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Applications of a change of measures technique for compound mixed renewal processes to the ruin problem

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Abstract In the present paper the change of measures technique for compound mixed renewal processes, developed in Tzaninis and Macheras [ArXiv:2007.05289 (2020) 1–25], is applied to the ruin problem in order to obtain an explicit formula for the probability of ruin in a mixed renewal risk model and to find upper and lower bounds for it.

Keywords Compound mixed renewal process, change of measures, progressively equivalent measures, regular conditional probabilities, ruin probability
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1 Introduction

The notion of the renewal risk model traces back to Sparre Andersen [2], and it has played a key role in classical Risk Theory as a natural generalization of the classical Cramér–Lundberg risk model. In its standard framework both claims and interarrival times form two sequences of i.i.d. random variables, which are also assumed to be mutually independent. However, these independence assumptions can be too restrictive as far as actuarial applications are considered, and generalizations to dependence scenarios must be developed.

In practice, insurance portfolios tend to be inhomogeneous and a mixture of smaller homogeneous ones that are identified by the realizations of a random variable (or vector) Θ. Such portfolios are usually modelled via (compound) mixed counting processes, with the typical example being that of the (compound) mixed Poisson one (MPP for short). In the case of (compound) MPPs, the sequence of the interarrival times W is P-conditionally independent and P-conditionally exponentially distributed, something that leads to an interesting dependence structure between the interarrival times and reveals a connection between (compound) MPPs and exchangeability.

Mixed renewal processes (MRPs for short) serve as a proper generalization of MPPs, which has not been much considered in the literature, but they are interesting from both theoretical and applied point of view. Their study is mathematically challenging, as they are not, in general, Markov processes (see [14], Theorem 3, and [19], Proposition 3.2) and they have a close connection to exchangeable stochastic processes (see [14], Corollary on p. 20, and [17], Theorem 4.2). As far as practical models are considered, MRPs seem to have interesting applications in both life insurance (see, e.g., [8]) and nonlife insurance mathematics (see, e.g., [29] and [30]). It is worth noticing that Segerdahl [29] was the first who considered MRPs, in an actuarial context, as an alternative to the classical Pólya–Lundberg process (i.e. a mixed Poisson-gamma one).

The change of measures technique has been successfully applied to various theories such as queues and fluid flows (Asmussen [3, 4], Palmowski and Rolski [21, 22]), ruin theory (Dassios and Embrechts [9], Asmussen [3], Asmussen and Albrecher [5], Schmidli [25–27]), simulation (Boogaert and De Waegenaere [6], Ridder [23]) and pricing of insurance risks (premium calculation principles) (Delbaen and Haezendonck [10], Lyberopoulos and Macheras [16], Macheras and Tzaninis [20, 31]). The process of interest is usually Markovian and, under a suitably chosen new probability measure, it is again a Markov process with some “nicer” desired properties.

In [31], the same problem was investigated for the class of compound MRPs. In the same paper, given an aggregate claims process S being a MRP under P, a full characterization of all probability measures Q, which are progressively equivalent to P and preserve the structure of S, but with some better desired properties, was provided, see [31] Theorem 4.5 and Corollary 4.8 as well as Proposition 4.15. In the present paper by utilizing the aforementioned change of measures technique an explicit formula and bounds for the probability of ruin in a mixed renewal risk model are obtained.

Part of Proposition 4.15 from [31] is Proposition 3.1, formulated for the purposes of the present paper and being the starting point for applications to the ruin problem. Proposition 3.1, as well as Proposition 3.2, extend the corresponding results for the renewal risk model (see, e.g., [27], Lemmas 8.4 and 8.6, respectively) to the compound MRPs.

A first consequence of Proposition 3.1 is Proposition 4.1, where an upper bound for the ruin probability within finite time is obtained. Another implication of Proposition 3.1 is Proposition 4.2, where it is proven, that if the net profit condition is fulfilled under an original measure P, and S is a compound MRP under P then under the new measure resulting from Proposition 3.1 the process S will be of the same type, except that the net profit condition will no longer be fulfilled and ruin will always occur within finite time. Thus, the ruin problem becomes easier to handle, since by Proposition 4.2 under the new measure the probability of ruin is equal to 1, something that gives the opportunity to express the ruin probability under P as a quantity under the new measure, see Theorem 4.1, and to find upper and lower bounds for it, see Corollary 4.2.

2 Preliminaries

Throughout this paper, unless stated otherwise, (Ω, Σ, P) *is an arbitrary but fixed probability space.* Given a topology $\mathfrak{T}$ on Ω write $\mathfrak{B}(\Omega)$ for its **Borel σ-algebra** on Ω, i.e. the σ-algebra generated by $\mathfrak{T}$. The measure theoretic terminology is standard and generally follows [7]. For the definitions of real-valued random variables and random variables, cf., e.g., [7], p. 308. Notation $P_X := P_X(\theta) := \mathbf{K}(\theta)$ denotes that X is distributed according to the law $\mathbf{K}(\theta)$, where $\theta \in D \subseteq \mathbb{R}^d$ $(d \in \mathbb{N})$ is the parameter of the distribution. Denote again by $\mathbf{K}(\theta)$ the distribution function induced by the probability distribution $\mathbf{K}(\theta)$. Notation $\mathbf{Ga}(b, a)$, where $a, b \in (0, \infty)$, stands for the law of gamma distribution (cf., e.g., [28], p. 180). In particular, $\mathbf{Ga}(b, 1) = \mathbf{Exp}(b)$ stands for the law of exponential distribution. For two real-valued random variables X and Y write $X = Y$ P-a.s. if $\{X \neq Y\}$ is a P-null set. If $A \subseteq \Omega$, then $A^c := \Omega \setminus A$, while χ_A denotes the indicator (or characteristic) function of the set A. For a map $f : D \to E$ and for a nonempty set $A \subseteq D$ denote by $f \upharpoonright A$ the restriction of f to A. Write $\mathbb{E}_P[X \mid \mathcal{F}]$ for a version of a conditional expectation (under P) of a P-integrable random variable X given a σ-subalgebra $\mathcal{F}$ of Σ. For $X := \chi_E$ with $E \in \Sigma$ set $P(E \mid \mathcal{F}) := \mathbb{E}_P[\chi_E \mid \mathcal{F}]$. For the unexplained terminology of Probability and Risk Theory, see [28].

Given two measurable spaces (Ω, Σ) and (Υ, H), a function k from $\Omega \times H$ into $[0, 1]$ is a **Σ-H-Markov kernel** if it has the following properties:

(k1) The set-function $B \mapsto k(\omega, B)$ is a probability measure on H for any fixed $\omega \in \Omega$.

(k2) The function $\omega \mapsto k(\omega, B)$ is Σ-measurable for any fixed $B \in H$.

In particular, given a real-valued random variable X on Ω and a d-dimensional random vector Θ on Ω, a **conditional distribution of X over Θ** is a $\sigma(\Theta)$-$\mathfrak{B}$-Markov kernel denoted by $P_{X|\Theta} := P_{X|\sigma(\Theta)}$ and satisfying for each $B \in \mathfrak{B}$ the condition

$$P_{X|\Theta}(\bullet, B) = P(X^{-1}[B] \mid \sigma(\Theta))(\bullet) \quad P \upharpoonright \sigma(\Theta) - \text{a.s.}$$

Clearly, for every $\mathfrak{B}_d$-$\mathfrak{B}$-Markov kernel k, the map $K(\Theta)$ from $\Omega \times \mathfrak{B}$ into $[0, 1]$ defined by means of

$$K(\Theta)(\omega, B) := (k(\bullet, B) \circ \Theta)(\omega) \quad \text{for any} \quad (\omega, B) \in \Omega \times \mathfrak{B}$$

is a $\sigma(\Theta)$-$\mathfrak{B}$-Markov kernel. Then for $\theta = \Theta(\omega)$ with $\omega \in \Omega$ the probability measures $k(\theta, \bullet)$ are distributions on $\mathfrak{B}$ and so one may write $\mathbf{K}(\theta)(\bullet)$ instead of $k(\theta, \bullet)$. Consequently, in this case $K(\Theta)$ will be denoted by $\mathbf{K}(\Theta)$.

For any real-valued random variables X, Y on Ω the conditional distributions $P_{X|\Theta}$ and $P_{Y|\Theta}$ are $P \upharpoonright \sigma(\Theta)$-equivalent (in symbols, $P_{X|\Theta} = P_{Y|\Theta}$ $P \upharpoonright \sigma(\Theta)$-a.s.), if there exists a P-null set $M \in \sigma(\Theta)$ such that for any $\omega \notin M$ and $B \in \mathfrak{B}$ the equality $P_{X|\Theta}(\omega, B) = P_{Y|\Theta}(\omega, B)$ holds true.

For the definition of a P-conditionally (stochastically) independent process over $\sigma(\Theta)$ as well as of a P-conditionally identically distributed process over $\sigma(\Theta)$, cf., e.g., [31], p. 4. Recall that a process is **P-conditionally (stochastically) independent or identically distributed given** Θ, if it is conditionally independent or identically distributed over the σ-algebra $\sigma(\Theta)$.

Henceforth, unless stated otherwise, Θ is a d-dimensional random vector on Ω with values in $D \subseteq \mathbb{R}^d$ ($d \in \mathbb{N}$). Furthermore, simply write "conditionally" in the place of "conditionally given Θ" whenever conditioning refers to Θ.

A family $N := \{N_t\}_{t\in\mathbb{R}_+}$ of random variables from (Ω, Σ) into $(\overline{\mathbb{R}}, \mathfrak{B}(\overline{\mathbb{R}}))$ is called a **counting** (or **claim number**) **process**, if there exists a P-null set $\Omega_N \in \Sigma$ such that the process N restricted on $\Omega \setminus \Omega_N$ takes values in $\mathbb{N}_0 \cup \{\infty\}$, has right-continuous paths, presents jumps of size (at most) one, vanishes at $t = 0$ and increases to infinity. Without loss of generality one may and do assume, that $\Omega_N = \emptyset$. Denote by $T := \{T_n\}_{n\in\mathbb{N}_0}$ and $W := \{W_n\}_{n\in\mathbb{N}}$ the **(claim) arrival process** and **(claim) interarrival process**, respectively (cf., e.g., [28], Section 1.1, p. 6, for the definitions) associated with N. Note also that every arrival process induces a counting process, and vice versa (cf., e.g., [28], Theorem 2.1.1).

