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Models of space-time random fields on the sphere

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Abstract General models of random fields on the sphere associated with nonlocal equations in time and space are studied. The properties of the corresponding angular power spectrum are discussed and asymptotic results in terms of random time changes are found.

Keywords Fractional equations, spherical Brownian motion, subordinators, random fields, Laplace–Beltrami operators, spherical harmonics
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1 Introduction

The models of spherical random fields are in great demand in various applied areas such as geophysics, geodesy, planetary sciences, astronomy, cosmology and others. In recent years one can observe the growing popularity of stochastic partial differential equations in modeling space-time random fields. Solutions to stochastic Cauchy problems for various classes of partial differential equations on the sphere admit exact

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series representations, which is important, in particular, for numerical approximation of such random fields, since this can be achieved effectively by truncating the corresponding expansions (see, e.g., [2, 8, 18] for the most recent progress on properties of truncated expansions).

Papers [14, 13] develop the approach to construct time dependent random fields on the sphere through coordinate-change and subordination. These models of random fields arise as solutions to partial differential equations with operators of a particular form and random initial condition represented by a Gaussian random field.

In the present paper we generalize results of paper [14] and consider random fields arising as solutions to the fractional equations of the form

$$\left(\gamma - \Psi(-\Delta_{\mathbb{S}_1^2}) + \mathfrak{D}_t^{\Phi}\right) X_t(\boldsymbol{x}) = 0,\ \boldsymbol{x} \in \mathbb{S}_1^2,\ t > 0,\ \gamma \geq 0, \tag{1.1}$$

subject to the initial condition $X_0(\boldsymbol{x}) = T(\boldsymbol{x})$, $\boldsymbol{x} \in \mathbb{S}_1^2 = \{\boldsymbol{x} \in R^3 : \|\boldsymbol{x}\| = 1\}$, with $T(\boldsymbol{x})$, $\boldsymbol{x} \in \mathbb{S}_1^2$, being a square integrable isotropic Gaussian random field.

In the above equation, the generalized Laplace–Beltrami operator $\Psi(-\Delta_{\mathbb{S}_1^2})$ is defined in terms of the transition semigroup of the subordinate rotational Brownian motion $B_t^{\Psi} = B_{H_t}$, where H_t is a subordinator with the Laplace exponent Ψ. This covers, in particular, the case of fractional Laplace operator $(-\Delta_{\mathbb{S}_1^2})^{\alpha}$, $\alpha \in (0, 1)$. The time derivative $\mathfrak{D}_t^{\Phi}$ is the generalized convolution-type derivative associated with the Bernštein function Φ, which reduces to the Caputo–Djrbashian (C-D) fractional derivative $\frac{\partial^\beta}{\partial t^\beta}$ for $\Phi(\lambda) = \lambda^\beta$, $\beta \in (0, 1)$.

Note that the generalized Laplace–Beltrami operator $\Psi(-\Delta_{\mathbb{S}_1^2})$, where Ψ is a Bernštein function, can be treated by means of the transition semigroup of the subordinate rotational Brownian motion B_t^{Ψ}. This gives the possibility of a deeper insight into the structure of the solution $X_t(\boldsymbol{x})$ to Equation (1.1). Namely, such approach permits us to obtain not only the Karhunen–Loève expansion for the solution, but also its representation as a coordinate-changed random field.

We show that solution to (1.1) is a time-varying random field with the following representation in terms of spherical harmonics Y_{lm}:

$$X_t(\boldsymbol{x}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} a_{lm}\, \widetilde{l}(t, \gamma + \Psi(\mu_l)) Y_{lm}(\boldsymbol{x}),\ \boldsymbol{x} \in \mathbb{S}_1^2,\ t > 0, \tag{1.2}$$

where $a_{lm} = \int_{\mathbb{S}_1^2} X_0(\boldsymbol{x}) Y_{lm}^*(\boldsymbol{x}) \mu(d\boldsymbol{x})$. The series expansion (1.2) is associated with the function Φ in the following way: $\widetilde{l}(t, \lambda) = \mathbf{E}[\exp(-\lambda L_t)]$ with L being the inverse process for the subordinator with the Laplace exponent Φ. We also represent $X_t(\boldsymbol{x})$ as a coordinate-changed random field.

The use of the generalized derivative $\mathfrak{D}_t^{\Phi}$ allows to construct more general models of random fields that those in the paper [14], where the C-D fractional derivative was used.

The equation similar to (1.1) for tangent random fields on the sphere was considered in the recent paper [4]. Namely, the authors study the equation with the C-D fractional derivative in time, fractional diffusion operator $\psi(-\Delta_{\mathbb{S}_1^2}) = (-\Delta_{\mathbb{S}_1^2})^{\alpha/2}(I -$

$\Delta_{\mathbb{S}_1^2})^{\gamma/2}$, $\alpha \in (0,2]$, $\alpha+\gamma \in [0,2]$, and driving fractional Brownian noise. The corresponding solution is given as an expansion in terms of vector spherical harmonics with random coefficients represented by stochastic integrals with respect to fractional Brownian motions.

The paper is organized as follows. Sections 2–3 make necessary preparations and provide a concise review on the operators used in equations and facts on isotropic random fields. The main results are stated in Section 4: we give the different representations for solutions to (1.1) and discuss their properties.

2 Generalized fractional operators

To define our models of space-time random fields, we will use partial differential equations with generalized fractional derivatives in time and space variables.

2.1 Generalized fractional Caputo–Djrbashian or convolution-type derivative

We first introduce the generalized fractional operator to act on the time variable.

Let us consider the subordinator H, that is, a nonnegative Lévy process with almost surely increasing paths. The process H is characterized by a Lévy measure Π on $(0,\infty)$ such that $\int_0^\infty (1\wedge z)\Pi(dz) < \infty$ and the corresponding Bernštein function Φ (called the Laplace exponent or symbol of H). That is,

$$\mathbf{E}[\exp(-\lambda H_t)] = \exp(-t\Phi(\lambda)),$$

where

$$\Phi(\lambda) = \int_0^\infty \left(1-e^{-\lambda z}\right)\Pi(dz), \quad \lambda \geq 0. \tag{2.1}$$

In the general case, the expression for Bernštein function (2.1) contains two more terms, namely, of the form $a+b\lambda$, but we consider now the case $a=b=0$.

We also recall that

$$\frac{\Phi(\lambda)}{\lambda} = \int_0^\infty e^{-\lambda z}\overline{\Pi}(z)dz, \qquad \overline{\Pi}(z) = \Pi((z,\infty)) \tag{2.2}$$

and $\overline{\Pi}$ is the so-called *tail of the Lévy measure*. For details, see the book [6].

