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Asymptotic normality of randomized periodogram for estimating quadratic variation in mixed Brownian–fractional Brownian model

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Abstract We study asymptotic normality of the randomized periodogram estimator of quadratic variation in the mixed Brownian–fractional Brownian model. In the semimartingale case, that is, where the Hurst parameter H of the fractional part satisfies $H \in (3/4, 1)$, the central limit theorem holds. In the nonsemimartingale case, that is, where $H \in (1/2, 3/4]$, the convergence toward the normal distribution with a nonzero mean still holds if $H = 3/4$, whereas for the other values, that is, $H \in (1/2, 3/4)$, the central convergence does not take place. We also provide Berry–Esseen estimates for the estimator.

Keywords Central limit theorem, multiple Wiener integrals, Malliavin calculus, fractional Brownian motion, quadratic variation, randomized periodogram

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1 Introduction and motivation

The quadratic variation, or the pathwise volatility, of stochastic processes is of paramount importance in mathematical finance. Indeed, it was the major discovery of the celebrated article by Black and Scholes [8] that the prices of financial derivatives depend only on the volatility of the underlying asset. In the Black–Scholes model of geometric Brownian motion, the volatility simply means the variance. Later the Brownian model was extended to more general semimartingale models. Delbaen and Schachermayer [10, 11] gave the final word on the pricing of financial derivatives with semimartingales. In all these models, the volatility simply meant the variance or the semimartingale quadratic variance. Now, due to the important article by Föllmer [13], it is clear that the variance is not the volatility. Instead, one should consider the pathwise quadratic variation. This revelation and its implications to mathematical finance has been studied, for example, in [6, 23].

An important class of pricing models is the mixed Brownian–fractional Brownian model. This is a model where the quadratic variation is determined by the Brownian part and the correlation structure is determined by the fractional Brownian part. Thus, this is a pricing model that captures the long-range dependence while leaving the Black–Scholes pricing formulas intact. The mixed Brownian–fractional Brownian model has been studied in the pricing context, for example, in [1, 5, 7].

By the hedging paradigm the prices and hedges of financial derivative depend only on the pathwise quadratic variation of the underlying process. Consequently, the statistical estimation of the quadratic variation is an important problem. One way to estimate the quadratic variation is to use directly its definition by the so-called *realized quadratic variation*. Although the consistency result (see Section 2.1) does not depend on a specific choice of the sampling scheme, the asymptotic distribution does. There are numerous articles that study the asymptotic behavior of realized quadratic variation; see [4, 3, 16, 14, 15] and references therein. Another approach, suggested by Dzhaparidze and Spreij [12], is to use the randomized periodogram estimator. In [12], the case of semimartingales was studied. In [2], the randomized periodogram estimator was studied for the mixed Brownian–fractional Brownian model, and the weak consistency of the estimator was proved. This article investigates the asymptotic normality of the randomized periodogram estimator for the mixed Brownian–fractional Brownian model.

The rest of the paper is organized as follows. In Section 2, we briefly introduce the two estimators for the quadratic variation already mentioned. In Section 3, we introduce the stochastic analysis for Gaussian processes needed for our results. In particular, we introduce the Föllmer pathwise calculus and Malliavin calculus. Section 4 contains our main results: the central limit theorem for the randomized periodogram estimator and an associated Berry–Esseen bound. Finally, some technical calculations are deferred into Appendix A.1 and Appendix A.2.

2 Two methods for estimating quadratic variation

2.1 Using discrete observations: realized quadratic variation

It is well known that (see [22, Chapter 6]) for a semimartingale X, the bracket $[X, X]$ can be identified with

$$[X,X]_t = \mathbb{P}\text{-}\lim_{|\pi|\to 0}\sum_{t_k\in\pi}(X_{t_k}-X_{t_{k-1}})^2,$$

where $\pi = \{t_k : 0 = t_0 < t_1 < \cdots < t_n = t\}$ is a partition of the interval $[0,t]$, $|\pi| = \max\{t_k - t_{k-1} : t_k \in \pi\}$, and $\mathbb{P}$-lim means convergence in probability. Statistically speaking, the sums of squared increments (*realized quadratic variation*) is a consistent estimator for the bracket as the volume of observations tends to infinity. Barndorff-Nielsen and Shephard [3] studied precision of the realized quadratic variation estimator for a special class of continuous semimartingales. They showed that sometimes the realized quadratic variation estimator can be a rather noisy estimator. So one should seek for new estimators of the quadratic variation.

2.2 Using continuous observations: randomized periodogram

Dzhaparidze and Spreij [12] suggested another characterization of the bracket $[X,X]$. Let $\mathbb{F}^X$ be the filtration of X, and τ be a finite stopping time. For $\lambda \in \mathbb{R}$, define the *periodogram* $I_\tau(X;\lambda)$ of X at τ by

$$\begin{aligned} I_\tau(X;\lambda) :&= \left|\int_0^\tau e^{i\lambda s}\mathrm{d}X_s\right|^2 \\ &= 2\,\mathbf{Re}\int_0^\tau\int_0^t e^{i\lambda(t-s)}\mathrm{d}X_s\mathrm{d}X_t + [X,X]_\tau \quad \text{(by Itô formula).} \end{aligned} \tag{1}$$

Let ξ be a symmetric random variable independent of the filtration $\mathbb{F}^X$ with density g_ξ and real characteristic function φ_ξ. For given $L > 0$, define the randomized periodogram by

$$\mathbb{E}_\xi I_\tau(X;L\xi) = \int_{\mathbb{R}} I_\tau(X;Lx)g_\xi(x)\mathrm{d}x. \tag{2}$$

If the characteristic function φ_ξ is of bounded variation, then Dzhaparidze and Spreij have shown that we have the following characterization of the bracket as $L \to \infty$:

$$\mathbb{E}_\xi I_\tau(X;L\xi) \xrightarrow{\mathbb{P}} [X,X]_\tau. \tag{3}$$

Recently, the convergence (3) is extended in [2] to some class of stochastic processes which contains nonsemimartingales in general. Let $W = \{W_t\}_{t\in[0,T]}$ be a standard Brownian motion, and $B^H = \{B_t^H\}_{t\in[0,T]}$ be a fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2},1)$, independent of the Brownian motion W. Define the mixed Brownian–fractional Brownian motion X_t by

$$X_t = W_t + B_t^H, \quad t \in [0,T].$$

Remark 1. It is known that (see [9]) the process X is an $(\mathbb{F}^X,\mathbb{P})$-semimartingale if $H \in (\frac{3}{4},1)$, and for $H \in (\frac{1}{2},\frac{3}{4}]$, X is not a semimartingale with respect to its own filtration $\mathbb{F}^X$. Moreover, in both cases, we have

$$[X,X]_t = t. \tag{4}$$

If the partitions in (4) are nested, that is, for each n, we have $\pi^{(n)} \subset \pi^{(n+1)}$, then the convergence can be strengthened to almost sure convergence. Hereafter, we always assume that the sequences of partitions are nested.

Given $\lambda \in \mathbb{R}$, define the periodogram of X at T as (1), that is,

$$\begin{aligned} I_T(X;\lambda) &= \left| \int_0^T e^{i\lambda t} \mathrm{d}X_t \right|^2 \\ &= \left| e^{i\lambda T} X_T - i\lambda \int_0^T X_t e^{i\lambda t} \mathrm{d}t \right|^2 \\ &= X_T^2 + X_T \int_0^T i\lambda \big(e^{i\lambda(T-t)} - e^{-i\lambda(T-t)} \big) X_t \mathrm{d}t + \lambda^2 \left| \int_0^T e^{i\lambda t} X_t \mathrm{d}t \right|^2 . \end{aligned}$$

Let $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ be another probability space. We identify the σ-algebra $\mathcal{F}$ with $\mathcal{F} \otimes \{\phi, \tilde{\Omega}\}$ on the product space $(\Omega \times \tilde{\Omega}, \mathcal{F} \otimes \tilde{\mathcal{F}}, \mathbb{P} \otimes \tilde{\mathbb{P}})$. Let $\xi : \tilde{\Omega} \to \mathbb{R}$ be a real symmetric random variable with density g_ξ and independent of the filtration $\mathbb{F}^X$. For any positive real number L, define the randomized periodogram $\mathbb{E}_\xi I_T(X; L\xi)$ as in (2) by

$$\mathbb{E}_\xi I_T(X; L\xi) := \int_{\mathbb{R}} I_T(X; Lx) g_\xi(x) \mathrm{d}x, \tag{5}$$

where the term $I_T(X; Lx)$ is understood as before. Azmoodeh and Valkeila [2] proved the following:

Theorem 1. *Assume that X is a mixed Brownian–fractional Brownian motion, $\mathbb{E}_\xi I_T(X; L\xi)$ be the randomized periodogram given by (5), and*

$$\mathbb{E}\xi^2 < \infty.$$

Then, as $L \to \infty$, we have

$$\mathbb{E}_\xi I_T(X; L\xi) \xrightarrow{\mathbb{P}} [X, X]_T.$$

3 Stochastic analysis for Gaussian processes

3.1 Pathwise Itô formula

Föllmer [13] obtained a pathwise calculus for continuous functions with finite quadratic variation. The next theorem essentially belongs to Föllmer. For a nice exposition and its use in finance, see Sondermann [24].

