N\mathbb{E}(W_1) = 1$, the choice $j = k_1 = 1$ in (17) yields

$$\int_0^{\delta a_N} t^{\alpha-1}\mathbb{E}(e^{-tS_{N-1}/a_N})\,\mathrm{d}t \;\sim\; \frac{a_N}{N\varphi_1(1/a_N)} \;\sim\; \frac{1}{\alpha}, \qquad N \to \infty, \tag{19}$$

where the last asymptotics holds, since $\varphi_1(1/a_N) \sim \alpha\Gamma(1-\alpha)a_N^{1-\alpha}\ell(a_N)$ and $a_N^\alpha/N \sim L(a_N) = \Gamma(1-\alpha)\ell(N)$. Note that in (19) it is important that $\delta < \infty$ because otherwise the integral on the left hand side of (19) could take the value ∞. For $j = 1$ and $k_1 = k \in \mathbb{N}$, (17) thus reduces to

$$\Phi_1^{(N)}(k) \sim \frac{N\varphi_k(1/a_N)}{\Gamma(k)a_N^k}\frac{1}{\alpha} \sim \frac{\Gamma(k-\alpha)}{\Gamma(k)\Gamma(1-\alpha)}, \qquad N \to \infty,$$

which shows that (18) holds for $j = 1$. In particular, $c_N = \Phi_1^{(N)}(2) \to 1-\alpha > 0$ as $N \to \infty$. The induction step from $j-1$ to j (≥ 2) works as follows. By the consistency relation (4) and the induction hypothesis,

$$\begin{aligned}\Phi_j^{(N)}(1,\ldots,1) &= \Phi_{j-1}^{(N)}(1,\ldots,1) - (j-1)\Phi_{j-1}^{(N)}(2,1,\ldots,1)\\ &\to \alpha^{j-2} - \alpha^{j-2}(1-\alpha) = \alpha^{j-1}.\end{aligned}$$

Thus, (18) holds for $k_1 = \cdots = k_j = 1$ and the choice $k_1 = \cdots = k_j = 1$ in (17) yields

$$\int_0^{\delta a_N} t^{j\alpha-1}\mathbb{E}(e^{-tS_{N-j}/a_N})\,\mathrm{d}t \sim \frac{\Gamma(j)a_N^j}{N^j(\varphi_1(1/a_N))^j}\alpha^{j-1} \sim \frac{\Gamma(j)}{\alpha}, \qquad N \to \infty.$$

Therefore, (17) reduces to

$$\begin{aligned}\Phi_j^{(N)}(k_1,\ldots,k_j) &\sim \frac{N^j\prod_{i=1}^j\varphi_{k_i}(1/a_N)}{\Gamma(k)a_N^k}\frac{\Gamma(j)}{\alpha}\\ &\sim \frac{N^j\Gamma(j)\prod_{i=1}^j(\alpha\Gamma(k_i-\alpha)a_N^{k_i-\alpha}\ell(a_N))}{\alpha\Gamma(k)a_N^k}\\ &= \alpha^{j-1}\frac{\Gamma(j)}{\Gamma(k)}\Big(\frac{N\Gamma(1-\alpha)\ell(a_N)}{a_N^\alpha}\Big)^j\prod_{i=1}^j\frac{\Gamma(k_i-\alpha)}{\Gamma(1-\alpha)}\\ &\to \alpha^{j-1}\frac{\Gamma(j)}{\Gamma(k)}\prod_{i=1}^j\frac{\Gamma(k_i-\alpha)}{\Gamma(1-\alpha)} =: \phi_j(k_1,\ldots,k_j),\end{aligned}$$

since $N\Gamma(1-\alpha)\ell(a_N) = NL(a_N) \sim a_N^\alpha$ as $N \to \infty$. The induction is complete.

In summary, $\Phi_j^{(N)}(k_1,\ldots,k_j) \to \phi_j(k_1,\ldots,k_j)$ as $N \to \infty$ for all $j, k_1, \ldots, k_j \in \mathbb{N}$. The quantities $\phi_j(k_1,\ldots,k_j)$ are (see [31, Eq. (16)] for the analogous formula for the rates of the continuous-time Poisson–Dirichlet coalescent) the transition probabilities of the discrete-time two-parameter Poisson–Dirichlet coalescent with parameters α and 0. The convergence result (v) of Theorem 1 therefore follows from [32, Theorem 2.1]. □

Let us now turn to the (boundary) case $\alpha = 1$, so we now assume that $\mathbb{P}(X > x) \sim x^{-1}\ell(x)$ as $x \to \infty$ for some function ℓ slowly varying at ∞.

