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Covariance between the forward recurrence time and the number of renewals

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In memory of my beloved father Ioannis Losidis

Abstract Recurrence times and the number of renewals in $(0, t]$ are fundamental quantities in renewal theory. Firstly, it is proved that the upper orthant order for the pair of the forward and backward recurrence times may result in NWUC (NBUC) interarrivals. It is also demonstrated that, under DFR interarrival times, the backward recurrence time is smaller than the forward recurrence time in the hazard rate order. Lastly, the sign of the covariance between the forward recurrence time and the number of renewals in $(0, t]$ at a fixed time point t and when $t \to \infty$ is studied assuming that the interarrival distribution belongs to certain ageing classes.

Keywords Renewal process, remaining term, forward recurrence time, number of renewals, covariance

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1 Introduction

A renewal process considers the sequence of partial sums $\{S_n : n = 0, 1, 2, \ldots\}$ of some independent, nonnegative random variables X_i, so that $S_0 = 0$ and, for $n \geq 1$, $S_n = X_1 + X_2 + \cdots + X_n$. We also assume that X_i $(i = 1, 2, \ldots)$ has a distribution function (d.f.) F. We assume throughout the paper that F is absolutely continuous and has density f. We write $\mu_k = \int_0^\infty x^k dF(x)$ for the k-th moment of F around zero; for simplicity, we write μ rather than μ_1 for the first moment. Also we assume that

F has a finite variance σ^2. Further, the equilibrium distribution associated with F, denoted by F_e, is defined as $F_e(t) = \mu^{-1} \int_0^t \overline{F}(y)dy$, where $\overline{F} = 1 - F$.

A renewal process is the counting process $\{N(t) : t \geq 0\}$ whereby $N(t) = \sup\{n : S_n \leq t\}$. $N(t)$ is the number of renewals that occurred in $(0, t]$ as defined above.

However, the distribution of $N(t)$ is not always easy to calculate, which makes the renewal function $U(t) = E[N(t)] = \sum_{n=1}^{\infty} \mathbb{P}(S_n \leq t)$, for $t \geq 0$, a quantity of interest. Assuming that F is absolutely continuous, the renewal function U is also absolutely continuous with a density u, the renewal density. We mention for future reference that a version of u satisfies the *renewal equation*

$$u(t) = f(t) + \int_0^t u(t-y) f(y)\, dy,$$

see, for example, Feller (1971, Ch. XI) [7].

In literature, there are several sources which study the forward recurrence time, see, for example, Brown (1980) [2] and (1981) [3], Shaked and Zhu (1992) [17], Gakis and Sivazlian (1992) [8] and (1994) [9] and most recently Losidis and Politis (2020) [14].

We define by $\gamma_t = S_{N(t)+1} - t$ the forward recurrence time and by $\delta_t = t - S_{N(t)}$ the backward recurrence time.

Gakis and Sivazlian (1992) [8] compute the joint distribution of the backward and forward recurrence times using the joint distribution $\mathbb{P}(\gamma_t \leq y, \delta_t \leq x, N(t) = n > 0)$, namely

$$\mathbb{P}(\gamma_t \leq y, \delta_t \leq x) = \sum_{n=0}^{\infty} \mathbb{P}(N(t) = n, \gamma_t \leq y, \delta_t \leq x).$$

The above relation triggered this paper's main topic: to investigate the relationship that governs recurrence times and $N(t)$. In literature, it seems that so far little attention has been given to this topic. Exceptions are Coleman (1982) [5] and Losidis and Politis (2020) [14]. More specifically, Coleman (1982) [5] proves that the covariance of the forward recurrence time and the number of renewals is given by

$$Cov(\gamma_t, N(t)) = tU(t) - \int_0^t U(z)dz - \mu\,(U(t))^2 + \mu \int_0^t U(t-z)u(z)dz. \quad (1)$$

He also proves that when $t \to \infty$

$$Cov(\gamma_t, N(t)) = -\frac{\mu_3}{6\mu^2} + \frac{\mu_2^2}{4\mu^3}. \quad (2)$$

Losidis and Politis (2020) [14], using the remaining term $Q(t)$ defined as

$$Q(t) = \frac{\mu_2}{2\mu^2} - 1 - \left(U(t) - \frac{t}{\mu}\right), \quad (3)$$

prove that the asymptotic covariance between the forward recurrence time and the number of renewals on $(0, t)$ is given by

$$\lim_{t\to\infty} Cov(\gamma_t, N(t)) = -\int_0^{\infty} Q(z)dz. \quad (4)$$

Assuming that $U(t)$ is absolutely continuous, $Q(t)$ is also continuous with $Q'(t) = q(t)$.

We define the failure rate by

$$r(z) = \frac{f(z)}{\overline{F}(z)} = -\frac{d}{dz} \ln \overline{F}(z), \tag{5}$$

and we say that the distribution F has decreasing (increasing) failure rate (denoted by DFR (IFR)) if the quantity $\overline{F}(t-x)/\overline{F}(t)$ is decreasing (increasing) in t $(t > 0)$ for all $x \geq 0$.