Theorem 2 ([24])**.** *Let $X : [0, T] \to \mathbb{R}$ be a continuous process with continuous quadratic variation $[X, X]_t$, and let $F \in C^2(\mathbb{R})$. Then for any $t \in [0, T]$, the limit of the Riemann–Stieltjes sums*

$$\lim_{|\pi| \to 0} \sum_{t_i \leq t} F_x(X_{t_{i-1}})(X_{t_i} - X_{t_{i-1}}) := \int_0^t F_x(X_s) \mathrm{d}X_s$$

exists almost surely. Moreover, we have

$$F(X_t) = F(X_0) + \int_0^t F_x(X_s) \mathrm{d}X_s + \frac{1}{2} \int_0^t F_{xx}(X_s) \mathrm{d}[X, X]_s. \tag{6}$$

The rest of the section contains the essential elements of Gaussian analysis and Malliavin calculus that are used in this paper. See, for instance, Refs. [17, 18] for further details. In what follows, we assume that all the random objects are defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

3.2 Isonormal Gaussian processes derived from covariance functions

Let $X = \{X_t\}_{t\in[0,T]}$ be a centered continuous Gaussian process on the interval $[0, T]$ with $X_0 = 0$ and continuous covariance function $R_X(s,t)$. We assume that $\mathcal{F}$ is generated by X. Denote by $\mathcal{E}$ the set of real-valued step functions on $[0, T]$, and let $\mathfrak{H}$ be the Hilbert space defined as the closure of $\mathcal{E}$ with respect to the scalar product

$$\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]}\rangle_{\mathfrak{H}} = R_X(t,s), \quad s,t \in [0,T].$$

For example, when X is a Brownian motion, $\mathfrak{H}$ reduces to the Hilbert space $L^2([0,T], \mathrm{d}t)$. However, in general, $\mathfrak{H}$ is not a space of functions, for example, when X is a fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$ (see [21]). The mapping $\mathbf{1}_{[0,t]} \longmapsto X_t$ can be extended to a linear isometry between $\mathfrak{H}$ and the Gaussian space $\mathcal{H}_1$ spanned by a Gaussian process X. We denote this isometry by $\varphi \longmapsto X(\varphi)$, and $\{X(\varphi);\ \varphi \in \mathfrak{H}\}$ is an isonormal Gaussian process in the sense of [18, Definition 1.1.1], that is, it is a Gaussian family with covariance function

$$\begin{aligned}\mathbb{E}\big(X(\varphi_1)X(\varphi_2)\big) &= \langle \varphi_1, \varphi_2\rangle_{\mathfrak{H}} \\ &= \int_{[0,T]^2} \varphi_1(s)\varphi_2(t)\mathrm{d}R_X(s,t), \quad \varphi_1, \varphi_2 \in \mathcal{E},\end{aligned}$$

where $\mathrm{d}R_X(s,t) = R_X(\mathrm{d}s, \mathrm{d}t)$ stands for the measure induced by the covariance function R_X on $[0,T]^2$. Let $\mathcal{S}$ be the space of smooth and cylindrical random variables of the form

$$F = f\big(X(\varphi_1), \dots, X(\varphi_n)\big), \tag{7}$$

where $f \in C_b^\infty(\mathbb{R}^n)$ (f and all its partial derivatives are bounded). For a random variable F of the form (7), we define its Malliavin derivative as the $\mathfrak{H}$-valued random variable

$$DF = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\big(X(\varphi_1), \dots, X(\varphi_n)\big)\varphi_i.$$

By iteration, the mth derivative $D^m F \in L^2(\Omega; \mathfrak{H}^{\otimes m})$ is defined for every $m \geq 2$. For $m \geq 1$, $\mathbb{D}^{m,2}$ denotes the closure of $\mathcal{S}$ with respect to the norm $\|\cdot\|_{m,2}$, defined by the relation

$$\|F\|_{m,2}^2 \ = \ \mathbb{E}\big[|F|^2\big] + \sum_{i=1}^{m} \mathbb{E}\big(\|D^i F\|_{\mathfrak{H}^{\otimes i}}^2\big).$$

Let δ be the adjoint of the operator D, also called the *divergence operator*. A random element $u \in L^2(\Omega, \mathfrak{H})$ belongs to the domain of δ, denoted $\mathrm{Dom}(\delta)$, if and only if it satisfies

$$\big|\mathbb{E}\langle DF, u\rangle_{\mathfrak{H}}\big| \leq c_u \,\|F\|_{L^2}$$

for any $F \in \mathbb{D}^{1,2}$, where c_u is a constant depending only on u. If $u \in \mathrm{Dom}(\delta)$, then the random variable $\delta(u)$ is defined by the duality relationship

$$\mathbb{E}\big(F\delta(u)\big) = \ E\langle DF, u\rangle_{\mathfrak{H}}, \tag{8}$$

which holds for every $F \in \mathbb{D}^{1,2}$. The divergence operator δ is also called the Skorokhod integral because when the Gaussian process X is a Brownian motion, it coincides with the anticipating stochastic integral introduced by Skorokhod [18]. We denote $\delta(u) = \int_0^T u_t \delta X_t$.

For every $q \geq 1$, the symbol $\mathcal{H}_q$ stands for the qth *Wiener chaos* of X, defined as the closed linear subspace of $L^2(\Omega)$ generated by the family $\{H_q(X(h)) : h \in \mathfrak{H}, \|h\|_{\mathfrak{H}} = 1\}$, where H_q is the qth Hermite polynomial defined as

$$H_q(x) = (-1)^q e^{\frac{x^2}{2}} \frac{d^q}{dx^q}\big(e^{-\frac{x^2}{2}}\big). \tag{9}$$

We write by convention $\mathcal{H}_0 = \mathbb{R}$. For any $q \geq 1$, the mapping $I_q^X(h^{\otimes q}) = H_q(X(h))$ can be extended to a linear isometry between the symmetric tensor product $\mathfrak{H}^{\odot q}$ (equipped with the modified norm $\sqrt{q!}\|\cdot\|_{\mathfrak{H}^{\otimes q}}$) and the qth Wiener chaos $\mathcal{H}_q$. For $q = 0$, we write by convention $I_0^X(c) = c$, $c \in \mathbb{R}$. For any $h \in \mathfrak{H}^{\odot q}$, the random variable $I_q^X(h)$ is called a multiple Wiener–Itô integral of order q. A crucial fact is that if $\mathfrak{H} = L^2(A, \mathcal{A}, \nu)$, where ν is a σ-finite and nonatomic measure on the measurable space $(A, \mathcal{A})$, then $\mathfrak{H}^{\odot q} = L_s^2(\nu^q)$, where $L_s^2(\nu^q)$ stands for the subspace of $L^2(\nu^q)$ composed of the symmetric functions. Moreover, for every $h \in \mathfrak{H}^{\odot q} = L_s^2(\nu^q)$, the random variable $I_q^X(h)$ coincides with the q-fold multiple Wiener–Itô integral of h with respect to the centered Gaussian measure (with control ν) generated by X (see [18]). We will also use the following central limit theorem for sequences living in a fixed Wiener chaos (see [20, 19]).

Theorem 3. *Let $\{F_n\}_{n\geq 1}$ be a sequence of random variables in the qth Wiener chaos, $q \geq 2$, such that $\lim_{n\to\infty} \mathbb{E}(F_n^2) = \sigma^2$. Then, as $n \to \infty$, the following asymptotic statements are equivalent:*

(i) *F_n converges in law to $\mathcal{N}(0, \sigma^2)$.*

(ii) *$\|DF_n\|_{\mathfrak{H}}^2$ converges in L^2 to $q\sigma^2$.*

To obtain Berry–Esseen-type estimate, we shall use the following result from [17, Corollary 5.2.10].

