Modern Stochastics: Theory and Applications 9 (2) (2022) 123–137
https://doi.org/10.15559/21-VMSTA195

Averaging principle for the one-dimensional parabolic equation driven by stochastic measure

Boris Manikin

Taras Shevchenko National University of Kyiv, Kyiv, Ukraine

bmanikin@gmail.com (B. Manikin)

Received: 2 June 2021, Revised: 27 November 2021, Accepted: 27 November 2021, Published online: 10 January 2022

Abstract A stochastic parabolic equation on $[0, T] \times \mathbb{R}$ driven by a general stochastic measure is considered. The averaging principle for the equation is established. The convergence rate is compared with other results on related topics.

Keywords Stochastic measure, averaging principle, mild solution, stochastic parabolic equation

2010 MSC 60G57

1 Introduction

In this paper we establish the averaging principle for the stochastic parabolic equation

$$\begin{cases} \mathcal{L}u_\varepsilon(t,x)dt + f(t/\varepsilon, x, u_\varepsilon(t,x))\,dt + \sigma(t/\varepsilon, x)\,d\mu(x) = 0\,, \\ u_\varepsilon(0,x) = u_0(x)\,, \end{cases} \tag{1}$$

where ε is a small positive parameter, $(t, x) \in [0, T] \times \mathbb{R}$, μ is a general stochastic measure on Borel σ-algebra on $\mathbb{R}$ (see Section 2), f, σ are measurable functions, $\mathcal{L}$ is the operator of the form

$$\mathcal{L}u(t,x) = a(t)\frac{\partial^2 u(t,x)}{\partial^2 x} + b(t)\frac{\partial u(t,x)}{\partial x} + c(t)u(t,x) - \frac{\partial u(t,x)}{\partial t}. \tag{2}$$

Here a, b, c are defined on $[0, T]$. We consider convergence $u_\varepsilon(t,x) \to \bar{u}(t,x)$, $\varepsilon \to 0^+$, where $\bar{u}$ is the solution of the averaged equation

$$\begin{cases} \mathcal{L}\bar{u}(t,x)dt + \bar{f}(x, \bar{u}(t,x))\,dt + \bar{\sigma}(x)\,d\mu(x) = 0\,, \\ \bar{u}(0,x) = u_0(x)\,. \end{cases} \tag{3}$$

Functions $\bar{f}, \bar{\sigma}$ are defined below. Note that we consider solutions to the formal equations (1) and (3) in the mild form.

Averaging is widely used to describe the asymptotic behaviour of both stochastic and deterministic systems. Stochastic parabolic equation with the random noise represented by a general stochastic measure was introduced in [1]. The averaging principle for two-time-scales system driven by two independent Wiener processes was studied, for example, in [7]. Some other equations driven by Wiener process are considered in [4, 6] and [20]. Different types of equations with general stochastic measures are investigated in [2, 3, 14, 19] and [16].

The rest of the paper is organized as follows. Section 2 contains the basic facts concerning stochastic measures and integrals with respect to them. Section 3 contains precise formulation of the problem, assumptions, auxiliary statements while the main result is proved in Section 4. Some examples are given in Section 5.

2 Preliminaries

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a complete probability space and $\mathcal{B}$ be a Borel σ-algebra on $\mathbb{R}$. Denote the set of all real-valued random variables defined on $(\Omega, \mathcal{F}, \mathsf{P})$ as $\mathsf{L}_0 = \mathsf{L}_0(\Omega, \mathcal{F}, \mathsf{P})$, convergence in L_0 means the convergence in probability.

Definition 1. A σ-additive mapping $\mu : \mathcal{B} \to \mathsf{L}_0$ is called *stochastic measure* (SM).

In other words, μ is a vector measure with values in L_0. We do not assume any martingale properties or moment existence for SM.

Consider some examples of SMs. If M_t is a square integrable martingale then $\mu(A) = \int_0^T \mathbf{1}_A(t)\,dM_t$ is an SM. α-stable random measure on $\mathcal{B}$ for $\alpha \in (0,1) \cup (1,2]$, as it is defined in [18, Sections 3.2–3.3], is an SM by Definition 1. For a fractional Brownian motion W_t^H with Hurst index $H > 1/2$ and a bounded measurable function $f : [0,T] \to \mathbb{R}$ we can define an SM $\mu(A) = \int_0^T f(t)\mathbf{1}_A(t)\,dW_t^H$, see [11, Theorem 1.1]. Some other examples can be found in [14].

In [10, Chapter 7] the definition of the integral $\int_A g\,d\mu$, where $g : \mathbb{R} \to \mathbb{R}$ is a deterministic measurable function, $A \in \mathcal{B}$ and μ is an SM, is given and its properties are studied. In particular, every bounded measurable g is integrable with respect to (w.r.t.) any μ. This integral was constructed and studied in [10] for μ defined on an arbitrary σ-algebra, but in our paper, we consider SM on Borel subsets of $\mathbb{R}$.

In the sequel, μ denotes a SM, C and $C(\omega)$ denote positive constant and positive random constant, respectively, whose exact values are not important ($C < \infty$, $C(\omega) < \infty$ a.s.).

We will use the following statement.

Lemma 1 (Lemma 3.1 in [12]). *Let $\phi_l : \mathbb{R} \to \mathbb{R}$, $l \geq 1$, be measurable functions such that $\tilde{\phi}(x) = \sum_{l=1}^{\infty} |\phi_l(x)|$ is integrable w.r.t. μ on $\mathbb{R}$. Then*

$$\sum_{l=1}^{\infty} \Big(\int_{\mathbb{R}} \phi_l \, d\mu \Big)^2 < \infty \quad a.s.$$

We consider the *Besov spaces* $B_{22}^{\alpha}([c,d])$, $0 < \alpha < 1$, with a standard norm

$$\|g\|_{B_{22}^{\alpha}([c,d])} = \|g\|_{\mathsf{L}_2([c,d])} + \Big(\int_0^{d-c} (w_2(g,r))^2 r^{-2\alpha-1} \, dr \Big)^{1/2},$$

where

$$w_2(g,r) = \sup_{0 \leq h \leq r} \Big(\int_c^{d-h} |g(y+h) - g(y)|^2 \, dy \Big)^{1/2}.$$

For any $j \in \mathbb{Z}$ and all $n \geq 0$, put

$$d_{kn}^{(j)} = j + k2^{-n}, \quad 0 \leq k \leq 2^n, \quad \Delta_{kn}^{(j)} = (d_{(k-1)n}^{(j)}, d_{kn}^{(j)}], \quad 1 \leq k \leq 2^n .$$

The following lemma is a key tool for estimates of the stochastic integral.