Furthermore, let $X := \{X_n\}_{n\in\mathbb{N}}$ be a sequence of positive real-valued random variables on Ω, and for any $t \geq 0$ define

$$S_t := \begin{cases} \sum_{k=1}^{N_t} X_k & \text{if } t > 0; \\ 0 & \text{if } t = 0. \end{cases}$$

Accordingly, the sequence X is said to be the **claim size process**, and the family $S := \{S_t\}_{t\in\mathbb{R}_+}$ of real-valued random variables on Ω is said to be the **aggregate claims process induced by** the pair (N, X). Recall that a pair (N, X) is called a **risk process**, if N is a counting process, X is P-i.i.d. and the processes N and X are P-independent (see [28], Chapter 6, Section 6.1).

3 A change of measures technique for compound mixed renewal processes

Recall that a counting process N is a **P-mixed renewal process with mixing parameter Θ and interarrival time conditional distribution $\mathbf{K}(\Theta)$** (written P-MRP($\mathbf{K}(\Theta)$) for short), if the induced interarrival process W is P-conditionally independent and

$$\forall\, n \in \mathbb{N} \qquad [P_{W_n|\Theta} = \mathbf{K}(\Theta) \quad \mathbf{P} \upharpoonright \sigma(\Theta)\text{-a.s.}]$$

(see also [17], Definition 3.1, or [19], Definition 3.2(b)). In particular, if the distribution P_Θ of Θ is degenerate at some point $\theta_0 \in D$, then the counting process N becomes a *P-renewal process with interarrival time distribution* $\mathbf{K}(\theta_0)$ (written P-RP($\mathbf{K}(\theta_0)$) for short).

Accordingly, an aggregate claims process S induced by a P-risk process (N, X) such that N is a P-MRP($\mathbf{K}(\Theta)$) is called a **compound mixed renewal process with**

parameters $\mathbf{K}(\Theta)$ and P_{X_1} (P-CMRP($\mathbf{K}(\Theta)$, P_{X_1}) for short). In particular, if P_Θ is degenerate at $\theta_0 \in D$, then S is called a **compound renewal process with parameters $\mathbf{K}(\theta_0)$ and P_{X_1}** (P-CRP($\mathbf{K}(\theta_0)$, P_{X_1}) for short).

Throughout what follows denote again by $\mathbf{K}(\Theta)$ and $\mathbf{K}(\theta)$ the conditional distribution function and the distribution function induced by the conditional probability distribution $\mathbf{K}(\Theta)$ and the probability distribution $\mathbf{K}(\theta)$, respectively.

Remark 3.1. **(a)** For any $n \in \mathbb{N}$ the interarrival times W_n of a P-MRP($\mathbf{K}(\Theta)$) remain P-identically distributed (see [31], Remark 2.1) but they fail to be P-independent. In fact, assuming that $\mathbb{E}_P^2[W_1 \mid \Theta] \in \mathcal{L}^1(P)$ and applying standard computations along with the fact that W is P-conditionally i.i.d., it follows that $\mathrm{Cov}_P(W_n, W_m) = \mathrm{Var}_P(\mathbb{E}_P[W_1 \mid \Theta]) > 0$ for any $n, m \in \mathbb{N}$ with $n \neq m$.

(b) Applying [17], Theorem 4.2, which holds true under some mild assumptions satisfied by the majority of the probability spaces appearing in applied Probability Theory, one gets that there exists a d-dimensional random vector Θ such that W is P-conditionally i.i.d. if and only if the sequence W is P-exchangeable (recall that a sequence of random variables is called exchangeable if the joint distribution of the sequence is invariant under the permutation of the indices); hence exchangeability seems to be an appealing way to introduce a dependence structure between the claim interarrival times of a counting process. Actually, exchangeability seems to be a natural assumption in the risk model context (see [1], Remark 2.7), implying that the assumption that N is a MRP may be seen also as a natural one, in order to model this kind of dependence between claim interarrival times.

The following conditions for the quadruplet (P, W, X, Θ) (or, if no confusion arises, for the probability measure P) will be useful throughout the paper:

(a1) the pair (W, X) is P-conditionally independent;

(a2) the random vector Θ and the process X are P-(unconditionally) independent.

Since conditioning is involved in the definition of (compound) mixed renewal processes, it is natural to expect that regular conditional probabilities (or disintegrations) will play a key. To this purpose recall the following definition (cf., e.g., [31], Definition 3.2).

Definition 3.1. The family $\{P_\theta\}_{\theta \in D}$ of probability measures on Σ is called a **regular conditional probability** (rcp for short) of P over P_Θ if

(d1) for each $E \in \Sigma$ the map $\theta \mapsto P_\theta(E)$ is $\mathfrak{B}(D)$-measurable;

(d2) $\int P_\theta(E)\, P_\Theta(d\theta) = P(E)$ for each $E \in \Sigma$.

The family $\{P_\theta\}_{\theta \in D}$ is consistent with Θ if, for each $B \in \mathfrak{B}(D)$, the equality $P_\theta(\Theta^{-1}(B)) = 1$ holds for P_Θ-almost every $\theta \in B$.

A rcp $\{P_\theta\}_{\theta \in D}$ of P over P_Θ consistent with Θ is **essentially unique**, if for any other rcp $\{\widetilde{P}_\theta\}_{\theta \in D}$ of P over P_Θ consistent with Θ there exists a P_Θ-null set $N \in \mathfrak{B}(D)$ such that for any $\theta \notin N$ the equality $P_\theta = \widetilde{P}_\theta$ holds true.

Regular conditional probabilities seem to have a bad reputation when it comes to applications, and that is probably due to the facts that their own existence is not always guaranteed (see [18], Examples 4 and 5) and their construction usually involves manipulations with Radon–Nikodým derivatives. Nevertheless, as the spaces used in applied Probability Theory are mainly Polish ones, such rcps always exist (see [11], Theorem 6), and in fact they can be explicitly constructed for the class of (compound) mixed renewal processes (see [31], Proposition 4.1).

From now on the family $\{P_\theta\}_{\theta \in D}$ *is a rcp of* P *over* P_Θ *consistent with* Θ.

Let S be the aggregate claims process induced by the counting process N and the claim size process X. Fix on arbitrary $u \in \Upsilon$ and $t \in \mathbb{R}_+$, and define the function $r_t^u : \Omega \times D \to \mathbb{R}$ by means of $r_t^u(\omega,\theta) := u + c(\theta)\cdot t - S_t(\omega)$ for any $(\omega,\theta) \in \Omega \times D$, where c is a positive $\mathfrak{B}(D)$-measurable function. For arbitrary but fixed $\theta \in D$, the process $r^u(\theta) := \{r_t^u(\theta)\}_{t\in\mathbb{R}_+}$ defined by $r_t^u(\theta) := r_t^u(\omega,\theta)$ for any $\omega \in \Omega$, is called the **reserve process** induced by the **initial reserve** u, the **premium intensity** or **premium rate** $c(\theta)$ and the aggregate claims process S (see [28], Section 7.1, pp. 155–156, for the definition).

Define the real-valued function $R_t^u(\Theta)$ on Ω by means of $R_t^u(\Theta) := r_t^u \circ (id_\Omega \times \Theta)$. The process $R^u(\Theta) := \{R_t^u(\Theta)\}_{t\in\mathbb{R}_+}$ is called the **reserve process** induced by the initial reserve u, the **stochastic premium intensity** or **stochastic premium rate** $c(\Theta)$ and the aggregate claims process S.

Remark 3.2. The most common choice for the premium rate in Risk Theory is that of a positive constant c (cf., e.g., [12], p. 215, [27], p. 83, or [28], p. 155). Nevertheless, mixed claim number processes are widely used in order to model claim counts in a portfolio of risks which is thought to be inhomogeneous and a mixture of smaller homogeneous ones which can be identified by the realization of the random vector Θ; hence the choice of a stochastic premium rate $c(\Theta)$ instead of a constant premium rate c seems natural, as the homogeneous portfolios may have different premium rates.

Remark 3.3. Assume that S is a P-CMRP($\mathbf{K}(\Theta)$, P_{X_1}), define the function $\kappa : D \times \mathbb{R}_+ \to \mathbb{R}$ by means of

$$\kappa(\theta,r) := \kappa_\theta(r) \quad \text{for any} \quad (\theta,r) \in D \times \mathbb{R}_+,$$

and for fixed $r \in \mathbb{R}_+$ denote by κ_Θ the random variable defined by the formula

$$\kappa_\Theta(r)(\omega) := \kappa_{\Theta(\omega)}(r) \quad \text{for any} \quad \omega \in \Omega.$$

According to [31], Proposition 3.3, there exists a P_Θ-null set $L_P \in \mathfrak{B}(D)$ such that S is a P_θ-CRP($\mathbf{K}(\theta)$, $(P_\theta)_{X_1}$) with $(P_\theta)_{X_1} = P_{X_1}$ for any $\theta \notin L_P$. For any $r \in \mathbb{R}_+$ such that $\mathbb{E}_P[e^{rX_1}] < \infty$ and any $\theta \in L_P^c$ let $\kappa_\theta(r)$ be the unique solution to the equation

$$M_{X_1}(r) \cdot (M_\theta)_{W_1}\big(-\kappa_\theta(r) - c(\theta)\cdot r\big) = 1, \tag{1}$$

where M_{X_1} and $(M_\theta)_{W_1}$ are the moment generating function of X_1 and W_1 under the measures P and P_θ, respectively (such a solution exists by, e.g., [24], Lemma 11.5.1(a)). Note that the latter condition is in fact a version of the well-known Cramér–Lundberg equation (cf., e.g., [27], p. 133). Condition (1), along with [31], Lemma 4.13, implies that $\kappa_\Theta(r)$ is the $P \upharpoonright \sigma(\Theta)$-a.s. unique solution to the equation

$$M_{X_1}(r) \cdot \mathbb{E}_P\big[e^{-\left(\kappa_\Theta(r)+c(\Theta)\cdot r\right)W_1} \mid \Theta\big] = 1 \quad P \upharpoonright \sigma(\Theta)\text{-a.s.} \tag{2}$$

$\mathcal{F} := \{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$, where $\mathcal{F}_t := \sigma(\mathcal{F}_t^S \cup \sigma(\Theta))$, denotes the canonical filtration generated by S and Θ; $\mathcal{F}_\infty^S := \sigma(\bigcup_{t\in\mathbb{R}_+} \mathcal{F}_t^S)$ and $\mathcal{F}_\infty := \sigma(\mathcal{F}_\infty^S \cup \sigma(\Theta))$. For the definition of a $(P, \mathcal{Z})$-martingale, where $\mathcal{Z} = \{\mathcal{Z}_t\}_{t\in\mathbb{R}_+}$ is a filtration for (Ω, Σ), cf., e.g., [28], p. 25. A $(P, \mathcal{Z})$-martingale $\{Z_t\}_{t\in\mathbb{R}_+}$ is **P-a.s. positive**, if Z_t is P-a.s. positive for each $t \geq 0$. For $\mathcal{Z} = \mathcal{F}$ write "martingale" instead of "$(P, \mathcal{Z})$-martingale", for simplicity.