Theorem 4. *Let $\{F_n\}_{n\geq 1}$ be a sequence of elements in the second Wiener chaos such that $\mathbb{E}(F_n^2) \to \sigma^2$ and $\operatorname{Var}\|DF_n\|_{\mathfrak{H}}^2 \to 0$ as $n \to \infty$. Then, $F_n \overset{law}{\to} Z \sim \mathcal{N}(0, \sigma^2)$, and*

$$\sup_{x\in\mathbb{R}} \big|\mathbb{P}(F_n < x) - \mathbb{P}(Z < x)\big| \leq \frac{2}{\mathbb{E}(F_n^2)}\sqrt{\operatorname{Var}\|DF_n\|_{\mathfrak{H}}^2} + \frac{2|\mathbb{E}(F_n^2) - \sigma^2|}{\max\{\mathbb{E}(F_n^2), \sigma^2\}}.$$

3.3 Isonormal Gaussian process associated with two Gaussian processes

In this subsection, we briefly describe how two Gaussian processes can be embedded into an isonormal Gaussian process. Let X_1 and X_2 be two independent centered continuous Gaussian processes with $X_1(0) = X_2(0) = 0$ and continuous covariance functions R_{X_1} and R_{X_2}, respectively. Assume that $\mathfrak{H}_1$ and $\mathfrak{H}_2$ denote the associated Hilbert spaces as explained in Section 3.2. The appropriate set $\tilde{\mathcal{E}}$ of elementary functions is the set of the functions that can be written as $\varphi(t, i) = \delta_{1i}\varphi_1(t) + \delta_{2i}\varphi_2(t)$ for

$(t, i) \in [0, T] \times \{1, 2\}$, where $\varphi_1, \varphi_2 \in \mathcal{E}$, and δ_{ij} is the Kronecker's delta. On the set $\tilde{\mathcal{E}}$, we define the inner product

$$\begin{aligned}\langle \varphi, \psi \rangle_{\tilde{\mathfrak{H}}} &:= \langle \varphi(\cdot, 1), \psi(\cdot, 1) \rangle_{\mathfrak{H}_1} + \langle \varphi(\cdot, 2), \psi(\cdot, 2) \rangle_{\mathfrak{H}_2} \\ &= \int_{[0,T]^2} \varphi(s, 1)\psi(t, 1)\mathrm{d}R_{X_1}(s, t) + \int_{[0,T]^2} \varphi(s, 2)\psi(t, 2)\mathrm{d}R_{X_2}(s, t),\end{aligned} \tag{10}$$

where $\mathrm{d}R_{X_i}(s, t) = R_{X_i}(\mathrm{d}s, \mathrm{d}t),\ i = 1, 2$.

Let $\mathfrak{H}$ denote the Hilbert space that is the completion of $\tilde{\mathcal{E}}$ with respect to the inner product (10). Notice that $\mathfrak{H} \cong \mathfrak{H}_1 \oplus \mathfrak{H}_2$, where $\mathfrak{H}_1 \oplus \mathfrak{H}_2$ is the direct sum of the Hilbert spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$, that is, it is a Hilbert space consisting of elements of the form of ordered pairs $(h_1, h_2) \in \mathfrak{H}_1 \times \mathfrak{H}_2$ equipped with the inner product $\langle (h_1, h_2), (g_1, g_2) \rangle_{\mathfrak{H}_1 \oplus \mathfrak{H}_2} := \langle h_1, g_1 \rangle_{\mathfrak{H}_1} + \langle h_2, g_2 \rangle_{\mathfrak{H}_2}$.

Now, for any $\varphi \in \tilde{\mathcal{E}}$, we define $X(\varphi) := X_1(\varphi(\cdot, 1)) + X_2(\varphi(\cdot, 2))$. Using the independence of X_1 and X_2, we infer that $\mathbb{E}(X(\varphi)X(\psi)) = \langle \varphi_1, \psi \rangle_{\mathfrak{H}}$ for all $\varphi, \psi \in \tilde{\mathcal{E}}$. Hence, the mapping X can be extended to an isometry on $\mathfrak{H}$, and therefore $\{X(h),\ h \in \mathfrak{H}\}$ defines an isonormal Gaussian process associated to the Gaussian processes X_1 and X_2.

3.4 Malliavin calculus with respect to (mixed Brownian) fractional Brownian motion

The fractional Brownian motion $B^H = \{B_t^H\}_{t \in \mathbb{R}}$ with Hurst parameter $H \in (0, 1)$ is a zero-mean Gaussian process with covariance function

$$\mathbb{E}\big(B_t^H B_s^H\big) = R_H(s, t) = \frac{1}{2}\big(|t|^{2H} + |s|^{2H} - |t - s|^{2H}\big). \tag{11}$$

Let $\mathfrak{H}$ denote the Hilbert space associated to the covariance function R_H; see Section 3.2. It is well known that for $H = \frac{1}{2}$, we have $\mathfrak{H} = L^2([0, T])$, whereas for $H > \frac{1}{2}$, we have $L^2([0, T]) \subset L^{\frac{1}{H}}([0, T]) \subset |\mathfrak{H}| \subset \mathfrak{H}$, where $|\mathfrak{H}|$ is defined as the linear space of measurable functions φ on $[0, T]$ such that

$$\|\varphi\|^2_{|\mathfrak{H}|} := \alpha_H \int_0^T \int_0^T \big|\varphi(s)\big|\big|\varphi(t)\big||t - s|^{2H-2}\mathrm{d}s\mathrm{d}t < \infty,$$

where $\alpha_H = H(2H - 1)$.

Proposition 1 ([18], Chapter 5)**.** *Let $\mathfrak{H}$ denote the Hilbert space associated to the covariance function R_H for $H \in (0, 1)$. If $H = \frac{1}{2}$, that is, B^H is a Brownian motion, then for any $\varphi, \psi \in \mathfrak{H} = L^2([0, T], \mathrm{d}t)$, the inner product of $\mathfrak{H}$ is given by the well-known Itô isometry*

$$\mathbb{E}\big(B^{\frac{1}{2}}(\varphi)B^{\frac{1}{2}}(\psi)\big) = \langle \varphi, \psi \rangle_{\mathfrak{H}} = \int_0^T \varphi(t)\psi(t)\mathrm{d}t.$$

If $H > \frac{1}{2}$, then for any $\varphi, \psi \in |\mathfrak{H}|$, we have

$$\mathbb{E}\big(B^H(\varphi)B^H(\psi)\big) = \langle \varphi, \psi \rangle_{\mathfrak{H}} = \alpha_H \int_0^T \int_0^T \varphi(s)\psi(t)|t - s|^{2H-2}\mathrm{d}s\mathrm{d}t. \tag{12}$$

The following proposition establishes the link between pathwise integral and Skorokhod integral in Malliavin calculus associated to fractional Brownian motion and will play an important role in our analysis.

Proposition 2 ([18]). *Let $u = \{u_t\}_{t\in[0,T]}$ be a stochastic process in the space $\mathbb{D}^{1,2}(|\mathfrak{H}|)$ such that almost surely*

$$\int_0^T \int_0^T |D_s u_t||t-s|^{2H-2}\mathrm{d}s\mathrm{d}t < \infty.$$

Then u is pathwise integrable, and we have

$$\int_0^T u_t \mathrm{d}B_t^H = \int_0^T u_t \delta B_t^H + \alpha_H \int_0^T \int_0^T D_s u_t |t-s|^{2H-2}\mathrm{d}s\mathrm{d}t.$$

For further use, we also need the following ancillary facts related to the isonormal Gaussian process derived from the covariance function of the mixed Brownian–fractional Brownian motion. Assume that $X = W + B^H$ stands for a mixed Brownian–fractional Brownian motion with $H > \frac{1}{2}$. We denote by $\mathfrak{H}$ the Hilbert space associated to the covariance function of the process X with inner product $\langle \cdot, \cdot \rangle_{\mathfrak{H}}$. Then a direct application of relation (10) and Proposition 1 yields the following facts. We recall that in what follows the notations I_1^X and I_2^X stand for multiple Wiener integrals of orders 1 and 2 with respect to isonormal Gaussian process X; see Section 3.2.

Lemma 1. *For any $\varphi_1, \varphi_2, \psi_1, \psi_2 \in L^2([0,T])$, we have*

$$\begin{aligned}\mathbb{E}\big(I_1^X(\varphi)I_1^X(\psi)\big) &= \langle \varphi, \psi \rangle_{\mathfrak{H}} \\ &= \int_0^T \varphi(t)\psi(t)\mathrm{d}t + \alpha_H \int_0^T \int_0^T \varphi(s)\psi(t)|t-s|^{2H-2}\mathrm{d}s\mathrm{d}t.\end{aligned}$$

Moreover,

$$\begin{aligned}&\mathbb{E}\big(I_2^X(\varphi_1\otimes\varphi_2)I_2^X(\psi_1\otimes\psi_2)\big) \\ &= 2\langle \varphi_1\otimes\varphi_2, \psi_1\otimes\psi_2\rangle_{\mathfrak{H}^{\otimes 2}} \\ &= \int_{[0,T]^2} \varphi_1(s_1)\psi_1(s_1)\varphi_2(s_2)\psi_2(s_2)\mathrm{d}s_1\mathrm{d}s_2 \\ &\quad + \alpha_H \int_{[0,T]^3} \varphi_1(s_1)\psi_1(s_1)\varphi_2(s_2)\psi_2(t_2)|t_2-s_2|^{2H-2}\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}t_2 \\ &\quad + \alpha_H \int_{[0,T]^3} \varphi_1(s_1)\psi_1(t_1)\varphi_2(s_1)\psi_2(s_1)|t_1-s_1|^{2H-2}\mathrm{d}s_1\mathrm{d}t_1\mathrm{d}s_1 \\ &\quad + \alpha_H^2 \int_{[0,T]^4} \varphi_1(s_1)\psi_1(t_1)\varphi_2(s_2)\psi_2(t_2) \\ &\quad \times |t_1-s_1|^{2H-2}|t_2-s_2|^{2H-2}\mathrm{d}s_1\mathrm{d}t_1\mathrm{d}s_2\mathrm{d}t_2.\end{aligned}$$

4 Main results

Throughout this section, we assume that $X = W + B^H$ is a mixed Brownian–fractional Brownian motion with $H > \frac{1}{2}$, unless otherwise stated. We denote by $\mathfrak{H}$ the Hilbert space associated to process X with inner product $\langle \cdot, \cdot \rangle_{\mathfrak{H}}$.