Lemma 2 (Lemma 3 in [13]). *Let Z be an arbitrary set, and the function $q(z,s) : Z \times [j, j+1] \to \mathbb{R}$ is such that all paths $q(z,\cdot)$ are continuous on $[j, j+1]$. Denote*

$$q_n(z,s) = \sum_{1 \leq k \leq 2^n} q\big(z, d_{(k-1)n}^{(j)}\big) \mathbf{1}_{\Delta_{kn}^{(j)}}(s).$$

Then the random function

$$\eta(z) = \int_{(j,j+1]} q(z,s) \, d\mu(s), \; z \in Z,$$

has a version

$$\begin{aligned} \widetilde{\eta}(z) = &\int_{(j,j+1]} q_0(z,s) \, d\mu(s) \\ &+ \sum_{n \geq 1} \Big(\int_{(j,j+1]} q_n(z,s) \, d\mu(s) - \int_{(j,j+1]} q_{n-1}(z,s) \, d\mu(s) \Big) \end{aligned} \tag{4}$$

such that for all $\beta > 0$, $\omega \in \Omega$, $z \in Z$

$$\begin{aligned} |\widetilde{\eta}(z)| \leq |q(z,j)\mu((j,j+1])| + &\Big\{ \sum_{n \geq 1} 2^{n\beta} \sum_{1 \leq k \leq 2^n} |q(z, d_{kn}^{(j)}) - q(z, d_{(k-1)n}^{(j)})|^2 \Big\}^{1/2} \\ &\times \Big\{ \sum_{n \geq 1} 2^{-n\beta} \sum_{1 \leq k \leq 2^n} |\mu(\Delta_{kn}^{(j)})|^2 \Big\}^{1/2}. \end{aligned} \tag{5}$$

Theorem 1.1 [9] implies that for $\alpha = (\beta + 1)/2$,

$$\Big\{ \sum_{n \geq 1} 2^{n\beta} \sum_{1 \leq k \leq 2^n} |q\big(z, d_{kn}^{(j)}\big) - q\big(z, d_{(k-1)n}^{(j)}\big)|^2 \Big\}^{1/2} \leq C \|q(z,\cdot)\|_{B_{22}^{\alpha}([j,j+1])}. \tag{6}$$

From Lemma 1 it follows that for each $\beta > 0$, $j \in \mathbb{Z}$

$$\sum_{n \geq 1} 2^{-n\beta} \sum_{1 \leq k \leq 2^n} |\mu(\Delta_{kn}^{(j)})|^2 < +\infty \quad \text{a.s.}$$

3 Formulation of the problem and auxiliary lemmas

We consider the mild solutions to (1), i.e. the measurable random functions $u_\varepsilon(t,x) = u_\varepsilon(t,x,\omega) : [0,T] \times \mathbb{R} \times \Omega \to \mathbb{R}$ such that the equations

$$u_\varepsilon(t,x) = \int_{\mathbb{R}} p(t,x-y;0)u_0(y)\,dy + \int_0^t ds \int_{\mathbb{R}} p(t,x-y;s) f(s/\varepsilon, y, u_\varepsilon(s,y))\,dy \\ + \int_{\mathbb{R}} d\mu(y) \int_0^t p(t,x-y;s)\sigma(s/\varepsilon,y)\,ds \tag{7}$$

hold a.s. for each $(t,x) \in [0,T] \times \mathbb{R}$. Here p is the fundamental solution of operator $\mathcal{L}$ from (2).

We will refer to the following assumptions on f, σ, u_0.

Assumption E1. $u_0 : \mathbb{R} \times \Omega \to \mathbb{R}$ is measurable and for all $y, y_1, y_2 \in \mathbb{R}$

$$|u_0(y,\omega)| \le C(\omega), \quad |u_0(y_1,\omega) - u_0(y_2,\omega)| \le L_{u_0}(\omega)|y_1 - y_2|^{\beta(u_0)},$$

where $C(\omega)$, $L_{u_0}(\omega)$ are random constants, $\beta(u_0) \ge 1/2$.

Assumption E2. $f : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is measurable, bounded, and

$$|f(s,y_1,v_1) - f(s,y_2,v_2)| \le L_f\big(|y_1 - y_2| + |v_1 - v_2|\big),$$

for some constant L_f and all $s \in \mathbb{R}_+$, $y_1, y_2, v_1, v_2 \in \mathbb{R}$.

Assumption E3. $\sigma : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is measurable, bounded, and

$$|\sigma(s,y_1) - \sigma(s,y_2)| \le L_\sigma |y_1 - y_2|^{\beta(\sigma)}$$

for some constants L_σ, $1/2 < \beta(\sigma) < 1$, and all $s \in \mathbb{R}_+$, $y_1, y_2 \in \mathbb{R}$.

Assumption E4. There exist the limits

$$\bar{f}(y,v) = \lim_{t\to\infty} \frac{1}{t}\int_0^t f(s,y,v)\,ds, \quad \bar{\sigma}(y) = \lim_{t\to\infty} \frac{1}{t}\int_0^t \sigma(s,y)\,ds.$$

Note that if f, σ satisfy the conditions E2 and E3, respectively, $\bar{f}$ and $\bar{\sigma}$ satisfy them as well. We will show that for $\bar{f}$; the proof for $\bar{\sigma}$ is analogous. $\bar{f}$ is measurable as a limit of measurable functions, its boundedness is obvious while Lipschitz condition follows from the inequalities

$$\big|\bar{f}(y_1,v_1) - \bar{f}(y_2,v_2)\big| = \lim_{t\to\infty}\Big|\frac{1}{t}\int_0^t \big(f(s,y_1,v_1) - f(s,y_2,v_2)\big)ds\Big| \\ \le \limsup_{t\to\infty} \frac{1}{t}\int_0^t \big|f(s,y_1,v_1) - f(s,y_2,v_2)\big|ds \le L_f\big(|y_1 - y_2| + |v_1 - v_2|\big).$$

Therefore, functions

$$H_f(r,y,v) = f(r,y,v) - \bar{f}(y,v), \\ H_\sigma(r,y) = \sigma(r,y) - \bar{\sigma}(y), \quad r \in \mathbb{R}_+, \; y, v \in \mathbb{R}$$

are bounded.

