

Spatial birth-and-death processes with a finite number of particles

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Abstract The aim of this work is to establish essential properties of spatial birth-and-death processes with general birth and death rates on \mathbb{R}^d . Spatial birth-and-death processes with time dependent rates are obtained as solutions to certain stochastic equations. The existence, uniqueness, uniqueness in law and the strong Markov property of unique solutions are proven when the integral of the birth rate over \mathbb{R}^d grows not faster than linearly with the number of particles of the system. Martingale properties of the constructed process provide a rigorous connection to the heuristic generator.

The pathwise behavior of an aggregation model is also studied. The probability of extinction and the growth rate of the number of particles under condition of nonextinction are estimated.

Keywords Spatial birth-and-death processes, stochastic equation, graphical representation, interacting particles

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1 Introduction

We consider spatial birth-and-death processes with time dependent birth and death rates. At each moment of time the system is represented as a finite collection of motionless particles in \mathbb{R}^d . The particles can also be interpreted as individuals. Existing particles may die and new particles may appear. Each particle is characterized by its location.

The state space of a spatial birth-and-death Markov process on \mathbb{R}^d with finite number of particles is the space of finite subsets of \mathbb{R}^d

$$\Gamma_0(\mathbb{R}^d) = \{\eta \subset \mathbb{R}^d : |\eta| < \infty\},$$

where $|\eta|$ is the number of points of η . $\Gamma_0 := \Gamma_0(\mathbb{R}^d)$ is also called the space of finite configurations.

Denote by $\mathcal{B}(\mathbb{R}^d)$ the Borel σ -algebra on \mathbb{R}^d . The evolution of the spatial birth-and-death process on \mathbb{R}^d admits the following description. Let $\mathbb{R}_+ := [0, +\infty)$. Two measurable functions characterize the development in time – the birth rate $b : \mathbb{R}^d \times \mathbb{R}_+ \times \Gamma_0(\mathbb{R}^d) \rightarrow [0, \infty)$ and the death rate $d : \mathbb{R}^d \times \mathbb{R}_+ \times \Gamma_0(\mathbb{R}^d) \rightarrow [0, \infty)$. If the system is in state $\eta \in \Gamma_0$ at time t , then the probability that a new particle appears (a “birth”) in a bounded set $B \in \mathcal{B}(\mathbb{R}^d)$ over time interval $[t; t + \Delta t]$ is

$$\Delta t \int_B b(x, t, \eta) dx + o(\Delta t),$$

the probability that a particle $x \in \eta$ is deleted from the configuration (a “death”) over time interval $[t; t + \Delta t]$ is

$$d(x, t, \eta) \Delta t + o(\Delta t),$$

and no two events happen simultaneously. By an event we mean a birth or a death. Using a slightly different terminology, we can say that the rate at which a birth occurs in B is $\int_B b(x, t, \eta) dx$, the rate at which a particle $x \in \eta$ dies is $d(x, t, \eta)$, and no two events happen at the same time.

Such processes, in which the birth and death rates depend on the spatial structure of the system, as opposed to classical \mathbb{Z}_+ -valued birth-and-death processes (see, e.g., [26, page 116], [1, page 109]), were first studied by Preston [38]. A heuristic description similar to that above appeared already there. Our description resembles the one in [24].

We say that the rates b and d , or the corresponding birth-and-death process, are time-homogeneous if b and d do not depend on time. By abuse of notation we write in this case $b(x, s, \eta) = b(x, \eta)$, $d(x, s, \eta) = d(x, \eta)$. The (heuristic) generator of a time-homogeneous spatial birth-and-death process should be of the form

$$LF(\eta) = \int_{x \in \mathbb{R}^d} b(x, \eta) [F(\eta \cup x) - F(\eta)] dx + \sum_{x \in \eta} d(x, \eta) (F(\eta \setminus x) - F(\eta)), \quad (1)$$

for F in an appropriate domain, where $\eta \cup x$ and $\eta \setminus x$ are shorthands for $\eta \cup \{x\}$ and $\eta \setminus \{x\}$, respectively.

The purpose of this paper is twofold. First, we would like to lay the groundwork for a rigorous analysis of spatial birth-and-death processes with a finite number of particles. To this end we provide construction and the basic properties of the obtained process, such as the strong Markov property, martingale properties, and a coupling result ensuring that under certain conditions one birth-and-death process dominates another. The approach of obtaining the process as a solution to a certain stochastic equation can be deemed as an equivalent of the graphical representation for classical interacting particle systems, for example, the contact process or the voter model. The similarity manifests itself in that in both cases the entire family of processes starting at different possibly random times from different possibly random initial conditions and with different birth or death rates can be constructed from a single ‘noise’ process. Furthermore, the construction automatically provides a coupling for the entire family. The latter was used in [7] in the proof of a shape theorem; see also [16, page 301], [32, pages 33–34 and elsewhere] for the role of the graphical representation in the analysis of discrete-space models.

Of course, the birth-and-death process with a finite number of particles with time-homogeneous birth and death rates can be relatively easily constructed as a pure jump type Markov process (see, e.g., [29, Chapter 12]). However constructing a coupling for the entire infinite family of processes as described above would be rather challenging in that framework. Additionally, the stochastic equation approach also allows us to naturally incorporate the case of time-inhomogeneous birth and death rates. Not much attention has been given to spatial time-inhomogeneous birth-and-death processes in the mathematical literature yet, even though such temporally variant models have been shown to perform better as predictors in ecological models, see, e.g., [12, 39]. Of particular interest are periodic rates reflecting seasonal changes. In [36] a nearest-neighbor birth-and-death process is fitted to describe the movement of sand dunes. In [40] spatial birth-and-death processes are used to describe the process of openings and closures of restaurants and stores in an area in Tokyo. The estimation of the birth and death rates is discussed in [31] in various settings. We note that in [31] the particles are allowed to move. In [10] (see also [11]) the dynamics and moment equations are investigated for a model of biological population which essentially is a birth and death process with rates that in our notation can be described by

$$b(x, \eta) = c_+ \sum_{y \in \eta} a_+(y - x), \quad d(x, \eta) = \mu + c_- \sum_{y \in \eta} a_-(y - x).$$

Here $\mu, c_+, c_- > 0$, and a_+ and a_- are some kernels with compact support. The interpretation is as follows:

- each individual reproduces independently of the others at a constant rate $c_+ > 0$, and the offspring is displaced with a given kernel a_+ ;
- each individual dies at a constant rate $\mu > 0$;
- additionally, each individual dies at rate governed by the kernel a_- due to competition.

Further extensions related to this model can be found in [23, 18, 20, 37, 22]. The works [23, 18] are focused on microscopic, or probabilistic, aspects, whereas

in [20, 37, 22] the approach is more macroscopic and analytical. Among exciting open problems for a continuous space birth-and-death process are questions related to the asymptotic shape (see [7] for a shape theorem for a spatial birth processes) and survival of the process started from a single point configuration.

Our second aim is to give a detailed asymptotic analysis for the aggregation model and to demonstrate that it behaves differently from the corresponding mesoscopic model [21]. We show certain fine asymptotic properties of the process, such as the finiteness of the total number of deaths over an infinite time interval and an exponential growth of the number of particles within a certain region.

A short literature overview. Garcia and Kurtz [24] obtained birth-and-death processes as solutions to certain stochastic integral equations for the case when the death rate $d \equiv 1$. The systems treated there involve an infinite number of particles. In the earlier work [33] of Garcia another approach was used: birth-and-death processes were obtained as projections of Poisson point processes. A further development of the projection method appears in [25]. Fournier and Méléard [23] used a similar equation for the construction of the Bolker–Pacala–Dieckmann–Law process with finitely many particles. Following ideas of [24] and [23], we construct the birth-and-death process described above as a solution to a stochastic equation.

Holley and Stroock [27] constructed the spatial birth-and-death process as a Markov family of unique solutions to the corresponding martingale problem. For the most part, they consider a process contained in a bounded volume, with bounded birth and death rates. They also proved the corresponding result for the nearest neighbor model in \mathbb{R}^1 with an infinite number of particles. Bezborodov et al. [9] construct and study infinite particle birth-and-death systems on the integer lattice with birth and death rates satisfying some general conditions. The approach taken in this paper somewhat resembles that in [9], however, in the continuous-space settings the death part of the stochastic equation cannot be designed by assigning to each place its own independent Poisson process as is done in [9]. Therefore the stochastic equation we use differs significantly from the one in [9].

Belavkin and Kolokoltsov [4] discuss, among other things, a general structure of a Feller semigroup on disjoint unions of Euclidean spaces (see also references therein for the construction of the Markov processes with a given generator). We note in this regard that time-homogeneous birth-and-death processes need not have the C_0 -Feller property. Eibeck and Wagner [17] discuss convergence of particle systems to limiting kinetic equations. In particular, they construct the stochastic process corresponding to the particle system as a minimal jump process, or pure jump type Markov process in the terminology of Kallenberg [29]. The jump kernel is assumed to be locally bounded.

The scheme proposed by Etheridge and Kurtz [19] covers a wide range of interactions and applies to discrete and continuous models. Their approach is based on, among other things, assigning a certain mark ('level') to each particle and letting this mark evolve according to some law. A critical event, such as birth or death, occurs when the level hit some threshold. Recurrence properties of birth-and-death processes and convergence to the invariant distribution are analyzed by Møller [35]. Shcherbakov and Volkov [41] consider the long term behavior of birth-and-death processes on a finite graph with constant death rate and the birth rate of a special expo-

nential form. Density bounds, the existence of an invariant measure, and certain return times are studied in [6]. A birth-and-death process with constant birth rate involving infinitely many particles was constructed in [2] using a completely different approach based on a comparison with a Poisson random connection graph. In [8] it is shown that the Lebesgue–Poisson measure is a maximal irreducible measure. Bezborodov et al. [7] prove a shape theorem for a wide class of continuous-space birth processes which match the above description with the death rate $d \equiv 0$. The stochastic equation used in [7] to construct the process is a special case of our equation (3). Age-dependent birth-and-death processes and their scaling limits are studied in [34].

In the aforementioned references as well as in the present work the system is represented by a Markov process. An alternative approach consists in using the concept of statistical dynamics that substitutes the notion of a Markov stochastic process. This approach is based on considering evolutions of measures and their correlation functions. For details, see, e.g., [20, 21], and references therein.

Finkelshtein et al. [21] consider different aspects of statistical dynamics for the aggregation model. In this model the death rate is given by

$$d(x, \eta) = \exp\left(-\sum_{y \in \eta \setminus x} \phi(x - y)\right),$$

where ϕ is a positive measurable function. For more details, see [21]. In this paper we present an analysis of the long time behavior of a microscopic version of this model. In particular, we estimate the probability of extinction and the speed of growth of the average number of particles.

The paper is organized as follows. Notation, definitions and results are given in Section 2. Proofs are collected in Sections 3 and 4, with two auxiliary results given in Section A.

2 The set-up and main results

2.1 Construction and basic properties

The state space of a continuous-time, continuous-space birth and death process with a finite number of particles is

$$\Gamma_0(\mathbb{R}^d) = \{\eta \subset \mathbb{R}^d : |\eta| < \infty\},$$

where $|\eta|$ is the number of points of η . $\Gamma_0(\mathbb{R}^d)$ is often called the *space of finite configurations*. The space of n -point configuration is $\Gamma_0^{(n)}(\mathbb{R}^d) := \{\eta \subset \mathbb{R}^d : |\eta| = n\} \subset \Gamma_0(\mathbb{R}^d)$. We will use Γ_0 and $\Gamma_0^{(n)}$ as shorthands for $\Gamma_0(\mathbb{R}^d)$ and $\Gamma_0^{(n)}(\mathbb{R}^d)$, respectively. For $\eta, \zeta \in \Gamma_0$, $|\eta| = |\zeta| > 0$, we define

$$\rho(\eta, \zeta) := \min_{\varsigma} \max_{x \in \eta} \{|\varsigma(x) - x|\}, \tag{2}$$

where minimum is taken over the set of all bijections $\varsigma : \eta \rightarrow \zeta$. Note that in (2) the notation $|\cdot|$ is used for the Euclidean distance in \mathbb{R}^d (not for the number of points as in $|\eta|$), which hopefully should not lead to ambiguity. Define a metric $\tilde{\rho}$ on $\Gamma_0(\mathbb{R}^d)$ by setting $\tilde{\rho}(\eta, \zeta) = 1 \wedge \rho(\eta, \zeta)$ if $|\eta| = |\zeta| > 0$, $\tilde{\rho}(\emptyset, \emptyset) = 0$, and $\tilde{\rho}(\eta, \zeta) = 1$

if $|\eta| \neq |\zeta|$. Denote by $\mathcal{B}(\Gamma_0)$ the Borel σ -algebra generated by $\tilde{\rho}$. For $\eta \in \Gamma_0$ and $a > 0$ set

$$\mathbf{B}_\rho(\eta, a) := \{\zeta \in \Gamma_0^{(|\eta|)} \mid \rho(\eta, \zeta) \leq a\}.$$

Note that

$$\mathcal{B}(\Gamma_0) = \sigma(\{\emptyset\}, \mathbf{B}_\rho(\eta, a), \eta \in \Gamma_0, a > 0).$$

Let X be a locally compact separable metric space (in this paper X will be a subset of \mathbb{R}^m for some $m \in \mathbb{N}$). Even though our solution process will stay in Γ_0 , we introduce now a more general configuration space to accommodate the driving process. Denote by $\Gamma(X)$ the space of locally finite subsets of X

$$\Gamma(X) = \{\gamma \subset X \mid |\gamma \cap K| < \infty \text{ for all compact } K\},$$

also called *the space of configurations over X* . The space $\Gamma(X)$ can be endowed with the σ -field $\mathcal{B}(X)$ generated by the projection maps

$$\Gamma(X) \ni \gamma \mapsto |\gamma \cap B| \in \mathbb{Z}_+$$

where B is an arbitrary bounded Borel subset of X .