Introduce the inverse process associated to H (and, so, associated to Φ) as

$$L_t = \inf\{s \geq 0 : H_s > t\}, \quad t > 0.$$

L is a nonnegative process with almost surely nondecreasing paths.

We assume that $\Pi((0,\infty)) = \infty$ and, therefore, we focus only on strictly increasing subordinators. For this case, the inverse process L turns out to be a continuous process. Under the additional assumption that $\overline{\Pi}(z)$, $z \geq 0$, is absolutely continous function, the inverse process L_t possesses the probability density function $l(s,t)$ for each $t > 0$ [23].

Definition 1. Convolution-type derivative associated with the function Φ given by (2.1) is defined for an absolutely continuous function u by the formula

$$\mathfrak{D}_t^\Phi u(t) = \int_0^t \frac{\partial}{\partial t} u(t-s)\overline{\Pi}(s)ds. \tag{2.3}$$

According to the definition, the generalized fractional operator is characterized by the Bernštein function Φ, and thus is associated with the processes H and L introduced above. This operator can be used to study the properties of subordinators and their inverses and write the governing equations for their densities (see [23]).

In the case where $\Phi(\lambda) = \lambda^\alpha$, $\alpha \in (0, 1)$, we have that

$$\mathfrak{D}_t^\Phi u(t) = \frac{d^\alpha}{dt^\alpha} u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{u'(s)}{(t-s)^\alpha} ds,$$

where $u' = du/ds$, that is, $\mathfrak{D}_t^\Phi u(t)$ coincides with the well-known Caputo–Djrbashian fractional derivative.

Similarly to the Caputo–Djrbashian fractional derivative, the convolution type derivative can be characterized (and alternatively defined) by means of its Laplace transform.

Let $M > 0$ and $w \geq 0$. Let $\mathcal{M}_w$ be the set of (piecewise) continuous functions on $[0, \infty)$ of exponential order w such that $|u(t)| \leq M e^{wt}$. Denote by $\widetilde{u}$ the Laplace transform of u. Then, we define the operator $\mathfrak{D}_t^\Phi : \mathcal{M}_w \mapsto \mathcal{M}_w$ such that

$$\int_0^\infty e^{-\lambda t} \mathfrak{D}_t^\Phi u(t)\, dt = \Phi(\lambda) \widetilde{u}(\lambda) - \frac{\Phi(\lambda)}{\lambda} u(0), \quad \lambda > w, \tag{2.4}$$

where Φ is given in (2.1). Since u is exponentially bounded, the integral $\widetilde{u}$ is absolutely convergent for $\lambda > w$. By Lerch's theorem (see, e.g., [25]) the inverse Laplace transforms u and $\mathfrak{D}_t^\Phi u$ are uniquely defined. Formula (2.4) can be rewritten as

$$\Phi(\lambda) \widetilde{u}(\lambda) - \frac{\Phi(\lambda)}{\lambda} u(0) = (\lambda \widetilde{u}(\lambda) - u(0)) \frac{\Phi(\lambda)}{\lambda}, \tag{2.5}$$

and thus, $\mathfrak{D}_t^\Phi$ can be regarded as a convolution involving the ordinary derivative and the inverse transform of (2.2) iff $u \in \mathcal{M}_w \cap C([0, \infty), \mathbb{R}_+)$ and $u' \in \mathcal{M}_w$.

The operator $\mathfrak{D}_t^\Phi$ have been introduced and studied in the papers [17, 11, 23].

In Section 4 we study random fields on the sphere governed by equations with convolution-type derivatives $\mathfrak{D}_t^\Phi$. The following well-known fact will be important.

Proposition 1. *Let L be the inverse process for a subordinator with Bernštein function Φ, and assume that $\Pi(0, \infty) = \infty$ and the tail $\overline{\Pi}(s) = \Pi(s, \infty)$ is absolutely continuous. For the process L_t, $t > 0$, we have the density $l(t, x) = \mathbf{P}(L_t \in dx)/dx$, $t, x > 0$, with the Laplace transform*

$$\tilde{l}(t, \lambda) = \int_0^\infty e^{-\lambda x} l(t, x) dx = \mathbf{E}[e^{-\lambda L(t)}] \tag{2.6}$$

which satisfies the equation

$$\mathfrak{D}_t^\Phi \tilde{l}(t, \lambda) = -\lambda \tilde{l}(t, \lambda), \tag{2.7}$$

thus, $\tilde{l}(t, \lambda)$ is an eigenfunction of the operator $\mathfrak{D}_t^\Phi$ corresponding to the eigenvalue λ.

Proposition 1 was proved by different approaches in [17, 23, 9].

Remark 1. For L being the inverse process for a stable subordinator, that is, $\Phi(\lambda) = \lambda^\alpha$, (2.7) reduces to the well-known fact that the Mittag-Leffler function is an eigenfunction of the Caputo–Djrbashian fractional derivative. Namely, for the Mittag-Leffler function $E_\alpha(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k+1)}$, $x \in R$, it holds $\frac{\partial^\alpha}{\partial t^\alpha} E_\alpha(-t^\alpha \lambda) = -\lambda E_\alpha(-t^\alpha \lambda)$ (see, e.g., [16]).

2.2 *Generalized fractional Laplacian on the sphere*

Let $f \in L^2(\mathbb{S}_1^2) = L^2(\mathbb{S}_1^2, \mu)$, where μ is the Lebesgue measure on the unit sphere $\mathbb{S}_1^2$:

$$\mu(d\boldsymbol{x}) = \mu(d\vartheta, d\varphi) = d\varphi\, d\vartheta\, \sin\vartheta$$

with $\boldsymbol{x} \in \mathbb{S}_1^2$ being represented as $\boldsymbol{x} = (\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta)$, $\vartheta \in [0, \pi]$, $\varphi \in [0, 2\pi)$.

The set of spherical harmonics $\{Y_{lm} : l \geq 0,\ m = -l, \ldots, +l\}$ represents an orthogonal basis for the space $L^2(\mathbb{S}_1^2)$. Recall that for a fixed integer l the spherical harmonics

$$Y_{lm}(\vartheta, \varphi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\, Q_{lm}(\cos\vartheta) e^{im\varphi}$$

(or linear combinations of them) solve the eigenvalue problem

$$\triangle_{\mathbb{S}_1^2} Y_{lm} = -\mu_l\, Y_{lm}, \quad l \geq 0,\ |m| \leq l, \tag{2.8}$$

for the spherical Laplace (Laplace–Beltrami) operator

$$\triangle_{\mathbb{S}_1^2} = \frac{1}{\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2} + \frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial}{\partial\vartheta}\right), \quad \vartheta \in [0, \pi],\ \varphi \in [0, 2\pi). \tag{2.9}$$

The eigenvalues are given by $\mu_l = l(l+1)$.