4.1 Central limit theorem

We start with the following fact, which is one of our key ingredients.

Lemma 2 ([2])**.** *Let $\mathbb{E}\xi^2 < \infty$. Then the randomized periodogram of the mixed Brownian–fractional Brownian motion X given by* (5) *satisfies*

$$\mathbb{E}_\xi I_T(X; L\xi) = [X, X]_T + 2\int_0^T \int_0^t \varphi_\xi\big(L(t-s)\big)\mathrm{d}X_s\mathrm{d}X_t, \tag{13}$$

where φ_ξ is the characteristic function of ξ, and the iterated stochastic integral in the right-hand side is understood pathwise, that is, as the limit of the Riemann–Stieltjes sums; see Section 3.1.

Our first aim is to transform the pathwise integral in (13) into the Skorokhod integral. This is the topic of the next lemma.

Lemma 3. *Let $u_t = \int_0^t \varphi_\xi(L(t-s))\mathrm{d}X_s$, where φ_ξ denotes the characteristic function of a symmetric random variable ξ. Then $u \in \mathrm{Dom}(\delta)$, and*

$$\int_0^T u_t\mathrm{d}X_t = \int_0^T u_t\delta X_t + \alpha_H\int_0^T\int_0^T D_s^{(B^H)}u_t|t-s|^{2H-2}\mathrm{d}s\mathrm{d}t,$$

where the stochastic integral in the right-hand side is the Skorokhod integral with respect to mixed Brownian–fractional Brownian motion X, and $D^{(B^H)}$ denotes the Malliavin derivative operator with respect to the fractional Brownian motion B^H.

Proof. First, note that

$$u_t = u_t^W + u_t^{B^H} = \int_0^t \varphi_\xi\big(L(t-s)\big)\mathrm{d}W_s + \int_0^t \varphi_\xi\big(L(t-s)\big)\mathrm{d}B_s^H.$$

Moreover, $\mathbb{E}(\int_0^T u_t^2\mathrm{d}t) < \infty$, so that $u_t \in \mathbb{D}^{1,2}$ for almost all $t \in [0,T]$ and $\mathbb{E}(\int_{[0,T]^2}(D_s u_t)^2\mathrm{d}s\mathrm{d}t) < \infty$. Hence, $u \in \mathrm{Dom}(\delta)$ by [18, Proposition 1.3.1]. On the other hand,

$$\begin{aligned}
\int_0^T u_t\mathrm{d}X_t &= \int_0^T u_t\mathrm{d}W_t + \int_0^T u_t\mathrm{d}B_t^H \\
&= \int_0^T u_t^W\mathrm{d}W_t + \int_0^T u_t^{B^H}\mathrm{d}W_t + \int_0^T u_t^W\mathrm{d}B_t^H + \int_0^T u_t^{B^H}\mathrm{d}B_t^H \\
&= \int_0^T u_t^W\delta W_t + \int_0^T u_t^{B^H}\delta W_t + \int_0^T u_t^W\delta B_t^H + \int_0^T u_t^{B^H}\delta B_t^H \\
&\quad + \alpha_H\int_0^T\int_0^T D_s^{(B^H)}u_t^{B^H}|t-s|^{2H-2}\mathrm{d}s\mathrm{d}t \\
&= \int_0^T u_t\delta W_t + \int_0^T u_t\delta B_t^H + \alpha_H\int_0^T\int_0^T D_s^{(B^H)}u_t|t-s|^{2H-2}\mathrm{d}s\mathrm{d}t,
\end{aligned}$$

where we have used the independence of W and B^H, Proposition 2, and the fact that for adapted integrands, the Skorokhod integral coincides with the Itô integral. To finish the proof, we use the very definition of Skorokhod integral and relation (8) to obtain that $\int_0^T u_t\delta W_t + \int_0^T u_t\delta B_t^H = \int_0^T u_t\delta X_t$. □

We will also pose the following assumption for characteristic function φ_ξ of a symmetric random variable ξ.

Assumption 1. *The characteristic function φ_ξ satisfies*

$$\int_0^\infty \left|\varphi_\xi(x)\right| \mathrm{d}x < \infty.$$

Remark 2. Note that Assumption 1 is satisfied for many distributions. Especially, if the characteristic function φ_ξ is positive and the density function $g_\xi(x)$ is differentiable, then we get by applying Fubini's theorem and integration by part that

$$\int_0^\infty \varphi_\xi(x)\mathrm{d}x = 2\int_0^\infty \int_0^\infty cos(yx) g_\xi(y)\mathrm{d}y\mathrm{d}x = \pi g_\xi(0) < \infty.$$

We continue with the following technical lemma, which in fact provides a correct normalization for our central limit theorems.

Lemma 4. *Consider the symmetric two-variable function $\psi_L(s,t) := \varphi_\xi(L|t-s|)$ on $[0,T]\times[0,T]$. Then $\psi_L \in \mathfrak{H}^{\otimes 2}$, and moreover, as $L\to\infty$, we have*

$$\lim_{L\to\infty} L\|\psi_L\|^2_{\mathfrak{H}^{\otimes 2}} = \sigma_T^2 < \infty, \tag{14}$$

where $\sigma_T^2 := 2\,T\int_0^\infty \varphi_\xi^2(x)\mathrm{d}x$ is independent of the Hurst parameter H.

Remark 3. We point it out that the variance σ_T^2 in Lemma 4 is finite. This is a simple consequence of Assumption 1 and the fact that the characteristic function φ_ξ is bounded by one over the real line.

Proof. Throughout the proof, C denotes unimportant constant depending on T and H, which may vary from line to line. First, note that clearly $\psi_L \in \mathfrak{H}^{\otimes 2}$ since ψ_L is a bounded function. In order to prove (14), we show that, as $L\to\infty$,

$$\|\psi_L\|^2_{\mathfrak{H}^{\otimes 2}} \sim \frac{1}{L}.$$

Next, by applying Lemma 1 we obtain $\|\psi_L\|^2_{\mathfrak{H}^{\otimes 2}} = A_1 + A_2 + A_3$, where

$$A_1 := \int_{[0,T]^2} \varphi_\xi^2\big(L|t-s|\big)\mathrm{d}t\mathrm{d}s, \tag{15}$$

$$A_2 := \alpha_H \int_{[0,T]^3} \varphi_\xi\big(L|t-u|\big)\varphi_\xi\big(L|s-u|\big)|t-s|^{2H-2}\mathrm{d}t\mathrm{d}s\mathrm{d}u, \tag{16}$$

$$A_3 := \alpha_H^2 \int_{[0,T]^4} \varphi_\xi\big(L|t-u|\big)\varphi_\xi\big(L|s-v|\big)|t-s|^{2H-2}|v-u|^{2H-2}\mathrm{d}u\mathrm{d}v\mathrm{d}t\mathrm{d}s. \tag{17}$$

First, we show that $A_1 \sim \frac{1}{L}$. By change of variables $y = \frac{L}{T}s$ and $x = \frac{L}{T}t$ we obtain

$$A_1 = \frac{T^2}{L^2}\int_0^L \int_0^L \varphi_\xi^2\big(T|x-y|\big)\mathrm{d}x\mathrm{d}y.$$

Now, by applying L'Hôpital's rule and some elementary computations we obtain that

$$\lim_{L\to\infty} L^{-1}\int_0^L\int_0^L \varphi_\xi^2\big(T|x-y|\big)\mathrm{d}x\mathrm{d}y = \lim_{L\to\infty} 2\int_0^L \varphi_\xi^2\big(T(L-x)\big)\mathrm{d}x$$
$$= \frac{2}{T}\int_0^\infty \varphi_\xi^2(y)\mathrm{d}y,$$

which is finite by Assumption 1. Consequently, we get

$$\lim_{L\to\infty} LA_1 = 2T\int_0^\infty \varphi_\xi^2(y)\mathrm{d}y,$$

or, in other words, $A_1 \sim L^{-1}$. To complete the proof, it is shown in Appendix B that $\lim_{L\to\infty} L(A_2+A_3)=0$. □

We also apply the following proposition. The proof is rather technical and is postponed to Appendix A.

Proposition 3. *Consider the symmetric two-variable function* $\psi_L(s,t) := \varphi_\xi(L|t-s|)$ *on* $[0,T]\times[0,T]$. *Denote*

$$\tilde{\psi}_L(t,s) = \frac{\psi_L(s,t)}{\sqrt{2}\|\psi_L\|_{\mathfrak{H}^{\otimes 2}}}.$$

Then, for any $H\in(\frac{1}{2},1)$, *as* $L\to\infty$, *we have*

$$I_2^X(\tilde{\psi}_L) \xrightarrow{law} \mathcal{N}(0,1).$$

Our main theorem is the following.