Convention. With a slight abuse of notation, we identify $\gamma \in \Gamma$ with the induced point measure on X , so that

$$\gamma(B) = |\gamma \cap B|.$$

This convention also applies to elements of Γ_0 and other point processes and is used throughout the paper.

For more details about the notions introduced here, see, e.g., [15], [29, Chapter 12] or [30]. Throughout this paper Γ_2 stands for $\Gamma((0, +\infty) \times \mathbb{R}_+)$. Let π be the distribution of a Poisson random measure on $(\Gamma_2, \mathcal{B}(\Gamma_2))$, with the intensity measure being the Lebesgue measure on $(0, +\infty) \times \mathbb{R}_+$ (here and throughout $\mathcal{B}(X)$ is the Borel σ -algebra of X). Let $\mathcal{B}_t(\Gamma_2)$ be the smallest sub- σ -algebra of $\mathcal{B}(\Gamma_2)$ such that for every $A_1 \in \mathcal{B}((0, t])$, $A_2 \in \mathcal{B}(\mathbb{R}_+)$ the map

$$\Gamma_2 \ni \gamma \mapsto \gamma(A_1 \times A_2) \in \mathbb{Z}_+ \cup \{+\infty\}$$

is $\mathcal{B}_t(\Gamma_2)$ -measurable. Similarly, define $\mathcal{B}_{>t}(\Gamma_2)$ as the smallest sub- σ -algebra of $\mathcal{B}(\Gamma_2)$ such that for every $A_1 \in \mathcal{B}((t, \infty))$, $A_2 \in \mathcal{B}(\mathbb{R}_+)$ the map

$$\Gamma_2 \ni \gamma \mapsto \gamma(A_1 \times A_2) \in \mathbb{Z}_+ \cup \{+\infty\}$$

is $\mathcal{B}_{>t}(\Gamma_2)$ -measurable.

Let η_0 be a (random) finite initial configuration, and let $\hat{\eta}_0$ be the point process on $\mathbb{R}^d \times \Gamma_2$ obtained by associating to each point in η_0 an independent Poisson point process on $\mathbb{R}_+ \times \mathbb{R}_+$, with the distribution π . That is, if $\eta_0 = \sum_{i=1}^{|\eta_0|} \delta_{x_i}$, then

$$\hat{\eta}_0 = \sum_{i=1}^{|\eta_0|} \delta_{(x_i, \gamma_i)},$$

where $\{\gamma_i\}$ is an independent collection of Poisson point processes on Γ_2 .

Let us now introduce a stochastic differential equation driven by a Poisson point process designed in such a way that its solution is going to be a spatial birth-and-death process with birth and death rates b and d . It is not unusual to construct processes with jumps as solution to certain stochastic equations [24, 23, 3]. A short discussion of the structure of (3) can be found in Remark 2.1.

Consider the stochastic equation with the Poisson noise

$$\begin{aligned} \eta_t(B) = & \int_{(0,t] \times B \times [0,\infty) \times \Gamma_2} I_{[0,b(x,s,\eta_{s-})]}(u) I \left\{ \int_{\substack{r \in (s,t], \\ v \geq 0}} I_{[0,d(x,r,\eta_{r-})]}(v) \gamma(dr, dv) = 0 \right\} \\ & \times N(ds, dx, du, d\gamma) \\ & + \int_{B \times \Gamma_2} I \left\{ \int_{\substack{r \in (0,t], \\ v \geq 0}} I_{[0,d(x,r,\eta_{r-})]}(v) \gamma(dr, dv) = 0 \right\} \hat{\eta}_0(dx, d\gamma), \end{aligned} \tag{3}$$

where $(\eta_t)_{t \geq 0}$ is a cadlag Γ_0 -valued solution process, N is a Poisson point process on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}_+ \times \Gamma_2$, the mean measure of N is $ds \times dx \times du \times \pi$. We require the processes N and $\hat{\eta}_0$ to be independent of each other. Equation (3) is understood in the sense that the equality holds a.s. for every bounded $B \in \mathcal{B}(\mathbb{R}^d)$ and $t \geq 0$.

Remark 2.1. In the first integral on the right-hand side of (3) x is the place and s is the time of birth of a new particle. This particle is alive as long as $\int_s^t I_{[0,d(x,r,\eta_{r-})]}(v) \gamma(dr, dv) = 0$, where $(x, s, u, \gamma) \in N$. Thus, γ is the process ‘responsible’ for death. The variable u is a randomizer controlling whether birth occurs at a given time and location. In other words, each point of the driving Poisson process N in space-time carries an extra mark $u \in \mathbb{R}_+$ (used to decide whether the potential birth actually occurs) and a further two-dimensional Poisson process $\gamma \in \Gamma_2$ (used to decide when it dies). The main difference to the equation considered by Garcia and Kurtz [24] lies in the death term. Adapted to our notation, the equation there is of the form

$$\begin{aligned} \eta_t(B) = & \int_{(0,t] \times B \times [0,\infty) \times [0,\infty)} I_{[0,b(x,\eta_{s-})]}(u) I \left\{ \int_{r \in (s,t]} d(x, \eta_{r-}) dv < r \right\} \\ & \times \tilde{N}(ds, dx, du, dr) \\ & + \int_{B \times [0,\infty)} I \left\{ \int_{r \in (0,t]} d(x, \eta_{r-}) dv < r \right\} \tilde{\eta}_0(dx, dr), \end{aligned} \tag{4}$$

where \tilde{N} is a Poisson point process on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}_+$ with mean measure $ds \times dx \times du \times e^{-r} dr$, and $\tilde{\eta}_0$ is obtained from η_0 by attaching an independent unit exponential to each point. At first glance, (3) is more complicated than (4), since the death mechanism requires a whole Poisson random measure on $[0; \infty)^2$ instead of just one exponential random variable. However, it is more difficult *a priori* to define a filtration $\{\tilde{\mathcal{F}}_t\}_{t \geq 0}$ such that a solution to (4), if unique, should be adapted to and possess the Markov property with respect to $\{\tilde{\mathcal{F}}_t\}_{t \geq 0}$. This makes working with martingale properties of a solution to (4) more convoluted.

Conditions on b, d and η_0 . The birth rate b and death rate d are measurable maps from $\mathbb{R}^d \times \mathbb{R}_+ \times \Gamma_0$ to $[0, \infty)$. We assume that the birth rate b satisfies the following conditions: sublinear growth on the second variable in the sense that

$$\int_{\mathbb{R}^d} \sup_{s>0} b(x, s, \eta) dx \leq c_1 |\eta| + c_2, \tag{5}$$

for some constants $c_1, c_2 > 0$, and that $b(x, \cdot, \eta)$ and $d(x, \cdot, \eta)$ are left-continuous for any $x \in \mathbb{R}^d$ and $\eta \in \Gamma_0$.

We also assume that

$$E|\eta_0| < \infty. \tag{6}$$

Remark 2.2. Note that we consider a very general death rate: apart from measurability, d is only required to be left-continuous in the second argument.

We say that N is *compatible* with a right-continuous complete filtration $\{\mathcal{F}_t\}$ if for every $t \geq 0$

$$N([0, q] \times B \times C \times \Xi)$$

is \mathcal{F}_t -measurable for any $q \in [0, t]$, $B \in \mathcal{B}(\mathbb{R}^d)$, $C \in \mathcal{B}(\mathbb{R}_+)$, and $\Xi \in \mathcal{B}_t(\Gamma_2)$, and also

$$N((t + q', t + q' + q''] \times B' \times C' \times \Xi')$$

is independent of \mathcal{F}_t for any $q'' > q' \geq 0$, $B' \in \mathcal{B}(\mathbb{R}^d)$, $C' \in \mathcal{B}(\mathbb{R}_+)$, and $\Xi' \in \mathcal{B}_{>t}(\Gamma_2)$. We say that $\hat{\eta}_0$ is compatible with $\{\mathcal{F}_t\}$ if for every $t \geq 0$

$$\hat{\eta}_0([0, q] \times \Xi)$$

is \mathcal{F}_t -measurable for any $q \in [0, t]$ and $\Xi \in \mathcal{B}_t(\Gamma_2)$, and also

$$\hat{\eta}_0((t + q', t + q' + q''] \times \Xi')$$

is independent of \mathcal{F}_t for any $q'' > q' \geq 0$ and $\Xi' \in \mathcal{B}_{>t}(\Gamma_2)$. Sometimes we will use the representations

$$N = \sum_{q \in \mathcal{I}} \delta_{(s_q, x_q, u_q, \gamma_q)}, \quad \hat{\eta}_0 = \sum_{q \in \mathcal{J}} \delta_{(x_q, \gamma_q)},$$

where \mathcal{I} and \mathcal{J} are some countable disjoint sets. Since N and $\hat{\eta}_0$ are independent and the intensity measure of N is nonatomic, the following holds a.s.: if $q \neq q'$, $q, q' \in \mathcal{I} \cup \mathcal{J}$, then $x_q \neq x_{q'}$.

Definition 2.3. A (weak) solution of equation (3) is a triple $((\eta_t)_{t \geq 0}, N), (\Omega, \mathcal{F}, P), (\{\mathcal{F}_t\}_{t \geq 0})$, where

- (i) (Ω, \mathcal{F}, P) is a probability space, and $\{\mathcal{F}_t\}_{t \geq 0}$ is an increasing, right-continuous and complete filtration of sub- σ -algebras of \mathcal{F} ,
- (ii) N is a Poisson point process on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}_+ \times \Gamma_2$ with intensity $ds \times dx \times du \times \pi$,
- (iii) η_0 is a random \mathcal{F}_0 -measurable element in Γ_0 satisfying (6),
- (iv) the processes N and $\hat{\eta}_0$ are independent, and are compatible with $\{\mathcal{F}_t\}_{t \geq 0}$,

- (v) $(\eta_t)_{t \geq 0}$ is a cadlag Γ_0 -valued process adapted to $\{\mathcal{F}_t\}_{t \geq 0}$, $\eta_t|_{t=0} = \eta_0$,
- (vi) all integrals in (3) are well-defined,

$$E \int_0^t ds \left[\int_{\mathbb{R}^d} b(x, s, \eta_{s-}) dx + \sum_{x \in \eta_{s-}} d(x, s, \eta_{s-}) \right] < \infty, \quad t > 0,$$

and

- (vii) equality (3) holds a.s. for all $t \in [0, \infty)$ and all Borel sets B .

Following standard convention, we also call the process $(\eta_t)_{t \geq 0}$ just a solution. Note that for any solution $(\eta_t)_{t \geq 0}$ to (3) a.s.

$$\bigcup_{t \geq 0} \eta_t \subset \{x_q \mid q \in \mathcal{I} \cup \mathcal{J}\}. \tag{7}$$

Let

$$\begin{aligned} \mathcal{F}_t^0 &= \sigma \{ \eta_0, N([0, q] \times B \times C \times \Xi), \\ & \quad q \in [0, t], B \in \mathcal{B}(\mathbb{R}^d), C \in \mathcal{B}(\mathbb{R}_+), \Xi \in \mathcal{B}(\Gamma_2) \}, \end{aligned} \tag{8}$$

and let \mathcal{F}_t be the completion of \mathcal{F}_t^0 under P . Note that $\{\mathcal{F}_t\}_{t \geq 0}$ is a right-continuous filtration, see Section A.2 in the Appendix.

Definition 2.4. A solution of (3) is called *strong* if $(\eta_t)_{t \geq 0}$ is adapted to $(\mathcal{F}_t, t \geq 0)$.

Remark 2.5. In the definition above we considered solutions as processes indexed by $t \in [0, \infty)$. The reformulations for the case $t \in [0, T]$, $0 < T < \infty$, are straightforward. This remark also applies to many of the results below.

For σ -algebras \mathcal{A}_1 and \mathcal{A}_2 , let $\mathcal{A}_1 \vee \mathcal{A}_2$ be the smallest σ -algebra containing both \mathcal{A}_1 and \mathcal{A}_2 .

Definition 2.6. We say that *pathwise uniqueness* holds for equation (3) and an initial distribution ν if, whenever the triples $((\eta_t)_{t \geq 0}, N)$, (Ω, \mathcal{F}, P) , $(\{\mathcal{F}_t\}_{t \geq 0})$ and $((\tilde{\eta}_t)_{t \geq 0}, N)$, (Ω, \mathcal{F}, P) , $(\{\tilde{\mathcal{F}}_t\}_{t \geq 0})$ are weak solutions of (3) with $P\{\eta_0 = \tilde{\eta}_0\} = 1$ and $Law(\eta_0) = \nu$, and such that N is compatible with $\{\mathcal{F}_t \vee \tilde{\mathcal{F}}_t\}_{t \in [0, T]}$, we have $P\{\eta_t = \tilde{\eta}_t, t \in [0, \infty)\} = 1$ (that is, the processes $\eta, \tilde{\eta}$ are indistinguishable).