Notations 3.1. **(a)** The class of all real-valued $\mathfrak{B}(\Upsilon)$-measurable functions γ such that $\mathbb{E}_P\left[e^{\gamma(X_1)}\right] = 1$ will be denoted by $\mathcal{F}_P := \mathcal{F}_{P,X_1,\ln}$. The class of all real-valued $\mathfrak{B}(D)$-measurable functions ξ on D such that $P_\Theta(\{\xi > 0\}) = 1$ and $\mathbb{E}_P[\xi(\Theta)] = 1$ is denoted by $\mathcal{R}_+(D) := \mathcal{R}_+(D, \mathfrak{B}(D), P_\Theta)$.

(b) Denote by $\mathfrak{M}^k(D)$ ($k \in \mathbb{N}$) the class of all $\mathfrak{B}(D)$-$\mathfrak{B}(\mathbb{R}^k)$-measurable functions on D. For each $\rho \in \mathfrak{M}^k(D)$, the class of all probability measures Q on Σ satisfying (a1) and (a2), being **progressively equivalent** to P, i.e. $Q \upharpoonright \mathcal{F}_t \sim P \upharpoonright \mathcal{F}_t$ for any $t \geq 0$ (in the sense of absolute continuity), and such that S is a Q-CMRP($\mathbf{\Lambda}(\rho(\Theta))$, Q_{X_1}) is denoted by $\mathcal{M}_{S,\mathbf{\Lambda}(\rho(\Theta))} := \mathcal{M}_{S,\mathbf{\Lambda}(\rho(\Theta)),P,X_1}$. In the special case $d = k$ and $\rho := id_D$ write $\mathcal{M}_{S,\mathbf{\Lambda}(\Theta)} := \mathcal{M}_{S,\mathbf{\Lambda}(\rho(\Theta))}$ for simplicity.

(c) For given $\rho \in \mathfrak{M}^k(D)$ and $\theta \in D$, denote by $\mathcal{M}_{S,\mathbf{\Lambda}(\rho(\theta))}$ the class of all probability measures Q_θ on Σ, such that $Q_\theta \upharpoonright \mathcal{F}_t \sim P_\theta \upharpoonright \mathcal{F}_t$ for any $t \in \mathbb{R}_+$ and S is a Q_θ-CRP($\mathbf{\Lambda}(\rho(\theta))$, $(Q_\theta)_{X_1}$).

Henceforth, $\Upsilon := (0, \infty)$, $\Omega := \Upsilon^{\mathbb{N}} \times \Upsilon^{\mathbb{N}} \times D$, $\Sigma := \mathfrak{B}(\Omega)$, $\mathbf{K}(\Theta)$ and $\{P_\theta\}_{\theta\in D}$ are as in [31], Proposition 4.1, $P \in \mathcal{M}_{S,\mathbf{K}(\Theta)}$, and assume that $\mathbb{E}_P[W_1|\Theta] \in \Upsilon$ $P \upharpoonright \sigma(\Theta)$-a.s.

The following proposition is a part of Proposition 4.15 from [31]. Since it is the basic tool for the proofs of the results, it is restated exactly in the form needed for the purposes of the present paper.

Proposition 3.1. *For any $r \in \mathbb{R}_+$ such that $\mathbb{E}_P[e^{rX_1}] < \infty$, and for any $\theta \notin L_P$, let $\kappa_\theta(r)$ be the unique solution to Equation (1), and let $\kappa_\Theta(r)$ be as in Remark 3.3. Fix on arbitrary $r \in \mathbb{R}_+$ as above and put $\rho_r(\Theta) := \kappa_\Theta(r) + c(\Theta) \cdot r$. For each pair $(\gamma, \xi) \in \mathcal{F}_P \times \mathcal{R}_+(D)$ with $\gamma(x) := r \cdot x - \ln \mathbb{E}_P[e^{r \cdot X_1}]$ for any $x \in \Upsilon$, there exists a unique probability measure $Q^r \in \mathcal{M}_{S,\mathbf{\Lambda}(\rho_r(\Theta))}$, where*

$$\mathbf{\Lambda}(\rho_r(\Theta))(B) := \frac{\mathbb{E}_P[\chi_{W_1^{-1}[B]} \cdot e^{-\rho_r(\Theta)\cdot W_1} \mid \Theta]}{\mathbb{E}_P[e^{-\rho_r(\Theta)\cdot W_1} \mid \Theta]} \quad P \upharpoonright \sigma(\Theta)\text{-a.s.}$$

for any $B \in \mathfrak{B}(\Upsilon)$, determined by the condition

$$Q^r(A) = \int_A M_t^{(\gamma,r)}(\Theta)\, dP \quad \text{for all } 0 \leq u \leq t \text{ and } A \in \mathcal{F}_u, \qquad (RRM_\xi)$$

with $M^{(\gamma,r)}(\Theta) := \{M_t^{(\gamma,r)}(\Theta)\}_{t\in\mathbb{R}_+}$ being a P-a.s. positive martingale, fulfilling the condition

$$M_t^{(\gamma,r)}(\Theta) = \xi(\Theta) \cdot \widetilde{M}_t^{(\gamma,r)}(\Theta) \quad P \upharpoonright \sigma(\Theta)\text{-a.s.}$$

Moreover, there exist an essentially unique rcp $\{Q_\theta^r\}_{\theta\in D}$ of Q^r over Q_Θ^r consistent with Θ and a P_Θ-null set $L_{**} \in \mathfrak{B}(D)$, satisfying for any $\theta \notin L_{**}$ the conditions

$Q^r_\theta \in \mathcal{M}_{S,\mathbf{\Lambda}(\rho_r(\theta))}$ and

$$Q^r_\theta(A) = \int_A \widetilde{M}^{(\gamma,r)}_t(\theta)\, dP_\theta \quad \textit{for all } 0 \le u \le t \textit{ and } A \in \mathcal{F}_u, \qquad (RRM_\theta)$$

with

$$\widetilde{M}^{(\gamma,r)}_t(\theta) = e^{r\cdot S_t - \rho_r(\theta)\cdot T_{N_t} + \ln \mathbb{E}_P[e^{r\cdot X_1}]} \cdot \frac{\int_{J_t}^{\infty} e^{-\rho_r(\theta)\cdot w}\,(P_\theta)_{W_1}(dw)}{1 - \mathbf{K}(\theta)(J_t)},$$

where $J_t := t - T_{N_t}$ and $\widetilde{M}^{(\gamma,r)}(\theta) := \{\widetilde{M}^{(\gamma,r)}_t(\theta)\}_{t\in\mathbb{R}_+}$ is a P_θ-a.s. positive martingale.

In the following example Proposition 3.1 is applied for an initial probability measure $P \in \mathcal{M}_{S,\mathbf{Exp}(\Theta)}$, where Θ is a positive real valued random variable, i.e. when S is a compound mixed Poisson process (cf., e.g., [16], p. 4, for its definition).

Example 3.1. Take $D := \Upsilon$ and assume that $P \in \mathcal{M}_{S,\mathbf{Exp}(\Theta)}$, where Θ is a positive real-valued random variable. For any $r \in \mathbb{R}_+$ such that $M_{X_1}(r) := \mathbb{E}_P[e^{rX_1}] < \infty$, consider the functions γ and ξ, defined by means of $\gamma(x) := r\cdot x - \ln M_{X_1}(r)$ for any $x \in \Upsilon$ and $\xi(\theta) := \frac{e^{-r\cdot\theta}}{\mathbb{E}_P[e^{-r\cdot\Theta}]}$ for any $\theta \in D$. An easy computation justifies that $(\gamma,\xi) \in \mathcal{F}_P \times \mathcal{R}_+(D)$. Thus, applying Proposition 3.1, there exist a probability measure $Q^r \in \mathcal{M}_{S,\mathbf{\Lambda}(\rho_r(\Theta))}$, determined by ($RRM_\xi$), an essentially unique rcp $\{Q^r_\theta\}_{\theta\in D}$ of Q^r over Q^r_Θ consistent with Θ and a P_Θ-null set $L_{**} \in \mathfrak{B}(D)$ such that conditions $Q^r_\theta \in \mathcal{M}_{S,\mathbf{\Lambda}(\rho_r(\theta))}$ and (RRM_θ) hold for any $\theta \notin L_{**}$.

Fix an arbitrary $\theta \notin L_{**}$. Since by [31], Proposition 3.3, the aggregate claims process S is a P_θ-CPP(θ, P_{X_1}) with $P_{X_1} = (P_\theta)_{X_1}$, it follows by [27], condition (5.3) on p. 89, that $\kappa_\theta(r) = \theta\cdot(M_{X_1}(r) - 1) - c(\theta)\cdot r$, implying that $\rho_r(\theta) := \kappa_\theta(r) + c(\theta)\cdot r = \theta\cdot(M_{X_1}(r) - 1)$. Thus,

$$\mathbf{\Lambda}(\rho_r(\theta))(B) = \frac{\mathbb{E}_{P_\theta}[\chi_{W_1^{-1}[B]}\cdot e^{-\rho_r(\theta)\cdot W_1}]}{\mathbb{E}_{P_\theta}[e^{-\rho_r(\theta)\cdot W_1}]} = \frac{\int_B e^{-\theta\cdot(M_{X_1}(r)-1)\cdot w}\,(P_\theta)_{W_1}(dw)}{\int e^{-\theta\cdot(M_{X_1}(r)-1)\cdot w}\,(P_\theta)_{W_1}(dw)}$$

for any $B \in \mathfrak{B}(\Upsilon)$. But since $(P_\theta)_{W_1} = \mathbf{Exp}(\theta)$, the latter equality becomes

$$\mathbf{\Lambda}(\rho_r(\theta))(B) = \int_B \theta\cdot M_{X_1}(r)\cdot e^{-\theta\cdot M_{X_1}(r)\cdot w}\,\lambda(dw),$$

where λ denotes the Lebesgue probability measure; hence $\mathbf{\Lambda}(\rho_r(\theta)) = \mathbf{Exp}(\theta\cdot M_{X_1}(r))$, or equivalently $\mathbf{\Lambda}(\rho_r(\Theta)) = \mathbf{Exp}(\Theta\cdot M_{X_1}(r))$ $P \upharpoonright \sigma(\Theta)$-a.s. by [19], Lemma 3.2, implying that $Q^r \in \mathcal{M}_{S,\mathbf{Exp}(\Theta\cdot M_{X_1}(r))}$. In this situation conditions (RRM_θ) and (RRM_ξ) become

$$Q^r_\theta(A) = \int_A e^{r\cdot S_t - \theta\cdot t\cdot(M_{X_1}(r)-1)}\, dP_\theta \quad \text{for all } 0 \le u \le t \text{ and } A \in \mathcal{F}_u$$

and

$$Q^r(A) = \int_A \frac{e^{r\cdot S_t - \Theta\cdot t\cdot(M_{X_1}(r)-1) - \Theta\cdot r}}{\mathbb{E}_P[e^{-r\cdot\Theta}]}\, dP \quad \text{for all } 0 \le u \le t \text{ and } A \in \mathcal{F}_u,$$

respectively.