The spherical harmonics are written in terms of the associated Legendre functions

$$Q_{lm}(z) = (-1)^m (1-z^2)^{m/2} \frac{d^m}{dz^m} Q_l(z),$$

the Legendre polynomials Q_l are given by the Rodrigues formula

$$Q_l(z) = \frac{1}{2^l l!}\frac{d^l}{dz^l}(z^2-1)^l. \tag{2.10}$$

For $f \in L^2(\mathbb{S}_1^2)$ we have the representation

$$f(\boldsymbol{x}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} f_{lm} Y_{lm}(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{S}_1^2,$$

which holds in the L^2 sense, where

$$f_{lm} = \int_{\mathbb{S}_1^2} f(\boldsymbol{x}) Y_{lm}^*(\boldsymbol{x})\, \mu(d\boldsymbol{x}) = \int_0^{2\pi}\int_0^{\pi} f(\vartheta, \varphi) Y_{lm}^*(\vartheta, \varphi)\, \sin\vartheta\, d\vartheta\, d\varphi,$$

$$|m| \leq l,\ l = 0, 1, 2, \ldots. \tag{2.11}$$

(see, e.g., the Peter–Weyl representation theorem on the sphere in [21]).

The angular power spectrum of f is defined as

$$f_l = \sum_{|m|\le l} |f_{lm}|^2 = \sum_{|m|\le l} \left| \int_{\mathbb{S}_1^2} f(\boldsymbol{x}) Y_{lm}^*(\boldsymbol{x}) \mu(d\boldsymbol{x}) \right|^2, \quad l = 0, 1, 2, \ldots. \tag{2.12}$$

We next define the generalized fractional Laplace operators on the sphere following [13, 14]. Let $F(t)$, $t \ge 0$, be a Lévy subordinator with the Laplace exponent

$$\Psi(\lambda) = b\lambda + \int_0^\infty \left(1 - e^{-\lambda z}\right) M(dz), \ b \ge 0, \ \lambda \ge 0,$$

with M being the corresponding Lévy measure.

Let B_t, $t \ge 0$, be a Brownian motion on the unit sphere $\mathbb{S}_1^2$. Its transition density can be written as follows (see [26]):

$$\begin{aligned} Pr\{\boldsymbol{x} + B_t \in \mu(d\boldsymbol{y})\}/\mu(d\boldsymbol{y}) &= Pr\{B_t \in \mu(d\boldsymbol{y}) \mid B_0 = \boldsymbol{x}\}/\mu(d\boldsymbol{y}) \\ &= \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} e^{-t\mu_l} Y_{lm}(\boldsymbol{y}) Y_{lm}^*(\boldsymbol{x}). \end{aligned} \tag{2.13}$$

Consider the initial-value problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta_{\mathbf{S}_1^2} u, & \boldsymbol{x} \in \mathbf{S}_1^2, \ t > 0, \\ u(\boldsymbol{x}, 0) = f(\boldsymbol{x}), \end{cases} \tag{2.14}$$

for $f \in L^2(\mathbf{S}_1^2)$. The solution to the above problem can be written as follows:

$$\begin{aligned} u(\boldsymbol{x}, t) = P_t f(\boldsymbol{x}) = \mathbb{E} f(\boldsymbol{x} + B_t) &= \int_{\mathbf{S}_1^2} f(\boldsymbol{y}) Pr\{\boldsymbol{x} + B_t \in \mu(d\boldsymbol{y})\} \\ &= \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} e^{-t\mu_l} Y_{lm}(\boldsymbol{x}) f_{lm}, \end{aligned} \tag{2.15}$$

that is, the solution is given by the transition semigroup of the rotational Brownian motion B_t, $t > 0$, with values in $\mathbf{S}_1^2$.

In [13] the following operator acting on $f \in L^2(\mathbf{S}_1^2)$ was introduced:

$$\Psi(-\Delta_{\mathbf{S}_1^2}) f(\boldsymbol{x}) := \int_0^\infty \left(P_t f(\boldsymbol{x}) - f(\boldsymbol{x})\right) M(dt). \tag{2.16}$$

It was shown in [13] (see also [14]) that

$$\Psi(-\Delta_{\mathbf{S}_1^2}) Y_{lm}(\boldsymbol{x}) = -\Psi(\mu_l) Y_{lm}(\boldsymbol{x}), \tag{2.17}$$

thus, the spherical harmonics are the eigenfunctions of the operator $\Psi(-\Delta_{\mathbf{S}_1^2})$ with the eigenvalues $-\Psi(\mu_l)$. This was shown by direct calculations using the semigroup approach and the spectral representation of $\Psi(-\Delta_{\mathbf{S}_1^2})$ (or Phillips representation) in (2.16).

Basing on (2.17), the action of the operator $\Psi(-\Delta_{\mathbb{S}^2_1})$ can be also defined by means of a series representation as given below.

Let us consider the space of functions

$$H^s(\mathbb{S}^2_1) = \left\{ f \in L^2(\mathbb{S}^2_1) : \sum_{l=0}^{\infty} (2l+1)^{2s} f_l < \infty \right\}, \tag{2.18}$$

where f_l is the angular spectrum of f (see (2.12)). The Sobolev space $H^s(\mathbb{S}^2_1)$ is the closure of the set of all spherical harmonics with respect to the norm

$$\|f\|_{s,2} = \sum_{l\geq 0} (2l+1)^{2s} f_l.$$

For further discussion, the interested reader can consult [15], [5, page 35].

Definition 2. Let $f \in H^s(\mathbb{S}^2_1)$ and $s > 5/4$. Then

$$\Psi(-\Delta_{\mathbb{S}^2_1}) f(\boldsymbol{x}) := \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} f_{lm} Y_{lm}(\boldsymbol{x}) \Psi(\mu_l). \tag{2.19}$$

Note that since Ψ is the symbol of a subordinator, as $l \to \infty$, we have that $\Psi(l)/l \to 0$ (we write $\Psi(l) < l$ for large l) and as $l \to 0$, $\Psi(l) \to 0$. The series in (2.19) converges absolutely and uniformly. This can be proved by considering that $f_l < l^{-2s}$ with $s > 5/4$ (indeed, $f \in H^s(\mathbb{S}^2_1)$) and for the harmonic eigenfunction we have: $\|Y_{lm}\|_\infty < l^{1/2}$ ([24]). Since $\Psi(\mu_l) < l^2$ we have the claimed convergence. For more details, see [14, 13].

3 Isotropic random fields on the unit-radius sphere

Let us consider a real-valued, zero-mean, isotropic Gaussian random field $T(\boldsymbol{x})$, $\boldsymbol{x} \in \mathbb{S}^2_1$, that is, we assume $\mathbb{E}T(\boldsymbol{x}) = 0$, $\mathbb{E}T^2(\boldsymbol{x}) < \infty$, and for any $g \in SO(3)$ (the special group of rotations in $\mathbb{R}^3$) we have: $\mathbb{E}T(g\boldsymbol{x}_1)T(g\boldsymbol{x}_2) = \mathbb{E}T(\boldsymbol{x}_1)T(\boldsymbol{x}_2)$, $\boldsymbol{x}, \boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{S}^2_1$.