Definition 2.7. We say that *joint uniqueness in law* holds for equation (3) with an initial distribution ν if any two (weak) solutions $((\eta_t), N)$ and $((\eta'_t), N')$ of (3), $Law(\eta_0) = Law(\eta'_0) = \nu$, have the same joint distribution:

$$Law((\eta_t), N) = Law((\eta'_t), N').$$

Theorem 2.8. *Pathwise uniqueness, strong existence and joint uniqueness in law hold for equation (3). If b and d are time-homogeneous, then the unique solution is a strong Markov process, and the family of push-forward measures $\{P_\alpha, \alpha \in \Gamma_0\}$ defined in Remark 3.3 constitutes a Markov process, or a Markov family of probability measures, on $D_{\Gamma_0}[0, \infty)$.*

We call the unique solution of (3) (or, sometimes, the corresponding family of measures on $D_{\Gamma_0}[0, \infty)$) a *(spatial) birth-and-death Markov process*.

Remark 2.9. For time-homogeneous b and d , the transition probabilities of the embedded Markov chain (see, e.g., [29, Chapter 12]) of the birth-and-death process are completely described by

$$\begin{aligned}
 Q(\eta, \{\eta \setminus \{x\}\}) &= \frac{d(x, \eta)}{(B + D)(\eta)}, \quad x \in \eta, \eta \in \Gamma_0, \\
 Q(\eta, \{\eta \cup \{x\}, x \in U\}) &= \frac{\int_{x \in U} b(x, \eta) dx}{(B + D)(\eta)}, \quad U \in \mathcal{B}(\mathbb{R}^d), \eta \in \Gamma_0,
 \end{aligned}
 \tag{9}$$

where $(B + D)(\eta) = \int_{\mathbb{R}^d} b(x, \eta) dx + \sum_{x \in \eta} d(x, \eta)$.

The following two propositions establish a rigorous relation between the unique solution to (3) and L defined by (1). To formulate the first of them, let us consider the class \mathcal{C}_b of cylindrical functions $F : \Gamma_0 \rightarrow \mathbb{R}_+$ with bounded increments. We say that F has bounded increments if

$$\sup_{\eta \in \Gamma_0, x \in \mathbb{R}^d} |F(\eta \cup \{x\}) - F(\eta)| < \infty.$$

We say that F is cylindrical if for some $R = R_F > 0$

$$F(\eta) = F(\zeta) \text{ whenever } \eta \cap \mathbf{B}(\mathbf{o}_d, R) = \zeta \cap \mathbf{B}(\mathbf{o}_d, R),$$

where $\mathbf{B}(x, R)$ is the closed ball of radius R around x , and \mathbf{o}_d is the origin in \mathbb{R}^d . We recall that the filtration $\{\mathcal{F}_t, t \geq 0\}$ is introduced before Definition 2.4.

Proposition 2.10. *Let $(\eta_t)_{t \geq 0}$ be a weak solution to (3). Then for any $F \in \mathcal{C}_b$ the process*

$$\begin{aligned}
 F(\eta_t) - \int_0^t \left\{ \int_{\mathbb{R}^d} b(x, s, \eta_{s-}) [F(\eta_{s-} \cup \{x\}) - F(\eta_{s-})] dx \right. \\
 \left. - \sum_{x \in \eta_{s-}} d(x, s, \eta_{s-}) [F(\eta_{s-} \setminus \{x\}) - F(\eta_{s-})] \right\} ds
 \end{aligned}
 \tag{10}$$

is an $\{\mathcal{F}_t, t \geq 0\}$ -martingale. In particular, the integral in (10) is well-defined a.s.

Remark 2.11. Assume that all conditions we imposed on b, d , and η_0 are satisfied, except (6). Then we cannot claim that $E|\eta_t| < \infty$ for $t \geq 0$. However, we would still get a unique solution on $[0, \infty)$ satisfying all the items of Definition 2.3 except (iii) and (vi). One way to see this is to consider a sequence of initial conditions $\{\eta_0^{(m)}\}_{m \in \mathbb{N}}$, $\eta_0^{(m)} \subset \eta_0$, such that a.s. $|\eta_0^{(m)}| \leq m$ and

$$P\{\eta_0^{(m)} = \eta_0 \text{ for sufficiently large } m\} = 1.$$

We are mostly interested in the case of a nonrandom initial condition, therefore we do not discuss the case when (6) is not satisfied in more detail.

Remark 2.12. The process that starts at a possibly random time τ from a possibly random configuration ζ_τ can be obtained from the equation

$$\begin{aligned} &\eta_{t+\tau}(B) \\ &= \int_{(\tau, \tau+t] \times B \times [0, \infty) \times \Gamma_2} I_{[0, b(x, s, \eta_{s-})]}(u) I \left\{ \int_{\substack{r \in (s, \tau+t], \\ v \geq 0}} I_{[0, d(x, r, \eta_{r-})]}(v) \gamma(dr, dv) = 0 \right\} \\ &\quad \times N(ds, dx, du, d\gamma) \\ &+ \int_{B \times \Gamma_2} I \left\{ \int_{\substack{r \in (\tau, \tau+t], \\ v \geq 0}} I_{[0, d(x, r, \eta_{r-})]}(v) \gamma(dr, dv) = 0 \right\} \hat{\zeta}_\tau(dx, d\gamma) + \zeta_\tau(B), \quad t \geq 0. \end{aligned} \tag{11}$$

This is the equation of the type (3) with the initial condition ζ_τ and the driving process \bar{N} being the shift of N by τ as defined in (57). We rely here on the strong Markov property of the driving process N in the sense of Proposition A.2. Of course, τ should be an $\{\mathcal{F}_t, t \geq 0\}$ -stopping time, and ζ_τ needs to be \mathcal{F}_τ -measurable as a map from $(\Omega, \mathcal{F}_\tau)$ to $(\Gamma_0, \mathcal{B}(\Gamma_0))$ and such that $E|\zeta_\tau| < \infty$. Considering different pairs (τ, ζ_τ) , we obtain a coupled family of the birth-and-death processes as mentioned in the introduction.

We also discuss a stochastic domination of one birth-and-death process by another. Consider two equations of the form (3),

$$\begin{aligned} &\xi_t^{(k)}(B) \\ &= \int_{(0, t] \times B \times [0, \infty) \times \Gamma_2} I_{[0, b_k(x, s, \xi_{s-}^{(k)})]}(u) I \left\{ \int_{r \in (s, t], v \geq 0} I_{[0, d_k(x, r, \xi_{r-}^{(k)})]}(v) \gamma(dr, dv) = 0 \right\} \\ &\quad \times N(ds, dx, du, d\gamma) \\ &+ \int_{B \times \Gamma_2} I \left\{ \int_{r \in (0, t], v \geq 0} I_{[0, d_k(x, r, \xi_{r-}^{(k)})]}(v) \gamma(dr, dv) = 0 \right\} \hat{\xi}_0^{(k)}(dx, d\gamma), \quad k = 1, 2. \end{aligned} \tag{12}$$

We require the initial conditions $\xi_0^{(k)}$ and the rates b_k to d_k to satisfy the conditions imposed on η_0, b , and d . Let $(\xi_t^{(k)})_{t \in [0, \infty)}$ be the unique strong solutions.

Proposition 2.13. Assume that a.s. $\xi_0^{(1)} \subset \xi_0^{(2)}$, and that for any two finite configurations $\eta^1 \subset \eta^2$,

$$b_1(x, s, \eta^1) \leq b_2(x, s, \eta^2), \quad x \in \mathbb{R}^d, s \geq 0, \tag{13}$$

and

$$d_1(x, s, \eta^1) \geq d_2(x, s, \eta^2), \quad x \in \eta^1, s \geq 0. \tag{14}$$

Then a.s.

$$\xi_t^{(1)} \subset \xi_t^{(2)}, \quad t \in [0, \infty). \tag{15}$$

2.2 Aggregation model

Here we consider a specific time-homogeneous model which we call an aggregation model. This model has a property that the death rate decreases as the number of neighbors grows. We treat here the death rate given below in (16), and, in addition to previous assumptions, we require the birth rate to grow linearly on the number of points in configuration in the sense of (17). We prove in Proposition 2.14 that the probability of extinction is small if the initial configuration has many points in some fixed Borel set $\Lambda \subset \mathbb{R}^d$. Propositions 2.15, 2.16 and Theorem 2.17 describe the pathwise behavior of the process.

Let

$$d(x, \eta) = \exp \left\{ - \sum_{y \in \eta} \varphi(x - y) \right\}, \tag{16}$$

where φ is a nonnegative measurable function. Our prime examples are $\varphi(z) = c > 0$, or $\varphi(z) = cI\{z \in \tilde{\Lambda}\}$, where $\tilde{\Lambda}$ is a Borel set such that $\Lambda - \Lambda \in \tilde{\Lambda}$. Theorem 2.8 ensures existence and uniqueness of solutions, and that the unique solution is a pure jump type Markov process.

More specifically, let Λ be a measurable nonempty subset of \mathbb{R}^d . Assume that the birth rate and the initial condition η_0 satisfy (5) and (6), and, besides that, the inequalities

$$\int_{\Lambda} b(x, \eta) dx \geq c|\eta \cap \Lambda|, \quad \eta \in \Gamma_0, \tag{17}$$

and

$$b(x, \eta^1) \leq b(x, \eta^2), \quad \eta^1, \eta^2 \in \Gamma_0, \eta^1 \subset \eta^2, \tag{18}$$

hold for some positive c . Note that Λ is of positive Lebesgue measure by (17). We assume also that

$$\inf_{x, y \in \Lambda} \varphi(x - y) \geq \log a, \tag{19}$$

where $a > 1$.

We say that the process $(\eta_t)_{t \geq 0}$ goes extinct if $\inf\{t \geq 0 : \eta_t = \emptyset\} < \infty$. This infimum is called the *time of extinction*.

We want to show that the probability of extinction decays exponentially fast as the number of points of initial configuration inside Λ grows. Also, we will give a few statements describing the pace of growth of the number of points in the system.

Proposition 2.14. *Let $\tilde{C} > 0$. Then there exists $m_0 = m_0(\tilde{C}) \in \mathbb{N}$ such that, whenever $m \geq m_0$,*

$$P_{\alpha} \{ (\eta_t)_{t \geq 0} \text{ goes extinct} \} \leq \tilde{C}^{-m}$$

for all α satisfying $|\alpha \cap \Lambda| = m$.

Proposition 2.15. *For all $\alpha \in \Gamma_0$,*

$$P_{\alpha} \left(\{ |\eta_t \cap \Lambda| \rightarrow \infty \} \cup \{ \exists t' : \forall t \geq t', |\eta_t \cap \Lambda| = \emptyset \} \right) = 1. \tag{20}$$

Note that we do not require $b(\cdot, \emptyset) \equiv 0$; if $\int_{\Lambda} b(x, \emptyset) dx > 0$, then (20) implies

$$P_{\alpha}\{|\eta_t \cap \Lambda| \rightarrow \infty\} = 1.$$

The next proposition is a consequence of the exponentially fast decay of the death rate.

Proposition 2.16. *With probability 1 only a finite number of deaths inside Λ occur:*

$$P_{\alpha}\left\{|\eta_t \cap \Lambda| - |\eta_{t-} \cap \Lambda| = -1 \text{ for infinitely many different } t \geq 0\right\} = 0, \quad \alpha \in \Gamma_0.$$

Theorem 2.17. *Let $\alpha \in \Gamma_0$. For P_{α} -almost all $\omega \in F := \{\lim_{t \rightarrow \infty} |\eta_t \cap \Lambda| = \infty\}$ we have*

$$\liminf_{t \rightarrow \infty} \frac{|\eta_t \cap \Lambda|}{e^{ct}} > 0. \tag{21}$$

Corollary 2.18. *For all configurations α with $\alpha \cap \Lambda \neq \emptyset$,*

$$\liminf_{t \rightarrow \infty} \frac{E_{\alpha} |\eta_t \cap \Lambda|}{e^{ct}} > 0. \tag{22}$$

Remark. If Λ has a finite volume and the birth rate is given constant within Λ , that is

$$b(x, \eta) = c_3 > 0, \quad x \in \Lambda,$$

then from the proofs we can conclude that Theorem 2.17 still holds provided that we replace (21) by

$$\liminf_{t \rightarrow \infty} \frac{|\eta_t \cap \Lambda|}{t} > 0. \tag{23}$$

These two growth estimates stand in contrast to the mesoscopic behavior of the system [21]. Theorem 5.3 in [21] says that for some values of parameters the solution to the mesoscopic equation starting from sufficiently small initial condition stays bounded. On the contrary, the microscopic system grows whenever it survives, and the density always grows.

3 Proof of Theorem 2.8 and Proposition 2.10

Let us start with the equation

$$\bar{\eta}_t(B) = \int_{(0,t] \times B \times [0,\infty) \times \Gamma_2} I_{[0, \bar{b}(x,s,\eta_{s-})]}(u) N(ds, dx, du, d\gamma) + \eta_0(B), \tag{24}$$

where $\bar{b}(x, \eta) := \sup_{s>0, \xi \subset \eta} b(x, s, \xi)$. Note that \bar{b} satisfies sublinear growth condition (5) if b does.

This equation is of the type (3), with \bar{b} being the birth rate and the zero function being the death rate, and all definitions of existence and uniqueness of solutions are applicable here. Later a unique solution of (24) will be used as a dominating process to a solution to (3).