A wider class compared to DFR (IFR) is the IMRL (DMRL). We define by $m(t) = \mu \overline{F}_e(t)/\overline{F}(t)$ the mean residual life. We say that F has increasing (decreasing) mean residual life (we denote IMRL (DMRL) respectively), if $m(t)$ is increasing (decreasing) function. For more details, see Willmot and Lin (2001) [18]. Another class is the new worse (better) than used in convex ordering, named as the NWUC (NBUC) class. We say F is NWUC (NBUC) if

$$\int_y^\infty \overline{F}(x+t)dx \geq (\leq) \overline{F}(t) \int_y^\infty \overline{F}(x)dx, \quad \text{for all } t, y \geq 0.$$

In this paper, we tackle the sign of the covariance between the forward recurrence time and the number of renewals in $(0, t]$, a topic that escaped attention so far.

The main novelty of this paper is that under the assumption of IMRL interarrival times, we prove that the covariance between the forward recurrence time and the number of renewals in $(0, t]$ is less or equal to zero.

We organise the remaining of this paper as follows.

In Section 2, initially, we discuss the upper orthant stochastic order for the family of *random pairs* (γ_t, δ_t) and the hazard rate order between the backward and the forward recurrence times in a renewal process. Next, we focus on the covariance between the forward recurrence time and the number of renewals in $(0, t]$ for $t < \infty$.

In Section 3, we study the covariance between γ_t and $N(t)$ for $t \to \infty$. In contrast to Losidis and Politis (2020) [14], we focus on the distribution function F of the interarrival times instead of the equilibrium distribution function F_e. The last section has some concluding remarks.

2 The covariance of the forward recurrence time and the number of renewals in $(0, t]$

Recurrence times find numerous real-life applications. In addition to traditional application areas, such as reliability or actuarial risk theory, renewal processes have been used for several decades in biostatistics, in models for the early detection of disease (see, e.g., Zelen and Lee (2002) [19]).

To give a more specific example, from medicine, we can define the forward recurrence time as the time until a person leaves the disease state after detection at time t (in other words, the forward recurrence time represents the time that a patient knows he has a condition until recovery). The backward recurrence time would be the time from when the person has had the disease until the diagnosis.

We use this example to illustrate the critical question: given the time passed since the person was ill (this could be considered as the latest renewal), how long until they get infected again from the same decease (this shall be the next renewal)? Mathematically, this is best formulated by considering the joint distribution of the (backward and forward) recurrence times associated with a renewal process.

It is known (see, for example, Daley and Vere-Jones (2003) [6]) that for any $0 \leq x \leq t$ and $y \geq 0$ it holds that

$$\mathbb{P}(\gamma_t > y, \delta_t > x) = \overline{F}(t+y) + \int_0^{t-x} \overline{F}(t+y-s)u(s)ds. \tag{6}$$

In a multivariate setting, there are various ways to generalise the concept of the usual, univariate stochastic ordering. One of them is the upper orthant ordering, as discussed, for example, in Shaked and Shanthikumar (2007, Ch. 6) [16]. More explicitly, let $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ and $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_n)$ be two n-dimensional random vectors. Then we say that $\mathbf{X}$ is smaller that $\mathbf{Y}$ in the upper orthant stochastic order, and we denote this by $\mathbf{X} \leq^{uo} \mathbf{Y}$ if for any $t_1, t_2, \ldots, t_n$,

$$\mathbb{P}\left(X_1 > t_1, X_2 > t_2, \ldots, X_n > t_n\right) \leq \mathbb{P}\left(Y_1 > t_1, Y_2 > t_2, \ldots, Y_n > t_n\right).$$

Li et al. (2000) [11] showed that if γ_t is increasing in $t \geq 0$ in convex ordering then F is NWUC. Next we present a result similar to that of Li et al. (2000) [11], for the family of *random pairs* (γ_t, δ_t).

Proposition 1. *If for all $0 \leq t \leq s$ it holds that $(\gamma_t, \delta_t) \leq^{uo} (\geq^{uo})(\gamma_s, \delta_s)$, then the distribution function of the interarrival times is NWUC (NBUC).*

Proof. It is known (Karlin and Taylor (1975), p. 193) [10] that

$$\mathbb{P}(\gamma_t > y) = \overline{F}(t+y) + \int_0^t \mathbb{P}(\gamma_{t-z} > y)dF(z). \tag{7}$$

It is easy to prove that the event $\{\gamma_t > y + x\}$ is equivalent to $\{\gamma_{t+x} > y, \delta_{t+x} > x\}$, and then from the last equation we get

$$\mathbb{P}(\gamma_{t+x} > y, \delta_{t+x} > x) = \overline{F}(t+x+y) + \int_0^t \mathbb{P}(\gamma_{t+x-z} > y, \delta_{t+x-z} > x)dF(z). \tag{8}$$