Remark 3.4. It is worth noticing that in the special case when P_{W_1} is absolutely continuous with respect to the Lebesgue measure λ restricted to $\mathfrak{B}([0,1])$, the martingale $L^r(\theta) := \{L^r_t(\theta)\}_{t\in\mathbb{R}_+}$ for $r \in \mathbb{R}_+$, appearing in [27], Lemma 8.4, coincides with the martingale $\widetilde{M}^{(\gamma,r)}(\theta)$ for any $\theta \notin L_{**}$, and for any $t \in \mathbb{R}_+$ condition

$$M_t^{(\gamma,r)}(\Theta) = \xi(\Theta) \cdot L_t^r(\Theta)$$

holds true $P \restriction \sigma(\Theta)$-a.s.

Lemma 3.1. *For any $r \in \mathbb{R}_+$, $\theta \notin L_{**}$, Q^r_θ and $\kappa_\theta(r)$ as in Proposition 3.1 condition*

$$\kappa'_\theta(r) = \frac{\mathbb{E}_{Q^r_\theta}[X_1]}{\mathbb{E}_{Q^r_\theta}[W_1]} - c(\theta),$$

holds true.

Proof. Fix an arbitrary $r \in \mathbb{R}_+$ and $\theta \notin L_{**}$ as in Proposition 3.1.

Since $L_P \subseteq L_{**}$ by [31], Theorem 4.5, it follows by [31], Proposition 3.3, that condition (1) can be rewritten in the form

$$(M_\theta)_{X_1}(r) \cdot (M_\theta)_{W_1}(-c(\theta)\cdot r - \kappa_\theta(r)) = 1. \tag{3}$$

Differentiation with respect to r gives

$$\begin{aligned}&\big((M_\theta)_{X_1}(r)\big)' \cdot (M_\theta)_{W_1}(-c(\theta)\cdot r - \kappa_\theta(r)\big)\\ &\quad + (M_\theta)_{X_1}(r) \cdot \big((M_\theta)_{W_1}(-c(\theta)\cdot r - \kappa_\theta(r))\big)' \cdot (-c(\theta) - \kappa'_\theta(r)) = 0\end{aligned} \tag{4}$$

for all r in a neighbourhood of 0. The expectations $\mathbb{E}_{Q^r_\theta}[X_1]$ and $\mathbb{E}_{Q^r_\theta}[W_1]$ are given by

$$\mathbb{E}_{Q^r_\theta}[X_1] = \frac{\mathbb{E}_{P_\theta}\left[X_1 \cdot e^{r\cdot X_1}\right]}{\mathbb{E}_{P_\theta}[e^{r\cdot X_1}]} = \frac{((M_\theta)_{X_1}(r))'}{(M_\theta)_{X_1}(r)}$$

and

$$\mathbb{E}_{Q^r_\theta}[W_1] = \frac{\mathbb{E}_{P_\theta}\left[W_1 \cdot e^{-(r\cdot c(\theta)+\kappa_\theta(r))\cdot W_1}\right]}{\mathbb{E}_{P_\theta}[e^{-(r\cdot c(\theta)+\kappa_\theta(r))\cdot W_1}]} = \frac{\big((M_\theta)_{W_1}(-r\cdot c(\theta) - \kappa_\theta(r))\big)'}{\big((M_\theta)_{W_1}(-r\cdot c(\theta) - \kappa_\theta(r)\big)},$$

respectively, implying along with condition (4) that

$$\begin{aligned}&\mathbb{E}_{Q^r_\theta}[X_1] \cdot (M_\theta)_{X_1}(r) \cdot (M_\theta)_{W_1}(-c(\theta)\cdot r - \kappa_\theta(r))\\ &\quad + (M_\theta)_{X_1}(r) \cdot \mathbb{E}_{Q^r_\theta}[W_1] \cdot (M_\theta)_{W_1}(-c(\theta)\cdot r - \kappa_\theta(r)) \cdot (-c(\theta) - \kappa'_\theta(r)) = 0.\end{aligned}$$

The latter together with condition (3) gives

$$\mathbb{E}_{Q^r_\theta}[X_1] + \mathbb{E}_{Q^r_\theta}[W_1] \cdot (-c(\theta) - \kappa'_\theta(r)) = 0,$$

completing the proof. □

The following proposition extends Lemma 8.6 of [27].

Proposition 3.2. *For any $r \in \mathbb{R}_+$, $\theta \notin L_{**}$, Q^r_θ and $\kappa_\theta(r)$ as in Proposition 3.1, the following statements hold true:*

(i) $\lim_{t\to\infty} \frac{r^u_t(\theta)-u}{t} = -\kappa'_\theta(r)$ Q^r_θ*-a.s.;*

(ii) $\lim_{t\to\infty} \frac{R^u_t(\Theta)-u}{t} = -\kappa'_\Theta(r)$ Q^r*-a.s.;*

(iii) if there exists a P_Θ-null set $\widehat{L}_1$ in $\mathfrak{B}(D)$ such that for any $\theta \notin \widehat{L}_1$ the condition $P_\theta = Q^r_\theta$ holds, then the measures P and Q^r are equivalent on $\mathcal{F}_\infty$;

(iv) if there exists a P_Θ-null set $\widehat{L}_2$ in $\mathfrak{B}(D)$ such that for any $\theta \notin \widehat{L}_2$ the condition $P_\theta \neq Q^r_\theta$ holds, then the measures P and Q^r are singular on $\mathcal{F}_\infty$, i.e. there exists a set $E \in \mathcal{F}_\infty$ such that $P(E) = 0$ if and only if $Q^r(E) = 1$.

Proof. Fix an arbitrary $r \in \mathbb{R}_+$ as in Proposition 3.1.

Ad (i): Fix an arbitrary $\theta \notin L_{**}$, and note that $L_P \subseteq L_{**}$ by [31] T,heorem 4.5. Since S is a Q^r_θ-CRP by [31], Proposition 3.3, the strong law of large numbers yields

$$\lim_{t\to\infty} \frac{S_t}{t} = \frac{\mathbb{E}_{Q^r_\theta}[X_1]}{\mathbb{E}_{Q^r_\theta}[W_1]} \quad Q^r_\theta\text{-a.s.}$$

(cf., e.g., [13], Section 1.2, Theorem 2.3), or equivalently that

$$\lim_{t\to\infty} \frac{r^u_t(\theta) - u}{t} = \lim_{t\to\infty} \frac{c(\theta)\cdot t - S_t}{t} = c(\theta) - \frac{\mathbb{E}_{Q^r_\theta}[X_1]}{\mathbb{E}_{Q^r_\theta}[W_1]} \qquad Q^r_\theta\text{-a.s.,}$$

implying along with Lemma 3.1, assertion (i).

Ad (ii): Consider the function $v := \chi_{\left\{\lim_{t\to\infty} \frac{r^u_t - u}{t} = -\kappa'_\bullet(r)\right\}} : \Omega \times D \to [0,1]$ and put $g := v \circ (id_\Omega \times \Theta) = \chi_{\left\{\lim_{t\to\infty} \frac{r^u_t(\Theta)-u}{t} = -\kappa'_\Theta(r)\right\}}$. Since $v \in \mathcal{L}^1(M)$, where $M := P \circ (id_\Omega \times \Theta)^{-1}$, apply [15], Proposition 3.8(i), to get that

$$\mathbb{E}_{Q^r}[g \mid \Theta] = \mathbb{E}_{Q^r_\bullet}\left[v^\bullet\right] \circ \Theta \quad Q^r \restriction \sigma(\Theta)\text{-a.s.}$$

or equivalently

$$\begin{aligned} &Q^r\left(\left\{ \lim_{t\to\infty} \frac{R^u_t(\Theta) - u}{t} = -\kappa'_\Theta(r) \right\} \mid \Theta \right) \\ &= Q^r_\bullet\left(\left\{ \lim_{t\to\infty} \frac{r^u_t(\bullet) - u}{t} = -\kappa'_\bullet(r) \right\} \right) \circ \Theta \quad Q^r \restriction \sigma(\Theta)\text{-a.s.} \end{aligned}$$

Then for any $F \in \mathfrak{B}(D)$ it follows that

$$\begin{aligned} &\int_{\Theta^{-1}[F]} Q^r\left(\left\{ \lim_{t\to\infty} \frac{R^u_t(\Theta) - u}{t} = -\kappa'_\Theta(r) \right\} \mid \Theta \right) dQ^r \\ &= \int_{F\cap L^c_{**}} Q^r_\theta\left(\left\{ \lim_{t\to\infty} \frac{r^u_t(\theta) - u}{t} = -\kappa'_\theta(r) \right\} \right) Q^r_\Theta(d\theta) \\ &= \int_{\Theta^{-1}[F]} dQ^r, \end{aligned}$$

where the last equality follows by (i); hence

$$Q^r\left(\left\{\lim_{t\to\infty}\frac{R_t^u(\Theta)-u}{t}=-\kappa'_\Theta(r)\right\}\mid\Theta\right)=1\quad Q^r\restriction\sigma(\Theta)\text{-a.s.},$$

implying that assertion (ii) holds true.

The proof of the statements (iii) and (iv) follow by Proposition 3.1 together with [31], Proposition 3.11. □

4 Applications to the ruin problem

In this section the change of measures technique for compound mixed renewal processes appearing in Proposition 3.1 is applied to the ruin problem. In the first result a bound for the finite time ruin probability is proven. In order to present it denote by $\tau := \tau_u$ the **ruin time** of the reserve process $R^u(\Theta)$ ($u \in \mathbb{R}_+$) (cf., e.g., [27], p. 84, for the definition) and by $\psi(u,t) := P(\{\inf_{v\le t} R_v^u(\Theta) < 0\}) = P(\{\tau \le t\})$ the finite time ruin probability (cf., e.g., [5], p. 115) for the reserve process $R^u(\Theta)$ with respect to P.