Proposition 3.1. *Under assumptions (5) and (6), strong existence and pathwise uniqueness hold for equation (24). In particular, the unique solution $(\bar{\eta}_t)_{t \geq 0}$ satisfies*

$$E|\bar{\eta}_t| < \infty, \quad t \geq 0. \tag{25}$$

Proof. For $\omega \in \{\int_{\mathbb{R}^d} \bar{b}(x, \eta_0) dx = 0\}$, set $\zeta_t \equiv \eta_0$, $\sigma_n = \infty$, $n \in \mathbb{N}$.

For $\omega \in F := \{\int_{\mathbb{R}^d} \bar{b}(x, \eta_0) dx > 0\}$, we define the sequence of random pairs $\{(\sigma_n, \zeta_{\sigma_n})\}$, where

$$\sigma_{n+1} = \inf \left\{ t > 0 : \int_{(\sigma_n, \sigma_n+t] \times B \times [0, \infty) \times \Gamma_2} I_{[0, \bar{b}(x, \zeta_{\sigma_n})]}(u) N(ds, dx, du, d\gamma) > 0 \right\} + \sigma_n,$$

$$\sigma_0 = 0,$$

and

$$\zeta_0 = \eta_0, \quad \zeta_{\sigma_{n+1}} = \zeta_{\sigma_n} \cup \{z_{n+1}\}$$

for $z_{n+1} = \{x \in \mathbb{R}^d : N(\{\sigma_{n+1}\} \times \{x\} \times [0, \bar{b}(x, \zeta_{\sigma_n})] \times \Gamma_2) > 0\}$. The positions z_n are uniquely determined almost surely on F . Furthermore, $\sigma_{n+1} > \sigma_n$ a.s., and σ_n are finite a.s. on F (in particular, because $\bar{b}(x, \zeta_{\sigma_n}) \geq \bar{b}(x, \eta_0)$). For $\omega \in F$, we define $\zeta_t = \zeta_{\sigma_n}$ for $t \in [\sigma_n, \sigma_{n+1})$. Then by induction on n it follows that σ_n is a stopping time for each $n \in \mathbb{N}$, and ζ_{σ_n} is $\mathcal{F}_{\sigma_n} \cap F$ -measurable. By direct substitution we see that $(\zeta_t)_{t \geq 0}$ is a strong solution to (24) on the time interval $t \in [0, \lim_{n \rightarrow \infty} \sigma_n)$. We are going to show that

$$\lim_{n \rightarrow \infty} \sigma_n = \infty \quad \text{a.s.} \tag{26}$$

This relation is evidently true on the complement of F . If $P(F) = 0$, then (26) is proven.

If $P(F) > 0$, define a probability measure on F , $Q(A) = \frac{P(A)}{P(F)}$, $A \in \mathcal{I} := \mathcal{F} \cap F$, and define $\mathcal{I}_t = \mathcal{F}_t \cap F$.

The process N is independent of F , therefore it is a Poisson point process on the probability space (F, \mathcal{I}, Q) with the same intensity, compatible with $\{\mathcal{I}_t\}_{t \geq 0}$. From now on and until it is specified otherwise, we work on the filtered probability space $(F, \mathcal{I}, \{\mathcal{I}_t\}_{t \geq 0}, Q)$. We use the same symbols for random processes and random variables, having in mind that we consider their restrictions to F .

The process $(\zeta_t)_{t \in [0, \lim_{n \rightarrow \infty} \sigma_n)}$ has the Markov property, because the process N has the strong Markov property and independent increments by Proposition (A.2) in the Appendix. Recall that for $\eta \in \Gamma_0$ and $x \in \mathbb{R}^d$, $\eta \cup x$ is a shorthand for $\eta \cup \{x\}$. Indeed, conditioning on \mathcal{I}_{σ_n} ,

$$E \left[I_{\{\zeta_{\sigma_{n+1}} = \zeta_{\sigma_n} \cup x \text{ for some } x \in B\}} \mid \mathcal{I}_{\sigma_n} \right] = \frac{\int_B \bar{b}(x, \zeta_{\sigma_n}) dx}{\int_{\mathbb{R}^d} \bar{b}(x, \zeta_{\sigma_n}) dx},$$

thus the chain $\{\zeta_{\sigma_n}\}_{n \in \mathbb{Z}_+}$ is a Markov chain, and, given $\{\zeta_{\sigma_n}\}_{n \in \mathbb{Z}_+}$, $\sigma_{n+1} - \sigma_n$ are distributed exponentially:

$$E \left\{ I_{\{\sigma_{n+1} - \sigma_n > a\}} \mid \{\zeta_{\sigma_n}\}_{n \in \mathbb{Z}_+} \right\} = \exp \left\{ -a \int_{\mathbb{R}^d} \bar{b}(x, \zeta_{\sigma_n}) dx \right\}.$$

Therefore, the random variables $\gamma_n = (\sigma_n - \sigma_{n-1})(\int_{\mathbb{R}^d} \bar{b}(x, \zeta_{\sigma_n}) dx)$ constitute a sequence of independent random variables exponentially distributed with parameter 1, independent of $\{\zeta_{\sigma_n}\}_{n \in \mathbb{Z}_+}$. Thus Theorem 12.18 in [29] implies that $(\zeta_t)_{t \in [0, \lim_{n \rightarrow \infty} \sigma_n]}$ is a pure jump type Markov process.

The jump rate of $(\zeta_t)_{t \in [0, \lim_{n \rightarrow \infty} \sigma_n]}$ is given by

$$c(\alpha) = \int_{\mathbb{R}^d} \bar{b}(x, \alpha) dx.$$

Condition (5) implies that $c(\alpha) \leq c_1|\alpha| + c_2$. Hence

$$c(\zeta_{\sigma_n}) \leq c_1|\zeta_{\sigma_n}| + c_2 = c_1|\zeta_0| + c_1n + c_2.$$

We see that $\sum_n \frac{1}{c(\zeta_{\sigma_n})} = \infty$ a.s., hence Proposition 12.19 in [29] implies that $\sigma_n \rightarrow \infty$.

Now we return again to our initial probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$. We have proved the existence of a strong solution. The uniqueness follows by induction on jumps of the process. Namely, let $(\tilde{\zeta}_t)_{t \geq 0}$ be another solution of (24). Since a.s.

$$\int_{(0, \sigma_1) \times \mathbb{R}^d \times [0, \infty) \times \Gamma_2} I_{[0,0]}(u) N(ds, dx, du, d\gamma) = 0,$$

(here $I_{[0,0]}(u) = I\{u = 0\}$) we have $\zeta_t = \tilde{\zeta}_t = \eta_0$ a.s. on the complement F^c for all $t \geq 0$. From (vii) of Definition 2.3 and the equality

$$\int_{(0, \sigma_1) \times \mathbb{R}^d \times [0, \infty) \times \Gamma_2} I_{[0, \bar{b}(x, \eta_0)]}(u) N(ds, dx, du, d\gamma) = 0,$$

it follows that $P(\{\tilde{\zeta} \text{ has a birth before } \sigma_1\} \cap F) = 0$. At the same time, the equality

$$\int_{\{\sigma_1\} \times \mathbb{R}^d \times [0, \infty) \times \Gamma_2} I_{[0, \bar{b}(x, \eta_0)]}(u) N(ds, dx, du, d\gamma) = 1,$$

which holds a.s. on F , yields that $\tilde{\zeta}$ has a birth at the moment σ_1 , and in the same point of space. Therefore, $\tilde{\zeta}$ coincides with ζ on $[0, \sigma_1]$ a.s. on F . Similar reasoning shows that they coincide up to σ_n a.s. on F , and, since $\sigma_n \rightarrow \infty$ a.s. on F ,

$$P\{\tilde{\zeta}_t = \zeta_t \text{ for all } t \geq 0\} = 1.$$

Thus, pathwise uniqueness holds.

Now we turn our attention to (25). Since $\zeta_t \equiv \eta_0$ on F^c , we can assume without loss of generality that $P(F) = 1$. We can write

$$|\zeta_t| = |\eta_0| + \sum_{n=1}^{\infty} I\{|\zeta_t| - |\eta_0| \geq n\}$$

$$= |\eta_0| + \sum_{n=1}^{\infty} I\{\sigma_n \leq t\}. \tag{27}$$

Since $\sigma_n = \sum_{i=1}^n \frac{\gamma_i}{\int_{\mathbb{R}^d} \bar{b}(x, \zeta_{\sigma_i}) dx}$, we have

$$\begin{aligned} \{\sigma_n \leq t\} &= \left\{ \sum_{i=1}^n \frac{\gamma_i}{\int_{\mathbb{R}^d} \bar{b}(x, \zeta_{\sigma_i}) dx} \leq t \right\} \subset \left\{ \sum_{i=1}^n \frac{\gamma_i}{c_1 |\zeta_{\sigma_i}| + c_2} \leq t \right\} \\ &\subset \left\{ \sum_{i=1}^n \frac{\gamma_i}{(c_1 + c_2)(|\eta_0| + i)} \leq t \right\} = \{Z_t - Z_0 \geq n\}, \end{aligned}$$

where (Z_t) is the Yule process, i.e., the birth process on \mathbb{Z}_+ with transition rates $p_{k,k+1} = (c_1 + c_2)k$, $p_{k,l} = 0$, $l \neq k + 1$, see, e.g., [1, Chapter 3, Section 5]. Here (Z_t) is defined as follows: $Z_t - Z_0 = n$ when

$$\sum_{i=1}^n \frac{\gamma_i}{(c_1 + c_2)(|\eta_0| + i)} \leq t < \sum_{i=1}^{n+1} \frac{\gamma_i}{(c_1 + c_2)(|\eta_0| + i)},$$

and $Z_0 = |\eta_0|$. Thus, we have $|\zeta_t| \leq Z_t$ a.s., hence $E|\zeta_t| \leq EZ_t < \infty$. The constructed solution is strong. \square

Proposition 3.2. *Under assumptions (5)–(6), pathwise uniqueness and strong existence hold for equation (3). The unique solution (η_t) satisfies*

$$E|\eta_t| < \infty, \quad t \geq 0. \tag{28}$$

Proof. Let us define stopping times with respect to $\{\mathcal{F}_t, t \geq 0\}$, $0 = \theta_0 \leq \theta_1 \leq \theta_2 \leq \theta_3 \leq \dots$, and the sequence of (random) configurations $\{\eta_{\theta_j}\}_{j \in \mathbb{N}}$ as follows: as long as

$$\theta_{n+1} = \theta_{n+1}^b \wedge \theta_{n+1}^d + \theta_n < \infty,$$

where

$$\theta_{n+1}^b = \inf \left\{ t > 0 : \int_{(\theta_n, \theta_n + t] \times \mathbb{R}^d \times [0, \infty) \times \Gamma_2} I_{[0, b(x, s, \eta_{\theta_n})]}(u) N(ds, dx, du, d\gamma) > 0 \right\},$$

$$\theta_{n+1}^d = \inf \left\{ t > 0 : \sum_{\substack{q \in \mathcal{I} \cup \mathcal{J}, \\ x_q \in \eta_{\theta_n}}} \int_{(\theta_n, \theta_n + t] \times [0, \infty)} I_{[0, d(x_q, r, \eta_{\theta_n})]}(v) \gamma_q(dr, dv) > 0 \right\},$$

we set $\eta_{\theta_{n+1}} = \eta_{\theta_n} \cup \{z_{n+1}\}$ if $\theta_{n+1}^b \leq \theta_{n+1}^d$, where $\{z_{n+1}\} = \{z \in \mathbb{R}^d : N(\{\theta_n + \theta_{n+1}^b\} \times \{z\} \times \mathbb{R}_+ \times \Gamma_2) > 0\}$; $\eta_{\theta_{n+1}} = \eta_{\theta_n} \setminus \{z_{n+1}\}$ if $\theta_{n+1}^b > \theta_{n+1}^d$, where $\{z_{n+1}\} = \{x_q \in \eta_{\theta_n} : \gamma_q(\{\theta_n + \theta_{n+1}^d\} \times \mathbb{R}_+) > 0\}$; the configuration $\eta_{\theta_0} = \eta_0$ is the initial condition of (3), $\eta_t = \eta_{\theta_n}$ for $t \in [\theta_n, \theta_{n+1})$. Note that

$$P\{\theta_{n+1}^b = \theta_{n+1}^d \mid \min\{\theta_{n+1}^b, \theta_{n+1}^d\} < \infty\} = 0,$$

the points z_n are a.s. uniquely determined, and

$$P\{z_{n+1} \in \eta_{\theta_n} \mid \theta_{n+1}^b \leq \theta_{n+1}^d\} = 0.$$

If for some n

$$\theta_{n+1} = \infty,$$

we set $\theta_{n+k} = \infty, k \in \mathbb{N}$, and $\eta_t = \eta_{\theta_n}, t \geq \theta_n$.

Random variables $\theta_n, n \in \mathbb{N}$, are stopping times with respect to the filtration $\{\mathcal{F}_t, t \geq 0\}$. By the strong Markov property of a Poisson point process, see Proposition A.2, we obtain that a.s. on $\{\theta_n < \infty\}$ the conditional distribution of θ_{n+1}^b given \mathcal{F}_{θ_n} is

$$P\left\{\theta_{n+1}^b > p \mid \mathcal{F}_{\theta_n}\right\} = \exp\left\{-\int_{\theta_n}^{\theta_n+p} ds \int_{\mathbb{R}^d} b(x, s, \eta_{\theta_n}) dx\right\},$$

and a.s. on $\{\theta_n < \infty\}$ the conditional distribution of θ_{n+1}^d given \mathcal{F}_{θ_n} is

$$P\left\{\theta_{n+1}^d > p \mid \mathcal{F}_{\theta_n}\right\} = \exp\left\{-\int_{\theta_n}^{\theta_n+p} ds \sum_{x \in \eta_{\theta_n}} d(x, s, \eta_{\theta_n})\right\}.$$

In particular, $\theta_n^b, \theta_n^d > 0, n \in \mathbb{N}$.