Assumption E5. Functions

$$G_f(r, y, v) = \int_0^r (f(s, y, v) - \bar{f}(y, v))\, ds,$$
$$G_\sigma(r, y) = \int_0^r (\sigma(s, y) - \bar{\sigma}(y))\, ds, \quad r \in \mathbb{R}_+, \; y, v \in \mathbb{R}$$

are bounded.

Assertion E5 holds, for example, if $f(s, y, v)$ and $\sigma(s, y)$ are periodic in s for each y, v, and the set of values of minimal period is bounded.

Assumption L. Functions $a(t)$, $b(t)$, $c(t)$ are continuous in $[0, T]$, and for some positive constants β, L, δ the following inequalities hold in $[0, T]$

$$|a(t_1) - a(t_2)| \le L\, |t_1 - t_2|^\beta, \quad a(t) \ge \delta.$$

[8, section 4, Theorem 1] shows that under assumption L the fundamental solution exists and

$$|p(t, x - y; s)| \le M(t - s)^{-\frac{1}{2}} e^{-\frac{\lambda |x-y|^2}{t-s}}, \tag{8}$$
$$\left|\frac{\partial p(t, x - y; s)}{\partial y}\right| \le M(t - s)^{-1} e^{-\frac{\lambda |x-y|^2}{t-s}}, \tag{9}$$
$$\left|\frac{\partial^2 p(t, x - y; s)}{\partial y^2}\right| \le M(t - s)^{-3/2} e^{-\frac{\lambda |x-y|^2}{t-s}}, \tag{10}$$
$$\left|\frac{\partial p(t, x - y; s)}{\partial t}\right| \le M(t - s)^{-3/2} e^{-\frac{\lambda |x-y|^2}{t-s}}, \tag{11}$$

where λ and M are positive constants. Moreover, $p(t, x - y; s)$ satisfies the equation

$$a(s)\frac{\partial^2 p}{\partial y^2} - b(s)\frac{\partial p}{\partial y} + c(s)p + \frac{\partial p}{\partial s} = 0$$

(see [8, (4.17)]). Using boundedness of a, b, c and estimates (8)–(10), we obtain that

$$\left|\frac{\partial p(t, x - y; s)}{\partial s}\right| \le |c(s)||p(t, x - y; s)| + |b(s)|\left|\frac{\partial p(t, x - y; s)}{\partial y}\right|$$
$$+|a(s)|\left|\frac{\partial^2 p(t, x - y; s)}{\partial y^2}\right| \le M(t - s)^{-3/2} e^{-\frac{\lambda |x-y|^2}{t-s}} \tag{12}$$

(without loss of generality we can say that constant M in (12) is the same as in (8)–(11)). Let the stochastic measure μ satisfy the following condition.

Assumption M. $|y|^\rho$ is integrable w.r.t. μ on $\mathbb{R}$ for some $\rho > 1/2$.

Consider the mild solution of (3):

$$\bar{u}(t, x) = \int_{\mathbb{R}} p(t, x - y; 0) u_0(y)\, dy + \int_0^t ds \int_{\mathbb{R}} p(t, x - y; s) \bar{f}(y, \bar{u}(s, y))\, dy$$
$$+ \int_{\mathbb{R}} d\mu(y) \int_0^t p(t, x - y; s) \bar{\sigma}(y)\, ds\,. \tag{13}$$

According to [1, Theorem], fulfillment of the conditions E1–E3, L and M imply that solutions of (7), (13) exist, are unique and have Hölder continuous versions on $[\tau, T] \times [-K, K]$ for each $\tau, K > 0$. Therefore, u_ε and $\bar{u}$ have continuous versions on $(0, T] \times \mathbb{R}$.

Following auxiliary lemmas are the analogues of [16, Lemma 4.1.–4.3.].

Lemma 3. *Let E3, L and M hold. Then for version* (4) *of*

$$\vartheta(x,t) = \int_{\mathbb{R}} d\mu(y) \int_0^t p(t, x-y; s)\sigma(s,y)\, ds, \quad t \in [0,T],$$

for any $\gamma < 1/4$ there exists a random constant $C(\omega) < \infty$ a.s. (that depends on γ, is independent of x) such that

$$|\vartheta(x,t_1) - \vartheta(x,t_2)| \le C(\omega)|t_1 - t_2|^\gamma \tag{14}$$

for all $t_1, t_2 \in [0,T]$, $x \in \mathbb{R}$.

Lemma 4. *Let Assumptions E1–E3, L, M hold, and u be a solution of equation*

$$\begin{aligned} u(t,x) &= \int_{\mathbb{R}} p(t, x-y; 0)u_0(y)\, dy \\ &+ \int_0^t ds \int_{\mathbb{R}} p(t, x-y; s) f(s, y, u(s,y))\, dy \\ &+ \int_{\mathbb{R}} d\mu(y) \int_0^t p(t, x-y; s)\sigma(s,y)\, ds\,. \end{aligned}$$

Then for the continuous version of u, each $0 < \gamma < 1/4$, some $C(\omega)$, and all $0 < t_1 < t_2 \le T$, $x \in \mathbb{R}$, it holds

$$\begin{aligned} &|u(t_1,x) - u(t_2,x)| \le C(\omega)(\ln t_2 - \ln t_1 \\ &+ t_2 \ln t_2 - t_1 \ln t_1 - (t_2 - t_1)\ln(t_2 - t_1) + (t_2 - t_1)^\gamma). \end{aligned}$$

These lemmas are proved similarly to corresponding lemmas in [16]. We refer to (9), (11) instead of [16, (3.2)] and use (8) to prove the analogues of [16, (4.2), (4.3)].