For the field T we can write the spectral representation

$$T(\boldsymbol{x}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} a_{lm} Y_{lm}(\boldsymbol{x}) = \sum_{l=0}^{\infty} T_l(\boldsymbol{x}), \tag{3.1}$$

where

$$a_{lm} = \int_{\mathbb{S}^2} T(\boldsymbol{x}) Y^*_{lm}(\boldsymbol{x}) \mu(d\boldsymbol{x}) \tag{3.2}$$

are random Fourier coefficients, $Y_{lm}(\boldsymbol{x})$ are spherical harmonics. Convergence in (3.1) holds in the mean square sense, both with respect to $L^2(dP \times \mu(d\boldsymbol{x}))$ and $L^2(dP)$ for fixed $\boldsymbol{x} \in \mathbb{S}^2_1$, $\mu(d\boldsymbol{x})$ is the Lebesgue measure on the unit sphere $\mathbb{S}^2_1$ (see, e.g., [21, 22]):

$$\lim_{L\to\infty} \mathbf{E} \left\| T(\boldsymbol{x}) - \sum_{l=0}^{L} \sum_{m=-l}^{+l} a_{lm} Y_{lm}(\boldsymbol{x}) \right\|^2_{L^2(\mathbb{S}^2_1)}$$

$$= \lim_{L\to\infty} \mathbf{E}\Big[\int_{\mathbf{S}^2} \big(T(\boldsymbol{x}) - \sum_{l=0}^{L} \sum_{m=-l}^{+l} a_{lm} Y_{lm}(\boldsymbol{x})\big)^2 \mu(d\boldsymbol{x}) \Big] = 0$$

and

$$\lim_{L\to\infty} \mathbf{E}\left(T(\boldsymbol{x}) - \sum_{l=0}^{L}\sum_{m=-l}^{+l} a_{lm}Y_{lm}(\boldsymbol{x})\right)^2 = 0.$$

Remark 2. The representation (3.1) can be deduced as a consequence of the stochastic Peter–Weyl theorem and holds, more generally, for square integrable strictly isotropic random fields, that is, random fields with finite dimensional distributions invariant with respect to rotations $g \in SO(3)$: $\{T(\boldsymbol{x}_1), \ldots, T(\boldsymbol{x}_n)\} \stackrel{d}{=} \{T(g\boldsymbol{x}_1), \ldots, T(g\boldsymbol{x}_n)\}$, where $\stackrel{d}{=}$ denotes equality in distribution (see, e.g., [21]).

The coefficients (3.2) are zero-mean Gaussian complex random variables such that

$$\mathbf{E}[a_{lm}a^*_{l'm'}] = \delta_l^{l'}\delta_m^{m'}C_l = \delta_l^{l'}\mathbf{E}|a_{lm}|^2, \tag{3.3}$$

where C_l, $l \geq 0$, is the angular power spectrum of the random field T which fully characterizes, under Gaussianity, the dependence structure of T. As usual, we denote by δ_a^b the Kronecker delta and "$*$" stands for complex conjugation. For a real-valued random field T, it holds:

$$a_{lm} = (-1)^m a_{l\,-m}, \;\; l \geq 1, \;\; -l \leq m \leq l,$$

due to the property of the spherical harmonics $Y^*_{lm}(\boldsymbol{x}) = (-1)^m Y_{l\,-m}(\boldsymbol{x})$. We refer to the book by Marinucci and Peccati [21] for a thorough presentation of results concerning this field.

In analogy with (2.18) one can also introduce the space of processes

$$\mathcal{H}^s(\mathbb{S}^2_1) = \left\{ T \text{ as in (3.1) with } \sum_{lm} (\mu_l)^s\, \mathbf{E}[|a_{lm}|^2] < \infty \right\}. \tag{3.4}$$

Notice that the summability condition in (3.4) can be written as

$$\sum_{l\geq 0} (\mu_l)^s \frac{(2l+1)}{4\pi} C_l < \infty \tag{3.5}$$

by taking into consideration (3.3). We also notice that $\mathcal{H}^2(\mathbb{S}^2_1) \subset L^2(\mathbb{S}^2_1)$ and in particular, the summability condition for T can be written as

$$C_l \sim l^{-\theta}, \quad \text{with} \quad \theta > 2. \tag{3.6}$$

(We use here the usual notation $g \sim f$ meaning that $\frac{g(z)}{f(z)} \to 1$ as $z \to \infty$.) The space of processes (3.4) is just a characterization of processes introduced in analogy with the characterization of functions in the space (2.18). The link between spaces is given by (3.5) under the assumption (3.6). We underline that such a condition is sufficient for summability (see, e.g., [8]). For the sake of simplicity, further on we assume that (3.6) holds true. We recall that other types of asymptotic behaviors are also possible.

The decay of the angular power spectrum is connected to the smoothness of the covariance. Sample Hölder continuity and sample differentiability was discussed in

[19] (see Theorem 4.7). The authors also provided a deep discussion about summability of the angular power spectrum and the formalization in terms of weighted Sobolev space.

Let us take an isotropic Gaussian random field introduced above as initial condition for the fractional Cauchy problem

$$\frac{\partial u(t,\boldsymbol{x})}{\partial t} + \Psi(-\Delta_{\mathbb{S}^2_1})u(t,\boldsymbol{x}) = 0, \quad u(0,\boldsymbol{x}) = T(\boldsymbol{x}), \tag{3.7}$$

where the fractional operator is introduced in Section 2.2, with Ψ being the Laplace exponent of the subordinator F. In [13] it was shown that solution to (3.7) is given by

$$u(t,\boldsymbol{x}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{+l} e^{-\Psi(\mu_l)t} a_{lm} Y_{lm}(\boldsymbol{x}) = \mathbf{E}[T(\boldsymbol{x} + B_{F_t})|\mathfrak{F}_T],$$

where $\mathfrak{F}_T$ is the σ-field generated by T and B is a rotational Brownian motion on the sphere $\mathbb{S}^2_1$ time-changed by the subordinator F.

Generalization of (3.7) by means of the use of the Caputo–Djrbashian fractional derivative with respect to time was studied in [14]. In the next section we study the further generalization using the convolution-type derivative defined in Section 2.1.

4 Models of random fields on the sphere

4.1 Nonlocal equations

Let us introduce now models of random fields on the sphere driven by equations with fractional operators. We consider fractional operators in time and space associated with Bernštein functions Φ and Ψ, respectively, as defined in Section 2 above.