We are going to show that a.s.

$$\theta_n \rightarrow \infty, \quad n \rightarrow \infty. \tag{29}$$

Denote by θ'_k the moment of the k -th birth. It is sufficient to show that $\theta'_k \rightarrow \infty, k \rightarrow \infty$, because only finitely many deaths may occur between any two births, since there are only finitely many particles. By induction on k' we can see that $\{\theta'_k\}_{k' \in \mathbb{N}} \subset \{\sigma_i\}_{i \in \mathbb{N}}$, where σ_i are the moments of births of $(\bar{\eta}_t)_{t \geq 0}$, the solution of (24), and $\eta_t \subset \bar{\eta}_t$ for all $t \in [0, \lim_n \theta_n)$. For instance, let us show that $(\bar{\eta}_t)_{t \geq 0}$ has a birth at θ'_1 . We have $\bar{\eta}_{\theta'_1-} \supset \bar{\eta}_0 = \eta_0$, and $\eta_{\theta'_1-} \subset \eta_t \mid_{t=0} = \eta_0$, hence for all $x \in \mathbb{R}^d$

$$\bar{b}(x, \bar{\eta}_{\theta'_1-}) \geq \bar{b}(x, \eta_{\theta'_1-}) \geq b(x, \theta'_1, \eta_{\theta'_1-})$$

The latter implies that at time moment θ'_1 a birth occurs for the process $(\bar{\eta}_t)_{t \geq 0}$ in the same point. Hence, $\eta_{\theta'_1} \subset \bar{\eta}_{\theta'_1}$, and we can go on. Since $\sigma_k \rightarrow \infty$ as $k \rightarrow \infty$, we also have $\theta'_k \rightarrow \infty$, and therefore $\theta_n \rightarrow \infty, n \rightarrow \infty$.

Let us now prove the inequality from item (vi) of Definition 2.3,

$$E \int_0^t ds \left[\int_{\mathbb{R}^d} b(x, s, \eta_{s-}) dx + \sum_{x \in \eta_{s-}} d(x, s, \eta_{s-}) \right] < \infty, \quad t > 0. \tag{30}$$

Denote the number of births and deaths before t by b_t and d_t respectively, i.e.

$$b_t = \#\{s : |\eta_s| - |\eta_{s-}| = 1\} = \int_{(0,t] \times \mathbb{R}^d \times [0,\infty) \times \Gamma_2} I_{[0,b(x,s,\eta_{s-})]}(u) N(ds, dx, du, d\gamma)$$

and

$$d_t = \#\{s : |\eta_s| - |\eta_{s-}| = -1\} = \int_{(0,t] \times [0,\infty)} \sum_{\substack{q \in \mathcal{I} \cup \mathcal{J} \\ x_q \in \eta_{r-}}} I_{[0,d(x_q,r,\eta_{r-})]}(v) \gamma_q(dr, dv).$$

Note that $|\eta_t| = b_t - d_t + |\eta_0|$ and θ_k are the moments of jumps for $c_t := b_t + d_t$, so that

$$c_t = \sum_{k \in \mathbb{N}} I\{\theta_k \leq t\}, \quad t \geq 0.$$

For $n \in \mathbb{N}$ define

$$\begin{aligned} c_t^{(n)} &= \int_{(0,t] \times \mathbb{R}^d \times [0,\infty) \times \Gamma_2} I_{[0,b(x,s,\eta_{s-}) \wedge n]}(u) I\{|x| \leq n\} N(ds, dx, du, d\gamma) \\ &+ \int_{(0,t] \times [0,\infty)} \sum_{\substack{q \in \mathcal{I} \cup \mathcal{J} \\ x_q \in \eta_{r-}}} I_{[0,d(x_q,r,\eta_{r-}) \wedge n]}(v) I\{|x| \leq n\} \gamma_q(dr, dv). \end{aligned}$$

Then

$$M_t^{(n)} = c_t^{(n)} - \int_0^t \int_{x:|x| \leq n} (b(x, s, \eta_{s-}) \wedge n) dx ds - \int_0^t \sum_{x \in \eta_{s-}, |x| \leq n} (d(x, s, \eta_{s-}) \wedge n) ds$$

is a martingale with respect to $\{\mathcal{F}_t\}$, see, e.g., [28, (3.8), Section 3, Chapter 2]. By the optional stopping theorem $EM_{\theta_1 \wedge t}^{(n)} = 0$, hence

$$\begin{aligned} E \int_0^{\theta_1 \wedge t} \left(\int_{x:|x| \leq n} b(x, s, \eta_{s-}) \wedge n dx + \sum_{x \in \eta_{s-}, |x| \leq n} d(x, s, \eta_{s-}) \wedge n \right) ds \\ = Ec_{t \wedge \theta_1}^{(n)} \leq P\{\theta_1 < t\} \leq 1. \end{aligned}$$

Similarly,

$$\begin{aligned} E \int_{\theta_m \wedge t}^{\theta_{m+1} \wedge t} \left(\int_{x:|x| \leq n} b(x, s, \eta_{s-}) \wedge n dx + \sum_{x \in \eta_{s-}, |x| \leq n} d(x, s, \eta_{s-}) \wedge n \right) ds \\ = Ec_{t \wedge \theta_{m+1}}^{(n)} - Ec_{t \wedge \theta_m}^{(n)} \leq P\{\theta_{m+1} < t\}. \end{aligned}$$

Consequently,

$$E \int_0^t \left(\int_{x:|x| \leq n} b(x, s, \eta_{s-}) \wedge n dx + \sum_{x \in \eta_{s-}, |x| \leq n} d(x, s, \eta_{s-}) \wedge n \right) ds$$

$$\begin{aligned} &\leq \sum_{m=1}^{\infty} E \int_{\theta_m \wedge t}^{\theta_{m+1} \wedge t} \left(\int_{x:|x| \leq n} b(x, s, \eta_{s-}) \wedge n \, dx + \sum_{x \in \eta_{s-}, |x| \leq n} d(x, s, \eta_{s-}) \wedge n \right) ds \\ &\leq \sum_{m=1}^{\infty} P\{\theta_m \leq t\} = \sum_{m=1}^{\infty} P\{c_t \geq m\} = E c_t. \end{aligned}$$

Letting $n \rightarrow \infty$, we get, by the monotone convergence theorem,

$$E \int_0^t \left(\int_{x \in \mathbb{R}^d} b(x, s, \eta_{s-}) dx + \sum_{x \in \eta_{s-}} d(x, s, \eta_{s-}) \right) ds \leq E c_t.$$

Only existing particles may disappear, hence the number of deaths d_t satisfies

$$d_t \leq b_t + |\eta_0|.$$

Thus,

$$E c_t \leq 2E b_t + E|\eta_0| \leq 2E|\bar{\eta}_t| + E|\eta_0| < \infty, \tag{31}$$

and (30) follows.

Since $\eta_t \subset \bar{\eta}_t$ a.s., Proposition 3.1 implies (28).

It follows from the above construction, (29), and (30) that (η_t) is a strong solution to (3). Similarly to the proof of Proposition 3.1, we can show by induction on n that equation (3) has a unique solution on $[0, \theta_n]$. Namely, each two solutions coincide on $[0, \theta_n]$ a.s. Thus, any solution coincides with (η_t) a.s. for all $t \in [0, \theta_n]$. \square

Remark 3.3. Assume that b and d are time-homogeneous. Let η_0 be a nonrandom initial condition, $\eta_0 \equiv \alpha$, $\alpha \in \Gamma_0$. The solution of (3) with $\eta_0 \equiv \alpha$ will be denoted as $(\eta(\alpha, t))_{t \geq 0}$. Let P_α be the push-forward of P under the mapping

$$\Omega \ni \omega \mapsto (\eta(\alpha, \cdot)) \in D_{\Gamma_0}[0, \infty). \tag{32}$$

It can be derived from the proof of Proposition 3.2 that, for fixed $\omega \in \Omega$, the unique solution is jointly measurable in (t, α) . Thus, the family $\{P_\alpha\}$ of probability measures on $D_{\Gamma_0}[0, \infty)$ is measurable in α , that is, for any Borel set $\mathcal{D} \subset D_{\Gamma_0}[0, \infty)$ the map $\Gamma_0 \ni \alpha \mapsto P_\alpha(\mathcal{D})$ is measurable. We will often use formulations related to the probability space $(D_{\Gamma_0}[0, \infty), \mathcal{B}(D_{\Gamma_0}[0, \infty)), P_\alpha)$; in this case, coordinate mappings will be denoted by η_t ,

$$\eta_t(x) = x(t), \quad x \in D_{\Gamma_0}[0, \infty).$$

The processes $(\eta_t)_{t \in [0, \infty)}$ and $(\eta(\alpha, \cdot))_{t \in [0, \infty)}$ have the same law (under P_α and P , respectively). As one would expect, the family of measures $\{P_\alpha, \alpha \in \Gamma_0\}$ is a Markov process, or a Markov family of probability measures; see Proposition 3.6 below. For a measure μ on Γ_0 , we define

$$P_\mu = \int P_\alpha \mu(d\alpha).$$

We denote by E_μ the expectation under P_μ .

Remark 3.4. We solved equation (3) ω -wisely. We can deduce from the proof of Proposition 3.2 that θ_n and z_n are measurable functions of η_0 and N in the sense that, for example, $\theta_1 = F_1(\eta_0, N)$ a.s. for a measurable function $F_1 : \Gamma_0 \times \Gamma(\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}_+ \times \Gamma_2) \rightarrow \mathbb{R}_+$. As a consequence, there is a functional dependence of the solution process and the “input”: the process $(\eta_t)_{t \geq 0}$ is some function of η_0 and N .

The following corollary is a consequence of Proposition 3.2 and Remark 3.4.

Corollary 3.5. *Joint uniqueness in law holds for equation (3) with initial distribution ν satisfying*

$$\int_{\Gamma_0} |\gamma| \nu(d\gamma) < \infty.$$

As usually, the Markov property of a solution follows from uniqueness.

Proposition 3.6 (The strong Markov property). *Let b and d be time-homogenous. The unique solution $(\eta_t)_{t \in [0, \infty)}$ of (3) is a strong Markov process in the following sense. Let τ be an a.s. finite $(\mathcal{F}_t, t \geq 0)$ -stopping time such that $E|\eta_\tau| < \infty$. Then*

$$P\{(\eta_{\tau+t}, t \geq 0) \in \mathcal{D}\} = E P_{\eta_\tau}(\mathcal{D}), \quad \mathcal{D} \in \mathcal{B}(D_{\Gamma_0}[0, \infty)). \tag{33}$$

Furthermore, for any $\mathcal{D} \in \mathcal{B}(D_{\Gamma_0}[0, \infty))$,

$$P\{(\eta_{\tau+t}, t \geq 0) \in \mathcal{D} \mid \mathcal{F}_\tau\} = P\{(\eta_{\tau+t}, t \geq 0) \in \mathcal{D} \mid \eta_\tau\}; \tag{34}$$

that is, given η_τ , $(\eta_{\tau+t}, t \geq 0)$ is conditionally independent of $(\mathcal{F}_t, t \geq 0)$.

Proof. For $t \geq 0$,

$$\begin{aligned} & \eta_{\tau+t}(B) \\ &= \int_{(\tau, \tau+t] \times B \times [0, \infty) \times \Gamma_2} I_{[0, b(x, \eta_{s-})]}(u) I \left\{ \int_{\substack{r \in (s, \tau+t], \\ v \geq 0}} I_{[0, d(x, \eta_{r-})]}(v) \gamma(dr, dv) = 0 \right\} \\ & \quad \times N(ds, dx, du, d\gamma) \\ & \quad + \int_{B \times \Gamma_2} I \left\{ \int_{\substack{r \in (\tau, \tau+t], \\ v \geq 0}} I_{[0, d(x, \eta_{r-})]}(v) \gamma(dr, dv) = 0 \right\} \hat{\eta}_\tau(dx, d\gamma) + \eta_\tau(B), \quad t \geq 0. \end{aligned} \tag{35}$$

where $\hat{\eta}_\tau = \sum_{\substack{q \in \mathcal{I} \cup \mathcal{J}, \\ x_q \in \eta_\tau}} (x_q, \gamma_q)$. Here we need the strong Markov property of the driving process as given in Proposition A.2. Note that (35) can be considered as an equation of the type (3) with the unique solution being $(\eta_{\tau+t})_{t \in [0, \infty)}$. From Proposition 3.2, Remark 3.4, and Corollary 3.5 we get (33). The conditional independence (34) follows from Remark 3.4. \square

Proof of Theorem 2.8. The theorem is a consequence of Proposition 3.2, Remark 3.3, and Proposition 3.6. In particular, the Markov property of $\{P_\alpha, \alpha \in \Gamma_0\}$ follows from Corollary 3.5. \square

Let N_1 be the image of N under the projection

$$(s, x, u, \gamma) \mapsto (s, x, u).$$

The process N_1 is a Poisson point process on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}_+$ with intensity measure $dsdxdu$.