Theorem 5. *Assume that the characteristic function* φ_ξ *of a symmetric random variable* ξ *satisfies Assumption 1 and let* σ_T^2 *be given by (14). Then, as* $L\to\infty$, *we have the following asymptotic statements:*

1. *if* $H\in(\frac{3}{4},1)$, *then*

$$\sqrt{L}\big(\mathbb{E}_\xi I_T(X;L\xi)-[X,X]_T\big) \xrightarrow{law} \mathcal{N}\big(0,\sigma_T^2\big).$$

2. *if* $H=\frac{3}{4}$, *then*

$$\sqrt{L}\big(\mathbb{E}_\xi I_T(X;L\xi)-[X,X]_T\big) \xrightarrow{law} \mathcal{N}\big(\mu,\sigma_T^2\big),$$

where $\mu = 2\alpha_H T\int_0^\infty \varphi_\xi(x)x^{2H-2}\mathrm{d}x$.

3. *if* $H\in(\frac{1}{2},\frac{3}{4})$, *then*

$$L^{2H-1}\big(\mathbb{E}I_T(X;L\xi)-[X,X]_T\big) \xrightarrow{\mathbb{P}} \mu,$$

where the real number μ *is given in item 2. Notice that when* $H\in(\frac{1}{2},\frac{3}{4})$, *we have* $2H-1<\frac{1}{2}$.

Proof. First, by applying Lemmas 2 and 3 we can write

$$\mathbb{E}I_T(X;L\xi)-[X,X]_T=I_2^X(\psi_L)+\alpha_H\int_0^T\int_0^T\varphi_\xi\big(L|t-s|\big)|t-s|^{2H-2}\mathrm{d}s\mathrm{d}t.$$

Consequently, we obtain

$$\begin{aligned}&\sqrt{L}\big(\mathbb{E}I_T(X;L\xi)-[X,X]_T\big)\\&=\sqrt{L}\,I_2^X(\psi_L)+\sqrt{L}\,\alpha_H\int_0^T\int_0^T\varphi_\xi\big(L|t-s|\big)|t-s|^{2H-2}\mathrm{d}s\mathrm{d}t\\&:=A_1+A_2.\end{aligned}$$

Now, thanks to Proposition 3, for any $H\in(\frac{1}{2},1)$, we have

$$A_1=\sqrt{L}\,\|\psi_L\|_{\mathfrak{H}^{\otimes 2}}I_2^X(\tilde{\psi}_L)\overset{\text{law}}{\to}\mathcal{N}\big(0,\sigma_T^2\big),$$

where σ_H^2 is given by (14). Hence, it remains to study the term A_2. Using change of variables $y=\frac{L}{T}s$ and $x=\frac{L}{T}t$, we obtain

$$\begin{aligned}&\int_0^T\int_0^T\varphi_\xi\big(L|t-s|\big)|t-s|^{2H-2}\mathrm{d}s\mathrm{d}t\\&=T^{2H}L^{-2H}\int_0^L\int_0^L\varphi_\xi\big(T|x-y|\big)|x-y|^{2H-2}\mathrm{d}x\mathrm{d}y,\end{aligned}$$

where by L'Hôpital's rule we obtain

$$\lim_{L\to\infty}L^{-1}\int_0^L\int_0^L\varphi_\xi\big(T|x-y|\big)|x-y|^{2H-2}\mathrm{d}x\mathrm{d}y=2T^{1-2H}\int_0^\infty\varphi_\xi(x)x^{2H-2}\mathrm{d}x.$$

Note also that the integral in the right-hand side of the last identity is finite by Assumption 1. Consequently, we obtain

$$\begin{aligned}&\lim_{L\to\infty}L^{2H-1}\alpha_H\int_0^T\int_0^T\varphi_\xi\big(L|t-s|\big)|t-s|^{2H-2}\mathrm{d}s\mathrm{d}t\\&=2\alpha_H T\int_0^\infty\varphi_\xi(x)x^{2H-2}\mathrm{d}x=\mu.\end{aligned}\tag{18}$$

Therefore,

$$\lim_{L\to\infty}A_2=\lim_{L\to\infty}L^{\frac{3}{2}-2H}\mu,$$

which converges to zero for $H\in(\frac{3}{4},1)$, and item 1 of the claim is proved. Similarly, for $H=\frac{3}{4}$, we obtain

$$\lim_{L\to\infty}A_2=\mu,$$

which proves item 2 of the claim. Finally, for item 3, from (18) we infer that, as $L\to\infty$,

$$L^{2H-1}\alpha_H\int_0^T\int_0^T\varphi_\xi\big(L|t-s|\big)|t-s|^{2H-2}\mathrm{d}s\mathrm{d}t\longrightarrow\mu.$$

Furthermore, for the term $I_2^X(\psi_L)$, we obtain

$$L^{2H-1}\, I_2^X(\psi_L) = L^{2H-\frac{3}{2}} \times \sqrt{L}\, I_2^X(\psi_L) \xrightarrow{\mathbb{P}} 0$$

as $L \to \infty$. This is because $H < \frac{3}{4}$ implies $2H - \frac{3}{2} < 0$ and moreover $\sqrt{L}\, I_2^X(\psi_L) \xrightarrow{\text{law}} \mathcal{N}(0,1)$ and $L^{2H-\frac{3}{2}} \to 0$. □

Corollary 1. *When $X = W$ is a standard Brownian motion, that is, if the fractional Brownian motion part drops, then with similar arguments as in Theorem 5, we obtain*

$$\sqrt{L}\big(\mathbb{E}_\xi I_T(X; L\,\xi) - [X,X]_T\big) \xrightarrow{law} \mathcal{N}\big(0, \sigma_T^2\big),$$

where $\sigma_T^2 = 2T \int_0^\infty \varphi_\xi^2(x)\mathrm{d}x$, and φ_ξ is the characteristic function of ξ.

Remark 4. Note that the proof of Theorem 5 reveals that in the case $H \in (\frac{1}{2}, \frac{3}{4})$, for any $\epsilon > \frac{3}{2} - 2H$, we have that, as $L \to \infty$,

$$\sqrt{L}\big(\mathbb{E} I_T(X; L\,\xi) - [X,X]_T\big) \xrightarrow{\mathbb{P}} \infty,$$

and, moreover,

$$L^{\frac{1}{2}-\epsilon}\big(\mathbb{E} I_T(X; L\,\xi) - [X,X]_T\big) \xrightarrow{\mathbb{P}} 0.$$

4.2 The Berry–Esseen estimates

As a consequence of the proof of Theorem 5, we also obtain the following Berry–Esseen bound for the semimartingale case.

Proposition 4. *Let all the assumptions of Theorem 5 hold, and let $H \in (\frac{3}{4}, 1)$. Furthermore, let $Z \sim \mathcal{N}(0, \sigma_T^2)$, where the variance σ_T^2 is given by (14). Then there exists a constant C (independent of L) such that for sufficiently large L, we have*

$$\sup_{x\in\mathbb{R}} \big|\mathbb{P}(\sqrt{L}\big(\mathbb{E}_\xi\big(I_T(X; L\,\xi) - [X,X]_T\big) < x\big) - \mathbb{P}(Z < x)\big| \le C\rho(L),$$

where

$$\rho(L) = \max\left\{L^{\frac{3}{2}-2H}, \int_L^\infty \varphi_\xi^2(Tz)\mathrm{d}z\right\}.$$

Proof. By proof of Theorem 5 we have

$$\begin{aligned}
&\sqrt{L}\big(\mathbb{E} I_T(X; L\xi) - [X,X]_T\big) \\
&= \sqrt{L}\, I_2^X(\psi_L) + \sqrt{L}\,\alpha_H \int_0^T \int_0^T \varphi_\xi\big(L|t-s|\big)|t-s|^{2H-2}\mathrm{d}s\mathrm{d}t \\
&=: A_1 + A_2,
\end{aligned}$$

where

$$A_1 = \sqrt{2L}\|\psi_L\|_{\mathfrak{H}^{\otimes 2}}\, I_2^X(\tilde{\psi}_L).$$

Now, we know that the deterministic term A_2 converges to zero with rate $L^{\frac{3}{2}-2H}$ and the term $A_1 \overset{\text{law}}{\to} \mathcal{N}(0, \sigma_T^2)$. Hence, in order to complete the proof, it is sufficient to show that

$$\sup_{x\in\mathbb{R}} \left|\mathbb{P}(A_1 < x) - \mathbb{P}(Z < x)\right| \leq C\rho(L).$$

Now, by using the proof of Proposition 3 in Appendix A we have

$$\sqrt{\operatorname{Var}\|DF_L\|_{\mathfrak{H}}^2} \leq L^{-\frac{1}{2}} \leq L^{\frac{3}{2}-2H}.$$