We suppose that the function Φ corresponds to the subordinator H, L is its inverse process possessing the density l with Laplace transform $\widetilde{l}$ as introduced in Proposition 1 in (2.6). In what follows, we assume that the conditions of Proposition 1 are valid.

As the initial condition for the fractional equations in the theorems below we consider the real-valued, zero-mean, isotropic Gaussian random field $T(\boldsymbol{x})$, $\boldsymbol{x} \in \mathbb{S}^2_1$, with the spectral representation (3.1).

Theorem 1. *The solution in $L^2(dP \times d\lambda)$ to the fractional equation*

$$\left(\gamma - \Psi(-\Delta_{\mathbb{S}^2_1}) + \mathfrak{D}^{\Phi}_t\right) X_t(\boldsymbol{x}) = 0, \quad \boldsymbol{x} \in \mathbb{S}^2_1,\ t \ge 0, \quad \gamma > 0, \tag{4.1}$$

with the initial condition $X_0(\boldsymbol{x}) = T(\boldsymbol{x})$ is a time-dependent random field on the sphere $\mathbb{S}^2_1$ written as

$$X_t(\boldsymbol{x}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{+l} a_{lm}\,\widetilde{l}(t, \gamma + \Psi(\mu_l))Y_{lm}(\boldsymbol{x}), \tag{4.2}$$

where

$$a_{lm} = \int_{\mathbb{S}^2_1} X_0(\boldsymbol{x})Y^*_{lm}(\boldsymbol{x})\mu(d\boldsymbol{x}). \tag{4.3}$$

Proof. The proof follows the similar lines of those of the proof of Theorem 1 in [14]. In fact, the proof is deduced basing on the common method of separation of variables, the essential component of which is the knowledge of eigenfunctions for the operators involved into the equation. We present the main steps.

For the generalized D-C convolution-type derivative we have (see Proposition 1):

$$\mathfrak{D}_t^{\Phi}\widetilde{l}(t,\lambda) = -\lambda\,\widetilde{l}(t,\lambda), \quad \lambda > 0, \tag{4.4}$$

with $\widetilde{l}(t,\lambda)$ being the Laplace transform of the inverse subordinator L_t defined in (2.6).

For the generalized Laplace operator $\Psi(-\Delta_{\mathbb{S}^2_1})$, we know that

$$\Psi(-\Delta_{\mathbb{S}^2_1})Y_{lm}(\boldsymbol{x}) = -\Psi(\mu_l)Y_{lm}(\boldsymbol{x}). \tag{4.5}$$

This fact was shown in [14] by direct calculations by using the semigroup approach and the spectral representation (2.16) of the operator $\Psi(-\Delta_{\mathbb{S}^2_1})$. Note that we can also deduce from the result by Dautray and Lions (see [12], pp. 116–120) that the operator $\Psi(-\Delta_{\mathbb{S}^2_1})$ has the eigenvalues $\Psi(\mu_l)$ and (4.5) holds.

Thus, assuming that (4.2) holds true, we have that

$$\left(\gamma - \Psi(-\Delta_{\mathbb{S}^2_1})\right) X_t(\boldsymbol{x}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{+l} a_{l,m}\,(\gamma + \Psi(\mu_l))\,\widetilde{l}(t,\gamma + \Psi(\mu_l))Y_{lm}(\boldsymbol{x}). \tag{4.6}$$

On the other hand, using (4.4), we obtain

$$\mathfrak{D}_t^{\Phi}\, X_t(\boldsymbol{x}) = -\sum_{l=0}^{\infty}\sum_{m=-l}^{+l} a_{l,m}\,(\gamma + \Psi(\mu_l))\,\widetilde{l}(t,\gamma + \Psi(\mu_l))Y_{l,m}(\boldsymbol{x}). \tag{4.7}$$

By summing up (4.6) and (4.7), we obtain (4.1) as claimed. □

We now show that a solution to the fractional equation (4.1) can be represented as a coordinate-changed random field. Introduce the time dependent random field on $\mathbb{S}^2_1$,

$$T_t(\boldsymbol{x}) = \sum_{l=0}^{\infty}\sum_{|m|\le l} a_{lm}e^{-t\mu_l}\,Y_{lm}(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{S}^2_1,\ t \ge 0.$$

Let L be the inverse process associated with the function Φ as introduced above, F being the subordinator with the Bernštein function Ψ.

Define $\tau_t = F_{L_t}$ ($F_{L_t} = F \circ L_t$) as the composition of F and L.

$$\mathbf{E}[e^{-\xi\tau_t}] = \mathbf{E}[e^{-\xi\gamma L_t - \Psi(\xi)L_t}] = \widetilde{l}(t,\xi\gamma + \Psi(\xi)), \quad t \ge 0, \quad \xi \ge 0. \tag{4.8}$$

Let us define the random fields on the sphere $\mathbb{S}^2_1$,

$$Y_t(\boldsymbol{x}) = \mathbf{E}[T_{\tau_t}(\boldsymbol{x})|\mathfrak{F}_T], \quad Z_t(\boldsymbol{x}) = \mathbf{E}[T(\boldsymbol{x} + B_{\tau_t})|\mathfrak{F}_T], \tag{4.9}$$

where $\mathfrak{F}_T$ is the σ-field generated by T and B is a rotational Brownian motion on the sphere $\mathbb{S}^2_1$. The random field Y is a time-changed random field, whereas the random field Z is obtained by a random change of the coordinates of T. We remark that

$$Y_t(\boldsymbol{x}) = \int_0^\infty T_s(\boldsymbol{x})\, \mathbf{P}(\tau_t \in ds), \quad Z_t(\boldsymbol{x}) = \int_{\mathbb{S}^2_1} T(\boldsymbol{y})\mathbf{P}_{\boldsymbol{x}}(B_{\tau_t} \in \mu(d\boldsymbol{y})). \tag{4.10}$$

Theorem 2. *Let us consider the solution $X_t(\boldsymbol{x})$, $\boldsymbol{x} \in \mathbb{S}^2_1$, $t \ge 0$, to Equation (4.1) with $\gamma = 0$ and the random fields (4.9), and let $\mathfrak{F}_T$ be the σ-field generated by $X_0 = T$ on $\mathbb{S}^2_1$. Then the following representation in $L^2(dP \times d\lambda)$ holds true:*

$$X_t(\boldsymbol{x}) = \mathbf{E}\left[T(\boldsymbol{x} + B_{F\circ L_t})\middle|\mathfrak{F}_T\right], \quad t \ge 0, \tag{4.11}$$

or equivalently

$$X_t(\boldsymbol{x}) = \mathbf{E}\left[T_{F\circ L_t}(\boldsymbol{x})\middle|\mathfrak{F}_T\right], \quad t \ge 0. \tag{4.12}$$