Proof of Theorem 1 (iv). The proof has much in common with that of part (v). The details are however slightly different. For all $x > 0$,

$$\mathbb{E}(X1_{\{X\le x\}}) = \int_0^\infty \mathbb{P}(X1_{\{X\le x\}} > t)\,\mathrm{d}t = \int_0^x \mathbb{P}(t < X \le x)\,\mathrm{d}t$$
$$= \int_0^x \big(\mathbb{P}(X > t) - \mathbb{P}(X > x)\big)\,\mathrm{d}t = \int_0^x \mathbb{P}(X > t)\,\mathrm{d}t - x\mathbb{P}(X > x).$$

Using that $\int_0^x \mathbb{P}(X > t)\,\mathrm{d}t \sim \int_1^x \ell(t)/t\,\mathrm{d}t = \ell^*(x)$, $x\mathbb{P}(X > x) \sim \ell(x)$ and $\ell(x)/\ell^*(x) \to 0$ as $x \to \infty$, it follows that $\mathbb{E}(X1_{\{X\le x\}}) \sim \ell^*(x)$ as $x \to \infty$. Recall that ℓ^* is slowly varying at ∞. Thus, Eq. (2.3c) of Bingham and Doney [3] holds with $n = 0$ and $\alpha = \beta = 1$ and $L := \ell^*$, which is equivalent (see [3, Theorem A, Eq. (2.1)]) to

$$1 - \psi(u) \sim u\ell^*(1/u), \qquad u \to 0$$

and as well (see [3, Theorem A, Eq. (2.4)]) equivalent to

$$\varphi_1(u) := \mathbb{E}(Xe^{-uX}) = -\psi'(u) \sim \ell^*(1/u), \qquad u \to 0.$$

For $k \in \mathbb{N} \setminus \{1\}$, the asymptotic relation

$$\varphi_k(u) := \mathbb{E}(X^k e^{-uX}) \sim \Gamma(k-1)u^{1-k}\ell(1/u), \qquad u \to 0, \tag{20}$$

is verified exactly as in the proof of part (v) of Theorem 1. In particular, φ_k is regularly varying at 0 with index $1 - k$, $k \in \mathbb{N}_0$.

We now turn to the joint moments of $W_1, \ldots, W_j$. Let $a_1, a_2, \ldots$ be positive real numbers satisfying $\ell^*(a_N) \sim a_N/N$ as $N \to \infty$. As in the proof of part (v) of Theorem 1, fix some $\delta \in (0, \infty)$. Again, the exact value of δ is irrelevant but it is important that δ is finite. Let $j, k_1, \ldots, k_j \in \mathbb{N}$. Define $k := k_1 + \cdots + k_j$. By Lemma 3, as $N \to \infty$,

$$\mathbb{E}(W_1^{k_1}\cdots W_j^{k_j}) \sim \frac{1}{\Gamma(k)}\int_0^\delta u^{k-1}\mathbb{E}(e^{-uS_{N-j}})\prod_{i=1}^j \varphi_{k_i}(u)\,\mathrm{d}u$$
$$= \frac{1}{\Gamma(k)a_N^k}\int_0^{\delta a_N} t^{k-1}\mathbb{E}(e^{-tS_{N-j}/a_N})\prod_{i=1}^j \varphi_{k_i}(t/a_N)\,\mathrm{d}t.$$

Corollary 1, applied to $x_N := 1/a_N$, $f_N(t) := t^{k-1}\mathbb{E}(e^{-tS_{N-j}/a_N})1_{(0,\delta a_N)}(t)$ and $\varphi := \prod_{i=1}^j \varphi_{k_i}$, which is regularly varying at 0 with index $\sum_{i=1}^j (1 - k_i) = j - k$, shows that, as $N \to \infty$,

$$\mathbb{E}(W_1^{k_1}\cdots W_j^{k_j}) \sim \frac{\prod_{i=1}^j \varphi_{k_i}(1/a_N)}{\Gamma(k)a_N^k}\int_0^{\delta a_N} t^{j-1}\mathbb{E}(e^{-tS_{N-j}/a_N})\,\mathrm{d}t. \tag{21}$$