Integrating the above with respect to y on (k, ∞) (for $k > 0$) we have

$$\begin{aligned}\int_k^\infty \mathbb{P}(\gamma_{t+x} > y, \delta_{t+x} > x)dy &= \int_k^\infty \overline{F}(t+x+y)dy \\ &\quad + \int_k^\infty \int_0^t \mathbb{P}(\gamma_{t+x-z} > y, \delta_{t+x-z} > x)dF(z)dy.\end{aligned} \tag{9}$$

Under the assumption that $(\gamma_t, \delta_t) \leq^{uo} (\gamma_s, \delta_s)$, the double integral of Eq. (9) could be written as

$$\begin{aligned}
&\int_k^\infty \int_0^{t-x} \mathbb{P}(\gamma_{t+x-z} > y, \delta_{t+x-z} > x) dF(z) dy \\
&\quad \le \int_k^\infty \int_0^{t} \mathbb{P}(\gamma_{t+x} > y, \delta_{t+x} > x) dF(z) dy \\
&\quad = F(t) \int_k^\infty \mathbb{P}(\gamma_{t+x} > y, \delta_{t+x} > x) dy.
\end{aligned}$$

Inserting the above into Eq. (9) yields

$$\int_k^\infty \mathbb{P}(\gamma_{t+x} > y, \delta_{t+x} > x) dy \overline{F}(t) \le \int_k^\infty \overline{F}(t+x+y) dy. \tag{10}$$

By assuming $(\gamma_t, \delta_t) \le^{uo} (\gamma_s, \delta_s)$ it holds that

$$\mathbb{P}(\gamma_x > y, \delta_x > x) = \overline{F}(x+y) \le \mathbb{P}(\gamma_{t+x} > y, \delta_{t+x} > x).$$

Inserting the above into Eq. (10) and setting $x = 0$ completes the proof.

When $(\gamma_t, \delta_t) \ge^{uo} (\gamma_s, \delta_s)$, we can prove that F is NBUC by reversing the inequalities above. □

Losidis et al. (2020) [15] proved that, under the assumption of DFR interarrival times, it holds that

$$\mathbb{P}(\gamma_t > x) \ge \mathbb{P}(\delta_t > x), \tag{11}$$

for $0 \le x \le t$.

Recall that for two random variables X and Y supported on $[0, \infty)$ with survival functions $P(X > x)$ and $P(Y > y)$, we say that X is less than or equal to Y in the usual stochastic order ($X \le_{st} Y$) if $P(X > x) \le P(Y > y)$ for all $t \ge 0$ (see Shaked and Shanthikumar (2007, Ch. 1)) [16]. In the terminology of stochastic orders, (11) means that δ_t is smaller than γ_t in the usual stochastic order. The next result, which in view of the last equation is an improvment of this, shows that δ_t is smaller that γ_t in the hazard rate order (for details on those, see Shaked and Shanthikumar (2007) [16]).

Proposition 2. *If the distribution F of the interarrival times is DFR, then*

$$r_{\gamma_t}(x) \le r_{\delta_t}(x), \quad \textit{for all } 0 \le x \le t.$$

Proof. Losidis et al. (2020) [15] prove that the survival function of the forward and backward recurrence time is given by

$$\mathbb{P}(\gamma_t > x) = \overline{F}_e(x) - \int_0^x \overline{F}(t+x-z) l(z) dz \tag{12}$$

and

$$\mathbb{P}(\delta_t > x) = \overline{F}_e(x) - \int_0^x \overline{F}(t-z) l(z) dz \tag{13}$$

with $l(t)$ defined as

$$l(t) = u(t) - \frac{1}{\mu}.$$

For simplicity we denote

$$\phi(0, x; t) = \int_0^x \overline{F}(z) l(t + x - z) dz. \tag{14}$$

The first derivative of $\phi(0, x; t)$ is equal to

$$\frac{d}{dx}\phi(0, x; t) = \overline{F}(x) l(t) + \int_0^x \overline{F}(z) \frac{d}{dx} l(t + x - z) dz. \tag{15}$$

We also define (for simplicity) with

$$\phi(x, 0; t) = \int_0^x \overline{F}(z) l(t - z) dz, \tag{16}$$

with derivative

$$\frac{d}{dx}\phi(x, 0; t) = \overline{F}(x) l(t - x). \tag{17}$$

Combining Eq. (5), Eq. (17) and Eq. (13), the failure rate of the backward recurrence time is given by

$$r_{\delta_t}(x) = \frac{\frac{\overline{F}(x)}{\mu} + \frac{d}{dx}\phi(x, 0; t)}{\mathbb{P}(\delta_t > x)} = \frac{\frac{\overline{F}(x)}{\mu} + \overline{F}(x) l(t - x)}{\overline{F}_e(x) + \phi(0, x; t)}. \tag{18}$$