Proposition 4.1. *Let $r \in \mathbb{R}_+$ be as in Proposition 3.1 and $\underline{y}, \overline{y} \in \mathbb{R}_+$ with $0 \le \underline{y} < \overline{y} < \infty$. The finite time ruin probability satisfies the condition*

$$\psi(u,\overline{y}\cdot u)-\psi(u,\underline{y}\cdot u)\le\mathbb{E}_P\left[e^{-R_\Theta(\underline{y},\overline{y})\cdot u}\right]\quad\textit{for any } u\in\mathbb{R}_+,$$

where $R_\Theta(\underline{y},\overline{y}) := \sup_{r\in\mathbb{R}_+}\min\{r-\kappa_\Theta(r)\cdot\underline{y}, r-\kappa_\Theta(r)\cdot\overline{y}\}$.

Proof. Let $r \in \mathbb{R}_+$ be as in Proposition 3.1, fix arbitrary $u, \underline{y}, \overline{y} \in \mathbb{R}_+$ with $0 \le \underline{y} < \overline{y} < \infty$, and consider the functions $\gamma(x) := r\cdot x - \ln\mathbb{E}_P\left[e^{r\cdot X_1}\right]$ for any $x \in \Upsilon$ and $\xi \in \mathcal{R}_+(D)$. Since $(\gamma,\xi) \in \mathcal{F}_P \times \mathcal{R}_+(D)$, it follows by Proposition 3.1 that there exists a unique probability measure $Q^r \in \mathcal{M}_{S,\mathbf{\Lambda}(\rho_r(\Theta))}$ determined by condition (RRM_ξ) and such that the family $M^{(\gamma,r)}(\Theta)$ is a P-a.s. positive martingale. But since τ is a stopping time for $\mathcal{F}$ it follows by [27], Lemma 8.1, that

$$\begin{aligned}
\psi(u,\overline{y}\cdot u)-\psi(u,\underline{y}\cdot u)&=\mathbb{E}_{Q^r}\left[\chi_{\{\underline{y}\cdot u\le\tau\le\overline{y}\cdot u\}}\cdot\frac{1}{M_\tau^{(\gamma,r)}(\Theta)}\right]\\
&\le\mathbb{E}_{Q^r}\left[\chi_{\{\underline{y}\cdot u\le\tau\le\overline{y}\cdot u\}}\cdot\frac{e^{-r\cdot u+\max\{\kappa_\Theta(r)\cdot\underline{y}\cdot u,\kappa_\Theta(r)\cdot\overline{y}\cdot u\}}}{\xi(\Theta)}\right]\\
&\le\mathbb{E}_{Q^r}\left[\frac{e^{-\min\{r-\kappa_\Theta(r)\cdot\underline{y},r-\kappa_\Theta(r)\cdot\overline{y}\}\cdot u}}{\xi(\Theta)}\right]\\
&=\mathbb{E}_P\left[e^{-\min\{r-\kappa_\Theta(r)\cdot\underline{y},r-\kappa_\Theta(r)\cdot\overline{y}\}\cdot u}\right],
\end{aligned}$$

where the first inequality follows by $R_\tau^u = u + c(\Theta)\cdot\tau - S_\tau < 0$, $J_\tau = \tau - T_{N_\tau} = 0$ and condition (2). Choosing now the exponent as small as possible, one gets

$$\psi(u,\overline{y}\cdot u)-\psi(u,\underline{y}\cdot u)\le\mathbb{E}_P\left[e^{-R_\Theta(\underline{y},\overline{y})\cdot u}\right]\quad\text{for any } u\in\mathbb{R}_+,$$

where $R_\Theta(\underline{y}, \overline{y}) := \sup_{r \in \mathbb{R}_+} \min\{r - \kappa_\Theta(r) \cdot \underline{y}, r - \kappa_\Theta(r) \cdot \overline{y}\}$, completing in this way the proof. □

Remark 4.1. In the special case when P_Θ is degenerate at some point $\theta_0 \in D$, the inequality appearing in Proposition 4.1 reduces to the well-known finite time Lundberg inequality in a Sparre Andersen risk model

$$\psi(u, \overline{y} \cdot u) - \psi(u, \underline{y} \cdot u) \leq e^{-R_{\theta_0}(\underline{y}, \overline{y}) \cdot u}$$

for any $u \in \mathbb{R}_+$ and $\underline{y}, \overline{y} \in \mathbb{R}_+$ with $0 \leq \underline{y} < \overline{y} < \infty$ (cf., e.g., [27], p. 147).

Recall that for any arbitrary but fixed $\theta \in D$ the function $\psi_\theta : \Upsilon \to [0, 1]$ defined by $\psi_\theta(u) := P_\theta(\{\inf_{t \in \mathbb{R}_+} r_t^u(\theta) < 0\})$ is called the **probability of ruin** for the reserve process $r^u(\theta)$ with respect to P_θ (see [28], Section 7.1, p. 158, for the definition). The function $\psi : \Upsilon \to [0, 1]$ defined by $\psi(u) := P(\{\inf_{t \in \mathbb{R}_+} R_t^u(\Theta) < 0\})$ is called the **probability of ruin** for the reserve process $R^u(\Theta)$ with respect to P.

When considering infinite time ruin probabilities in a mixed renewal risk model one has to assume that the **conditional net profit condition**

$$c(\Theta) > \frac{\mathbb{E}_P[X_1]}{\mathbb{E}_P[W_1 \mid \Theta]} \quad P \restriction \sigma(\Theta)\text{-a.s.} \tag{NPC$_\Theta$}$$

holds, in order to avoid a P-a.s. ruin (see [30], Lemma 5.3). Note that since S is a P-CMRP($\mathbf{K}(\Theta)$, P_{X_1}) and conditions (a1) and (a2) are valid, one may apply [31], Proposition 3.2, along with [15], Lemma 3.5, in order to show that condition (NPC$_\Theta$) is equivalent to

$$c(\theta) > \frac{\mathbb{E}_{P_\theta}[X_1]}{\mathbb{E}_{P_\theta}[W_1]} \quad \text{for any } \theta \notin L_{**}, \tag{NPC$_\theta$}$$

where $L_{**} \in \mathfrak{B}(D)$ is the P_Θ-null set appearing in Proposition 3.1. Fix an arbitrary $\theta \notin L_{**}$. If condition (NPC$_\theta$) holds true for any $\theta \notin L_{**}$ and $r \in \mathbb{R}_+$ is as in Proposition 3.1, it follows by, e.g., [27], p. 133, that there exists an **adjustment coefficient** $R(\theta) \in \Upsilon$ with respect to P_θ.

Throughout what follows assume that condition (NPC$_\Theta$) *holds true and that* $R(\theta) \in \Upsilon$ *is an adjustment coefficient with respect to* P_θ *for any* $\theta \notin L_{**}$.

The next result is an immediate consequence of Proposition 4.1 and extends the celebrated Lundberg inequality to the case of CMRPs.

Corollary 4.1. *The inequality*

$$\psi(u) \leq \mathbb{E}_P\left[e^{-R(\Theta) \cdot u}\right] \quad \textit{for any } u \in \mathbb{R}_+$$

holds true.

The proof follows immediately by Proposition 4.1 for $\underline{y} = 0$, $\overline{y} \to \infty$ and $r = R(\theta)$.

Remark 4.2. Unfortunately, one cannot prove Corollary 4.1 directly from Proposition 3.1, as this would imply that the function γ should also depend on θ, resulting to a claim size process that violates condition (a2) under Q.

Example 4.1. Take $D := (0, 1)$, let Θ be a real-valued random variable on Ω, and assume that $P \in \mathcal{M}_{S,\mathbf{Exp}(\Theta)}$ such that $P_{X_1} = \mathbf{Exp}(\eta)$, $\eta \in (1, \infty)$, and $P_\Theta = \mathbf{Beta}(1, 2)$ (cf., e.g., [28], p. 179, for the definition of a beta distribution). Since conditions (a1) and (a2) hold true, it follows by [31], Proposition 3.3, that there exists a P_Θ-null set $L_P \in \mathfrak{B}(D)$ such that $P_\theta \in \mathcal{M}_{S,\mathbf{Exp}(\theta)}$ with $(P_\theta)_{X_1} = P_{X_1}$ for any $\theta \notin L_P$, implying that

$$\frac{\mathbb{E}_{P_\theta}[X_1]}{\mathbb{E}_{P_\theta}[W_1]} = \frac{\frac{1}{\eta}}{\frac{1}{\theta}} = \frac{\theta}{\eta} \quad \text{for any } \theta \notin L_P.$$

Put $c(\theta) := \frac{\theta}{\eta-\theta}$ for any $\theta \in D$. Fix an arbitrary $\theta \notin L_{**}$. Since $P_\theta \in \mathcal{M}_{S,\mathbf{Exp}(\theta)}$ with $(P_\theta)_{X_1} = \mathbf{Exp}(\eta)$ and condition (NPC_θ) is fulfilled, one can apply [28], Theorem 7.4.5, to obtain that $R(\theta) = \eta - \frac{\theta}{c(\theta)} = \theta$. Applying now Corollary 4.1 it follows that

$$\psi(u) \leq \mathbb{E}_P\left[e^{-\Theta\cdot u}\right] = \frac{e^{-u} + u - 1}{u^2} \quad \text{for any } u \in \Upsilon.$$

It should be clear by Proposition 4.1 and Corollary 4.1 that one cannot, in general, expect to obtain exponential bounds in the case of CMRPs, as the adjustment coefficient depends on the random vector Θ. However, applying Proposition 3.1 one can obtain an explicit formula for the probability of ruin in infinite time, see Theorem 4.1, which can lead to exponential bounds, see Corollary 4.2. In order to formulate Theorem 4.1, one needs to establish the validity of the following proposition, which is a consequence of Proposition 3.1, and allows one to construct a probability measure Q^{R^*}, being singular to the original probability measure P and such that ruin occurs Q^{R^*}-a.s.