Since $\mathbb{E}(W_1) = 1/N$, the asymptotic relation (21) turns for $j = k_1 = 1$ into

$$\frac{1}{N} \sim a_N^{-1}\varphi_1(1/a_N)\int_0^{\delta a_N} \mathbb{E}(e^{-tS_{N-1}/a_N})\,\mathrm{d}t$$
$$\sim a_N^{-1}\ell^*(a_N)\int_0^{\delta a_N} \mathbb{E}(e^{-tS_{N-1}/a_N})\,\mathrm{d}t,$$

or, equivalently,

$$\int_0^{\delta a_N} \mathbb{E}(e^{-tS_{N-1}/a_N})\,\mathrm{d}t \ \sim\ \frac{a_N}{N\ell^*(a_N)} \ \sim\ 1, \qquad N \to \infty.$$

Therefore, for $j = 1$ and $k = k_1 \in \mathbb{N} \setminus \{1\}$, (21) reduces to

$$\mathbb{E}(W_1^k) \ \sim\ \frac{\varphi_k(1/a_N)}{\Gamma(k)a_N^k} \ \sim\ \frac{\ell(a_N)}{(k-1)a_N}, \qquad N \to \infty,$$

since $\varphi_k(1/a_N) \sim \Gamma(k-1)a_N^{k-1}\ell(a_N)$ by (20). Thus, the coalescence probability c_N satisfies

$$c_N \ =\ N\mathbb{E}(W_1^2) \ \sim\ \frac{N\ell(a_N)}{a_N} \ \sim\ \frac{\ell(a_N)}{\ell^*(a_N)} \ \to\ 0, \qquad N \to \infty,$$

and

$$\frac{\Phi_1^{(N)}(k)}{c_N} \ =\ \frac{\mathbb{E}(W_1^k)}{\mathbb{E}(W_1^2)} \ \to\ \frac{1}{k-1} \ =\ \int_{[0,1]} x^{k-2}\,\Lambda(\mathrm{d}x), \qquad k \in \mathbb{N} \setminus \{1\},$$

where Λ denotes the uniform distribution on $[0, 1]$. To see that simultaneous multiple collisions cannot occur in the limit, note that $(N)_2\mathbb{E}(W_1W_2) = 1 - c_N \sim 1$ as $N \to \infty$, or, equivalently, $\mathbb{E}(W_1W_2) \sim 1/N^2$ as $N \to \infty$. Thus, (21) reduces for $j = 2$ and $k_1 = k_2 = 1$ to

$$\begin{aligned}\frac{1}{N^2} \ &\sim\ \frac{\varphi_1^2(1/a_N)}{a_N^2} \int_0^{\delta a_N} t\mathbb{E}(e^{-tS_{N-2}/a_N})\,\mathrm{d}t \\ &\sim\ \Big(\frac{\ell^*(a_N)}{a_N}\Big)^2 \int_0^{\delta a_N} t\mathbb{E}(e^{-tS_{N-2}/a_N})\,\mathrm{d}t,\end{aligned}$$

or, equivalently,

$$\int_0^{\delta a_N} t\mathbb{E}(e^{-tS_{N-2}/a_N})\,\mathrm{d}t \ \sim\ \Big(\frac{a_N}{N\ell^*(a_N)}\Big)^2 \ \sim\ 1, \qquad N \to \infty. \tag{22}$$

Therefore, for $j = 2$ and $k_1, k_2 \in \mathbb{N} \setminus \{1\}$, (21) reduces to

$$\mathbb{E}(W_1^{k_1}W_2^{k_2}) \ \sim\ \frac{\varphi_{k_1}(1/a_N)\varphi_{k_2}(1/a_N)}{\Gamma(k)a_N^k} \ \sim\ \frac{\Gamma(k_1-1)\Gamma(k_2-1)}{\Gamma(k)}\Big(\frac{\ell(a_N)}{a_N}\Big)^2,$$