Brown (1980) [2] proves that if the distribution F of the interarrival times is DFR, then the renewal density $u(t)$ is decreasing function of t. In view of Eq. (2), the same holds for $l(t)$, thus for $0 \leq z \leq t$ we have

$$l(0) \geq l(t - z) \geq l(t).$$

Then, Eq. (15) yields

$$\frac{d}{dx}\phi(0, x; t) \leq \overline{F}(x) l(t) \leq \overline{F}(x) l(t - x) = \frac{d}{dx}\phi(x, 0; t).$$

Next, using Eq. (12) and Eq. (5) we calculate the failure rate of the forward recurrence time as follows

$$r_{\gamma_t}(x) = \frac{\overline{F}(x)}{\mu \mathbb{P}(\gamma_t > x)} + \frac{\frac{d}{dx}\phi(0, x; t)}{\mathbb{P}(\gamma_t > x)} \leq \frac{\overline{F}(x)}{\mu \mathbb{P}(\gamma_t > x)} + \frac{\frac{d}{dx}\phi(x, 0; t)}{\mathbb{P}(\gamma_t > x)}. \tag{19}$$

Inserting (11) into (19) we have

$$r_{\gamma_t}(x) \leq \frac{\overline{F}(x)}{\mu \mathbb{P}(\gamma_t > x)} + \frac{\frac{d}{dx}\phi(x, 0; t)}{\mathbb{P}(\gamma_t > x)} \leq \frac{\overline{F}(x)}{\mu \mathbb{P}(\delta_t > x)} + \frac{\frac{d}{dx}\phi(x, 0; t)}{\mathbb{P}(\delta_t > x)} = r_{\delta_t}(x), \tag{20}$$

which completes the proof. □

Next, we present a relation for the conditional tail functions of the forward and backward recurrence times.

Corollary 1. *If the distribution F of the interarrival times is DFR, then*

$$\mathbb{P}(\gamma_t > x + t | \gamma_t > x) \geq \mathbb{P}(\delta_t > x + t | \delta_t > x), \quad \textit{for all } 0 \leq x \leq t.$$

Proof. Integrating both sides of Eq. (5) over z from 0 to t gives

$$\overline{F}(t) = exp\left(-\int_0^t r(z)dz\right), t \geq 0. \tag{21}$$

From Eq. (21) it follows that

$$\mathbb{P}(\gamma_t > x + y | \gamma_t > x) = exp\left(-\int_x^{x+y} r_{\gamma_t}(z)dz\right),$$

and using (20) it yields

$$\mathbb{P}(\gamma_t > x + y | \gamma_t > x) \geq exp\left(-\int_x^{x+y} r_{\delta_t}(z)dz\right) = \mathbb{P}(\delta_t > x + y | \delta_t > x),$$

which completes the proof. □

Continuing the example we began in this section, one could enquire the following: is there a connection between the number of occasions we may get sick from a virus and the period it takes to recover since we tested positive to it? Mathematically this could be approached by the covariance between the forward recurrence time and the number of renewals.

In the following result, initially, we provide an alternative formula for the $Cov(\gamma_t, N(t))$ in terms of the remaining term $Q(t)$, which enables us to show that, assuming IMRL interarrival times, $Cov(\gamma_t, N(t)) \leq 0$.

Theorem 2.1. *If the distribution F of the interarrival times is IMRL, then*

$$Cov(\gamma_t, N(t)) \leq 0, \quad \textit{for all } t \geq 0.$$

Proof. We start our proof by inserting Eq. (3) into Eq. (1). The first part of Eq. (1) is equal to

$$tU(t) = Q(0)t + \frac{t^2}{\mu} - tQ(t). \tag{22}$$

The second component of Eq. (1) is written as

$$-\int_0^t U(x)dx = -\left(Q(0)t + \frac{t^2}{2\mu} - \int_0^t Q(x)dx\right) = -Q(0)t - \frac{t^2}{2\mu} + \int_0^t Q(x)dx. \tag{23}$$

The third part of Eq. (1) is equal to

$$\begin{aligned} -\mu U(t)^2 &= -\mu\left(Q(0)^2 + (\frac{t}{\mu} - Q(t))^2 + 2Q(0)(\frac{t}{\mu} - Q(t))\right) \\ &= -\mu Q(0)^2 - \frac{t^2}{\mu} - \mu Q(t)^2 + 2tQ(t) - 2Q(0)t + 2\mu Q(0)Q(t) \end{aligned} \tag{24}$$