Proposition 4.2. *Let $r \in \mathbb{R}_+$ and L_{**} be as in Proposition 3.1. If $\sup_{\theta \in L_{**}^c} R(\theta) =: R^*$ exists in Υ and $\mathbb{E}_P\left[e^{R^*\cdot X_1}\right] < \infty$, then for any pair (γ, ξ) as in Proposition 3.1 there exist a unique probability measure Q^{R^*} determined by condition (RRM_ξ) and a rcp $\{Q_\theta^{R^*}\}_{\theta\in D}$ of Q^{R^*} over $Q_\Theta^{R^*}$ consistent with Θ satisfying condition (RRM_θ) for any $\theta \notin L_{**}$, and such that for any $u > 0$ the probabilities of ruin $\psi^{Q_\theta^{R^*}}(u)$ and $\psi^{Q^{R^*}}(u)$ with respect to $Q_\theta^{R^*}$ and Q^{R^*}, respectively, are equal to 1.*

Proof. Fix an arbitrary $\theta \notin L_{**}$ and assume that $R^* \in \Upsilon$ and $\mathbb{E}_P\left[e^{R^*\cdot X_1}\right] < \infty$. By Proposition 3.1 it follows that there exist a unique probability measure $Q^{R^*} \in \mathcal{M}_{S,\mathbf{\Lambda}(\rho_{R^*}(\Theta))}$ determined by condition (RRM_ξ) and a rcp $\{Q_\theta^{R^*}\}_{\theta\in D}$ of Q^{R^*} over $Q_\Theta^{R^*}$ consistent with Θ satisfying conditions $Q_\theta^{R^*} \in \mathcal{M}_{S,\mathbf{\Lambda}(\rho_{R^*}(\theta))}$ and (RRM_θ). Because $\kappa_\theta''(r) > 0$ by, e.g., [27], p. 133, it follows that the function κ_θ is strictly convex, or equivalently that κ_θ' is strictly increasing. Thus, since by, e.g., [27], p. 133, conditions $\kappa_\theta(0) = \kappa_\theta(R(\theta)) = 0$ and $\kappa_\theta'(0) < 0$ are valid, it follows that there exists a point $r_0 \in (0, R(\theta))$ such that $\kappa_\theta'(r_0) = 0$; hence $\kappa_\theta'(r) > 0$ for any $r > r_0$. Because $r_0 < R(\theta) \leq R^*$ one deduces that $\kappa_\theta'(R^*) > 0$. The latter, along with Lemma 3.1, yields that

$$0 < \frac{\mathbb{E}_{Q_\theta^{R^*}}[X_1]}{\mathbb{E}_{Q_\theta^{R^*}}[W_1]} - c(\theta) \Longleftrightarrow c(\theta) < \frac{\mathbb{E}_{Q_\theta^{R^*}}[X_1]}{\mathbb{E}_{Q_\theta^{R^*}}[W_1]},$$

implying that the net profit condition is violated with respect to $Q_\theta^{R^*}$; hence by [28], Corollary 7.1.4, one has

$$\psi^{Q_\theta^{R^*}}(u) = Q_\theta^{R^*}(\{\inf_{t\in\mathbb{R}_+} r_t^u(\theta) < 0\}) = 1 \quad \text{for any } \ u > 0,$$

implying along with [31], Remark 3.4(b), that

$$\psi^{Q^{R^*}}(u) = Q^{R^*}(\{\inf_{t\in\mathbb{R}_+} R_t^u < 0\}) = \int_D \psi^{Q_\theta^{R^*}}(u)\, Q_\Theta^{R^*}(d\theta) = 1 \quad \text{for any } \ u > 0,$$

completing the whole proof. □

In the following example, the assumptions $R^* < \infty$ and $\mathbb{E}_P\big[e^{R^* X_1}\big] < \infty$ of Proposition 4.2 hold.

Example 4.2. Assume that S is a P-CMPP(Θ) such that $P_{X_1} = \mathbf{Exp}(\eta)$, $\eta \in \Upsilon$, and $P_\Theta = \mathbf{Beta}(a, b)$, $a, b \in (0, \infty)$. According to [31], Proposition 3.3, there exists a P_Θ-null set $L_P \in \mathfrak{B}((0, 1))$ such that S is a P_θ-CPP(θ) for any $\theta \notin L_P$. Fix an arbitrary $\theta \notin L_P$ and assume that $c(\theta) = \frac{2\cdot\theta}{\eta\cdot(1+\theta)}$. By [28], Theorem 7.4.5, it follows that $R(\theta) = \eta - \frac{\theta}{c(\theta)} = \frac{\eta\cdot(1-\theta)}{2} \in (0, \eta)$ is an adjustment coefficient with respect to P_θ. Note that $R^* \in \Upsilon$, since $\sup_{\theta\in L_P^c} R(\theta) = \sup_{\theta\in L_P^c} \frac{\eta\cdot(1-\theta)}{2} = \frac{\eta}{2}$. Furthermore, $\mathbb{E}_P\big[e^{R^* X_1}\big] = \frac{\eta}{\eta - R^*} < \infty$.

Theorem 4.1. *Let (γ, ξ) be as in Proposition 3.1, $u \in \Upsilon$ and $\theta \notin L_{**}$. Under the assumptions of Proposition 4.2 the following hold:*

(i) $\psi_\theta(u) = \mathbb{E}_{Q_\theta^{R^*}}\big[e^{R^* r_\tau^u(\theta) + \kappa_\theta(R^*)\cdot\tau}\big] \cdot e^{-R^* u}$;

(ii) $\psi(u) = \mathbb{E}_{Q^{R^*}}\left[\frac{e^{R^*\cdot R_\tau^u(\Theta)+\kappa_\Theta(R^*)\cdot\tau}}{\xi(\Theta)}\right] \cdot e^{-R^*\cdot u}$.

Proof. Fix an arbitrary $u \in \Upsilon$.

Ad (i): Let $\theta \notin L_{**}$ be arbitrary but fixed. Since by Proposition 3.1 the family $\widetilde{M}^{(\gamma,R^*)}(\theta)$ is P_θ-a.s. positive martingale and τ is a stopping time for $\mathcal{F}$, one may apply [27], Lemma 8.1, to get

$$\begin{aligned}\psi_\theta(u) &= \int_{\{\tau<\infty\}} \frac{1}{\widetilde{M}_\tau^{(\gamma,R^*)}(\theta)}\, dQ_\theta^{R^*} \\ &= \mathbb{E}_{Q_\theta^{R^*}}\left[\chi_{\{\tau<\infty\}} \cdot e^{R^*\cdot r_\tau^u(\theta)+\kappa_\theta(R^*)\cdot\tau}\right] \cdot e^{-R^*\cdot u},\end{aligned}$$

where the second equality follows from condition (3) for $r = R^*$ and the fact that $J_\tau = 0$. Because the probability of ruin with respect to $Q_\theta^{R^*}$ is equal to 1, by Proposition 4.2, the previous condition yields

$$\psi_\theta(u) = \mathbb{E}_{Q_\theta^{R^*}}\left[e^{R^*\cdot r_\tau^u(\theta)+\kappa_\theta(R^*)\cdot\tau}\right] \cdot e^{-R^*\cdot u},$$

that is assertion (i) holds true.

Ad (ii): Assertion (i) together with [31], Remark 3.4(b), implies

$$\begin{aligned}\psi(u) &= \int \psi_\theta(u)\, P_\Theta(d\theta)\\ &= \int \mathbb{E}_{Q_\theta^{R^*}}\left[\frac{e^{R^*\cdot r_\tau^u(\theta)+\kappa_\theta(R^*)\cdot\tau}}{\xi(\theta)}\right] Q_\Theta^{R^*}(d\theta)\cdot e^{-R^*\cdot u}\\ &= \int \mathbb{E}_{Q^{R^*}}\left[\frac{e^{R^*\cdot R_\tau^u(\Theta)+\kappa_\Theta(R^*)\cdot\tau}}{\xi(\Theta)} \mid \Theta\right] dQ^{R^*}\cdot e^{-R^*\cdot u},\end{aligned}$$

where the last equality follows from [15], Proposition 3.8; hence

$$\psi(u) = \mathbb{E}_{Q^{R^*}}\left[\frac{e^{R^*\cdot R_\tau^u(\Theta)+\kappa_\Theta(R^*)\cdot\tau}}{\xi(\Theta)}\right]\cdot e^{-R^*\cdot u},$$

that is assertion (ii) holds true. □

In the example that follows, an explicit formula for the probability of ruin in the interesting case of constant premiums is obtained by applying Theorem 4.1.

Example 4.3. Let $r \in \mathbb{R}_+$ be as in Proposition 3.1, fix an arbitrary $u > 0$, assume that $c(\theta) = c \in \Upsilon$ for any $\theta \in D$ and put $G := \left\{\theta \in D : c > \mathbb{E}_{P_\theta}[X_1]/\mathbb{E}_{P_\theta}[W_1]\right\}$. Since the net profit condition holds true for any $\theta \in L_{**}^c \cap G$ there exists an adjustment coefficient $R(\theta) \in \Upsilon$ with respect to P_θ for any $\theta \in L_{**}^c \cap G$. Assume that $R^* = \sup_{\theta\in L_{**}^c\cap G} R(\theta) \in \Upsilon$ and $\mathbb{E}_P[e^{R^*\cdot X_1}] < \infty$, and let (γ, ξ) be a pair of functions as in Proposition 3.1. It then follows as in Proposition 4.2, that there exist a unique probability measure Q^{R^*} determined by condition (RRM_ξ) and a rcp $\{Q_\theta^{R^*}\}_{\theta\in D}$ of Q^{R^*} over $Q_\Theta^{R^*}$ consistent with Θ satisfying condition (RRM_θ) for any $\theta \notin L_{**}$, and such that ruin occurs $Q_\theta^{R^*}$-a.s. for any $\theta \in L_{**}^c \cap G$; hence

$$\begin{aligned}&\psi(u)\\ &= \int \psi_\theta(u)\, P_\Theta(d\theta)\\ &= \int_{L_{**}^c\cap G} \psi_\theta(u)\, P_\Theta(d\theta) + \int_{L_{**}^c\cap G^c} \psi_\theta(u)\, P_\Theta(d\theta)\\ &= \mathbb{E}_{Q^{R^*}}\left[\chi_{\Theta^{-1}(G)}\cdot\frac{e^{R^*\cdot R_\tau^u(\Theta)+\kappa_\Theta(R^*)\cdot\tau}}{\xi(\Theta)}\right]\cdot e^{-R^*\cdot u} + \mathbb{E}_{Q^{R^*}}\left[\chi_{\Theta^{-1}(G^c)}\cdot\frac{1}{\xi(\Theta)}\right],\end{aligned}$$

where the third equality follows by Theorem 4.1 and the fact that $\psi_\theta(u) = 1$ for any $\theta \in L_{**}^c \cap G^c$. It is clear that $\lim_{u\to\infty}\psi(u) = \mathbb{E}_{Q^{R^*}}\left[\chi_{\Theta^{-1}(G^c)}\cdot\frac{1}{\xi(\Theta)}\right] \geq 0$, a condition implying that in the situation of constant premium intensities, there might be cases where no matter how big is the initial capital u, the ruin probability ψ cannot be reduced beyond a certain level. The previous condition also reveals a connection between ψ and the choice of ξ, which clearly implies that a careful selection of ξ must be undertaken.