where the last asymptotics holds since $\varphi_{k_i}(1/a_N) \sim \Gamma(k_i - 1)a_N^{k_i-1}\ell(a_N)$ by (20). In particular, $\Phi_2^{(N)}(2, 2) = (N)_2\mathbb{E}(W_1^2W_2^2) \sim (N\ell(a_N)/a_N)^2/6 \sim c_N^2/6$. For $j, k_1, \ldots, k_j \in \mathbb{N} \setminus \{1\}$ it follows from the monotonicity property (5) that $\Phi_j^{(N)}(k_1, \ldots, k_j) \le \Phi_2^{(N)}(2, 2) = O(c_N^2)$, and, therefore, $\Phi_j^{(N)}(k_1, \ldots, k_j)/c_N \to 0$ as $N \to \infty$, which shows that simultaneous multiple collisions cannot occur in the limit.

To summarize, by [32, Theorem 2.1], the model is in the domain of attraction of the Λ-coalescent with Λ the uniform distribution on $[0, 1]$, which is the Bolthausen–Sznitman coalescent. □

Remark 4. Suppose that (6) holds with $\alpha \in (0,1)$. Using the same techniques as in the previous proof, it follows for all $j \in \mathbb{N}$ and $k_1, \ldots, k_j \geq 2$ that

$$\begin{aligned}\Phi_j^{(N)}(k_1,\ldots,k_j) &= (N)_j \mathbb{E}(W_1^{k_1}\cdots W_j^{k_j})\\ &\sim \frac{\Gamma(j)\Gamma(k_1-1)\cdots\Gamma(k_j-1)}{\Gamma(k)} c_N^j, \qquad N\to\infty,\end{aligned}$$

where $c_N \sim N\ell(a_N)/a_N \sim \ell(a_N)/\ell^*(a_N) \to 0$ as $N \to \infty$. Thanks to the monotonicity property (5) this formula is only needed for $j \in \{1,2\}$ in the previous proof.

We finally turn to the case $\alpha = 0$ corresponding to the last part (vi) of Theorem 1.

Proof of Theorem 1 (vi). Let Q_N denote the distribution of $X_2 + \cdots + X_N \stackrel{d}{=} S_{N-1}$. For all $p > 0$,

$$\mathbb{E}(W_1^p) = \mathbb{E}(W_1^p 1_{\{X_2+\cdots+X_N\leq N\}}) + \int_{(N,\infty)} \mathbb{E}\bigg(\Big(\frac{X}{X+x}\Big)^p\bigg)\, Q_N(\mathrm{d}x). \tag{23}$$

From Lemma 1 it follows that there exists $q \in (0,1)$ such that

$$\mathbb{E}(W_1^p 1_{\{X_2+\cdots+X_N\leq N\}}) \leq \mathbb{P}(S_{N-1}\leq N) \leq q^N$$

for all sufficiently large N. By Lemma 2, $\mathbb{E}((X/(X+x))^p) \sim \ell(x)$ as $x \to \infty$, which implies that

$$\int_{(N,\infty)} \mathbb{E}\bigg(\Big(\frac{X}{X+x}\Big)^p\bigg)\, Q_N(\mathrm{d}x) \sim \int_{(N,\infty)} \ell(x)\, Q_N(\mathrm{d}x), \qquad N\to\infty. \tag{24}$$

Note that the integral on the right hand side of (24) does not depend on the parameter p. For $p = 1$, taking $\mathbb{E}(W_1) = 1/N$ into account, Eq. (23), multiplied by N, turns into

$$1 = N\mathbb{E}(W_1 1_{\{X_2+\cdots+X_N\leq N\}}) + N\int_{(N,\infty)} \mathbb{E}\Big(\frac{X}{X+x}\Big)\, Q_N(\mathrm{d}x).$$

Noting that, for all sufficiently large N,

$$N\mathbb{E}(W_1 1_{\{X_2+\cdots+X_N\leq N\}}) \leq N\mathbb{P}(S_{N-1}\leq N) \leq Nq^N \to 0, \qquad N\to\infty,$$

it follows that $\lim_{N\to\infty} N\int_{(N,\infty)} \mathbb{E}(X/(X+x))\, Q_N(\mathrm{d}x) = 1$, or, equivalently,

$$\frac{1}{N} \sim \int_{(N,\infty)} \mathbb{E}\Big(\frac{X}{X+x}\Big)\, Q_N(\mathrm{d}x) \sim \int_{(N,\infty)} \ell(x)\, Q_N(\mathrm{d}x), \qquad N\to\infty,$$