Lastly, the forth part of Eq. (1) could be written as

$$\begin{aligned}
&\mu \int_0^t (U(t-x)u(x))\,dx \\
&\quad = \mu \int_0^t \left(Q(0) + \frac{t-x}{\mu} - Q(t-x) \right) \left(\frac{1}{\mu} - q(x) \right) dx \\
&\quad = Q(0)t - \mu Q(0)(Q(t) - Q(0)) + \frac{t^2}{\mu} - \int_0^t \frac{x}{\mu} dx - t(Q(t) - Q(0)) \\
&\qquad + \int_0^t xq(x)dz - \int_0^t Q(t-x)dx + \mu \int_0^t Q(t-x)q(x)dx.
\end{aligned} \tag{25}$$

By substituting Eqs. (22)–(25) into Eq. (1) and after some algebra, it follows that

$$\begin{aligned}
&Cov(\gamma_t, N(t)) \\
&\quad = \int_0^t zq(z)dz + \left(-\mu Q(t)^2 + \mu Q(0)Q(t) + \mu \int_0^t Q(t-x)q(x)dx \right).
\end{aligned} \tag{26}$$

By setting

$$b(t) = -\mu Q(t)^2 + \mu Q(0)Q(t) + \mu \int_0^t Q(t-x)q(x)dx, \tag{27}$$

and inserting the above into Eq. (26) we have

$$Cov(\gamma_t, N(t)) = \int_0^t zq(z)dz + b(t). \tag{28}$$

Brown (1980) [2] proved that under the assumption of IMRL interarrival times the quantity $U(t) - t\mu^{-1}$ is bounded as follows

$$U(t) - \frac{t}{\mu} \leq \frac{\mu_2}{2\mu^2} - 1,$$

and it is an increasing function of t. In view of Eq. (3) and provided that F is IMRL, $Q(t)$ is more than or equal to zero and is also a decreasing function of t ($q(t) \leq 0$). In this case

$$Q(t-x)q(x) \leq Q(t)q(x),$$

following which Eq. (27) gives

$$\begin{aligned}
b(t) &= -\mu Q(t)^2 + \mu Q(0)Q(t) + \mu \int_0^t Q(t-x)q(x)dx \\
&\leq -\mu Q(t)^2 + \mu Q(0)Q(t) + \mu Q(t)(Q(t) - Q(0)) = 0.
\end{aligned}$$

By inserting the inequality above into Eq. (28), it holds that

$$Cov(\gamma_t, N(t)) \leq \int_0^t zq(z)dz \leq 0,$$

which completes the proof. $\square$

In the next result we study the difference $Cov(\gamma_t, N(t) - \lim_{t\to\infty} Cov(\gamma_t, N(t))$. More specifically, we define

$$S(t) = Cov(\gamma_t, N(t)) - \lim_{t\to\infty} Cov(\gamma_t, N(t)). \tag{29}$$

If $S(t) \geq (\leq) 0$, then the limit of $Cov(\gamma_t, N(t))$ as $t \to \infty$ serves as a bound for the $Cov(\gamma_t, N(t))$. In the sequel we present conditions under which the asymptotic covariance between the forward recurrence time and the number of renewals can be used as an upper bound for $Cov(\gamma_t, N(t))$. We begin with a formula which calculates the difference $Cov(\gamma_t, N(t)) - \lim_{t\to\infty} Cov(\gamma_t, N(t))$.

Corollary 2. *The quantity $S(t)$ can be calculated by*

$$S(t) = tQ(t) + \int_t^\infty Q(z)dz - \mu Q(t)^2 + \mu Q(0)Q(t) + \mu \int_0^t Q(t-x)q(x)dx. \tag{30}$$

Proof. The first integral of Eq. (26) can be expressed as

$$\int_0^t zq(z)dz = tQ(t) - \int_0^t Q(z)dz = tQ(t) - \int_0^\infty Q(z)dz + \int_t^\infty Q(z)dz.$$

Inserting the above into Eq. (26) we have

$$\begin{aligned} Cov(\gamma_t, N(t)) = &- \int_0^\infty Q(z)dz + tQ(t) + \int_t^\infty Q(z)dz - \mu Q(t)^2 + \mu Q(0)Q(t) \\ &+ \mu \int_0^t Q(t-x)q(x)dx. \end{aligned} \tag{31}$$

In view of Eq. (4) we obtain

$$\begin{aligned} &Cov(\gamma_t, N(t)) - \lim_{t\to\infty} Cov(\gamma_t, N(t)) \\ &\quad = tQ(t) + \int_t^\infty Q(z)dz - \mu Q(t)^2 + \mu Q(0)Q(t) \\ &\quad + \mu \int_0^t Q(t-x)q(x)dx, \end{aligned} \tag{32}$$

which is the desired result. □

Conversely to the IMRL assumption of interarrival times ($Q(t) \geq 0$ and $q(t) \leq 0$), we next study the case when $Q(t)$ is less than or equal to zero and also is an increasing function. In this case we prove that the asymptotic covariance between the forward recurrence time and the number of renewals can be used as an upper bound for the $Cov(\gamma_t, N(t))$.