The following result shows that Theorem 4.1, along with Proposition 3.1, yields upper and lower bounds of the probability of ruin under P.

Corollary 4.2. *In the situation of Theorem 4.1, the following hold true:*

(i) $\psi_\theta(u) \geq \mathbb{E}_{Q_\theta^{R^*}}\left[e^{R^*\cdot r_\tau^u(\theta)}\right] \cdot e^{-R^*\cdot u}$;

(ii) $\psi(u) \geq \mathbb{E}_{Q^{R^*}}\left[\frac{e^{R^*\cdot R_\tau^u(\Theta)}}{\xi(\Theta)}\right] \cdot e^{-R^*\cdot u}$.

In particular, if condition $\mathbb{E}_P\left[e^{R^*\Theta}\right] < \infty$ *holds and if the function* $\xi : D \to \mathbb{R}$ *is defined by means of*

$$\xi(\theta) := \frac{e^{R^*\theta}}{\mathbb{E}_P\left[e^{R^*\Theta}\right]} \quad \text{for any} \quad \theta \in D,$$

then there exist a unique probability measure $\nu^{R^*} \in \mathcal{M}_{S,\mathbf{\Lambda}(\rho(\Theta))}$ *determined by condition* (RRM_ξ) *with* R^* *in the place of* r *and a rcp* $\{\nu_\theta^{R^*}\}_{\theta\in D}$ *of* ν^{R^*} *over* $\nu_\Theta^{R^*}$ *consistent with* Θ *satisfying conditions* $\nu_\theta^{R^*} \in \mathcal{M}_{S,\mathbf{\Lambda}(\rho(\theta))}$ *and* (RRM_θ) *for any* $\theta \notin L_{**}$*, and such that*

$$\psi(u) \leq \mathbb{E}_P\left[e^{R^*\Theta}\right] \cdot \mathbb{E}_{\nu^{R^*}}\left[e^{R^*\cdot R_\tau^u(\Theta)+\kappa_\Theta(R^*)\cdot\tau}\right] \cdot e^{-R^*\cdot u}.$$

Proof. Because $\kappa_\theta(R^*) > 0$ for any $\theta \notin L_{**}$, statements (i) and (ii) follow by statements (i) and (ii) of Theorem 4.1, respectively.

In particular, if condition $\mathbb{E}_P[e^{R^*\Theta}] < \infty$ holds and ξ is defined as above then $\xi \in \mathcal{R}_+(D)$, implying according to Proposition 3.1 that there exist a unique probability measure $\nu^{R^*} \in \mathcal{M}_{S,\mathbf{\Lambda}(\rho(\Theta))}$ determined by condition (RRM_ξ) and a rcp $\{\nu_\theta^{R^*}\}_{\theta\in D}$ of ν^{R^*} over $\nu_\Theta^{R^*}$ consistent with Θ satisfying conditions $\nu_\theta^{R^*} \in \mathcal{M}_{S,\mathbf{\Lambda}(\rho(\theta))}$ and (RRM_θ) for any $\theta \notin L_{**}$. Applying now Theorem 4.1 one has

$$\begin{aligned}\psi(u) &= \mathbb{E}_{\nu^{R^*}}\left[\frac{e^{R^*\cdot R_\tau^u(\Theta)+\kappa_\Theta(R^*)\cdot\tau}}{\xi(\Theta)}\right] \cdot e^{-R^*\cdot u} \\ &\leq \mathbb{E}_P\left[e^{R^*\Theta}\right] \cdot \mathbb{E}_{\nu^{R^*}}\left[e^{R^*\cdot R_\tau^u(\Theta)+\kappa_\Theta(R^*)\cdot\tau}\right] \cdot e^{-R^*\cdot u},\end{aligned}$$

completing in this way the proof. □

It is worth noting that in the Cramér–Lundberg risk model one can construct exponential martingales, and using the stopping theorem one is able to prove upper bounds for the ruin probabilities. However, this technique does not give the opportunity to prove a lower bound. A method to find also lower bounds for the ruin probabilities is the “change of measure technique” for a compound mixed renewal process S developed above.

Remark 4.3. Perhaps the most researched mixed renewal process is the mixed Poisson one. The most common definition encountered in the literature is that of a mixed Poisson process with mixing probability distribution U on $\mathfrak{B}(\Upsilon)$ (cf., e.g., [12], Definition 4.2) (written MPP(U) for short). Since every MPP(Θ) is a MPP(U) with

$U = P_\Theta$ (see [18], Theorem 3.1) all the previous results can be transferred to that case. On the other hand, existing results for the class of MPP(U) (cf., e.g., [12], Subsection 9.2.1, for ruin probabilities and mixed Poisson processes) cannot, in general, be transferred to the case of MPP(Θ), since it is not always possible, given a MPP(U), to construct a positive real-valued random variable Θ such that $P_\Theta = U$. Furthermore, even if one assumes the existence of Θ, it is not in general possible to construct an rcp of P over U consistent with Θ, as the probability measure P may be nonperfect (see [11], Theorem 4).

In the next two examples, an explicit computation for the ruin probabilities of the reserve process with respect to the probability measures P and P_θ is undertaken by applying the change of measures technique of Proposition 3.1 and Theorem 4.1.

Example 4.4. Take $D := (1, 2)$, let Θ be a real-valued random variable on Ω, and assume that $P \in \mathcal{M}_{S,\mathbf{Ga}(\Theta,2)}$, such that $P_{X_1} = \mathbf{Ga}(2, 2)$ and $P_\Theta = \mathbf{U}(1, 2)$. Since conditions (a1) and (a2) hold true, it follows by [31], Proposition 3.3, that there exists a P_Θ-null set $L_P \in \mathfrak{B}((1, 2))$ such that $P_\theta \in \mathcal{M}_{S,\mathbf{Ga}(\theta,2)}$ with $(P_\theta)_{X_1} = P_{X_1}$ for any $\theta \notin L_P$, implying that

$$\frac{\mathbb{E}_{P_\theta}[X_1]}{\mathbb{E}_{P_\theta}[W_1]} = \frac{1}{\frac{2}{\theta}} = \frac{\theta}{2} \quad \text{for any } \theta \notin L_P.$$

Put $c(\theta) := \theta + 1$ for any $\theta \in D$. As a first step the function κ_θ must be explicitly determined. For any $r \in (0, 2)$ and $\theta \notin L_{**}$, applying condition (1) and an easy computation one gets

$$\begin{aligned} &M_{X_1}(r) \cdot (M_\theta)_{W_1}\big(-\kappa_\theta(r) - c(\theta) \cdot r\big) = 1 \\ &\qquad \Longleftrightarrow \left(\frac{2}{2-r}\right)^2 \cdot \left(\frac{\theta}{\theta + \kappa_\theta(r) + c(\theta) \cdot r}\right)^2 = 1 \end{aligned}$$

which is equivalent to

$$\kappa_\theta(r) = \frac{r \cdot (r \cdot \theta + r - \theta - 2)}{2 - r} \tag{5}$$

or

$$\kappa_\theta(r) = \frac{r^2(\theta + 1) - r(\theta + 2) - 4\theta}{2 - r}, \tag{6}$$

respectively. Since condition (NPC$_\theta$) is valid for any $\theta \notin L_{**}$ and $\mathbb{E}_P[e^{r \cdot X_1}] < \infty$ for any $r \in (0, 2)$, it follows by, e.g., [27], p. 133, that there exists an adjustment coefficient $R(\theta)$ with respect to P_θ for any $\theta \notin L_{**}$ being the solution to the equation $\kappa_\theta(r) = 0$ on $(0, 2)$. The latter along with Equations (5) and (6) yields $R(\theta) = \frac{\theta+2}{\theta+1} \in (0, 2)$ and

$$R(\theta) = \frac{\theta + 2 + (17\theta^2 + 20\theta + 4)^{\frac{1}{2}}}{2(\theta + 1)} > 2$$

or

$$R(\theta) = \frac{\theta + 2 - (17\theta^2 + 20\theta + 4)^{\frac{1}{2}}}{2(\theta + 1)} < 0,$$

respectively; hence $R(\theta) = \frac{\theta+2}{\theta+1}$ is the solution to (5) in $(0, 2)$.

But since $R(\theta)$ is a strictly decreasing function of θ it follows that $R^* = \sup_{\theta \in L^c_{**}} R(\theta) = \frac{3}{2} \in (0,2)$, implying that $\mathbb{E}_P[e^{R^* \cdot X_1}] = \frac{2}{2-\frac{3}{2}} = 4 < \infty$ as well as that

$$\kappa_\theta(R^*) = \frac{3}{2} \cdot (\theta - 1) \quad \text{for any } \ \theta \notin L_{**}.$$

Put $\gamma(x) := R^* \cdot x - \ln \mathbb{E}_P[e^{R^* \cdot X_1}]$ for any $x \in \Upsilon$. By Proposition 4.2, for any $\xi \in \mathcal{R}_+(D)$ there exist a unique probability measure $Q^{R^*} \in \mathcal{M}_{S,\mathbf{\Lambda}(\rho(\Theta))}$ determined by condition (RRM_ξ) and a rcp $\{Q^{R^*}_\theta\}_{\theta \in D}$ of Q^{R^*} over $Q^{R^*}_\Theta$ consistent with Θ satisfying conditions $Q^{R^*}_\theta \in \mathcal{M}_{S,\mathbf{\Lambda}(\rho(\theta))}$ and (RRM_θ) for any $\theta \notin L_{**}$, and such that for any $u > 0$ the probabilities of ruin $\psi^{Q^{R^*}_\theta}(u)$ and $\psi^{Q^{R^*}}(u)$ with respect to $Q^{R^*}_\theta$ and Q^{R^*}, respectively, are equal to 1. It then follows by Theorem 4.1 that for any $u > 0$ and $\theta \notin L_{**}$, the ruin probabilities $\psi(u)$ and $\psi_\theta(u)$ satisfy conditions

$$\psi_\theta(u) = \mathbb{E}_{Q^{R^*}_\theta}\left[e^{R^* r^u_\tau(\theta) + \kappa_\theta(R^*) \cdot \tau}\right] \cdot e^{-R^* u} = \mathbb{E}_{Q^{R^*}_\theta}\left[e^{\frac{3}{2} \cdot r^u_\tau(\theta) + \frac{3}{2} \cdot (\theta - 1) \cdot \tau}\right] \cdot e^{-\frac{3}{2} \cdot u}$$

and

$$\psi(u) = \mathbb{E}_{Q^{R^*}}\left[\frac{e^{R^* \cdot R^u_\tau + \kappa_\Theta(R^*) \cdot \tau}}{\xi(\Theta)}\right] \cdot e^{-R^* \cdot u} = \mathbb{E}_{Q^{R^*}}\left[\frac{e^{\frac{3}{2} \cdot R^u_\tau + \frac{3}{2} \cdot (\Theta - 1) \cdot \tau}}{\xi(\Theta)}\right] \cdot e^{-\frac{3}{2} \cdot u}.$$