Proof of Proposition 2.10. We have

$$\begin{aligned} \eta_t(B) &= \int_{(0,t] \times B \times [0,\infty) \times \Gamma_2} I_{[0,b(x,s,\eta_{s-})]}(u) I \left\{ \int_{\substack{r \in (s,t], \\ v \geq 0}} I_{[0,d(x,r,\eta_{r-})]}(v) \gamma(dr, dv) = 0 \right\} \\ &\quad \times N(ds, dx, du, d\gamma) \\ &\quad + \int_{B \times \Gamma_2} I \left\{ \int_{\substack{r \in (0,t], \\ v \geq 0}} I_{[0,d(x,r,\eta_{r-})]}(v) \gamma(dr, dv) = 0 \right\} \hat{\eta}_0(dx, d\gamma) \\ &= \int_{(0,t] \times B \times [0,\infty)} I_{[0,b(x,s,\eta_{s-})]}(u) N_1(ds, dx, du) + \eta_0(B) \\ &\quad - \sum_{q \in \mathcal{I} \cup \mathcal{J}} \int_{(0,t] \times [0,\infty)} I\{x_q \in \eta_{r-}\} I_{[0,d(x_q,r,\eta_{r-})]}(v) \gamma_q(dr, dv). \end{aligned}$$

Recall that $\eta \cup x$ and $\eta \setminus x$ are shorthands for $\eta \cup \{x\}$ and $\eta \setminus \{x\}$, respectively. By Ito's formula ([28, Chapter 2, Theorem 5.1]), for $F \in \mathcal{C}_b$,

$$\begin{aligned} F(\eta_t) - F(\eta_0) &= \sum_{s \leq t} (F(\eta_s) - F(\eta_{s-})) \\ &= \int_{(0,t] \times \mathbf{B}(\mathbf{o}_d, R_F) \times [0,\infty)} I_{[0,b(x,s,\eta_{s-})]}(u) \{F(\eta_{s-} \cup x) - F(\eta_{s-})\} \\ &\quad \times N_1(ds, dx, du) \\ &\quad + \sum_{q \in \mathcal{I} \cup \mathcal{J}} \int_{(0,t] \times [0,\infty)} I\{x_q \in \eta_{r-}\} I_{[0,d(x_q,r,\eta_{r-})]} \\ &\quad \times (v) \{F(\eta_{r-} \setminus x) - F(\eta_{r-})\} \gamma_q(dr, dv). \end{aligned}$$

We can write

$$\begin{aligned} &\int_{(0,t] \times \mathbf{B}(\mathbf{o}_d, R_F) \times [0,\infty)} I_{[0,b(x,s,\eta_{s-})]}(u) \{F(\eta_{s-} \cup x) - F(\eta_{s-})\} N_1(ds, dx, du) \\ &= \int_{(0,t] \times \mathbf{B}(\mathbf{o}_d, R_F)} b(x, s, \eta_{s-}) \{F(\eta_{s-} \cup x) - F(\eta_{s-})\} dx ds \\ &+ \int_{(0,t] \times \mathbf{B}(\mathbf{o}_d, R_F) \times [0,\infty)} I_{[0,b(x,s,\eta_{s-})]}(u) \{F(\eta_{s-} \cup x) - F(\eta_{s-})\} \tilde{N}_1(ds, dx, du), \end{aligned}$$

where $\tilde{N} = N - dsdxdu$. Since $F \in \mathcal{C}_b$, the process

$$\int_{(0,t] \times \mathbf{B}(\mathbf{o}_d, R_F) \times [0, \infty)} I_{[0, b(x, s, \eta_{s-})]}(u) \{F(\eta_{s-} \cup x) - F(\eta_{s-})\} \tilde{N}_1(ds, dx, du)$$

is a martingale by item (vi) of Definition 2.3, see, e.g., [28, Section 3 of Chapter 2]. Similarly,

$$\sum_{q \in \mathcal{I} \cup \mathcal{J}} \int_{(0,t] \times [0, \infty)} I\{x_q \in \eta_{r-}\} I_{[0, d(x_q, r, \eta_{r-})]}(v) \{F(\eta_{r-} \setminus x) - F(\eta_{r-})\} \gamma_q(dr, dv)$$

can be decomposed into a sum of

$$\int_{(0,t]} \sum_{x \in \eta_{r-}} d(x, r, \eta_{r-}) \{F(\eta_{r-} \setminus x) - F(\eta_{r-})\} dr$$

and a martingale. The desired statement follows. □

Proof of Proposition 2.13. Let τ_1, τ_2, \dots be consecutive jump moments of the process $(\xi_t^{(1)}, \xi_t^{(2)})$. We will show by induction that each moment of birth for $(\xi_t^{(1)})_{t \in [0, \infty)}$ is a moment of birth for $(\xi_t^{(2)})_{t \in [0, \infty)}$, too, and each moment of death for $(\xi_t^{(2)})_{t \in [0, \infty)}$ is a moment of death for $(\xi_t^{(1)})_{t \in [0, \infty)}$ if the dying particle is in $(\xi_t^{(1)})_{t \in [0, \infty)}$. Moreover, in both cases the birth or the death occurs at exactly the same place. Here a moment of birth is a random time at which a new particle appears, a moment of death is a random time at which an existing particle disappears from the configuration. The statement formulated here is in fact equivalent to (15).

Here we deal only with the base case, the induction step is done in the same way. We have nothing to show if τ_1 is a moment of a birth of $(\xi_t^{(2)})_{t \in [0, \infty)}$ or a moment of death of $(\xi_t^{(1)})_{t \in [0, \infty)}$. Assume that a new particle is born for $(\xi_t^{(1)})_{t \in [0, \infty)}$ at τ_1 ,

$$\xi_{\tau_1}^{(1)} \setminus \xi_{\tau_1-}^{(1)} = \{x_1\}.$$

The process $(\xi_t^{(1)})_{t \in [0, \infty)}$ satisfies (3), therefore a.s. $N_1(\{x\} \times \{\tau_1\} \times [0, b_1(x_1, \tau_1, \xi_{\tau_1-}^{(1)})]) = 1$. Since

$$\xi_{\tau_1-}^{(1)} = \xi_0^{(1)} \subset \xi_0^{(2)} = \xi_{\tau_1-}^{(2)}, \tag{36}$$

by (13) we have $b_1(x_1, \tau_1, \xi_{\tau_1-}^{(1)}) \subset b_2(x_1, \tau_1, \xi_{\tau_1-}^{(2)})$, and hence

$$N(\{x\} \times \{\tau_1\} \times [0, b_2(x_1, \tau_1, \xi_{\tau_1-}^{(2)})] \times \Gamma_2) = 1,$$

hence

$$\xi_{\tau_1}^{(2)} \setminus \xi_{\tau_1-}^{(2)} = \{x_1\}.$$

Now let τ_1 be a moment of death for $(\xi_t^{(2)})_{t \in [0, \infty)}$, and let $\xi_{\tau_1-}^{(2)} \setminus \xi_{\tau_1}^{(2)} = \{x_q\}$ for some $q \in \mathcal{I} \cup \mathcal{J}$ (such a q always exists because of (7), and is unique). If $x_q \notin \xi_{\tau_1-}^{(1)}$,

we have nothing to prove. Hence we also assume $x_q \notin \xi_{\tau_1-}^{(1)}$. We have a.s. $\gamma_q(\{\tau_1\} \times [0, d_2(x_q, \tau_1, \xi_{\tau_1-}^{(2)})]) = 1$. By (36) and (14), $d_1(x_q, \tau_1, \xi_{\tau_1-}^{(1)}) \geq d_2(x_q, \tau_1, \xi_{\tau_1-}^{(2)})$, hence

$$\gamma_q(\{\tau_1\} \times [0, d_1(x_q, \tau_1, \xi_{\tau_1-}^{(1)})]) = 1.$$

It follows that $\xi_{\tau_1-}^{(1)} \setminus \xi_{\tau_1}^{(1)} = \{x_q\}$. □

4 Aggregation model: proofs

The main idea behind our analysis in this section is to couple the process $(\eta_t)_{t \geq 0}$ with another birth-and-death process, to which we can apply Lemma A.1.

To do so, let us introduce another pair of the birth and death rates, b_1, d_1 , and an initial condition $\xi_0 = \eta_0 \cap \Lambda$, such that $b_1(x, \eta) = d_1(x, \eta) = 0$ for $x \notin \Lambda$, $d_1(x, \eta) = a^{-|\eta|}$ for $x \in \Lambda$, $b_1(x, \eta) \leq b(x, \eta)$ for all x, η , and for some constant $c > 0$

$$\int_{\Lambda} b_1(x, \eta) dx = c|\eta \cap \Lambda|, \quad \eta \in \Gamma_0.$$

It follows from (17) that there exists a function b_1 satisfying these assumptions.

Functions b_1, d_1 satisfy conditions of Theorem 2.8. Furthermore, the conditions of Proposition 2.13 are satisfied here: for $\eta^1, \eta^2 \in \Gamma_0, \eta^1 \subset \eta^2$ we have

$$b_1(x, \eta^1) \leq b(x, \eta^1) \leq b(x, \eta^2)$$

as well as

$$d_1(x, \eta^1) \geq d(x, \eta^1) \geq d(x, \eta^2).$$

Denote by $(\xi_t)_{t \geq 0}$ the unique solution of (3) with the birth and death rates b_1, d_1 and initial condition ξ_0 . By Proposition 2.13, $\xi_t \subset \eta_t$ hold a.s. for all $t \geq 0$.

In this section we will work on the canonical probability space

$$(D_{\Gamma_0}[0, \infty) \times D_{\Gamma_0}[0, \infty), \mathcal{B}(D_{\Gamma_0}[0, \infty) \times D_{\Gamma_0}[0, \infty)), P_{\alpha}),$$

where P_{α} is the push-forward of the measure P under

$$\Omega \ni \omega \mapsto (\eta(\alpha, \cdot), (\xi(\alpha, \cdot)) \in D_{\Gamma_0}[0, \infty) \times D_{\Gamma_0}[0, \infty).$$

Consider the embedded Markov chain of the process $(\xi_t)_{t \geq 0}, Y_k := \xi_{\tau_k}$, where τ_k are the moments of jumps of (ξ_t) . It turns out that the process $u = \{u_k\}_{k \in \mathbb{N}}$, where $u_k := |Y_k|$, is a Markov chain, too. Indeed, the equality

$$P_{\alpha_1}\{|Y_1| = k\} = P_{\alpha_2}\{|Y_1| = k\}, \quad k \in \mathbb{N}, \alpha \in \Gamma_0.$$

holds when $|\alpha_1 \cap \Lambda| = |\alpha_2 \cap \Lambda|$, since both sides are equal to

$$\begin{cases} \frac{c}{c+a^{-|\alpha_1 \cap \Lambda|}} & \text{if } k = |\alpha_1 \cap \Lambda| + 1, \\ \frac{a^{-|\alpha_1 \cap \Lambda|}}{c+a^{-|\alpha_1 \cap \Lambda|}} & \text{if } k = |\alpha_1 \cap \Lambda| - 1, \\ 0 & \text{in other cases.} \end{cases}$$

Therefore, Lemma A.1 is applicable here, with $f(\cdot) = |\cdot|$.

Proof of Proposition 2.14. Having in mind the inclusion $\xi_t \subset \eta_t$ (P_α -a.s.), we will prove this proposition for (ξ_t) .

It follows from (9) that the transition probabilities for the Markov chain $\{u_k\}_{k \in \mathbb{Z}_+}$ are given by

$$p_{i,j} = P_\alpha\{u_k = j \mid u_{k-1} = i\} = \begin{cases} \frac{c}{c+a^{-i}} & \text{if } j = i + 1, \\ \frac{a^{-i}}{c+a^{-i}} & \text{if } j = i - 1, \\ 0 & \text{in other cases,} \end{cases} \quad (37)$$

for $i \in \mathbb{N}$, $j \in \mathbb{Z}_+$, and $p_{0,j} = I_{\{j=0\}}$.