Lemma 1. *If $Q(t) \leq 0$ and $q(t) \geq 0$, then $S(t) \leq 0$, for all $t \geq 0$.*

Proof. We begin with the assumption that $Q(t)$ is less than or equal to zero for every $t \geq 0$ and is also an increasing function. In this case

$$Q(0) \leq Q(t-x) \leq Q(t).$$

Multiplying the above inequality by $q(x)$ and integrating with respect to x on $(0, t)$ we have

$$Q(0)\int_0^t q(x)dx \le \int_0^t Q(t-x)q(x)dx \le Q(t)\int_0^t q(x)dx.$$

Substituting

$$\int_0^t q(x)dx = Q(t) - Q(0),$$

yields that

$$Q(0)\left(Q(t) - Q(0)\right) \le \int_0^t Q(t-x)q(x)dx \le Q(t)\left(Q(t) - Q(0)\right).$$

Inserting the upper bound given by the above inequality into (26) yields

$$Cov(\gamma_t, N(t)) \le \int_0^t zq(z)dz.$$

Under the assumption that $q(t) \ge 0$ one may derive that

$$Cov(\gamma_t, N(t)) \le \int_0^t zq(z)dz \le \int_0^\infty zq(z)dz = -\int_0^\infty Q(z)dz,$$

and in view of Eq. (4) the proof is completed. □

Remark 1. Brown (1987) [4] proves that under the assumption of IFR interarrival times it holds that

$$U(t) \ge \frac{t}{\mu} + \frac{\sigma^2}{\mu^2} - 1.$$

Now we define with $Q_I(t)$ the difference,

$$Q_I(t) = \frac{\sigma^2}{\mu^2} - 1 - \left(U(t) - \frac{t}{\mu}\right) \le 0,$$

with $\frac{d}{dt}Q_I(t) = \mu^{-1} - u(t) = q(t)$.

It can be proved that Eq. (26) is still valid for the case of IFR interarrival times. More specifically,

$$\begin{aligned} &Cov(\gamma_t, N(t)) \\ &\quad = \int_0^t zq(z)dz + \left(-\mu Q_I(t)^2 + \mu Q_I(0)Q_I(t) + \mu\int_0^t Q_I(t-x)q(x)dx\right), \end{aligned}$$

and using, as an additional assumption, that $Q_I(t)$ is increasing, it follows that

$$Cov(\gamma_t, N(t)) \le \lim_{t\to\infty} Cov(\gamma_t, N(t)).$$

3 The asymptotic covariance of the forward recurrence time and the number of renewals

Losidis and Politis (2020) [14] study the covariance between the forward recurrence time and the number of renewals up to time t in an ordinary renewal process when $t \to \infty$, and link the sign of the asymptotic covariance with the equilibrium distribution F_e. However, it would be preferable to link the sign with the distribution F of the interarrival times instead of the F_e.

Next, we use stochastic order results between γ_t and γ_∞ in order to determinate the sign of $\lim_{t\to\infty} Cov(\gamma_t, N(t))$.

More precisely, as $t \to \infty$, the random variable γ_t converges in distribution to the variable γ_∞, with $\lim_{t\to\infty} \mathbb{E}(\gamma_t) = \mathbb{E}(\gamma_\infty) = \mu_2/(2\mu)$. The next result presents a formula for $\lim_{t\to\infty} Cov(\gamma_t, N(t))$ in terms of the variables γ_t and γ_∞.

Proposition 3. *The asymptotic covariance between the forward recurrence time γ_t and the number of renewals $N(t)$ is given by the formula*

$$\lim_{t\to\infty} Cov(\gamma_t, N(t)) = \frac{1}{\mu}\int_0^\infty (\mathbb{E}(\gamma_x) - \mathbb{E}(\gamma_\infty))\, dx. \tag{33}$$

Proof. By Wald's identity, the mean forward recurrence time is given by

$$\mathbb{E}(\gamma_x) = \mu\left(1 + U(x) - \frac{x}{\mu}\right).$$

Inserting Eq. (3) in the above one we have

$$-Q(x) = \frac{1}{\mu}\left(\mathbb{E}(\gamma_x) - \frac{\mu_2}{2\mu}\right) = \frac{1}{\mu}\left(\mathbb{E}(\gamma_x) - \mathbb{E}(\gamma_\infty)\right).$$

The proof is completed by integrating the last equation over $(0, \infty)$ with respect to x. □

Shaked and Zhu (1992) [17] proved that the renewal function is convex (concave) if, and only if, the excess lifetime γ_t is stochastically decreasing (increasing). They also proved that if the renewal function is convex (concave), then the distribution F of the interarrival time is NBUE (NWUE). Combining this with Eq. (33) and under the assumption that renewal function $U(t)$ is convex (concave), then we get that the distribution F of the interarrival times is NBUE (NWUE) and the asymptotic covariance between the forward recurrence time and the number of renewals is more (less) or equal to zero.