Proof. From (2.15) we have that $\mathbf{E}Y_{lm}(\boldsymbol{x} + B_t) = e^{-t\mu_l}Y_{lm}(\boldsymbol{x})$ (see [13]) and therefore

$$\begin{aligned}\mathbf{E}[T(\boldsymbol{x} + B_{\tau_t})|\mathfrak{F}_T] &= \sum_{lm} a_{lm}\mathbf{E}[Y_{lm}(\boldsymbol{x} + B_{\tau_t})|\mathfrak{F}_T] \\ &= \sum_{lm} a_{lm}\mathbf{E}[e^{-\mu_l\tau_t}]Y_{lm}(\boldsymbol{x}) = \mathbf{E}[T_{\tau_t}(\boldsymbol{x})|\mathfrak{F}_T],\end{aligned}$$

that is, the representation (4.2) in $L^2(dP \times d\lambda)$ and the right-hand sides of Equations (4.11) and (4.12) coincide. On the other hand, we can write:

$$\begin{aligned}Z_t(\boldsymbol{x}) = \mathbf{E}[T(\boldsymbol{x} + B_{\tau_t})|\mathfrak{F}_T] &= \mathbf{E}\Big[\sum_{lm} a_{lm}Y_{lm}(\boldsymbol{x} + B_{\tau_t})|\mathfrak{F}_T\Big] \\ &= \sum_{lm} a_{lm}\mathbf{E}[Y_{lm}(\boldsymbol{x} + B_{\tau_t})] = \sum_{lm} a_{lm}Y_{lm}(\boldsymbol{x})\mathbf{E}e^{-\mu_l\tau_t} \\ &= \sum_{lm} a_{lm}Y_{lm}(\boldsymbol{x})\widetilde{l}(t, \Psi(\mu_l)) = X_t(\boldsymbol{x}).\end{aligned}$$

In the calculations above we used that a_{lm} are measurable w.r.t. $\mathfrak{F}_T$, B_{τ_t} is independent of $\mathfrak{F}_T$, $\mathbf{E}[Y_{lm}(\boldsymbol{x} + B_{\tau_t})] = Y_{lm}(\boldsymbol{x})\mathbf{E}e^{-\mu_l\tau_t}$ (see [13]), and $\mathbf{E}e^{-\mu_l\tau_t} = \mathbf{E}e^{-\mu_l F(L_t)} = \mathbf{E}e^{-\Psi(\mu_l)L_t} = \widetilde{l}(t, \Psi(\mu_l))$. The proof is concluded. □

Remark 3. In the case where L_t is an inverse stable subordinator, that is, $\Phi(s) = s^\beta$, the derivative $\mathfrak{D}^\Phi_t$ becomes the C-D fractional derivative, $\widetilde{l}(t, \mu)$ is given by the Mittag-Leffler function $\widetilde{l}(t, \mu) = E_\beta(-t^\beta\mu)$, Theorem 1 reduces to first part of Theorem 1 in [14]. From Theorem 2 it follows that some correction is needed for the second part of Theorem 1 in [14]. Namely, the representation (3.9) therein should be stated for $\gamma = 0$. Indeed, if $\gamma \neq 0$, the same arguments as in the above proof of Theorem 2 and the use of (4.8) will lead to the expression of $Z_t(\boldsymbol{x})$ in the following form: $Z_t(\boldsymbol{x}) = \sum_{lm} a_{lm}Y_{lm}(\boldsymbol{x})\widetilde{l}(t, \mu_l\gamma + \Psi(\mu_l))$. This expression differs form the representation (4.2) for $X_t(\boldsymbol{x})$ if $\gamma \neq 0$.

Remark 4. One particular case is $\Phi = \Psi$, that is, both space and time derivatives in Equation (4.1) are related to the same Bernštein function.

Remark 5. Equation (4.1) can be considered, in particular, with the following fractional diffusion operator

$$\psi(-\Delta_{\mathbb{S}^2_1}) := (-\Delta_{\mathbb{S}^2_1})^{\alpha/2}(I - \Delta_{\mathbb{S}^2_1})^{\gamma/2}, \tag{4.13}$$

where $\psi(t) := t^{\alpha/2}(1+t)^{\gamma/2}$, $\alpha \in (0, 2]$, $\gamma > 0$, and the representation (4.2) holds true. It should be noted that there is a Lévy subordinator with Laplace exponent $\psi(t)$ (see Theorem 2 in [3]). For the fractional operator (4.13) the eigenvalues are $\psi(\mu_l) = \mu_l^{\alpha/2}(1+\mu_l)^{\gamma/2}$ (see [12], pp. 119–120). Therefore, the representation of the solution of the form (4.2) holds with such $\psi(\mu_l)$ inserted instead of $\Psi(\mu_l)$.

Equation (4.1) can be considered for more general functions Ψ, not only for Bernštein functions. Indeed, the proof relies on two main facts given by (4.4) and (4.5), that is, we need to know the eigenfunctions and eigenvalues for the operators. However, if Ψ is a Bernštein function, then it is possible to have a deeper insight into the structure of the field $X_t(x)$ and obtain not only its Karhunen–Loève expansion, but also its representation as a coordinate-changed random field as stated in Theorem 2.

Remark 6. The random field (4.2) obtained as solution to the fractional Cauchy problem (4.1) can serve to construct more involved models, in particular, can be used as an initial condition for fractional SPDE (see, e.g., [2]). Possible further extensions can be also achieved via introducing into the model (4.1) a driving (fractional) Brownian noise, similarly to the studies undertaken in [2, 4].

Example. Consider the tempered stable subordinator H, with the Bernštein function

$$\Phi(\lambda) = (\lambda + \beta)^\alpha - \beta^\alpha, \quad \alpha \in (0, 1), \beta > 0. \tag{4.14}$$

The corresponding Lévy measure and its tail are given by the formulas

$$\Pi(dz) = \frac{1}{\Gamma(1-\alpha)}\alpha e^{-\beta z} z^{-\alpha-1} dz; \quad \overline{\Pi(z)} = \frac{1}{\Gamma(1-\alpha)}\alpha\beta^\alpha \Gamma(-\alpha, z),$$

correspondingly, where $\Gamma(-\alpha, z) = \int_z^\infty e^{-v} v^{-\alpha-1} dv$ is the incomplete Gamma function.

The generalized C-D convolution-type derivative (2.3) for Φ, given by (4.14), becomes

$$\mathfrak{D}_t^\Phi u(t) = \frac{\alpha\beta^\alpha}{\Gamma(1-\alpha)} \int_0^t \frac{\partial}{\partial t} u(t-s)\Gamma(-\alpha, s) ds. \tag{4.15}$$

We can consider Equation (4.1) with such derivative in time, and then in the representation of the solution (4.2) we will have the Laplace transform $\widetilde{l}(t, \lambda)$ of the density of the inverse tempered stable subordinator, the formula of which is given, e.g., in [1]. As we can see from the results below, $\widetilde{l}(t, \lambda)$ appears also in the expressions for the moments of the fields (4.2). Therefore, we obtain the model of random fields on the sphere with different representation and different properties than that considered in [14].