where the last asymptotics holds by (24) for $p = 1$. Therefore, for every $p > 0$ the integral in (24) is asymptotically equal to $1/N$ and it follows from (23) that $N\mathbb{E}(W_1^p) \to 1$ as $N \to \infty$ for all $p > 0$. In particular, $c_N = N\mathbb{E}(W_1^2) \to 1$ as $N \to \infty$. Moreover, $\Phi_2^{(N)}(2,2) = (N)_2\mathbb{E}(W_1^2W_2^2) \leq (N)_2\mathbb{E}(W_1W_2) = 1 - c_N \to 0$ as $N \to \infty$. Thus, for all $j, k_1, \ldots, k_j \in \mathbb{N}\setminus\{1\}$, $\Phi_j^{(N)}(k_1,\ldots,k_j) \leq \Phi_2^{(N)}(2,2) \to 0$ as $N \to \infty$, which shows that simultaneous multiple collisions cannot occur in the limit. By [32, Theorem 2.1], the model is in the domain of attraction of the discrete-time star-shaped coalescent. □

A Appendix

For convenience we present the following version of the monotone density theorem.

Lemma 4. *Let* $x_0 \in (0,\infty]$ *and assume that* $G:(0,x_0)\to\mathbb{R}$ *has the form* $G(x) = \int_{(x,x_0)} g(y)\,\lambda(\mathrm{d}y)$ *for some measurable function* $g:(0,x_0)\to\mathbb{R}$. *If* $G(x)\sim x^{-\rho}\ell(x)$ *as* $x\to 0$ *for some constant* $\rho\in[0,\infty)$ *and some function* ℓ *slowly varying at* 0 *and if* g *is monotone in some right neighborhood of* 0*, then* $\lim_{x\to 0} x^{\rho+1}g(x)/\ell(x) = \rho$.

Remark 5. Note that $G'(x) = -g(x)$. The statement of the lemma is hence equivalent to $\lim_{x\to 0} xG'(x)/G(x) = -\rho$.

The following proof of Lemma 4 almost exactly coincides with the proofs known for standard versions of the monotone density theorem (see, for example, Bingham, Goldie and Teugels [2, Theorem 1.7.2] or Feller [12, p. 446]. The proof is provided, since the monotone density theorem in the form of Lemma 4 is heavily used throughout the proofs in Section 4.

Proof of Lemma 4. Suppose first that g is nonincreasing in some right neighborhood of 0. If $0<a<b<\infty$, then, for all $x\in(0,x_0/b)$, $G(ax)-G(bx) = \int_{(ax,bx]} g(y)\,\lambda(\mathrm{d}y)$ so, for x small enough,

$$\frac{(b-a)xg(bx)}{x^{-\rho}\ell(x)} \le \frac{G(ax)-G(bx)}{x^{-\rho}\ell(x)} \le \frac{(b-a)xg(ax)}{x^{-\rho}\ell(x)}.$$

The middle fraction is

$$\frac{G(ax)}{(ax)^{-\rho}\ell(ax)}a^{-\rho}\frac{\ell(ax)}{\ell(x)} - \frac{G(bx)}{(bx)^{-\rho}\ell(bx)}b^{-\rho}\frac{\ell(bx)}{\ell(x)} \to a^{-\rho}-b^{-\rho}, \qquad x\to 0,$$

so the first inequality above yields

$$\limsup_{x\to 0}\frac{g(bx)}{x^{-\rho-1}\ell(x)} \le \frac{a^{-\rho}-b^{-\rho}}{b-a}.$$

Taking $b:=1$ and letting $a\uparrow 1$ gives

$$\limsup_{x\to 0}\frac{g(x)}{x^{-\rho-1}\ell(x)} \le \lim_{a\to 1}\frac{a^{-\rho}-1}{1-a} = \rho.$$

By a similar treatment of the right inequality with $a:=1$ and $b\downarrow 1$ we find that the lim inf is at least ρ, and the conclusion follows. The argument when g is nondecreasing in some right neighborhood of 0 is similar. □

The following two results are extended versions of Theorem 2 and Theorem 3 of Karamata [22] adapted to our purposes. Lemma 5 provides conditions under which a slowly varying part inside an integral can be moved in front of the integral without changing the asymptotics of the integral. Corollary 1 is a similar result for the regularly varying case. The results are slightly more general than those provided in [22], since the functions g_N and f_N arising in the statements are allowed to depend on N, which is not the case in the formulation of [22].