Finally, using the notation of the proof of Lemma 4, we have

$$E\big(F_n^2\big) = L\,\|\psi_L\|_{\mathfrak{H}^{\otimes 2}}^2 = L \times (A_1 + A_2 + A_3),$$

where $A_2 + A_3 \leq CL^{-2H}$. Consequently,

$$L \times (A_2 + A_3) \leq CL^{1-2H} \leq CL^{\frac{3}{2}-2H}.$$

To complete the proof, we have

$$\begin{aligned} LA_1 &= \frac{T^2}{L}\int_0^L\int_0^L \varphi_\xi^2\big(T|x-y|\big)\mathrm{d}y\mathrm{d}x = \frac{T^2}{L}\int_0^L\int_{-x}^{L-x} \varphi_\xi^2(Tz)\mathrm{d}z\mathrm{d}x \\ &= \frac{T^2}{L}\int_{-L}^L\int_{-z}^{L-z} \varphi_\xi^2(Tz)\mathrm{d}x\mathrm{d}z = T^2\int_{-L}^L \varphi_\xi^2(Tz)\mathrm{d}z \\ &= 2T^2\int_0^L \varphi_\xi^2(Tz)\mathrm{d}z. \end{aligned}$$

This gives us

$$LA_1 - \sigma_T^2 = 2T^2\int_L^\infty \varphi_\xi^2(Tz)\mathrm{d}z.$$

Now, the claim follows by an application of Theorem 4. □

Remark 5. In many cases of interest, the leading term in $\rho(L)$ is the polynomial term $L^{\frac{3}{2}-2H}$, which reveals that the role of the particular choice of φ_ξ affects only to the constant. In particular, if φ_ξ admits an exponential decay, that is, $|\varphi_\xi(t)| \leq C_1e^{-C_2t}$ for some constants $C_1, C_2 > 0$, then $\int_L^\infty \varphi_\xi^2(Tz)\mathrm{d}z \leq C_3e^{-C_4L} \leq CL^{\frac{3}{2}-2H}$ for some constants $C_3, C_4, C > 0$. As examples, this is the case if ξ is a standard normal random variable with characteristic function $\varphi_\xi(t) = e^{-\frac{t^2}{2}}$ or if ξ is a standard Cauchy random variable with characteristic function $\varphi_\xi(t) = e^{-|t|}$.

Remark 6. Consider the case $X = W$, that is, X is a standard Brownian motion. In this case, the correction term A_2 in the proof of Theorem 5 disappears, and we have

$$\mathbb{E}\big(F_L^2\big) - \sigma_T^2 = 2T^2\int_L^\infty \varphi_\xi^2(Tx)\mathrm{d}x.$$

Furthermore, by applying L'Hôpital's rule twice and some elementary computations it can be shown that

$$\mathbb{E}\big[\|DF_L\|_{\mathfrak{H}}^2 - \mathbb{E}\|DF_L\|_{\mathfrak{H}}^2\big]^2 \leq \left|\varphi_\xi(TL)\right| L^{-1}.$$

Consequently, in this case, we obtain the Berry–Esseen bound

$$\sup_{x\in\mathbb{R}} \left|\mathbb{P}(\sqrt{L}\big(\mathbb{E}_\xi\big(I_T(X; L\,\xi) - [X,X]_T\big) < x\big) - \mathbb{P}(Z < x)\right| \le C\rho(L),$$

where

$$\rho(L) = \max\left\{\sqrt{\left|\varphi_\xi(TL)\right|L^{-1}}, \int_L^\infty \varphi_\xi^2(Tz)\mathrm{d}z\right\},$$

which is in fact better in many cases of interest. For example, if φ_ξ admits an exponential decay, then we obtain $\rho(L) \le e^{-cL}$ for some constant c.

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A Appendix section

A.1 Proof of Proposition 3

Denote $F_L = I_2^X(\tilde{\psi}_L)$ and note that by the definition of $\tilde{\psi}_L$ we have $\mathbb{E}(F_L^2) = 1$. Hence, it is sufficient to prove that, as $L \to \infty$,

$$\mathbb{E}\big[\|DF_L\|_{\mathfrak{H}}^2 - \mathbb{E}\|DF_L\|_{\mathfrak{H}}^2\big]^2 \to 0.$$

Now, using the definition of the Malliavin derivative, we get

$$D_sF_L = 2\,I_1^X\big(\tilde{\psi}_L(s,\cdot)\big) = \frac{\sqrt{2}}{\|\psi_L\|_{\mathfrak{H}^{\otimes 2}}}\, I_1^X\big(\varphi_\xi(L|s-\cdot|)\big).$$

For the rest of the proof, C denotes unimportant constants, which may vary from line to line. Furthermore, we also use the short notation

$$K(\mathrm{d}s, \mathrm{d}t) = \delta_0(t-s)\mathrm{d}s\mathrm{d}t + \alpha_H|t-s|^{2H-2}\mathrm{d}s\mathrm{d}t,$$

where δ_0 denotes the Kronecker delta function, to denote the measure associated to the Hilbert space $\mathfrak{H}$ generated by the mixed Brownian–fractional Brownian motion X. Furthermore, without loss of generality, we assume that $\varphi_\xi \ge 0$. Indeed, otherwise we simply approximate the integral by taking absolute values inside the integral, which is consistent with Assumption 1. Now we have

$$\|D_sF_L\|_{\mathfrak{H}}^2 = \frac{C}{\|\psi_L\|_{\mathfrak{H}^{\otimes 2}}^2}\int_0^T\int_0^T I_1^X\big(\varphi_\xi\big(L|u-\cdot|\big)\big)I_1^X\big(\varphi_\xi\big(L|v-\cdot|\big)\big)K(\mathrm{d}u, \mathrm{d}v).$$

Next, using the multiplication formula for multiple Wiener integrals, we see that

$$\begin{aligned}
&I_1^X\big(\varphi_\xi\big(L|u-\cdot|\big)\big)I_1^X\big(\varphi_\xi\big(L|v-\cdot|\big)\big)\\
&\quad= \big\langle\varphi_\xi\big(L|u-\cdot|\big), \varphi_\xi\big(L|v-\cdot|\big)\big\rangle_{\mathfrak{H}} + I_2^X\big(\varphi_\xi\big(L|u-\cdot|\big)\tilde{\otimes}\varphi_\xi\big(L|v-\cdot|\big)\big)\\
&\quad=: J_1(u,v) + J_2(u,v),
\end{aligned}$$

where the term J_1 is deterministic, and J_2 has expectation zero. Hence, we need to show that

$$\mathbb{E}\left[\frac{1}{\|\psi_L\|^2_{\mathfrak{H}^{\otimes 2}}}\int_0^T\int_0^T J_2(u,v)K(\mathrm{d}u,\mathrm{d}v)\right]^2 \to 0. \tag{19}$$

Therefore, by applying Fubini's theorem it suffices to show that, as $L \to \infty$,

$$\frac{1}{\|\psi_L\|^4_{\mathfrak{H}^{\otimes 2}}}\int_{[0,T]^4}\mathbb{E}\big[J_2(u_1,v_1)J_2(u_2,v_2)\big]K(\mathrm{d}u_1,\mathrm{d}v_1)K(\mathrm{d}u_2,\mathrm{d}v_2) \to 0. \tag{20}$$

First, using isometry (iii) [18, p. 9] , we get that

$$\begin{aligned}
&\mathbb{E}\big[J_2(u_1,v_1)J_2(u_2,v_2)\big]\\
&\quad = 2\int_{[0,T]^4}\big(\varphi_\xi\big(L|u_1-\cdot|\big)\tilde{\otimes}\varphi_\xi\big(L|v_1-\cdot|\big)\big)(x_1,y_1)\\
&\qquad \times \big(\varphi_\xi\big(L|u_2-\cdot|\big)\tilde{\otimes}\varphi_\xi\big(L|v_2-\cdot|\big)\big)(x_2,y_2)\,K(\mathrm{d}x_1,\mathrm{d}x_2)K(\mathrm{d}y_1,\mathrm{d}y_2).
\end{aligned}$$

By plugging into (20) we obtain that it suffices to have

$$\begin{aligned}
&\frac{1}{\|\psi_L\|^4_{\mathfrak{H}^{\otimes 2}}}\int_{[0,T]^8}\big(\varphi_\xi\big(L|u_1-\cdot|\big)\tilde{\otimes}\varphi_\xi\big(L|v_1-\cdot|\big)\big)(x_1,y_1)\\
&\quad \times \big(\varphi_\xi\big(L|u_2-\cdot|\big)\tilde{\otimes}\varphi_\xi\big(L|v_2-\cdot|\big)\big)(x_2,y_2)\\
&\quad \times K(\mathrm{d}x_1,\mathrm{d}x_2)K(\mathrm{d}y_1,\mathrm{d}y_2)K(\mathrm{d}u_1,\mathrm{d}v_1)K(\mathrm{d}u_2,\mathrm{d}v_2) \to 0.
\end{aligned} \tag{21}$$