Example 4.5. Take $D := \Upsilon$, let Θ be a real-valued random variable on Ω, and assume that $P \in \mathcal{M}_{S,\mathbf{Ga}(\Theta,2)}$, such that $P_{X_1} = \mathbf{Ga}(2,2)$ and $P_\Theta = \mathbf{Ga}(b,a)$, where $(b,a) \in \Upsilon^2$. Since conditions (a1) and (a2) hold true, it follows by [31], Proposition 3.3, that there exists a P_Θ-null set $L_P \in \mathfrak{B}(D)$ such that $P_\theta \in \mathcal{M}_{S,\mathbf{Ga}(\theta,2)}$ with $P_{X_1} = (P_\theta)_{X_1}$ for any $\theta \notin L_P$, implying that

$$\frac{\mathbb{E}_{P_\theta}[X_1]}{\mathbb{E}_{P_\theta}[W_1]} = \frac{1}{\frac{2}{\theta}} = \frac{\theta}{2} \quad \text{for any } \ \theta \notin L_P.$$

Put $c(\theta) := \theta$ for any $\theta \in D$. For any $r \in (0,2)$ and $\theta \notin L_{**}$, it follows as in Example 4.4 that there exists an adjustment coefficient $R(\theta) \in (0,2)$ with respect to P_θ being the solution to the equation

$$\kappa_\theta(r) = \frac{r \cdot \theta \cdot (r-1)}{2-r} = 0 \tag{7}$$

for any $\theta \notin L_{**}$. Thus, $R^* = \sup_{\theta \in L^c_{**}} R(\theta) = 1 \in (0,2)$, implying that $\mathbb{E}_P[e^{R^* \cdot X_1}] = \frac{2}{2-1} = 2 < \infty$.

Put $\gamma(x) := R^* \cdot x - \ln \mathbb{E}_P[e^{R^* \cdot X_1}]$ for any $x \in \Upsilon$. By Proposition 4.2 for any $\xi \in \mathcal{R}_+(D)$ there exist a unique probability measure $Q^{R^*} \in \mathcal{M}_{S,\mathbf{\Lambda}(\rho(\Theta))}$ determined by condition (RRM_ξ) and a rcp $\{Q^{R^*}_\theta\}_{\theta \in D}$ of Q^{R^*} over $Q^{R^*}_\Theta$ consistent with Θ satisfying conditions $Q^{R^*}_\theta \in \mathcal{M}_{S,\mathbf{\Lambda}(\rho(\theta))}$ and (RRM_θ) for any $\theta \notin L_{**}$, and such that for any $u > 0$ the probabilities of ruin $\psi^{Q^{R^*}_\theta}(u)$ and $\psi^{Q^{R^*}}(u)$ with respect to $Q^{R^*}_\theta$ and Q^{R^*}, respectively, are equal to 1. It then follows by Theorem 4.1 that for any $u > 0$ and $\theta \notin L_{**}$, the ruin probabilities $\psi(u)$ and $\psi_\theta(u)$ satisfy conditions

$$\psi_\theta(u) = \mathbb{E}_{Q^{R^*}_\theta}\left[e^{R^* r^u_\tau(\theta) + \kappa_\theta(R^*) \cdot \tau}\right] \cdot e^{-R^* u} = \mathbb{E}_{Q^{R^*}_\theta}\left[e^{r^u_\tau(\theta)}\right] \cdot e^{-u} \leq e^{-u}$$

and

$$\psi(u) = \mathbb{E}_{Q^{R^*}}\left[\frac{e^{R^*\cdot R_\tau^u + \kappa_\Theta(R^*)\cdot\tau}}{\xi(\Theta)}\right]\cdot e^{-R^*\cdot u} = \mathbb{E}_{Q^{R^*}}\left[\frac{e^{R_\tau^u}}{\xi(\Theta)}\right]\cdot e^{-u} \le e^{-u},$$

where the inequalities follow by Proposition 4.2.

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References

[1] Albrecher, H., Constantinescu, C., Loisel, S.: Explicit ruin formulas for models with dependence among risks. Insur. Math. Econ. **48**, 265–270 (2011) MR2799308. https://doi.org/10.1016/j.insmatheco.2010.11.007

[2] Andersen, E.S.: On the collective theory of risk in the case of contagion between the claims. In: Transactions XVth International Congress of Actuaries, vol. 2, pp. 219–229 (1957)

[3] Asmussen, S.: Busy period analysis, rare events and transient behaviour in fluid models. J. Appl. Math. Stoch. Anal. **7**, 269–299 (1994) MR1301702. https://doi.org/10.1155/S1048953394000262

[4] Asmussen, S.: Stationary distribution for fluid models with or without Brownian noise. Stoch. Models **11**, 21–49 (1995) MR1316767. https://doi.org/10.1080/15326349508807330

[5] Asmussen, S., Albrecher, H.: Ruin Probabilities, 2nd edn. World Scientific Publishing, London (2010) MR2766220. https://doi.org/10.1142/9789814282536

[6] Boogaert, P., De Waegenaere, A.: Simulation of ruin probabilities. Stoch. Models **9**, 95–99 (1990) MR1084493. https://doi.org/10.1016/0167-6687(90)90020-E

[7] Cohn, D.L.: Measure Theory, 2nd edn. Birkhäuser Advanced Texts (2013) MR3098996. https://doi.org/10.1007/978-1-4614-6956-8

[8] Cook, R.J., Ng, E.T.M., Mukherjee, J., Vaughan, D.: Two-state mixed renewal processes for chronic disease. Stat. Med. **18**, 175–188 (1999)

[9] Dassios, A., Embrechts, P.: Martingales and insurance risk. Stoch. Models **5**, 181–217 (1989) MR1000630. https://doi.org/10.1080/15326348908807105

[10] Delbaen, F., Haezendock, J.: A martingale approach to premium calculation principles in an arbitrage free market. Insur. Math. Econ. **8**, 269–277 (1989) MR1029895. https://doi.org/10.1016/0167-6687(89)90002-4

[11] Faden, A.M.: The existence of regular conditional probabilities: Necessary and sufficient conditions. Ann. Probab. **13**, 288–298 (1985) MR0770643

[12] Grandell, J.: Mixed Poisson Processes. Chapman & Hall (1997) MR1463943. https://doi.org/10.1007/978-1-4899-3117-7

[13] Gut, A.: Stopped Random Walks: Limit Theorems and Applications, 2nd edn. Springer, New York (2009) MR2489436. https://doi.org/10.1007/978-0-387-87835-5

[14] Huang, W.J.: On the characterization of point processes with the exchangeable and Markov properties. Sankhya, Ser. A **52**, 16–27 (1990) MR1176273

[15] Lyberopoulos, D.P., Macheras, N.D.: Some characterizations of mixed Poisson processes. Sankhya, Ser. A **74**, 57–79 (2012) MR3010292. https://doi.org/10.1007/s13171-012-0011-y

[16] Lyberopoulos, D.P., Macheras, N.D.: A characterization of martingale-equivalent compound mixed Poisson processes. ArXiv:1905.07629, 1–28 (2019)

[17] Lyberopoulos, D.P., Macheras, N.D.: Some characterizations of mixed renewal processes. Math. Slovaca, to appear, 1–28 (2021)

[18] Lyberopoulos, D.P., Macheras, N.D., Tzaninis, S.M.: On the equivalence of various definitions of mixed Poisson processes. Math. Slovaca **69**, 453–468 (2019) MR3925926. https://doi.org/10.1515/ms-2017-0238

[19] Macheras, N.D., Tzaninis, S.M.: Some characterizations for Markov processes as mixed renewal processes. Math. Slovaca **68**, 1477–1494 (2018) MR3881559. https://doi.org/10.1515/ms-2017-0196

[20] Macheras, N.D., Tzaninis, S.M.: A characterization of equivalent martingale measures in a renewal risk model with applications to premium calculation principles. Mod. Stoch. Theory Appl. **7**, 43–60 (2020) MR4085675. https://doi.org/10.15559/20-vmsta148

[21] Palmowski, Z., Rolski, T.: A note on martingale inequalities for fluid models. Stat. Probab. Lett. **31**, 13–21 (1996) MR1421764. https://doi.org/10.1016/S0167-7152(96)00007-7

[22] Palmowski, Z., Rolski, T.: Superposition of alternating on-off flows and a fluid model. Ann. Appl. Probab. **8**, 524–541 (1998) MR1624957. https://doi.org/10.1214/aoap/1028903537

[23] Ridder, A.: Fast simulation of Markov fluid models. J. Appl. Probab. **33**, 786–804 (1996) MR1401475. https://doi.org/10.1017/s002190020010021x

[24] Rolski, T., Schmidli, H., Schmidt, V., Teugels, J.L.: Stochastic Processes for Insurance and Finance. Wiley, Chichester (1999) MR1680267. https://doi.org/10.1002/9780470317044

[25] Schmidli, H.: Lundberg inequalities for a Cox model with a piecewise constant intensity. J. Appl. Probab. **33**, 196–210 (1996) MR1371967. https://doi.org/10.1017/s0021900200103857

[26] Schmidli, H.: An extension to the renewal theorem and an application to risk theory. Ann. Appl. Probab. **7**, 121–133 (1997) MR1428752. https://doi.org/10.1214/aoap/1034625255

[27] Schmidli, H.: Risk Theory. Springer (2017) MR3753610. https://doi.org/10.1007/978-3-319-72005-0

[28] Schmidt, K.D.: Lectures on Risk Theory. B.G. Teubner, Stuttgart (1996) MR1402016. https://doi.org/10.1007/978-3-322-90570-3

[29] Segerdahl, C.-G.: Stochastic processes and practical working models or why is the Pólya process approach defective in modern practice and how to cope with its deficiencies? Scand. Actuar. J. **3–4**, 146–166 (1970) MR0350918. https://doi.org/10.1080/03461238.1970.10405661

[30] Tzaninis, S.M., Macheras, N.D.: A characterization of equivalent martingale probability measures in a mixed renewal risk model with applications in Risk Theory. ArXiv:2007.09051, 1–23 (2020)

[31] Tzaninis, S.M., Macheras, N.D.: A characterization of progressively equivalent probability measures preserving the structure of a compound mixed renewal process. ArXiv:2007.05289, 1–25 (2020)