Lemma 5. *Let $h(r,y,z) : \mathbb{R}_+ \times \mathbb{R} \times Z \to \mathbb{R}$, $\bar{h}(y,z) : \mathbb{R} \times Z \to \mathbb{R}$ be measurable for each fixed z, the functions*

$$H(r,y,z) = h(r,y,z) - \bar{h}(y,z), G(r,y,z) = \int_0^r (h(v,y,z) - \bar{h}(y,z))dv$$

be bounded on $\mathbb{R}_+ \times \mathbb{R} \times Z$, the functions φ_1, φ_2 be measurable on $[0,T]$ and the inequality $0 \le \varphi_1(t) \le \varphi_2(t) \le t$ hold on $[0,T]$. Then

$$\Big| \int_{\mathbb{R}} dy \int_{\varphi_1(t)}^{\varphi_2(t)} p(t, x-y; s) H(s/\varepsilon, y, z) ds \Big| \le C\varepsilon |\ln \varepsilon|, \tag{15}$$

for all $x \in \mathbb{R}$, $t \in [0,T]$, $\varepsilon > 0$, where the constant C does not depend on φ_1, φ_2.

Proof. Assume that $\varphi_2(t) - \varphi_1(t) \geq \varepsilon$. In this case we can rewrite the inner integral as following: $\int_{\varphi_1(t)}^{\varphi_2(t)} = \int_{\varphi_1(t)}^{\varphi_2(t)-\varepsilon} + \int_{\varphi_2(t)-\varepsilon}^{\varphi_2(t)}$. We have that

$$\Big|\int_{\mathbb{R}} dy \int_{\varphi_2(t)-\varepsilon}^{\varphi_2(t)} p(t, x-y; s) H(s/\varepsilon, y, z) ds\Big| \\ \leq C \int_{\varphi_2(t)-\varepsilon}^{\varphi_2(t)} ds \int_{\mathbb{R}} |p(t, x-y; s)| dy \leq C\varepsilon. \tag{16}$$

On the other hand,

$$\Big|\int_{\mathbb{R}} dy \int_{\varphi_1(t)}^{\varphi_2(t)-\varepsilon} p(t, x-y; s) H(s/\varepsilon, y, z) ds\Big| \\ = C\varepsilon \Big| \int_{\mathbb{R}} \Big(p(t, x-y; s) G(s/\varepsilon, y, z)\Big|_{\varphi_1(t)}^{\varphi_2(t)-\varepsilon} \\ - \int_{\varphi_1(t)}^{\varphi_2(t)-\varepsilon} \frac{\partial p(t, x-y; s)}{\partial s} G(s/\varepsilon, y, z) ds \Big) dy \Big| \leq C\varepsilon \Big(\int_{\mathbb{R}} \big| p(t, x-y; \varphi_1(t)) \big| dy \\ + \int_{\mathbb{R}} \big| p(t, x-y; \varphi_2(t)-\varepsilon) \big| dy + \int_{\varphi_1(t)}^{\varphi_2(t)-\varepsilon} ds \int_{\mathbb{R}} \frac{dy}{(t-s)^{3/2}} e^{-\frac{\lambda|x-y|^2}{t-s}} \Big) \\ \leq C\varepsilon + C\varepsilon \int_{\varphi_1(t)}^{\varphi_2(t)-\varepsilon} \frac{ds}{t-s} = C\varepsilon\big(1 - \ln(\varepsilon + t - \varphi_2(t)) + \ln(t - \varphi_1(t))\big) \\ \leq C\varepsilon\big(1 + |\ln T| \vee |\ln \varepsilon|\big) \leq C\varepsilon |\ln \varepsilon|,$$

and we get (15). If $\varphi_2(t) - \varphi_1(t) < \varepsilon$, we obtain (15) similarly to (16). □

The following lemma is an analogue of [15, Lemma 3].

Lemma 6. *Let $h(r, y) : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, $\bar{h}(y) : \mathbb{R} \to \mathbb{R}$ be measurable, the functions*

$$H(r, y) = h(r, y) - \bar{h}(y), \quad G(r, y) = \int_0^r (h(v, y) - \bar{h}(y)) dv$$

be bounded on $\mathbb{R}_+ \times \mathbb{R}$. Then

$$\Big|\int_0^t p(t, x-y; s) H(s/\varepsilon, y) ds\Big| \leq C\sqrt{\varepsilon}, \tag{17}$$

for all $x \in \mathbb{R}$, $t \in [0, T]$, $\varepsilon > 0$.

Proof. If $t \geq \varepsilon$, we can use the decomposition $\int_0^t = \int_0^{t-\varepsilon} + \int_{t-\varepsilon}^t$.

For the first summand,

$$\Big|\int_0^{t-\varepsilon} p(t, x-y; s) H(s/\varepsilon, y) ds\Big| = \varepsilon \Big| p(t, x-y; s) G(s/\varepsilon, y)\Big|_0^{t-\varepsilon} \\ - \int_0^{t-\varepsilon} \frac{\partial p(t, x-y; s)}{\partial s} G(s/\varepsilon, y) ds \Big| \overset{G(0,y)=0}{\leq} C\varepsilon \Big(|p(t, x-y; t-\varepsilon)| \\ + \int_0^{t-\varepsilon} \Big| \frac{\partial p(t, x-y; s)}{\partial s} \Big| ds \Big) \overset{(8),\,(12)}{\leq} C\sqrt{\varepsilon} + C\varepsilon \int_0^{t-\varepsilon} ds (t-s)^{-3/2} \leq C\sqrt{\varepsilon}. \tag{18}$$

For the second summand,

$$\Big|\int_{t-\varepsilon}^{t} p(t, x-y; s)H(s/\varepsilon, y)ds\Big| \le C\int_{t-\varepsilon}^{t}(t-s)^{-1/2}ds = C\sqrt{\varepsilon}. \tag{19}$$

From (18), (19) we obtain (17). If $t < \varepsilon$, (17) is a corollary of (19). $\square$

4 The main result

We are ready to formulate the main result of the paper.