The next result gives expressions for the higher-order moments of the solution (4.2).

Proposition 2. *For $n \in \mathbb{N}$, the higher-order moments of (4.2) are given by*

$$\mathbf{E}[(X_t(\boldsymbol{x}))^n] = \sum_{l_1=0}^{\infty} \cdots \sum_{l_n=0}^{\infty} \Big(\prod_{j=1}^{n} \widetilde{l}(t, \gamma + \Psi(\mu_{l_j})) \Big) \sqrt{\frac{\prod_{j=1}^{n}(2l_j+1)}{(4\pi)^n}} \mathbf{E}[a_{l_1 0} \cdots a_{l_n 0}]. \tag{4.16}$$

Proof. We follow the proof of Proposition 1 in [13]. We have

$$\mathbf{E}[(X_t(\boldsymbol{x}))] = \sum_{l_1=0}^{\infty} \cdots \sum_{l_n=0}^{\infty} \mathbf{E}\left[\prod_{j=1}^{n} T_{l_j}(\boldsymbol{x}) \widetilde{l}(t, \gamma + \Psi(\mu_{l_j})) \right],$$

where

$$\mathbf{E}\left[\prod_{j=1}^{n} T_{l_j}(\boldsymbol{x}) \right] = \sum_{m_1=-l_1}^{+l_1} \cdots \sum_{m_n=-l_n}^{+l_n} \mathbf{E}[a_{l_1 m_1} \cdots a_{l_n m_n}] \prod_{j=1}^{n} Y_{l_j m_j}(\boldsymbol{x}).$$

Since the random field T is isotropic, we take advantage of the property that $T_l(\boldsymbol{x}) \stackrel{law}{=} T(\boldsymbol{x}_N)$ where $\boldsymbol{x}_N = (0,0)$ is the North Pole and that $Y_{lm}(\boldsymbol{x}_N) = 0$ for $m \neq 0$ and $Y_{l0}(\boldsymbol{x}_N) = \sqrt{(2l+1)/4\pi}$ (see [24]). We obtain that

$$\mathbf{E}\left[\prod_{j=1}^{n} T_{l_j}(\boldsymbol{x}) \right] = \sqrt{\frac{\prod_{j=1}^{n}(2l_j+1)}{(4\pi)^n}} \mathbf{E}[a_{l_1 0} \cdots a_{l_n 0}].$$

By collecting all pieces together we get the claimed result. □

Note, that $\mathbf{E}[(X_t(g\boldsymbol{x}))^n] = \mathbf{E}[(X_t(\boldsymbol{x}))^n]$, $\forall\, g \in SO(3)$.

4.2 *Angular power spectrum*

Under isotropy, the harmonic coefficients $\{a_{lm} : l \geq 0,\ |m| \leq l\}$ appearing in (3.1) are such that the power spectrum $\{C_l = \mathbf{E}|a_{lm}|^2 : l \geq 0\}$ associated with the random field T depends uniquely on the frequency l. The variance of T can be written as

$$\mathbf{E}[T(\boldsymbol{x})]^2 = \sum_{l=0}^{\infty} \frac{2l+1}{4\pi} C_l \quad \text{for all } \boldsymbol{x} \in \mathbb{S}_1^2, \tag{4.17}$$

and thus, to ensure $\mathbf{E}[T(\boldsymbol{x})]^2 < \infty$, we can require, in particular, the power spectrum to be such that $C_l \sim l^{-\theta}$ as $l \to \infty$ with $\theta > 2$. As we can see from (4.17), the correlation structure of T is strictly related to the collection $\{C_l : l \geq 0\}$ of the angular power spectrum.

An interesting review on the characterization of random fields on the sphere $\mathbb{S}^d$, $d \geq 2$, is given in [7]. The authors consider the Karhunen–Loève expansion in terms of spherical harmonics and Gegenbauer polynomials and discuss questions of integrability and path-continuity in the (space) fractional case. Many authors focuse on the connection between covariance structure, summability and regularity (see, e.g., [19] and references therein). Our next theorem is concerned with subordination of random fields.

Theorem 3. *For the representation* (4.2) *of the solution to* (4.1) *the following propositions hold:*

(i) $X_0 \in \mathcal{H}^s(\mathbb{S}_1^2)$, $s > 3$.

(ii) $\forall t \geq 0$

$$\mathbf{E}[(X_t(\boldsymbol{x}))^2] = \sum_l \frac{2l+1}{4\pi} C_l(t) = \sum_l C_l^*(t), \tag{4.18}$$

where $C_l^*(t) \sim (\overline{\Pi}(t))^2 l^{-\theta-3}$, $\theta > 2$, $t > 0$, *if* $C_l \sim l^{-\theta}$, *as* $l \to \infty$.

(iii) For the angular power spectrum we have that, for $t \geq 0$, $l \geq 0$, $|m| \leq l$,

$$C_l(t) = \mathbf{E}|a_{lm}(t)|^2 \leq C_l \sup_{\sigma \in (0,1)} \Gamma(1+\sigma)\,(\gamma + \Psi(\mu_l))^{-\sigma}\,\mathbf{E}[(L_t)^{-\sigma}].$$

(iv) For $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{S}_1^2$, $t, s \geq 0$, *for all* $g \in SO(3)$, *we have that*

$$\mathbb{E}[X_t(g\boldsymbol{x})\,X_s(g\boldsymbol{y})] = \sum_l \frac{2l+1}{4\pi} C_l\,\big(\widetilde{l}(t, \gamma+\Psi(\mu_l))\big)\big(\widetilde{l}(s, \gamma+\Psi(\mu_l))\big)\,Q_l(\langle \boldsymbol{x}, \boldsymbol{y}\rangle),$$

where Q_l *is given by* (2.10), $\langle \boldsymbol{x}, \boldsymbol{y}\rangle = \cos d(x, y)$ *with* $d(\boldsymbol{x}, \boldsymbol{y})$ *being the spherical distance between the points* $\boldsymbol{x}, \boldsymbol{y}$.