Lemma 5. *Let $L:(0,\infty)\to(0,\infty)$ be slowly varying at 0 (or ∞), let $(x_N)_{N\in\mathbb{N}}$ be a sequence of positive real numbers satisfying $x_N\to 0$ (or $x_N\to\infty$) as $N\to\infty$. Furthermore, let $g_N:(0,\infty)\to[0,\infty)$ be nonnegative, integrable functions with $0<\int_0^\infty g_N(t)\,\mathrm{d}t<\infty$ for all $N\in\mathbb{N}$ and such that, for some $a>0$ and some $\eta>0$,*

$$\int_0^a t^{-\eta}g_N(t)\,\mathrm{d}t\;<\;\infty\quad\textit{and}\quad\int_a^\infty t^{\eta}g_N(t)\,\mathrm{d}t\;<\;\infty$$

for all $N\in\mathbb{N}$. Then, as $N\to\infty$,

$$\int_0^\infty L(x_Nt)g_N(t)\,\mathrm{d}t\;\sim\;L(x_N)\int_0^\infty g_N(t)\,\mathrm{d}t.$$

Proof. Define $P(x):=x^\eta L(x)$ and $Q(x):=x^{-\eta}L(x)$, $x>0$. Note that P is regularly varying with index η and Q is regularly varying with index $-\eta$. By [2, Theorem 1.5.2], $P(x_Nt)/P(x_N)\to t^\eta$ as $N\to\infty$ uniformly in $t\in(0,a]$ and $Q(x_Nt)/Q(x_N)\to t^{-\eta}$ as $N\to\infty$ uniformly in $t\in[a,\infty)$. Thus, for every $\varepsilon>0$ there exists $N_0=N_0(\varepsilon)\in\mathbb{N}$ such that, for all $N\in\mathbb{N}$ with $N>N_0$,

$$P(x_N)(1-\varepsilon)\;\le\;t^{-\eta}P(x_Nt)\;\le\;P(x_N)(1+\varepsilon)\quad\text{for all } t\in(0,a]$$

and

$$Q(x_N)(1-\varepsilon)\;\le\;t^{\eta}Q(x_Nt)\;\le\;Q(x_N)(1+\varepsilon)\quad\text{for all } t\in[a,\infty).$$

For all $N\in\mathbb{N}$ with $N>N_0$ it follows that

$$\begin{aligned}
&\int_0^\infty L(x_Nt)g_N(t)\,\mathrm{d}t\\
&\quad= x_N^{-\eta}\int_0^a t^{-\eta}P(x_Nt)g_N(t)\,\mathrm{d}t+x_N^{\eta}\int_a^\infty t^{\eta}Q(x_Nt)g_N(t)\,\mathrm{d}t\\
&\quad\le x_N^{-\eta}P(x_N)(1+\varepsilon)\int_0^a g_N(t)\,\mathrm{d}t+x_N^{\eta}Q(x_N)(1+\varepsilon)\int_a^\infty g_N(t)\,\mathrm{d}t\\
&\quad= (1+\varepsilon)L(x_N)\int_0^\infty g_N(t)\,\mathrm{d}t
\end{aligned}$$

and, analogously, $\int_0^\infty L(x_Nt)g_N(t)\,\mathrm{d}t\ge(1-\varepsilon)L(x_N)\int_0^\infty g_N(t)\,\mathrm{d}t$. □

Corollary 1. *Let $\varphi:(0,\infty)\to(0,\infty)$ be regularly varying at 0 (or ∞) with index $\gamma\in\mathbb{R}$ and let $(x_N)_{N\in\mathbb{N}}$ be a sequence of positive real numbers satisfying $x_N\to 0$ (or $x_N\to\infty$) as $N\to\infty$. Furthermore, let $f_N:(0,\infty)\to[0,\infty)$, $N\in\mathbb{N}$, be functions such that $0<\int_0^\infty t^\eta f_N(t)\,\mathrm{d}t<\infty$ for all $N\in\mathbb{N}$ and all η in some neighborhood of γ, i.e. for all $\eta\in(\gamma-\varepsilon,\gamma+\varepsilon)$ for some $\varepsilon>0$. Then, as $N\to\infty$,*

$$\int_0^\infty \varphi(x_Nt)f_N(t)\,\mathrm{d}t\;\sim\;\varphi(x_N)\int_0^\infty t^\gamma f_N(t)\,\mathrm{d}t.$$