Theorem 1. *Let Assumptions E1–E5, L, M hold. Then for continuous versions of* u_ε *and* $\bar{u}$*, for any* $0 < \gamma_1 < \min\Big\{\frac{1}{5}, \frac{1}{2}\Big(1 - \frac{1}{2\beta(\sigma)}\Big)\Big\}$ *we have*

$$\sup_{\varepsilon>0, t\in[0,T], x\in\mathbb{R}} \varepsilon^{-\gamma_1}|u_\varepsilon(t,x) - \bar{u}(t,x)| < +\infty \quad a.s. \tag{20}$$

Proof. For each $(t,x) \in [0,T]\times\mathbb{R}$ we take versions of stochastic integrals $\int_{\mathbb{R}} d\mu(y)\int_0^t p(t,x-y;s)\sigma(s/\varepsilon,y)\,ds$ and $\int_{\mathbb{R}} d\mu(y)\int_0^t p(t,x-y;s)\bar{\sigma}(y)\,ds$ that are defined by Lemma 2. We obtain

$$\begin{aligned}
&|u_\varepsilon(t,x) - \bar{u}(t,x)|\\
&\quad\le \Big|\int_0^t ds\int_{\mathbb{R}} p(t,x-y;s)(f(s/\varepsilon,y,u_\varepsilon(s,y)) - f(s/\varepsilon,y,\bar{u}(s,y))\,dy\Big|\\
&\qquad + \Big|\int_0^t ds\int_{\mathbb{R}} p(t,x-y;s)(f(s/\varepsilon,y,\bar{u}(s,y)) - \bar{f}(y,\bar{u}(s,y)))\,dy\Big| + |\xi_\varepsilon|\\
&\quad =: I_1 + I_2 + |\xi_\varepsilon|,
\end{aligned} \tag{21}$$

where

$$\begin{aligned}
\xi_\varepsilon = &\int_{\mathbb{R}} d\mu(y)\int_0^t p(t,x-y;s)\sigma(s/\varepsilon,y)ds\\
&-\int_{\mathbb{R}} d\mu(y)\int_0^t p(t,x-y;s)\bar{\sigma}(y)ds.
\end{aligned}$$

To estimate the second term, we divide $[0,T]$ into n segments of length $\Delta = T/n$ and rewrite I_2 as

$$\begin{aligned}
I_2 = \Big|\sum_{k=0}^{n-1}&\int_{(k\Delta\wedge t,(k+1)\Delta\wedge t]} ds\int_{\mathbb{R}} p(t,x-y;s)\\
&\times\big(f(s/\varepsilon,y,\bar{u}(s,y)) - \bar{f}(y,\bar{u}(s,y))\big)\,dy\Big|.
\end{aligned}$$

Thus,

$$\begin{aligned}
I_2 \le \sum_{k=0}^{n-1}\Big(\Big|&\int_{(k\Delta\wedge t,(k+1)\Delta\wedge t]} ds\int_{\mathbb{R}} p(t,x-y;s)\big(f(s/\varepsilon,y,\bar{u}(s,y))\\
&- f(s/\varepsilon,y,\bar{u}(k\Delta,y))\big)\,dy\Big|
\end{aligned}$$

$$
\begin{aligned}
&+\Big|\int_{(k\Delta\wedge t,(k+1)\Delta\wedge t]} ds \int_{\mathbb{R}} p(t,x-y;s)\big(f(s/\varepsilon,y,\bar u(k\Delta,y))\\
&-\bar f(y,\bar u(k\Delta,y))\big)\,dy\Big|\\
&+\Big|\int_{(k\Delta\wedge t,(k+1)\Delta\wedge t]} ds \int_{\mathbb{R}} p(t,x-y;s)\big(\bar f(y,\bar u(k\Delta,y))\\
&-\bar f(y,\bar u(s,y))\big)\,dy\Big|\Big)\\
&=: \sum_{k=0}^{n-1}(I_{21}^{(k)}+I_{22}^{(k)}+I_{23}^{(k)}).
\end{aligned}
$$

Applying Lemma 5 for $h(r,y,z)=f(r,y,\bar u(z,y)), \bar h(y,z)=\bar f(y,\bar u(z,y)), \varphi_1(t)=k\Delta\wedge t, \varphi_2(t)=(k+1)\Delta\wedge t, k\in\{0,\dots,n-1\}$, we obtain

$$
I_{22}^{(k)}\le C\varepsilon|\ln\varepsilon|.
$$

We estimate $\sum_{k=1}^{n-1} I_{21}^{(k)}$ and $I_{21}^{(0)}$ separately. For $\sum_{k=1}^{n-1} I_{21}^{(k)}$ we use Lemma 4:

$$
\begin{aligned}
\sum_{k=1}^{n-1} I_{21}^{(k)} &\le L_f \sum_{k=1}^{n-1}\int_{(k\Delta\wedge t,(k+1)\Delta\wedge t]} ds\int_{\mathbb{R}} |p(t,x-y;s)||\bar u(s,y)-\bar u(k\Delta,y)|\,dy\\
&\le L_f C(\omega)\sum_{k=1}^{n-1}\int_{\mathbb{R}} dy\int_{(k\Delta\wedge t,(k+1)\Delta\wedge t]} |p(t,x-y;s)|(\ln s-\ln k\Delta\\
&\quad + s\ln s - k\Delta\ln k\Delta-(s-k\Delta)\ln(s-k\Delta)+(s-k\Delta)^{\gamma})ds. \qquad (22)
\end{aligned}
$$

(recall that $0<\gamma<1/4$). Note that for each $k\in\{1,\dots,n-1\}$ the function

$$
f_k(s)=\ln s-\ln k\Delta+s\ln s-k\Delta\ln k\Delta-(s-k\Delta)\ln(s-k\Delta)+(s-k\Delta)^{\gamma}
$$

is increasing on $[k\Delta,(k+1)\Delta]$. Therefore, we can estimate the sum in (22) in the following way:

$$
\begin{aligned}
&\sum_{k=1}^{n-1}\int_{\mathbb{R}} dy\int_{(k\Delta\wedge t,(k+1)\Delta\wedge t]} |p(t,x-y;s)| f_k(s)\,ds\\
&\le \sum_{k=1}^{n-1} f_k((k+1)\Delta)\int_{(k\Delta\wedge t,(k+1)\Delta\wedge t]} ds\int_{\mathbb{R}} |p(t,x-y;s)|\,dy\\
&\le C\sum_{k=1}^{n-1} f_k((k+1)\Delta)\Delta = C\Delta\sum_{k=1}^{n-1}\Big((\ln(k+1)\Delta-\ln k\Delta)\\
&\quad +((k+1)\Delta\ln(k+1)\Delta-k\Delta\ln k\Delta)-\Delta\ln\Delta+\Delta^{\gamma}\Big)\\
&= C\Delta\Big(\ln n\Delta-\ln\Delta+n\Delta\ln n\Delta-\Delta\ln\Delta-(n-1)\Delta\ln\Delta+(n-1)\Delta^{\gamma}\Big)\\
&\overset{n\Delta=T}{=} CT(T+1)\frac{\ln n}{n}+\frac{n-1}{n}T\Big(\frac{T}{n}\Big)^{\gamma}\le Cn^{-\gamma}.
\end{aligned}
$$