Corollary 3. *If the distribution F of the interarrival times is IMRL, then*

$$\lim_{t\to\infty} Cov(\gamma_t, N(t)) \le 0.$$

Proof. Brown (1980) [2] proves that if the distribution function F of the interarrival times is IMRL, then the expected forward recurrence time at t, $\mathbb{E}(\gamma_t)$, is increasing in $t \ge 0$, which means $\mathbb{E}(\gamma_t) \le \mathbb{E}(\gamma_\infty)$. Proof is completed in view of Eq. (33). □

By Proposition 3 it is clear that, if $\mathbb{E}(\gamma_\infty) \geq (\leq)(\mathbb{E}(\gamma_t)$ then

$$\lim_{t\to\infty} Cov(\gamma_t, N(t)) \leq (\geq) 0.$$

However, it is known (see, e.g., Brown (1980) [2] or Shaked and Zhu (1992) [17]) that, in contrast to the IMRL case, the condition that $m(t)$ is a decreasing function does not guarantee that $\mathbb{E}(\gamma_t)$ is decreasing, and Corollary 3 does not hold for the reverse inequality, i.e. assuming DMRL interarrival times. However, it can be proven considering IFR interarrival times. More specifically, the following theorem holds.

Theorem 3.1. *If the distribution F of the interarrival times is IFR, then*

$$\lim_{t\to\infty} Cov(\gamma_t, N(t)) \geq 0.$$

Proof. Barlow et al. (1963) [1] prove that

$$\left(\frac{\mu_{i+r}\Gamma(i+1)}{\mu_i\Gamma(i+r+1)}\right)^s \leq \left(\frac{\mu_{i+s}\Gamma(i+1)}{\mu_i\Gamma(i+s+1)}\right)^r.$$

Setting $s = i = 1$ and $r = 2$ we have $\mu_3/(6\mu) \leq \mu_2^2/(4\mu^2)$, and in view of Eq. (2) we derive the desired result. □

Corollary 4. *If the distribution F of the interarrival times is DFR, then*

$$\lim_{t\to\infty} Cov(\gamma_t, N(t)) \geq \lim_{t\to\infty} Cov(\delta_t, N(t)).$$

Proof. Losidis et al. (2020) [15] proved that under the assumption of DFR interarrival times $\mathbb{E}(\gamma_t) \geq \mathbb{E}(\delta_t)$. Gakis and Sivazlian (1994) [9] proved that $\lim_{t\to\infty} E(\gamma_t) = E(\gamma_\infty) = \mu_2/(2\mu) = E(\delta_\infty)$. Combining the above, Eq. (33) gives

$$\lim_{t\to\infty} Cov(\gamma_t, N(t)) = \frac{1}{\mu}\int_0^\infty (\mathbb{E}(\gamma_x) - \mathbb{E}(\gamma_\infty))\, dx \geq \frac{1}{\mu}\int_0^\infty (\mathbb{E}(\delta_x) - \mathbb{E}(\delta_\infty))\, dx.$$

In view of Eq. (33) the proof is completed. □

Proposition 4. *If the variance of the forward recurrence time is increasing (decreasing) function of t, then*

$$\lim_{t\to\infty} Cov(\gamma_t, N(t)) \leq (\geq) 0.$$

Proof. Coleman (1982) [5] proves that the variance of the forward recurrence time can be calculated as

$$Var(\gamma_t) = \mu_2(1 + U(t)) - \mu^2(1 + U(t))^2 + 2\mu\left(tU(t) - \int_0^t U(x)dx\right)$$

The first derivative of $Var(\gamma_t)$ gives

$$\begin{aligned}\frac{d}{dt}Var(\gamma_t) &= \mu_2 u(t) - 2\mu^2\,(1 + U(t))\,u(t) + 2\mu t u(t)\\ &= 2\mu^2 u(t)\left(\frac{\mu_2}{2\mu^2} - 1 - U(t) + \frac{t}{\mu}\right).\end{aligned}$$

Inserting Eq. (3) into the above gives

$$\frac{d}{dt}Var(\gamma_t) = 2\mu^2 u(t) Q(t),$$

from which it yields that if the variance of the forward recurrence time is increasing (decreasing), then $Q(t) \geq (\leq) 0$, and in view of (4) the proof is completed. $\square$

For the remaining of this section, we focus on the asymptotic covariance between the variables γ_t^r (for $r = 1, 2, \ldots$) and $N(t)$, namely $\lim_{t\to\infty} Cov(\gamma_t^r, N(t))$.