Proof. Define $L(x) := x^{-\gamma}\varphi(x)$ for $x > 0$, and $g_N(t) := t^\gamma f_N(t)$ for $t > 0$. Choose $\eta := \varepsilon/2 > 0$. Then, for any $a > 0$,

$$\int_0^a t^{-\eta} g_N(t)\,\mathrm{d}t \;=\; \int_0^a t^{\gamma-\eta} f_N(t)\,\mathrm{d}t \;\le\; \int_0^\infty t^{\gamma-\eta} f_N(t)\,\mathrm{d}t \;<\; \infty$$

by assumption and as well

$$\int_a^\infty t^{\eta} g_N(t)\,\mathrm{d}t \;=\; \int_a^\infty t^{\gamma+\eta} f_N(t)\,\mathrm{d}t \;\le\; \int_0^\infty t^{\gamma+\eta} f_N(t)\,\mathrm{d}t \;<\; \infty$$

by assumption. Thus, Lemma 5 is applicable, which yields the result. □

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References

[1] Athreya, K.B.: Rates of decay for the survival probability of a mutant gene. J. Math. Biol. **30**(6), 577–581 (1992). MR1173109

[2] Bingham, N.H., Goldie, C.M., Teugels, J.L.: Regular Variation. Cambridge University Press, Cambridge (1987). MR0898871

[3] Bingham, N.H., Doney, R.A.: Asymptotic properties of supercritical branching processes. I. The Galton–Watson process. Adv. Appl. Probab. **6**(4), 711–731 (1974). MR0362525

[4] Boenkost, F., González Casanova, A., Pokalyuk, C., Wakolbinger, A.: Haldane's formula in Cannings models: the case of moderately weak selection. Electron. J. Probab. **26** (Paper no. 4), 1–36 (2021). MR4216517

[5] Boenkost, F., González Casanova, A., Pokalyuk, C., Wakolbinger, A.: Haldane's formula in Cannings models: the case of moderately strong selection. J. Math. Biol. **83**(6–7), Article number 70 (2021). 31 pages. MR4348421

[6] Cannings, C.: The latent roots of certain Markov chains arising in genetics: a new approach. I. Haploid models. Adv. Appl. Probab. **6**(2), 260–290 (1974). MR0343949

[7] Cannings, C.: The latent roots of certain Markov chains arising in genetics: a new approach. II. Further haploid models. Adv. Appl. Probab. **7**(2), 264–282 (1975). MR0371430

[8] Cordero, F., González Casanova, A., Schweinsberg, J., Wilke-Berenguer, M.: Λ-coalescents arising in populations with dormancy (2020). arXiv preprint 2009.09418

[9] Cortines, A.: The genealogy of a solvable population model under selection with dynamics related to directed polymers. Bernoulli **22**(4), 2209–2236 (2016). MR3498028

[10] Dembo, A., Zeitouni, O.: Large Deviations Techniques and Applications. Springer (2010). MR2571413

[11] Eldon, B., Wakeley, J.: Coalescent processes when the distribution of offspring number among individuals is highly skewed. Genetics **172**(4), 2621–2633 (2006)

[12] Feller, W.: An Introduction to Probability Theory and Its Applications. Vol. II, 2nd edn. Wiley, New York (1971). MR0270403

[13] Griffiths, R.C., Spanò, D.: Orthogonal polynomial kernels and canonical correlations for Dirichlet measures. Bernoulli **19**(2), 548–598 (2013). MR3037164

[14] Haldane, J.B.S.: A mathematical theory of neutral and artificial selection, Part V. Selection and mutation. Proc. Camb. Philol. Soc. **23**(7), 838–844 (1927)

[15] Huillet, T.: Pareto genealogies arising from a Poisson branching evolution model with selection. J. Math. Biol. **68**(3), 727–761 (2014). MR3152761