Since the zero is a trap and it is accessible from all other states, there are no recurrent states except zero, and the process u has only two possible types of behavior on infinity:

$$P_\alpha\{\exists l \in \mathbb{N} \text{ s.t. } u_l = 0 \text{ or } \lim_{m \rightarrow \infty} u_m = \infty\} = 1.$$

We will now use properties of countable state space Markov chains, see, e.g., [14, § 12, chapter 1]. Chung considers there Markov chain with a reflecting barrier at 0, but we may still apply those results, adapting them correspondingly. Denote $\varrho_m = \prod_{k=1}^m \frac{p_{k,k-1}}{p_{k,k+1}}$. Then the probability $P_\alpha\{\exists k \in \mathbb{N} \text{ s.t. } u_k = 0\}$ equals to 1 if and only if $\sum_{j=1}^\infty \varrho_j = \infty$, whichever initial condition α , $|\alpha \cap \Lambda| > 0$, we have. Moreover, if $\sum_{j=1}^\infty \varrho_j < \infty$ and $P_\alpha\{u_0 = q\} = 1$ (or, equivalently, $|\alpha \cap \Lambda| = q$, $q \in \mathbb{N}$), then $p_q := P_\alpha\{\exists k \in \mathbb{N} \text{ s.t. } u_k = 0\} = \frac{\sum_{j=q}^\infty \varrho_j}{1 + \sum_{j=1}^\infty \varrho_j}$. From (37) we see that in our case $\varrho_j = c^{-j} a^{-\frac{j(j+1)}{2}}$, and

$$p_q = \frac{\sum_{j=q}^\infty c^{-j} a^{-\frac{j(j+1)}{2}}}{1 + \sum_{j=1}^\infty c^{-j} a^{-\frac{j(j+1)}{2}}} \leq \frac{\sum_{j=q}^\infty c^{-j} a^{-\frac{j^2}{2}}}{1 + \sum_{j=1}^\infty c^{-j} a^{-\frac{j^2}{2}}}. \quad (38)$$

Now, for arbitrary $C > 1$ choose $q \in \mathbb{N}$ for which $c^{-1} a^{-\frac{q}{2}} < C^{-1}$. For $j > q$ we have $c^{-j} a^{-\frac{j^2}{2}} < c^{-j} a^{-\frac{jq}{2}} = (c^{-1} a^{-\frac{q}{2}})^j < C^{-j}$, and

$$\sum_{j=q}^\infty c^{-j} a^{-\frac{j^2}{2}} < \sum_{j=q}^\infty C^{-j} = \frac{C^{-q}}{1 - C^{-1}},$$

so that the statement of the proposition for $(\xi_t)_{t \geq 0}$ follows from (38). □

Note that for (η_t) the events comprising number of particles going to infinity and extinction are not exclusive, in particular not if $\int_\Lambda b(x, \emptyset) dx > 0$. However, it holds that

$$P\left(\{|\xi_t| = 0 \text{ for sufficiently large } t\} \cup \{|\xi_t| \rightarrow \infty, t \rightarrow \infty\}\right) = 1 \quad (39)$$

and

$$P\left(\{|\xi_t| = 0 \text{ for sufficiently large } t\} \cap \{|\xi_t| \rightarrow \infty, t \rightarrow \infty\}\right) = 0. \quad (40)$$

The following equality is also taken from [14, § 12, chapter 1]; for $q > s, q, s \in \mathbb{N}$, and all β with $|\beta \cap \Lambda| = q$,

$$P_\beta\{\exists k \in \mathbb{N} : |u_k| = s\} = \frac{\sum_{j=q}^{\infty} \varrho_j(s)}{1 + \sum_{j=s+1}^{\infty} \varrho_j(s)},$$

where $\varrho_m(s) = \prod_{k=s+1}^m \frac{p_{k,k-1}}{p_{k,k+1}} = c^{-(m-s)} a^{-\frac{1}{2}(m-s)(m+s+1)}$; in our case

$$P_\beta\{\exists k \in \mathbb{N} : |u_k| = s\} = \frac{\sum_{j=q}^{\infty} c^{-(j-s)} a^{-\frac{1}{2}(j-s)(j+s+1)}}{1 + \sum_{j=s+1}^{\infty} c^{-(j-s)} a^{-\frac{1}{2}(j-s)(j+s+1)}} := c_{q,s} < 1. \quad (41)$$

Note that

$$c_{q+1,1} \rightarrow 0, \quad q \rightarrow \infty. \quad (42)$$

Proof of Proposition 2.15. Let $(X_k)_{k \in \mathbb{Z}_+}$ be the embedded chain of $(\eta_t)_{t \geq 0}$. First we will show that for all $m \in \mathbb{N}$ and $\alpha \in \Gamma_0$,

$$P_\alpha\{|X_k \cap \Lambda| = m \text{ infinitely often}\} = 0. \quad (43)$$

Let $\beta \in \Gamma_0, |\beta \cap \Lambda| = m, m \in \mathbb{N}$ (the case of $m = 0$ is similar, and we do not write it down). Denote $\bar{k} = \min\{k \in \mathbb{N} : X_k \cap \Lambda \neq X_0 \cap \Lambda\}$. Since $\xi_t \subset \eta_t$ holds P_β -a.s.,

$$\begin{aligned} P_\beta\{|X_k \cap \Lambda| > m, \forall k \geq \bar{k}\} &\geq P_\beta\{|Y_k \cap \Lambda| > m, \forall k \geq 1\} \\ &= P_\beta\{u_k > m, \forall k \geq 1\}. \end{aligned} \quad (44)$$

By (41), the probability $P_\beta\{u_k > m, \forall k \geq 1\}$ is positive and does not depend on $\beta, |\beta \cap \Lambda| = m$:

$$s_m := P_\beta\{u_k > m, \forall k \geq 1\} \geq p_{m,m+1}(1 - c_{m+1,m}) > 0. \quad (45)$$

Define $k_i^m, i \in \mathbb{N}$, subsequently by $k_{j+1}^m = \min\{k > k_j^m : |X_k \cap \Lambda| = m \text{ and } \exists \bar{k} < k : |X_{\bar{k}} \cap \Lambda| \neq m\}, k_0^m = 0$. Note that for all β

$$P_\beta\left\{\exists n_0 : |X_n \cap \Lambda| = m \text{ for all } n \geq n_0\right\} = 0.$$

By the strong Markov property,

$$\begin{aligned} P_\alpha\left\{|X_k \cap \Lambda| = m \text{ infinitely often}\right\} &\leq P_\alpha\left\{k_j^m < \infty, \forall j \in \mathbb{N}\right\} \\ &= \prod_{j=1}^{\infty} P_\alpha\{k_{j+1}^m < \infty \mid k_j^m < \infty\} = 0, \end{aligned} \quad (46)$$

by (44) and (45). Indeed, if $P_\alpha\{k_j^m < \infty\} > 0$, then

$$\begin{aligned} P_\alpha\{k_{j+1}^m < \infty \mid k_j^m < \infty\} &= \frac{E_\alpha I_{\{k_j^m < \infty\}} P_{X_{k_j^m}}\{k_1^m < \infty\}}{E_\alpha I_{\{k_j^m < \infty\}}} \\ &\leq \frac{E_\alpha I_{\{k_j^m < \infty\}} (1 - P_{X_{k_j^m}}\{|X_k \cap \Lambda| > m, \forall k \geq \tilde{k}\})}{E_\alpha I_{\{k_j^m < \infty\}}} \\ &\leq \frac{E_\alpha I_{\{k_j^m < \infty\}} (1 - P_{X_{k_j^m}}\{u_k > m, \forall k \geq 1\})}{E_\alpha I_{\{k_j^m < \infty\}}} = 1 - s_m. \end{aligned}$$

Note that $1 - s_m < 1$ does not depend on j , hence (46) follows. Having proved (43), we observe that

$$\begin{aligned} &\{|\eta_t \cap \Lambda| \rightarrow \infty\} \cup \{\exists t' : \forall t \geq t', |\eta_t \cap \Lambda| = \emptyset\} \\ &= \left(\bigcup_{m=1}^{\infty} \{ |X_k \cap \Lambda| = m \text{ infinitely often} \} \right)^c. \end{aligned} \tag{47}$$

Note that if for some element of probability space $\omega \in \Omega$ the process $(\eta_t)_{t \geq 0}$ is stuck in a trap γ , $\gamma \cap \Lambda = \emptyset$, then ω belongs to the set on the left-hand side of (47) and does not belong to the set $\{ |X_k \cap \Lambda| = m \text{ infinitely often} \}$, $m \in \mathbb{N}$.

The statement of the proposition follows from (43) and (47). □

Proof of Proposition 2.16. Define $\tilde{\eta}_t := \eta_t \cap \Lambda$ and let $\tilde{X}_k = \tilde{\eta}_{\zeta_k}$, where ζ_k is the ordered sequence of jumps of $(\tilde{\eta}_t)_{t \geq 0}$. Of course, the process $\{\tilde{\eta}_t\}_{t \geq 0}$ is not Markov in general, and neither is $\{\tilde{X}_k\}_{k \in \mathbb{N}}$. However, for all $\alpha \in \Gamma_0(\mathbb{R}^d)$ the inequality

$$P_\alpha\{|\tilde{X}_1| - |\tilde{X}_0| = 1\} \geq p_{|\alpha \cap \Lambda|, |\alpha \cap \Lambda| + 1}$$

holds, because for every $\zeta \in \Gamma_0$, $\zeta \cap \Lambda = m$, the integral of the birth rate $b(\cdot, \zeta)$ over Λ is larger than cm , and the cumulative death rate in Λ , $\sum_{x \in \zeta \cap \Lambda} d(x, \zeta)$, is less than ma^{-m} .

The probability of the event that absolutely no death occurs is positive, even when the initial configuration contains only one point inside Λ :

$$\begin{aligned} P_\alpha \left\{ |\tilde{\eta}_t| - |\tilde{\eta}_{t-}| \geq 0 \text{ for all } t \geq 0 \right\} &= P_\alpha \left\{ |\tilde{X}_{k+1}| - |\tilde{X}_k| = 1 \text{ for all } k \in \mathbb{N} \right\} \\ &= \prod_{k \in \mathbb{N}} P_\alpha \left\{ |\tilde{X}_{k+1}| - |\tilde{X}_k| = 1 \mid |\tilde{X}_k| - |\tilde{X}_{k-1}| = 1, \dots, |\tilde{X}_1| - |\tilde{X}_0| = 1 \right\} \\ &\geq \prod_{k \in \mathbb{N}} \inf_{\substack{\zeta \in \Gamma_0(\mathbb{R}^d), \\ |\zeta \cap \Lambda| = |\alpha \cap \Lambda| + k}} P_\zeta \{ |\tilde{X}_1| - |\tilde{X}_0| = 1 \} \\ &\geq \prod_{i=|\alpha|}^{\infty} p_{i,i+1} = \prod_{i=|\alpha|}^{\infty} \frac{c}{c + a^{-i}} = \prod_{i=|\alpha|}^{\infty} \left(1 - \frac{a^{-i}}{c + a^{-i}} \right) > 0, \end{aligned}$$

because the series $\sum_{i=|\alpha|}^{\infty} \frac{a^{-i}}{c+a^{-i}}$ converges. In particular, $\prod_{i=m}^{\infty} p_{i,i+1} \rightarrow 1$ as m goes to ∞ . Also,

$$P_{\alpha_n} \{ |\tilde{\eta}_t| - |\tilde{\eta}_{t-}| \geq 0 \text{ for all } t \geq 0 \} \rightarrow 1, \quad |\alpha_n \cap \Lambda| \rightarrow \infty. \tag{48}$$

It is clear that only a.s. finite number of deaths inside Λ occurs on $\{\exists t' : \forall t \geq t', |\eta_t \cap \Lambda| = \emptyset\}$. By Proposition 2.15, it remains to show that only a.s. finite number of deaths inside Λ occurs on $\{|\eta_t \cap \Lambda| \rightarrow \infty\} = \{|\tilde{\eta}_t| \rightarrow \infty\}$. Let us introduce the stopping times $\sigma_n = \inf\{s \in \mathbb{R} : |\tilde{\eta}_s| \geq n\}$, which are finite on $\{|\tilde{\eta}_t| \rightarrow \infty\}$. Only a finite number of events (births and deaths) occur until arbitrary finite time P_{β} -a.s. for all $\beta \in \Gamma_0$, hence for $n \in \mathbb{N}$

$$\begin{aligned} &P_{\alpha} \left(\{|\tilde{\eta}_t| - |\tilde{\eta}_{t-}| \geq 0 \text{ for all but finitely many } t \geq 0\} \cap \{|\tilde{\eta}_t| \rightarrow \infty\} \right) \\ &\geq P_{\alpha} \left(\{|\tilde{\eta}_t| - |\tilde{\eta}_{t-}| \geq 0 \text{ for all } t \geq \sigma_n\} \cap \{|\tilde{\eta}_t| \rightarrow \infty\} \right) \\ &= E_{\alpha} \left[I_{\{|\tilde{\eta}_t| \rightarrow \infty\}} P_{\eta_{\sigma_n}} \{|\tilde{\eta}_t| - |\tilde{\eta}_{t-}| \geq 0 \text{ for all } t \geq 0\} \right]. \end{aligned}$$

From $|\eta_{\sigma_n}| \geq n$ we have by (48)

$$P_{\eta_{\sigma_n}} \{ |\tilde{\eta}_t| - |\tilde{\eta}_{t-}| \geq 0 \text{ for all } t \geq 0 \} \rightarrow 1, \quad n \rightarrow \infty.$$

Therefore,

$$P_{\alpha} \left(\{|\tilde{\eta}_t| - |\tilde{\eta}_{t-}| \geq 0 \text{ for all but finitely many } t \geq 0\} \cap \{|\tilde{\eta}_t| \rightarrow \infty\} \right) = P_{\alpha} \{ |\tilde{\eta}_t| \rightarrow \infty \}.$$

□

Proposition 2.16 is also applicable to $(\xi)_{t \geq 0}$, since b_1, d_1 satisfy all the conditions imposed on b, d .

Proof of Theorem 2.17. First we prove the theorem for $(\xi)_{t \geq 0}$: we prove that for P_{α} -almost all $\omega \in F_1 := \{\lim_{t \rightarrow \infty} |\xi_t \cap \Lambda| = \infty\}$,

$$\liminf_{t \rightarrow \infty} \frac{|\xi_t \cap \Lambda|}{e^{ct}} > 0. \tag{49}$$

Without loss of generality we assume $u_0 = |\alpha \cap \Lambda| > 0$. Let $0 = \tau_0 < \tau_1 < \tau_2 < \dots$ be the moments of jumps of $(\xi_t)_{t \geq 0}$, so that $\xi_{\tau_k} = Y_k$. We recall that the random variables $u_n = |Y_n|$ make up a Markov chain by Lemma A.1. Note that a.s. on F_1 , $u_n > 0$ for all $n \in \mathbb{N}$. Denote $\psi(n) = cn + na^{-n}$. Then

$$\int_{\Lambda} b_1(x, Y_k) dx + \sum_{x \in Y_k} d_1(x, Y_k) = c|Y_k| + |Y_k|a^{-|Y_k|} = \psi(u_k).$$

By Theorem 12.17 in [29] there exists a sequence of independent unit exponentials $\{\gamma_k\}_{k \in \mathbb{N}}$, such that $\gamma_k = \psi(u_k)(\tau_k - \tau_{k-1})$ a.s. on $\{\tau_k < \infty\} \supset F_1$, which is independent of Y . In particular, $\{\gamma_k\}_{k \in \mathbb{N}}$ is independent of $\{u_k\}_{k \in \mathbb{Z}_+}$.