Now we need to estimate

$$
\begin{aligned}
|\bar{u}(t,x)-\bar{u}(0,x)| &= |\bar{u}(t,x)-u_0(x)| \\
&\le \Big|\int_{\mathbb{R}} p(t,x-y;0)u_0(y)\,dy - u_0(x)\Big| \\
&+ \Big|\int_0^t ds \int_{\mathbb{R}} p(t,x-y;s)\bar{f}(y,\bar{u}(s,y))\,dy\Big| \\
&+ \Big|\int_{\mathbb{R}} d\mu(y)\int_0^t p(t,x-y;s)\bar{\sigma}(y)\,ds\Big| =: I_{211}^{(0)} + I_{212}^{(0)} + I_{213}^{(0)}.
\end{aligned}
$$

Note that

$$
I_{211}^{(0)} \le Ct^{\lambda_1} + \Big|\int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t\,a(0)}} e^{-\frac{(x-y)^2}{4ta(0)}} u_0(y)dy - u_0(x)\Big|, \tag{23}
$$

where $\lambda_1 > 0$ (see (4.4) and the proof of (4.64) in [8]). Using that

$$
\int_{\mathbb{R}} \frac{dy}{\sqrt{4\pi t\,a(0)}} e^{-\frac{(x-y)^2}{4ta(0)}} = 1,
$$

we obtain

$$
\begin{aligned}
\Big|\int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t\,a(0)}} e^{-\frac{(x-y)^2}{4ta(0)}} u_0(y)dy - u_0(x)\Big| &\le \int_{\mathbb{R}} \frac{\big|u_0(y)-u_0(x)\big|}{\sqrt{4\pi t\,a(0)}} e^{-\frac{(x-y)^2}{4ta(0)}}\,dy \\
&\le Ct^{-1/2}\int_{\mathbb{R}} |x-y|^{\beta(u_0)} e^{-\frac{(x-y)^2}{4ta(0)}}\,dy \\
&\overset{v=(x-y)/2\sqrt{ta(0)}}{=} Ct^{\beta(u_0)} \int_0^{+\infty} e^{-v^2} v^{\beta(u_0)} dv = Ct^{\beta(u_0)}.
\end{aligned} \tag{24}
$$

On the other hand,

$$
I_{212}^{(0)} \le C\int_0^t ds \int_{\mathbb{R}} |p(t,x-y;s)|\,dy \le Ct, \tag{25}
$$

$$
I_{213}^{(0)} \overset{(14)}{\le} C(\omega)t^{\gamma}. \tag{26}
$$

Denote $\gamma_2 = \gamma \wedge \lambda_1$. From (23), (24), (25), (26) it follows that $|\bar{u}(t,x)-\bar{u}(0,x)| \le Ct^{\gamma_2}$. Therefore,

$$
\begin{aligned}
I_{21}^{(0)} &\le L_f \int_{(0,\Delta\wedge t]} ds \int_{\mathbb{R}} |p(t,x-y;s)||\bar{u}(s,y)-\bar{u}(0,y)|\,dy \\
&\le C\int_{(0,\Delta\wedge t]} s^{\gamma_2}\,ds \le Cn^{-1-\gamma_2},
\end{aligned}
$$

and we can see that $\sum_{k=0}^{n-1} I_{21}^{(k)} \le C(\omega)n^{-\gamma}$, where $\gamma < 1/4$. Using similar arguments we can prove that $\sum_{k=0}^{n-1} I_{23}^{(k)} \le C(\omega)n^{-\gamma}$. Thus we obtain

$$
I_2 \le C(\omega)(n|\varepsilon \ln \varepsilon| + n^{-\gamma}).
$$

Function $g(x)=x|\varepsilon\ln\varepsilon|+x^{-\gamma}$, $x>0$, has the minimum value

$$g(x_*)=|\varepsilon\ln\varepsilon|^{\gamma/(\gamma+1)}(\gamma+1)\gamma^{-\gamma/(\gamma+1)},\quad x_*=(\gamma/|\varepsilon\ln\varepsilon|)^{1/(\gamma+1)}.$$

We have $\frac{g(x+1)}{g(x)}\le C$, therefore there exists a positive integer $n_*=[x_*]+1$ such that

$$g(n_*)\le C|\varepsilon\ln\varepsilon|^{\gamma/(\gamma+1)}.$$

Recall that $\gamma_1<1/5$. Therefore, we can take $\gamma<1/4$ such that $\gamma/(\gamma+1)>\gamma_1$ and obtain

$$I_2\le C(\omega)\varepsilon^{\gamma_1}. \tag{27}$$

Now we estimate ξ_ε. Denote

$$q(z,y)=\int_0^t p(t,x-y;s)(\sigma(s/\varepsilon,y)-\bar\sigma(y))ds.$$

We will estimate $\|q(z,\cdot)\|_{B^\alpha_{22}((j,j+1])}$. Consider

$$\begin{aligned}q(z,y+h)-q(z,y)&=J_1+J_2:=\\&=\int_0^t p(t,x-y;s)\big(\sigma(s/\varepsilon,y+h)-\sigma(s/\varepsilon,y)-\bar\sigma(y+h)+\bar\sigma(y)\big)\,ds\\&+\int_0^t\big(p(t,x-y-h;s)-p(t,x-y;s)\big)\big(\sigma(s/\varepsilon,y+h)-\sigma(s/\varepsilon,y)\big)\,ds.\end{aligned}$$

Using E3, we obtain

$$|J_1|\le 2L_\sigma h^{\beta(\sigma)}\int_0^t|p(t,x-y;s)|ds\le Ch^{\beta(\sigma)}.$$