The rest of the proof is based on similar arguments as the proof of Lemma 4. Indeed, again by the symmetric property of measures $K(\mathrm{d}x,\mathrm{d}y)$ and functions $\varphi_\xi(L|u_1-\cdot|)\tilde{\otimes}\varphi_\xi(L|v_1-\cdot|)$ we obtain five different terms, denoted by A_k, $k=1,2,3,4,5$, of the forms

$$\begin{aligned}
A_1 &= \int_{[0,T]^4}\varphi_\xi\big(L|u-x|\big)\varphi_\xi\big(L|u-y|\big)\varphi_\xi\big(L|v-x|\big)\varphi_\xi\big(L|y-v|\big)\mathrm{d}x\mathrm{d}y\mathrm{d}v\mathrm{d}u,\\
A_2 &= \alpha_H\int_{[0,T]^5}\varphi_\xi\big(L|u-x_1|\big)\varphi_\xi\big(L|u-y|\big)\varphi_\xi\big(L|v-x_2|\big)\varphi_\xi\big(L|y-v|\big)\\
&\quad \times |x_1-x_2|^{2H-2}\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}y\mathrm{d}v\mathrm{d}u,\\
A_3 &= \alpha_H^2\int_{[0,T]^6}\varphi_\xi\big(L|u-x_1|\big)\varphi_\xi\big(L|u-y_1|\big)\varphi_\xi\big(L|v-x_2|\big)\varphi_\xi\big(L|y_2-v|\big)\\
&\quad \times |x_1-x_2|^{2H-2}|y_1-y_2|^{2H-2}\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}v\mathrm{d}u,\\
A_4 &= \alpha_H^3\int_{[0,T]^7}\varphi_\xi\big(L|u_1-x_1|\big)\varphi_\xi\big(L|v_1-y_1|\big)\varphi_\xi\big(L|v-x_2|\big)\varphi_\xi\big(L|y_2-v|\big)\\
&\quad \times |x_1-x_2|^{2H-2}|y_1-y_2|^{2H-2}|u_1-v_1|^{2H-2}\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}v_1\mathrm{d}u_1\mathrm{d}v,\\
A_5 &= \alpha_H^4\int_{[0,T]^8}\varphi_\xi\big(L|u_1-x_1|\big)\varphi_\xi\big(L|v_1-y_1|\big)\varphi_\xi\big(L|u_2-x_2|\big)\\
&\quad \times \varphi_\xi\big(L|y_2-v_2|\big)|x_1-x_2|^{2H-2}|y_1-y_2|^{2H-2}|u_1-v_1|^{2H-2}\\
&\quad \times |u_2-v_2|^{2H-2}\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}v_1\mathrm{d}u_1\mathrm{d}v_2\mathrm{d}u_2.
\end{aligned}$$

Next, we prove that $A_3 \leq CL^{-3}$. First, by change of variables we obtain

$$A_3 = CL^{-4H-2} \int_{[0,L]^6} \varphi_\xi\big(T|u-x_1|\big)\varphi_\xi\big(T|u-y_1|\big)\varphi_\xi\big(T|v-x_2|\big)\varphi_\xi\big(T|y_2-v|\big) \\ \times |x_1-x_2|^{2H-2}|y_1-y_2|^{2H-2}\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}v\mathrm{d}u.$$

Note that Assumption 1 implies that $\int_0^L \varphi_\xi(T|x-y|)\mathrm{d}x \leq C$, where the constant C does not depend on L and y. Similarly, we have

$$\int_0^L |x-y|^{2H-2}\mathrm{d}x \leq CL^{2H-1},$$

where again the constant C is independent of L and y. Moreover, we have $\varphi_\xi(T|u-v|) \leq 1$ for any $u, v \in \mathbb{R}$. Hence, we can estimate

$$\begin{aligned} A_3 &\leq CL^{-4H-2} \int_{[0,L]^6} \varphi_\xi\big(T|u-x_1|\big)\,\varphi_\xi\big(T|u-y_1|\big)\,\varphi_\xi\big(T|v-x_2|\big) \\ &\quad \times \varphi_\xi\big(T|y_2-v|\big)\,|x_1-x_2|^{2H-2}|y_1-y_2|^{2H-2}\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}v\mathrm{d}u \\ &\leq CL^{-4H-2} \int_{[0,L]^6} 1 \times \varphi_\xi\big(T|u-y_1|\big)\,\varphi_\xi\big(T|v-x_2|\big)\,\varphi_\xi\big(T|y_2-v|\big) \\ &\quad \times |x_1-x_2|^{2H-2}|y_1-y_2|^{2H-2}\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}v\mathrm{d}u \\ &= CL^{-4H-2} \int_{[0,L]^4} \varphi_\xi\big(T|v-x_2|\big)\,\varphi_\xi\big(T|y_2-v|\big)\,|y_1-y_2|^{2H-2} \\ &\quad \times \left(\int_{[0,L]^2} \varphi_\xi\big(T|u-y_1|\big)\,|x_1-x_2|^{2H-2}\mathrm{d}u\mathrm{d}x_1\right)\mathrm{d}x_2\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}v \\ &\leq CL^{-4H-2} \times L^{2H-1} \int_{[0,L]^4} \varphi_\xi\big(T|v-x_2|\big)\,\varphi_\xi\big(T|y_2-v|\big) \\ &\quad \times |y_1-y_2|^{2H-2}\mathrm{d}x_2\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}v \\ &= CL^{-2H-3} \int_{[0,L]^3} \varphi_\xi\big(T|y_2-v|\big)\,|y_1-y_2|^{2H-2} \\ &\quad \times \left(\int_0^L \varphi_\xi\big(T|v-x_2|\big)\mathrm{d}x_2\right)\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}v \\ &\leq CL^{-2H-3} \int_{[0,L]^2} |y_1-y_2|^{2H-2}\left(\int_0^L \varphi_\xi\big(T|y_2-v|\big)\mathrm{d}v\right)\mathrm{d}y_1\mathrm{d}y_2 \\ &\leq CL^{-2H-3} \int_{[0,L]^2} |y_1-y_2|^{2H-2}\mathrm{d}y_1\mathrm{d}y_2 = CL^{-3}. \end{aligned}$$

To conclude, treating A_1, A_2, A_4, and A_5 similarly, we deduce that

$$\sum_{k=1}^{5} |A_k| \leq CL^{-3}.$$

Hence, by applying $\|\psi_L\|^2_{\mathfrak{H}^{\otimes 2}} \sim L^{-1}$ we obtain (21), which completes the proof.

A.2 Analysis of the variance

We have

$$A_2 = \alpha_H \int_{[0,T]^3} \varphi_\xi\big(L|t-u|\big)\varphi_\xi\big(L|s-u|\big)|t-s|^{2H-2}\mathrm{d}t\mathrm{d}s\mathrm{d}u, \tag{22}$$

$$A_3 = \alpha_H^2 \int_{[0,T]^4} \varphi_\xi\big(L|t-u|\big)\varphi_\xi\big(L|s-v|\big)|t-s|^{2H-2}|v-u|^{2H-2}\mathrm{d}u\mathrm{d}v\mathrm{d}t\mathrm{d}s, \tag{23}$$

which, by change of variable, leads to

$$\begin{aligned} A_2 &= \alpha_H T^{2H+1} L^{-2H-1} \int_{[0,L]^3} \varphi_\xi\big(T|t-u|\big)\varphi_\xi\big(T|s-u|\big)|t-s|^{2H-2}\mathrm{d}t\mathrm{d}s\mathrm{d}u, \\ A_3 &= \alpha_H^2 T^{4H} L^{-4H} \\ &\quad\times \int_{[0,L]^4} \varphi_\xi\big(T|t-u|\big)\varphi_\xi\big(T|s-v|\big)|t-s|^{2H-2}|v-u|^{2H-2}\mathrm{d}u\mathrm{d}v\mathrm{d}t\mathrm{d}s. \end{aligned}$$

We begin with the term A_2. Denote

$$\tilde{A}_2(L) = \int_{[0,L]^3} \varphi_\xi\big(T|t-u|\big)\varphi_\xi\big(T|s-u|\big)|t-s|^{2H-2}\mathrm{d}t\mathrm{d}s\mathrm{d}u.$$

By differentiating we get

$$\begin{aligned} \frac{\mathrm{d}\tilde{A}_2}{\mathrm{d}L}(L) &= 2\int_{[0,L]^2} \varphi_\xi\big(T|L-u|\big)\varphi_\xi\big(T|u-v|\big)|L-v|^{2H-2}\mathrm{d}v\mathrm{d}u \\ &\quad + \int_{[0,L]^2} \varphi_\xi\big(T|L-u|\big)\varphi_\xi\big(T|L-v|\big)|u-v|^{2H-2}\mathrm{d}u\mathrm{d}v \\ &=: J_1 + J_2. \end{aligned}$$