[16] Huillet, T., Möhle, M.: Population genetics models with skewed fertilities: a forward and backward analysis. Stoch. Models **27**(3), 521–554 (2011). MR2827443

[17] Huillet, T., Möhle, M.: Correction on 'Population genetics models with skewed fertilities: a forward and backward analysis'. Stoch. Models **28**(3), 527–532 (2012). MR2959453

[18] Huillet, T., Möhle, M.: On the extended Moran model and its relation to coalescents with multiple collisions. Theor. Popul. Biol. **87**, 5–14 (2013)

[19] Huillet, T., Möhle, M.: Asymptotics of symmetric compound Poisson population models. Comb. Probab. Comput. **24**(1), 216–253 (2015). MR3318045

[20] Karamata, J.: Neuer Beweis und Verallgemeinerung der Tauberschen Sätze, welche die Laplacesche und Stieltjessche Transformation betreffen. J. Reine Angew. Math. **164**, 27–39 (1931). MR1581248

[21] Karamata, J.: Neuer Beweis und Verallgemeinerung einiger Tauberian-Sätze. Math. Z. **33**(1), 294–299 (1931). MR1545213

[22] Karamata, J.: Some theorems concerning slowly varying functions (1962). Technical Report #369, The Univ. of Wisconsin

[23] Karlin, S., McGregor, J.: Direct product branching processes and related Markov chains. Proc. Natl. Acad. Sci. USA **51**, 598–602 (1964). MR0163362

[24] Karlin, S., McGregor, J.: Direct product branching processes and related induced Markov chains. I. Calculations of rates of approach to homozygosity. In: Proc. Internat. Res. Sem., Statist. Lab., Univ. California, Berkeley, Calif., 1963, pp. 111–145. Springer (1965). MR0217892

[25] Kingman, J.F.C.: The coalescent. Stoch. Process. Appl. **13**(3), 235–248 (1982). MR0671034

[26] Kingman, J.F.C.: Exchangeability and the evolution of large populations. In: Exchangeability in Probability and Statistics (Rome, 1981), pp. 97–112. North-Holland, Amsterdam–New York (1982). MR675968

[27] Kingman, J.F.C.: On the genealogy of large populations. J. Appl. Probab. **Special Vol. 19A**, 27–43 (1982). Essays in statistical science. MR633178

[28] Kozubowski, T.J., Podgórski, K.: A generalized Sibuya distribution. Ann. Inst. Stat. Math. **70**(4), 855–887 (2018). MR3830290

[29] Möhle, M.: Total variation distances and rates of convergence for ancestral coalescent processes in exchangeable population models. Adv. Appl. Probab. **32**(4), 983–993 (2000). MR1808909

[30] Möhle, M.: On sampling distributions for coalescent processes with simultaneous multiple collisions. Bernoulli **12**(1), 35–53 (2006). MR2202319

[31] Möhle, M.: Asymptotic results for coalescent processes without proper frequencies and applications to the two-parameter Poisson–Dirichlet coalescent. Stoch. Process. Appl. **120**(11), 2159–2173 (2010). MR2684740

[32] Möhle, M., Sagitov, S.: A classification of coalescent processes for haploid exchangeable population models. Ann. Probab. **29**(4), 1547–1562 (2001). MR1880231

[33] Pitman, J.: Coalescents with multiple collisions. Ann. Probab. **27**(4), 1870–1902 (1999). MR1742892

[34] Sagitov, S.: The general coalescent with asynchronous mergers of ancestral lines. J. Appl. Probab. **36**(4), 1116–1125 (1999). MR1742154

[35] Schweinsberg, J.: Coalescents with simultaneous multiple collisions. Electron. J. Probab. **5** (Paper no. 12), 1–50 (2000). MR1781024

[36] Schweinsberg, J.: A necessary and sufficient condition for the Λ-coalescent to come down from infinity. Electron. Commun. Probab. **5**, 1–11 (2000). MR1736720

[37] Schweinsberg, J.: Coalescent processes obtained from supercritical Galton–Watson processes. Stoch. Process. Appl. **106**(1), 107–139 (2003). MR1983046

[38] Yule, G.U.: A mathematical theory of evolution based on the conclusions of Dr. J.C. Willis, F.R.S. Philos. Trans. R. Soc. Lond. Ser. B **213**, 21–87 (1925)