Proof. (i) For the process $X_t(\boldsymbol{x})$ introduced in the previous section we have that

$$\begin{aligned}
\left|\Big(\gamma - \Psi(-\Delta_{\mathbb{S}_1^2})\Big)\,X_t(\boldsymbol{x})\right| &\leq \sum_{lm} |a_{l,m}||\gamma + \Psi(\mu_l)||\widetilde{l}(t, \gamma + \Psi(\mu_l))||Y_{l,m}(\boldsymbol{x})| \\
&\leq \sum_{lm} |a_{l,m}||\gamma + \Psi(\mu_l)||Y_{l,m}(\boldsymbol{x})| \\
&\leq |\gamma| \sum_{lm} |a_{l,m}||Y_{l,m}(\boldsymbol{x})| + \sum_{lm} |a_{l,m}||\Psi(\mu_l)||Y_{l,m}(\boldsymbol{x})|
\end{aligned}$$

and (recall that $|Y_{lm}| \leq l^{1/2}$)

$$\begin{aligned}
&\sum_{lm} \mathbf{E}|a_{l,m}|^2|Y_{l,m}(\boldsymbol{x})|^2 \leq \sum_l \frac{2l+1}{4\pi}\, l\, C_l, \\
&\sum_{lm} \mathbf{E}|a_{l,m}|^2|\Psi(\mu_l)|^2|Y_{l,m}(\boldsymbol{x})|^2 \leq \sum_l \frac{2l+1}{4\pi} |\Psi(\mu_l)|^2\, l\, C_l.
\end{aligned}$$

Thus, from (2.19), (3.5) we obtain $\big(\gamma - \Psi(-\Delta_{\mathbb{S}_1^2})\big)X_t(\boldsymbol{x}) \in \mathcal{H}^{2s}(\mathbb{S}_1^2)$, $2s > 6$.

(ii)

$$\mathbf{E}[(X_t(\boldsymbol{x}))^2] = \sum_{l \geq 0} \frac{2l+1}{4\pi} \big(\widetilde{l}(t, \gamma + \Psi(\mu_l))\big)^2 C_l$$

is finite if $\forall\, t$, $\big(\widetilde{l}(t, \gamma + \Psi(\mu_l))\big)^2 C_l \sim l^{-\theta(\gamma)}$, with $\theta(\gamma) > 2$.

The angular power spectrum of $X_t(\boldsymbol{x})$ can be written as

$$C_l(t) = \left|\mathbf{E}[e^{-(\gamma+\Psi(\mu_l))L_t}]\right|^2 C_l$$

from which we get the Laplace transform

$$\varphi(\lambda, l) := \int_0^\infty e^{-\lambda t} \sqrt{C_l(t)}\, dt = \frac{\Phi(\lambda)}{\lambda} \frac{\sqrt{C_l}}{\gamma + \Psi(\mu_l) + \Phi(\lambda)}. \tag{4.19}$$

If $\sqrt{C_l}/\Psi(\mu_l) \to d_\Psi \geq 0$ as $l \to \infty$, then

$$\varphi(\lambda, l) \to \frac{\Phi(\lambda)}{\lambda} d_\Psi = \int_0^\infty e^{-\lambda t} \left(d_\Psi \overline{\Pi}(t)\right) dt \quad \text{as} \quad l \to \infty,$$

where $\overline{\Pi}$ has been defined in (2.2). Since $\Psi(\mu_l) \to \infty$ as $l \to \infty$, we get that

$$\varphi(\lambda, l) \sim \frac{\Phi(\lambda)}{\lambda} \frac{\sqrt{C_l}}{\Psi(l^2)} = \int_0^\infty e^{-\lambda t} \left(\frac{\sqrt{C_l}}{\Psi(l^2)}\, \overline{\Pi}(t)\right) dt.$$

Thus, we conclude that, for $t > 0$,

$$C_l(t) \sim \left(\frac{\overline{\Pi}(t)}{\Psi(l^2)}\right)^2 C_l.$$

(iii) We notice that

$$\mathbf{E}[e^{-qL_t}] = \int_0^1 \mathbf{P}(e^{-qL_t} > s) ds \leq \int_0^1 \frac{\mathbf{E}[f(e^{-qL_t})]}{f(s)} ds$$

for f nonnegative and nondecreasing in the set $\{e^{-qL_t} > s\}$. By choosing $f(s) = (-\ln s)^{-\sigma}$, $\sigma \in (0,1)$, $s \in (0,1)$, we obtain

$$\mathbf{E}[e^{-qL_t}] \leq \mathbf{E}[(qL_t)^{-\sigma}] \int_0^1 \frac{ds}{(-\ln s)^{-\sigma}} = q^{-\sigma}\, \Gamma(1+\sigma) \mathbf{E}[(L_t)^{-\sigma}].$$

From (4.19) we get the result.

(iv) The expression for the covariance is derived following the same arguments as in [14], Theorem 3. □

Remark 7. Let us consider the special case $\Phi(\lambda) = \lambda^\beta$ and $\Psi(\xi) = \xi^\alpha$. First, we have

$$C_l(t) \leq \Gamma(1+\sigma)\, (\gamma + (\mu_l)^\alpha)^{-\sigma}\, \mathbf{E}[(L_t)^{-\sigma}]\, C_l.$$

Since

$$\int_0^\infty e^{-\lambda t}\mathbf{E}[(L_t)^{-\sigma}]\,dt = \frac{(\Phi(\lambda))^\sigma}{\lambda}\Gamma(1-\sigma) = \Gamma(1-\sigma)\,\lambda^{\beta\sigma-1},$$

we obtain that $\mathbf{E}[(L_t)^{-\sigma}] = \frac{\Gamma(1-\sigma)}{\Gamma(1-\beta\sigma)}\,t^{-\beta\sigma}$. Thus, $\forall\, t>0$,

$$C_l(t) \le C_l \sup_\sigma \frac{\sigma\pi}{\sin\sigma\pi}\,(\gamma+(\mu_l)^\alpha)^{-\sigma}\,\frac{t^{-\beta\sigma}}{\Gamma(1-\beta\sigma)}, \quad l\ge 0.$$

Remark 8 (High-resolution or high-frequency analysis)**.** The convergence rate of (4.17) depends on C_l, in particular, on the high-frequency behaviour of C_l and therefore, on the high-frequency resolution of T. In (3.1), $T_l(x) = \sum_{|m|\le l} a_{lm}Y_{lm}(x)$ represents the l-th frequency component of T, and in real data, we get more and more information (or resolution) as l increases. In physical experiments, when we measure, e.g., the CMB radiation, the power spectrum of the spherical random fields T is usually unknown and we are interested in the empirical counterpart of the angular power spectrum (see [21]) $\widehat{C_l} = \frac{1}{2l+1}\sum_{m=-l}^{+l}|a_{lm}|^2$. Thus, we may be interested in the high-frequency consistency of $\{\widehat{C_l}\,:\,l\ge 0\}$ or the high-frequency ergodicity of T. In the high-frequency analysis, the behavior of angular power spectrum of the fields considered in this paper depends on Ψ and Φ and, therefore, we introduce a large class of models in which many important practical aspects can be captured.

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