For J_2 we use boundedness of σ:

$$\begin{aligned}|J_2|&\le C\int_0^t|p(t,x-y-h;s)-p(t,x-y;s)|ds\\&=C\int_0^t\Big|\int_{x-y-h}^{x-y}\frac{\partial p(t,v;s)}{\partial v}\,dv\Big|ds\overset{(9)}{\le}C\int_0^t\frac{ds}{t-s}\int_{x-y-h}^{x-y}e^{-\frac{\lambda v^2}{t-s}}dv\le\\&C\int_{-h/2}^{h/2}dr\int_0^t\frac{1}{t-s}e^{-\frac{\lambda r^2}{t-s}}ds\overset{z=\lambda r^2/(t-s)}{=}C\int_{-h/2}^{h/2}dr\int_{\lambda r^2/t}^{\infty}\frac{e^{-z}}{z}dz\\&\le C\int_{-h/2}^{h/2}dr\Big(\int_{\lambda r^2/t}^1\frac1z dz+\int_1^\infty e^{-z}dz\Big)\le C\int_{-h/2}^{h/2}(1+|\ln|r||)\,dr\\&=Ch+2C(r-r\ln r)\Big|_0^{h/2}\le Ch^{\beta(\sigma)}\end{aligned}\tag{28}$$

(see formula 4.10 from [17]). Here we used the inequality

$$\int_{x-y-h}^{x-y}e^{-\frac{\lambda v^2}{t-s}}dv\le\int_{-h/2}^{h/2}e^{-\frac{\lambda r^2}{t-s}}dr.$$

On the other hand, Lemma 6 implies that

$$|q(z, y+h) - q(z, y)| \le |q(z, y+h)| + |q(z, y)| \le C\varepsilon^{1/2}. \tag{29}$$

From (28) and (29) it follows that for all $\theta \in [0, 1]$

$$|q(z, y+h) - q(z, y)| \le Ch^{\beta(\sigma)(1-\theta)}\varepsilon^{\theta/2},$$
$$(\omega_2(q, r))^2 \le Ch^{2\beta(\sigma)(1-\theta)}\varepsilon^{\theta},$$
$$\|q(z, \cdot)\|_{L_2((j,j+1])} \le C\sqrt{\varepsilon} \le C\varepsilon^{\theta/2},$$

and

$$\|q(z, \cdot)\|_{B^{\alpha}_{22}((j,j+1])} < \varepsilon^{\theta/2},$$

if integral

$$\int_0^1 v^{2\beta(\sigma)(1-\theta)-2\alpha-1}dv$$

is finite. That holds true if and only if

$$\alpha < \beta(\sigma)(1-\theta) \Leftrightarrow \theta < 1 - \frac{\alpha}{\beta(\sigma)}. \tag{30}$$

For $\gamma_1 < \frac{1}{2}\Big(1 - \frac{1}{2\beta(\sigma)}\Big)$ we can choose $\theta = 2\gamma_1$, $1 > \alpha > 1/2$ such that (30) holds.

Using Lemma 2, we get

$$\begin{aligned}
|\xi_\varepsilon| = \Big|\int_{\mathbb{R}} q(z, y)\, d\mu(y)\Big| &\le \sum_{j\in\mathbb{Z}}\Big|\int_{(j,j+1]} q(z, y)\, d\mu(y)\Big| \\
&\overset{(5),\,(6)}{\le} \sum_{j\in\mathbb{Z}} |q(z, j)\mu((j, j+1])| \\
&+C\sum_{j\in\mathbb{Z}}\|q(z,\cdot)\|_{B^{\alpha}_{22}([j,j+1])}\Big\{\sum_{n\ge1}2^{n(1-2\alpha)}\sum_{1\le k\le 2^n}|\mu(\Delta^{(j)}_{kn})|^2\Big\}^{1/2} \\
&\le C\varepsilon^{\gamma_1}\Big[\sum_{j\in\mathbb{Z}}|\mu((j, j+1])| + \sum_{j\in\mathbb{Z}}\Big\{\sum_{n\ge1}2^{n(1-2\alpha)}\sum_{1\le k\le 2^n}|\mu(\Delta^{(j)}_{kn})|^2\Big\}^{1/2}\Big] \\
&\le C\varepsilon^{\gamma_1}\Big[\Big(\sum_{j\in\mathbb{Z}}(|j|+1)^{2\rho}(\mu((j, j+1]))^2\Big)^{1/2}\Big(\sum_{j\in\mathbb{Z}}(|j|+1)^{-2\rho}\Big)^{1/2} \\
&+\Big(\sum_{n\ge1}2^{n(1-2\alpha)}\sum_{j\in\mathbb{Z}}(|j|+1)^{2\rho}\sum_{1\le k\le 2^n}|\mu(\Delta^{(j)}_{kn})|^2\Big)^{1/2}\Big(\sum_{j\in\mathbb{Z}}(|j|+1)^{-2\rho}\Big)^{1/2}\Big],
\end{aligned}$$

where $\rho > 1/2$ is taken from Assumption M, the sums with SMs have the form $\sum_{l=1}^{\infty}\Big(\int_{\mathbb{R}}\phi_l\, d\mu\Big)^2$, where

$$\{\phi_l(y),\ l\ge1\} = \{(|j|+1)^{\rho}\,\mathbf{1}_{(j,j+1]}(y),\ j\in\mathbb{Z}\},$$
$$\{\phi_l(y),\ l\ge1\} = \{(|j|+1)^{\rho}2^{n(1-2\alpha)/2}\mathbf{1}_{\Delta^{(j)}_{kn}}(y),\ j\in\mathbb{Z},\ n\ge1,\ 1\le k\le 2^n\}.$$

From the inequalities

$$\sum_{l=1}^{\infty} |\phi_l(y)| \le C(1 + |y|^{\rho}), \quad \sum_{j\in\mathbb{Z}} (|j| + 1)^{-2\rho} < \infty,$$

and Lemma 1 it follows that

$$|\xi_\varepsilon| \le C\varepsilon^{\gamma_1} \quad \text{a.s.} \tag{31}$$

From (21), (27) and (31) it follows that

$$|u_\varepsilon(t, x) - \bar{u}(t, x)| \le C\varepsilon^{\gamma_1} \\ + \int_0^t ds \int_{\mathbb{R}} |p(t, x - y; s)||f(s/\varepsilon, y, u_\varepsilon(s, y)) - f(s/\varepsilon, y, \bar{u}(s, y))\, dy| \quad \text{a.s.}$$