Gakis and Sivazlian (1994) [9] study the asymptotic covariance between the forward and backward recurrence times, denoted as

$$\lim_{t\to\infty} Cov(\gamma_t, \delta_t) = \frac{\mu_3}{6\mu} - \frac{\mu_2^2}{4\mu^2}. \tag{34}$$

Combining (2) and (34) yields that

$$\lim_{t\to\infty} Cov(\gamma_t, N(t)) = -\frac{1}{\mu}\lim_{t\to\infty} Cov(\gamma_t, \delta_t). \tag{35}$$

Substituting γ_t with γ_t^r in Eq. (35) gives

$$\lim_{t\to\infty} Cov(\gamma_t^r, N(t)) = -\frac{1}{\mu}\lim_{t\to\infty} Cov(\gamma_t^r, \delta_t). \tag{36}$$

Next, we show that this quantity plays a key role for known bounds for the renewal function $U(t)$ under the assumption of IMRL interarrival times.

Brown (1980) [2], assuming that the distribution function F of the interarrival times is IMRL, gives the following lower bound for the renewal function:

$$U(t) \geq \frac{t}{\mu} + \frac{\mu_2}{2\mu^2} - 1 - \min_{0\leq r\leq k} c_r t^{-r},$$

for $r = 1, 2, \ldots, k$, with

$$c_r = q_r - \mu^{-1} r! \sum_{s-1}^{r-1} \frac{c_s}{s!}\frac{\mu_{r+1-s}}{(r+1-s)!},$$

where

$$q_r = \frac{\mu_{r+2}}{(r+1)(r+2)\mu^2} - \frac{\mu_2\mu_{r+1}}{2(r+1)\mu^3}.$$

Lemma 2. *It holds that*

$$\lim_{t\to\infty} Cov(\gamma_t^r, N(t)) = -q_r.$$

Proof. Losidis and Politis (2019) [13] prove that for any $r, s = 1, 2, \ldots$ it holds that

$$\lim_{t\to\infty} Cov(\gamma_t^r, \delta_t^s) = \frac{r!\mu_{r+s+1}}{\mu(r+s+1)!} - \frac{\mu_{r+1}\mu_{s+1}}{\mu^2(r+1)(s+1)}.$$

Inserting the above equation (for $s = 1$) into Eq. (36) yields that

$$\lim_{t\to\infty} Cov(\gamma_t^r, N(t)) = -\frac{1}{\mu}\left(\frac{r!\mu_{r+2}}{\mu(r+2)!} - \frac{\mu_{r+1}\mu_2}{2\mu^2(r+1)}\right) = -q_r, \tag{37}$$

which completes the proof. □

Next, we present an alternative formula for the $\lim_{t\to\infty} Cov(\gamma_t^r, N(t))$. More specifically, the following proposition holds.

Proposition 5. *Let r be any positive integer such that $\mu_{r+2} < \infty$. As $t \to \infty$, the covariance of the forward recurrence time and the number of renewals in $(0, t]$ is given by*

$$\lim_{t\to\infty} Cov(\gamma_t^r, N(t)) = \frac{1}{(r+1)}\frac{\mu_2\mu_{r+1}}{2\mu^3(r+1)}\left(r - C_r\right),$$

with

$$C_r = \frac{\frac{\mu_{r+2}}{\mu(r+2)} - \frac{\mu_{r+1}\mu_2}{2\mu^2(r+1)}}{\frac{\mu_{r+1}\mu_2}{2\mu^2(r+1)}} \tag{38}$$

Proof. From Eq. (38) we have

$$\frac{\mu_{r+2}}{\mu(r+2)} = \frac{\mu_2\mu_{r+1}}{2\mu^2(r+1)}\left(1 + C_r\right).$$

Inserting the above equation into Eq. (37) and after some algebra we derive the desired result. □

The following corollary is immediate.

Corollary 5. *In an ordinary renewal process, it holds that* $\lim_{t\to\infty} Cov(\gamma_t^r, N(t)) \geq (\leq) 0$ *if an only if $C_r \leq (\geq) r$.*

4 Concluding remarks

1. In Section 2, we provide an alternative formula (see Eq. (26)) for the covariance $Cov(\gamma_t, N(t))$ in terms of the remaining term $Q(t)$. This alternative formula has some advantages. First, it gives a direct connection with the asymptotic covariance in Eq. (2). Second, it enables us to show that, assuming IMRL interarrival times, $Cov(\gamma_t, N(t)) \leq 0$.

2. The exact computation of $Cov(\gamma_t, N(t))$ requires the knowledge of the underlying renewal function, which may not be available. For this reason, we construct bounds for $Cov(\gamma_t, N(t))$. An additional benefit of Eq. (26) compared to the already known Eq. (1) by Coleman (1982) [5] is its structure. More specifically, a refurbishment of Eq. (26) gives Eq. (32) which includes the covariance and its limit to infinity. By employing the bounds of Losidis and Politis (2017) [12] for the renewal function $U(t)$ and the renewal density $u(t)$, respectively, we are able to propose bounds for the covariance with the correct asymptotic behaviour, e.g., the covariance $Cov(\gamma_t, N(t))$ tends to its limit as $t \to \infty$.

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