First, we analyze the term J_1. Similarly to Appendix A, we assume that $\varphi_\xi \geq 0$. Hence, we have

$$\begin{aligned} \frac{1}{2}J_1 &= \int_{[0,L]^2} \varphi_\xi\big(T|L-u|\big)\varphi_\xi\big(T|u-v|\big)|L-v|^{2H-2}\mathrm{d}v\mathrm{d}u \\ &= \int_{[0,L]^2} \varphi_\xi(Tu)\varphi_\xi\big(T|u-v|\big)v^{2H-2}\mathrm{d}v\mathrm{d}u \\ &= \int_0^L \int_1^L \varphi_\xi(Tu)\varphi_\xi\big(T|u-v|\big)v^{2H-2}\mathrm{d}v\mathrm{d}u \\ &\quad + \int_0^L \int_0^1 \varphi_\xi(Tu)\varphi_\xi\big(T|u-v|\big)v^{2H-2}\mathrm{d}v\mathrm{d}u \\ &\leq \int_0^L \int_1^L \varphi_\xi(Tu)\varphi_\xi\big(T|u-v|\big)\mathrm{d}v\mathrm{d}u + \int_0^L \int_0^1 \varphi_\xi(Tu)v^{2H-2}\mathrm{d}v\mathrm{d}u \\ &\leq C. \end{aligned}$$

For the term J_2, we write

$$\begin{aligned}
J_2 &= \int_{[0,L]^2} \varphi_\xi\big(T|L-u|\big)\varphi_\xi\big(T|L-v|\big)|u-v|^{2H-2}\mathrm{d}u\mathrm{d}v \\
&= \int_{[0,L]^2} \varphi_\xi(Tu)\varphi_\xi(Tv)|u-v|^{2H-2}\mathrm{d}u\mathrm{d}v \\
&= 2\int_0^L\int_0^t \varphi_\xi(Tu)\varphi_\xi(Tv)(v-u)^{2H-2}\mathrm{d}u\mathrm{d}v \\
&= 2\bigg(\int_0^1\int_0^t + \int_1^L\int_0^{t-1} + \int_1^L\int_{t-1}^t\bigg)\varphi_\xi(Tu)\varphi_\xi(Tv)(v-u)^{2H-2}\mathrm{d}u\mathrm{d}v \\
&=: J_{2,1} + J_{2,2} + J_{2,3}.
\end{aligned}$$

Now, it is straightforward to show that $J_{2,1} + J_{2,2} \le C$. Consequently, as $L \to \infty$, we obtain

$$A_2 \sim L^{-2H-1}\tilde{A}_2 \sim L^{-2H}(J_1 + J_{2,1} + J_{2,2} + J_{2,3}),$$

where

$$L^{-2H}(J_1 + J_{2,1} + J_{2,2}) \sim L^{-2H}.$$

For the term $J_{2,3}$, we write

$$J_{2,3}(L) = \int_1^L\int_{t-1}^t \varphi_\xi(Tu)\varphi_\xi(Tv)(v-u)^{2H-2}\mathrm{d}u\mathrm{d}v,$$

so that

$$\begin{aligned}
\frac{\mathrm{d}J_{2,3}}{\mathrm{d}L}(L) &= \int_{L-1}^L \varphi_\xi(TL)\varphi_\xi(Tv)(L-v)^{2H-2}\mathrm{d}v \\
&\le C\varphi_\xi(TL).
\end{aligned}$$

Hence, by L'Hôpital's rule we have $L^{-2H}J_{2,3} \sim L^{1-2H}\varphi_\xi(TL)$. On the other hand, we have $\varphi_\xi(TL) = o(L^{2H-2})$ since φ is integrable by Assumption 1. Hence, $L^{-2H}J_{2,3} = o(L^{-1})$, which shows that $\lim_{L\to\infty} LA_2 = 0$. Consequently, A_2 does not affect the variance. The term A_3 is easier and can be treated with similar elementary computations together with L'Hôpital's rule. As a consequence, we obtain $A_3 \sim L^{-2H}$, so that $\lim_{L\to\infty} LA_3 = 0$. Hence, A_3 does not affect the variance either, which justifies (14).

References

[1] Androshchuk, T., Mishura, Y.: Mixed Brownian–fractional Brownian model: Absence of arbitrage and related topics. Stochastics **78**(5), 281–300 (2006). MR2270939 (2007k:60198). doi:10.1080/17442500600859317

[2] Azmoodeh, E., Valkeila, E.: Characterization of the quadratic variation of mixed Brownian–fractional Brownian motion. Stat. Inference Stoch. Process. **16**(2), 97–112 (2013). MR3071882. doi:10.1007/s11203-013-9079-9

[3] Barndorff-Nielsen, E.O., Shephard, N.: Estimating quadratic variation using realized variance. J. Appl. Econ. **17**, 457–477 (2002)

[4] Barndorff-Nielsen, O.E., Shephard, N.: Econometric analysis of realized volatility and its use in estimating stochastic volatility models. J. R. Stat. Soc., Ser. B, Stat. Methodol. **64**(2), 253–280 (2002). MR1904704 (2003b:60105). doi:10.1111/1467-9868.00336

[5] Bender, C., Sottinen, T., Valkeila, E.: Arbitrage with fractional Brownian motion? Theory Stoch. Process. **13**(1–2), 23–34 (2007). MR2343807 (2008d:91038)

[6] Bender, C., Sottinen, T., Valkeila, E.: Pricing by hedging and no-arbitrage beyond semimartingales. Finance Stoch. **12**(4), 441–468 (2008). MR2447408 (2009j:91078). doi:10.1007/s00780-008-0074-8

[7] Bender, C., Sottinen, T., Valkeila, E.: Fractional processes as models in stochastic finance. In: Advanced Mathematical Methods for Finance, pp. 75–103. Springer, Berlin (2011). MR2792076. doi:10.1007/978-3-642-18412-3_3

[8] Black, F., Scholes, M.: The pricing of options and corporate liabilities [reprint of J. Polit. Econ. **81** (1973), no. 3, 637–654]. In: Financial Risk Measurement and Management. Internat. Lib. Crit. Writ. Econ., vol. 267, pp. 100–117. Edward Elgar, Cheltenham (2012). MR3235225

[9] Cheridito, P.: Mixed fractional Brownian motion. Bernoulli **7**(6), 913–934 (2001). MR1873835. doi:10.2307/3318626

[10] Delbaen, F., Schachermayer, W.: A general version of the fundamental theorem of asset pricing. Math. Ann. **300**(3), 463–520 (1994). MR1304434 (95m:90022b). doi:10.1007/BF01450498

[11] Delbaen, F., Schachermayer, W.: The fundamental theorem of asset pricing for unbounded stochastic processes. Math. Ann. **312**(2), 215–250 (1998). MR1671792 (99k:60117). doi:10.1007/s002080050220

[12] Dzhaparidze, K., Spreij, P.: Spectral characterization of the optional quadratic variation process. Stoch. Process. Appl. **54**(1), 165–174 (1994). MR1302700. doi:10.1016/0304-4149(94)00011-5

[13] Föllmer, H.: Calcul d'Itô sans probabilités. In: Séminaire de Probabilités, XV (Univ. Strasbourg, Strasbourg, 1979/1980) (French). Lecture Notes in Math., vol. 850, pp. 143–150. Springer, Berlin (1981). MR0622559 (82j:60098)

[14] Fukasawa, M.: Central limit theorem for the realized volatility based on tick time sampling. Finance Stoch. **14**(2), 209–233 (2010). MR2607763 (2011e:62235). doi:10.1007/s00780-008-0087-3

[15] Hayashi, T., Jacod, J., Yoshida, N.: Irregular sampling and central limit theorems for power variations: The continuous case. Ann. Inst. Henri Poincaré, Probab. Stat. **47**(4), 1197–1218 (2011). MR2884231. doi:10.1214/11-AIHP432

[16] Jacod, J.: Limit of random measures associated with the increments of a Brownian semimartingale. Preprint number 120, Laboratoire de Probabilités, Université Pierre et Marie Curie, Paris (1994)

[17] Nourdin, I., Peccati, G.: Normal Approximations Using Malliavin Calculus: From Stein's Method to Universality. Cambridge University Press (2012). MR2962301. doi:10.1017/CBO9781139084659

[18] Nualart, D.: The Malliavin Calculus and Related Topics. Probability and Its Applications. Springer (2006). MR2200233

[19] Nualart, D., Ortiz-Latorre, S.: Central limit theorems for multiple stochastic integrals and Malliavin calculus. Stoch. Process. Appl. **118**(4), 614–628 (2008). MR2394845. doi:10.1016/j.spa.2007.05.004

[20] Nualart, D., Peccati, G.: Central limit theorems for sequences of multiple stochastic integrals. Ann. Probab. **33**(1), 177–193 (2005). MR2118863. doi:10.1214/009117904000000621

[21] Pipiras, V., Taqqu, M.: Integration questions related to fractional Brownian motion. Probab. Theory Relat. Fields **118**(2), 251–291 (2000). MR1790083. doi:10.1007/s440-000-8016-7

[22] Protter, P.: Stochastic Integration and Differential Equations. Springer, Berlin (2004). MR2020294

[23] Schoenmakers, J.G.M., Kloeden, P.E.: Robust option replication for a Black–Scholes model extended with nondeterministic trends. J. Appl. Math. Stoch. Anal. **12**(2), 113–120 (1999). MR1701227 (2000g:91030). doi:10.1155/S104895339900012X

[24] Sondermann, D.: Introduction to Stochastic Calculus for Finance: A New Didactic Approach. Springer, Berlin (2006). MR2254170