From boundedness of f it follows that $\sup_{x\in\mathbb{R}} |u_\varepsilon(t, x) - \bar{u}(t, x)| < \infty$ a.s. On the other hand, using (8) and the Lipschitz condition on f, we get

$$\sup_{x\in\mathbb{R}} |u_\varepsilon(t, x) - \bar{u}(t, x)| \le C\varepsilon^{\gamma_1} + C \int_0^t \sup_{x\in\mathbb{R}} |u_\varepsilon(t, x) - \bar{u}(t, x)|\, ds.$$

From the Gronwall inequality we obtain

$$\sup_{x\in\mathbb{R}} |u_\varepsilon(t, x) - \bar{u}(t, x)| \le C(\omega)\varepsilon^{\gamma_1},$$

where $C(\omega)$ is independent of t and ε. Thus, we obtain (20). □

5 Examples

Example 1. Let $a(t) = a^2 > 0$, $b(t) = c(t) = 0$ for each $t \in [0, T]$. Then

$$\mathcal{L}u(t, x) = a^2 \frac{\partial^2 u(t, x)}{\partial^2 x} - \frac{\partial u(t, x)}{\partial t},$$

and (1), (3) are the heat equations. Note that the averaging principle for the heat equation was considered in [16], and the same order of strong convergence was obtained.

Example 2. Let $a(t) = 1$, $b(t) = 0$, $c(t) = -1$, $t \in [0, T]$. Then

$$\mathcal{L}u(t, x) = \frac{\partial^2 u(t, x)}{\partial^2 x} - u(t, x) - \frac{\partial u(t, x)}{\partial t},$$

and (1), (3) are the so-called cable equations. The cable equation describes potential changes along the branch of the dendritic tree (see [5]). The averaging principle for the cable equation on $[0, L]$ with $f \equiv 0$ was established in [2] with the better order of strong convergence ($\gamma_1 < \frac{1}{2}(1 - \frac{1}{2\beta(\sigma)})$).

References

[1] Bodnarchuk, I.M.: Regularity of the mild solution of a parabolic equation with stochastic measure. Ukr. Math. J. **69**, 1–18 (2017). MR3631616. https://doi.org/10.1007/s11253-017-1344-4

[2] Bodnarchuk, I.M.: Averaging principle for a stochastic cable equation. Mod. Stoch. Theory Appl. **7**, 449–467 (2020). MR4195646. https://doi.org/10.15559/20-vmsta168

[3] Bodnarchuk, I.M., Radchenko, V.M.: The equation for vibrations of a fixed string driven by a general stochastic measure. Theory Probab. Math. Stat. **101**, 1–11 (2020). MR3824679. https://doi.org/10.1090/tpms/1108

[4] Brehier, C.-E.: Strong and weak orders in averaging for SPDEs. Stoch. Process. Appl. **122**, 2553–2593 (2012). MR2926167. https://doi.org/10.1016/j.spa.2012.04.007

[5] Bressloff, P.C.: Waves in Neutral Media: From Single Neurons to Neural Fields. Lecture Notes on Mathematical Modelling in the Life Sciences. Springer, New York (2014). MR3136844. https://doi.org/10.1007/978-1-4614-8866-8

[6] Duan, J., Wang, W.: Effective Dynamics of Stochastic Partial Differential Equations. Elsevier, Amsterdam (2014). MR3289240

[7] Fu, H., Liu, J.: Strong convergence in stochastic averaging principle for two timescales stochastic partial differential equations. J. Math. Anal. Appl. **384**, 70–86 (2011). MR2822851. https://doi.org/10.1016/j.jmaa.2011.02.076

[8] Ilyin, A.M., Kalashnikov, A.S., Oleynik, O.A.: Linear second-order partial differential equations of the parabolic type. J. Math. Sci. **108**(4), 435–542 (2002). MR1875963. https://doi.org/10.1023/A:1013156322602

[9] Kamont, A.: A discrete characterization of Besov spaces. Approx. Theory Appl. (N.S.) **13**(2), 63–77 (1997). MR1750304

[10] Kwapien, S., Woyczynski, W.A.: Random Series and Stochastic Integrals: Single and Multiple. Birkhäuser, Boston (1992). MR1167198. https://doi.org/10.1007/978-1-4612-0425-1

[11] Memin, T., Mishura, Y., Valkeia, E.: Inequalities for the moment of wiener integrals with respect to a fractional Brownian motion. Stat. Probab. Lett. **51**, 197–206 (2001). MR1822771. https://doi.org/10.1016/S0167-7152(00)00157-7

[12] Radchenko, V.M.: Mild solution of the heat equation with a general stochastic measure. Stud. Math. **194**(3), 231–251 (2009). MR2539554. https://doi.org/10.4064/sm194-3-2

[13] Radchenko, V.M.: Evolution equations driven by general stochastic measures in Hilbert space. Theory Probab. Appl. **59**(2), 328–339 (2015). MR3416054. https://doi.org/10.1137/S0040585X97T987119

[14] Radchenko, V.M.: Averaging principle for equation driven by a stochastic measure. Stochastics **91**(6), 905–915 (2019). MR3985803. https://doi.org/10.1080/17442508.2018.1559320

[15] Radchenko, V.M.: Averaging principle for the heat equation driven by a general stochastic measure. Stat. Probab. Lett. **146**, 224–230 (2019). MR3885229. https://doi.org/10.1016/j.spl.2018.11.024

[16] Radchenko, V.M.: Strong convergence rate in averaging principle for the heat equation driven by a general stochastic measure. Commun. Stoch. Anal. **13**, 1 (2019). https://doi.org/10.31390/cosa.13.2.01

[17] Radchenko, V.M., Manikin, B.I.: Approximation of the solution to the parabolic equation driven by stochastic measure. Theory Probab. Math. Stat. **102**, 145–156 (2020). https://doi.org/10.1090/tpms/1131

[18] Samorodnitsky, G., Taqqu, M.: Stable Non-Gaussian Random Processes. Chapman and Hall, London (1994). MR1280932

[19] Shen, G., Wu, J.-L., Yin, X.: Averaging principle for fractional heat equations driven by stochastic measures. Appl. Math. Lett. **106**, 106404 (2020). MR4090373. https://doi.org/10.1016/j.aml.2020.106404

[20] Wang, W., Roberts, A.: Average and deviation for slow-fast stochastic partial differential equations. J. Differ. Equ. **253**, 1265–1286 (2012). MR2927381. https://doi.org/10.1016/j.jde.2012.05.011