From Proposition 2.16 we know that only a finite number of deaths inside Λ occur a.s. In particular, there exists a positive finite random variable \mathbf{m} such that the inequalities

$$u_0 + n \geq u_n \geq u_0 + n - \mathbf{m}(\omega), \quad n \in \mathbb{N}, \tag{50}$$

hold a.s. on F_1 .

A.s. on F_1

$$\tau_n = \sum_{k=1}^{n-1} (\tau_{k+1} - \tau_k) = \sum_{k=1}^{n-1} \frac{\gamma_k}{\psi(u_k)} \geq \sum_{k=1}^{n-1} \frac{\gamma_k}{u_0 + ck}.$$

Due to Kolmogorov’s two-series theorem, the series $\sum_{k=1}^{\infty} \frac{\gamma_k}{u_0+ck}$ is divergent a.s. (we recall that $E\gamma_k = D\gamma_k = 1$). Hence $\tau_n \rightarrow \infty$ a.s.

We will show below that a.s. on F_1

$$c\tau_n \leq \ln n + c\tilde{\gamma}, \quad n \in \mathbb{N}, \tag{51}$$

where $\tilde{\gamma}$ is some finite a.s. on F_1 random variable. Using (51), we obtain

$$\begin{aligned} P_\alpha(F_1) &\leq P_\alpha \left\{ |\xi_t| \geq \frac{e^{ct}}{(\mathbf{m} + 1)e^{c\tilde{\gamma}}}, t \geq 0 \right\} = P_\alpha \left\{ |\xi_{\tau_n}| \geq \frac{e^{c\tau_{n+1}}}{(\mathbf{m} + 1)e^{c\tilde{\gamma}}}, n \in \mathbb{N} \right\} \\ &= P_\alpha \left\{ u_n \geq \frac{1}{\mathbf{m} + 1} e^{c\tau_{n+1} - c\tilde{\gamma}}, n \in \mathbb{N} \right\} \\ &= P_\alpha \left\{ \ln(u_n) + \ln(\mathbf{m} + 1) \geq c\tau_{n+1} - c\tilde{\gamma}, n \in \mathbb{N} \right\} \leq P_\alpha(F_1). \end{aligned}$$

Therefore, a.s. on F_1 , $|\xi_t| \geq \frac{e^{ct}}{(\mathbf{m}+1)e^{c\tilde{\gamma}}}$ for all $t \geq 0$, and hence (49) holds.

Inequality (51) follows from the convergence a.s. on F_1 of the series

$$\sum_{k=1}^{\infty} \left(\frac{\gamma_k}{\psi(u_k)} - \frac{1}{ck} \right). \tag{52}$$

To establish the convergence of (52), we note that

$$\sum_{k=1}^{\infty} \left(\frac{\gamma_k}{\psi(u_k)} - \frac{\gamma_k}{cu_k} \right) \tag{53}$$

converges a.s. on F_1 by Kolmogorov’s two-series theorem:

$$\begin{aligned} - \sum_{k=1}^{\infty} \left(\frac{\gamma_k}{\psi(u_k)} - \frac{\gamma_k}{cu_k} \right) &= \sum_{k=1}^{\infty} \gamma_k \frac{u_k a^{-u_k}}{cu_k \psi(u_k)} \leq \frac{1}{c^2} \sum_{k=1}^{\infty} \gamma_k \frac{a^{-u_k}}{u_k} \\ &= \frac{1}{c^2} \sum_{k=1}^{\mathbf{m}} + \frac{1}{c^2} \sum_{k=\mathbf{m}+1}^{\infty} \leq \frac{1}{c^2} \sum_{k=1}^{\mathbf{m}} \gamma_k \frac{a^{-u_k}}{u_k} + \frac{1}{c^2} \sum_{j=1}^{\infty} \gamma_k \frac{a^{-j}}{j} < \infty. \end{aligned}$$

The series

$$\sum_{k=1}^{\infty} \left(\frac{\gamma_k}{cu_k} - \frac{1}{cu_k} \right) = \sum_{k=1}^{\infty} \frac{\gamma_k - 1}{cu_k} \tag{54}$$

converges a.s. on F_1 , too, by Kolmogorov's theorem, (50), and since $\{\gamma_k\}$ is independent of $\{u_k\}$: using conditioning on $\{u_k\}$ we get

$$\begin{aligned} & P_\alpha \left(\left\{ \sum_{k=1}^\infty \frac{\gamma_k - 1}{cu_k} \text{ converges} \right\} \cap F_1 \right) \\ &= E_\alpha P_\alpha \left[\left\{ \sum_{k=1}^\infty \frac{\gamma_k - 1}{cu_k} \text{ converges} \right\} \cap F_1 \mid \{u_k\} \right] \\ &= E_\alpha P_\alpha [F_1 \mid \{u_k\}] = P_\alpha(F_1). \end{aligned}$$

Finally, by (50)

$$\sum_{k=1}^\infty \left(\frac{1}{cu_k} - \frac{1}{ck} \right) \tag{55}$$

also converges a.s. on F_1 .

The a.s convergence of the series in (52) follows from the fact that (53), (54), and (55) converge.

We have thus proved that (49) holds a.s. on F_1 . To establish the statement of the theorem, note that $\tilde{\sigma}_n = \inf\{t > 0 : |\eta_t| \geq n\}$ is finite on F and a.s.

$$\left\{ \liminf_{t \rightarrow \infty} \frac{|\eta_t \cap \Lambda|}{e^{ct}} = 0, |\eta_t| \rightarrow \infty \right\} \subset \left\{ \liminf_{t \rightarrow \infty} \frac{|\xi_t|}{e^{ct}} = 0 \right\}.$$

It follows from (39) and (40) that

$$P_\beta \left\{ \liminf_{t \rightarrow \infty} \frac{|\xi_t|}{e^{ct}} = 0 \right\} = P_\beta \{(\xi_t)_{t \geq 0} \text{ goes extinct}\}, \quad \beta \in \Gamma_0.$$

Therefore, by Proposition 2.14 and the strong Markov property

$$\begin{aligned} P_\alpha \left\{ \liminf_{t \rightarrow \infty} \frac{|\eta_t \cap \Lambda|}{e^{ct}} = 0, |\eta_t| \rightarrow \infty \right\} &= E_\alpha P_{\eta_{\tilde{\sigma}_n}} \left\{ \liminf_{t \rightarrow \infty} \frac{|\eta_t \cap \Lambda|}{e^{ct}} = 0, |\eta_t| \rightarrow \infty \right\} \\ &\leq E_\alpha P_{\eta_{\tilde{\sigma}_n}} \left\{ \liminf_{t \rightarrow \infty} \frac{|\xi_t|}{e^{ct}} = 0 \right\} \leq \tilde{C}^{-n}, \end{aligned}$$

where \tilde{C} is the constant from Proposition 2.14. Since n is arbitrary,

$$P_\alpha \left\{ \liminf_{t \rightarrow \infty} \frac{|\eta_t \cap \Lambda|}{e^{ct}} = 0, |\eta_t| \rightarrow \infty \right\} = 0.$$

□

Proof of Corollary 2.18. Let us fix a configuration α , $\alpha \cap \Lambda \neq \emptyset$. We saw in the proof of Theorem 2.17 that for P_α -almost all $\omega \in F_1$ we have

$$|\xi_t| \geq \frac{1}{(\mathbf{m} + 1)e^{c\tilde{\gamma}}} e^{ct}, \quad t \geq 0,$$

where \mathbf{m} and $\tilde{\gamma}$ are random variables a.s. finite on F_1 . Let G_k be the set $\{\omega : \frac{1}{(\mathbf{m}+1)e^{c\tilde{\gamma}}} \geq \frac{1}{k}\}$, $k \in \mathbb{N}$. Then $\bigcup_{k \in \mathbb{N}} G_k \supset F_1$, and, since $P_\alpha(F_1) > 0$,

$$P_\alpha(G_k \cap F_1) > 0$$

for some $k \in \mathbb{N}$. Hence

$$E_\alpha|\eta_t \cap \Lambda| \geq E_\alpha|\xi_t|I_{G_k \cap F_1} \geq \frac{1}{k}e^{ct}P_\alpha(G_k \cap F_1).$$

□

A Appendix

A.1 Markovian functions of a Markov chain

Let $(S, \mathcal{B}(S))$ be a Polish (state) space. Consider a (time-homogeneous) Markov chain on $(S, \mathcal{B}(S))$ as a family of probability measures on S^∞ . Specifically, on the measurable space $(\bar{\Omega}, \mathcal{F}) = (S^\infty, \mathcal{B}(S^\infty))$ consider a family of probability measures $\{P_s\}_{s \in S}$ such that for the coordinate mappings

$$\begin{aligned} X_n &: \bar{\Omega} \rightarrow S, \\ X_n(s_1, s_2, \dots) &= s_n, \end{aligned}$$

the process $X := \{X_n\}_{n \in \mathbb{Z}_+}$ is a Markov chain satisfying for all $s \in S$

$$P_s\{X_0 = s\} = 1,$$

$$P_s\{X_{n+m_j} \in A_j, j = 1, \dots, l \mid \mathcal{F}_n\} = P_{X_n}\{X_{m_j} \in A_j, j = 1, \dots, l\}.$$

Here $A_j \in \mathcal{B}(S)$, $m_j \in \mathbb{N}$, $l \in \mathbb{N}$, $\mathcal{F}_n = \sigma\{X_1, \dots, X_n\}$. The space S is separable, hence there exists a transition probability kernel $Q : S \times \mathcal{B}(S) \rightarrow [0, 1]$ such that

$$Q(s, A) = P_s\{X_1 \in A\}, \quad s \in S, A \in \mathcal{B}(S).$$

Consider a transformation of the chain X , $Y_n = f(X_n)$, where $f : S \rightarrow \mathbb{R}$ is a Borel-measurable function. Here we formulate sufficient conditions for $Y = \{Y_n\}_{n \in \mathbb{Z}_+}$ to be a Markov chain. A very similar question was discussed by Burke and Rosenblatt [13] for discrete space Markov chains. The following lemma is proven in [7, Section 4] (sadly the formulation in [7] contains a typo; the formulation below is taken from [5]).

Lemma A.1. *Assume that for any bounded Borel function $h : S \rightarrow S$*

$$E_s h(Y_1) = E_q h(Y_1) \text{ whenever } f(s) = f(q), \tag{56}$$

Then Y is a Markov chain.

Condition (56) is the equality of distributions of Y_1 under two different measures, P_s and P_q . Clearly, this result holds for a Markov chain which is not necessarily defined on a canonical state space, because the property of a process to be a Markov chain depends on its distribution only.

A.2 Strong Markov property of the driving process

Let N be compatible with a right-continuous complete filtration $\{\mathcal{F}_t\}_{t \geq 0}$, and τ be a finite a.s. $\{\mathcal{F}_t\}_{t \geq 0}$ -stopping time. For $\gamma \in \Gamma_2$, $\gamma = \sum_i \delta_{(s_i, u_i)}$, let $\theta_\tau \gamma = \sum_{i: s_i > \tau} \delta_{(s_i - \tau, u_i)}$. Also, for $\Xi \in \mathcal{B}(\Gamma_2)$ we define the shift

$$\theta_\tau \Xi = \{\gamma \in \Gamma_2 \mid \theta_\tau \gamma \in \Xi\}$$

Introduce another point process \bar{N} on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}_+ \times \Gamma_2$,

$$\bar{N}([0, s] \times U \times \Xi) = N((\tau, \tau + s] \times U \times \theta_\tau \Xi), \quad s > 0, U \in \mathcal{B}(\mathbb{R}^d \times \mathbb{R}_+), \Xi \in \mathcal{B}(\Gamma_2). \tag{57}$$

Proposition A.2. *The process \bar{N} is a Poisson point process with intensity measure $ds \times dx \times du \times \pi$, independent of \mathcal{F}_τ .*

Proof. To prove the proposition, it suffices to show that

(i) for any $b > a > 0$, open bounded $U \subset \mathbb{R}^d \times \mathbb{R}_+$ and open $\Xi \subset \Gamma_2$, $\bar{N}((a, b) \times U \times \Xi)$ is a Poisson random variable with mean $(b - a) \times l(U) \times \pi(\Xi)$, where l is the Lebesgue measure on $\mathbb{R}^d \times \mathbb{R}_+$, and

(ii) for any $b_k > a_k > 0$, $k = 1, \dots, m$, open bounded $U_k \subset \mathbb{R}^d$ and open $\Xi_k \subset \Gamma_2$ such that $((a_i, b_i) \times U_i \times \Xi_i) \cap ((a_j, b_j) \times U_j \times \Xi_j) = \emptyset$, $i \neq j$, the collection $\{\bar{N}((a_k, b_k) \times U_k \times \Xi_k)\}_{k=1, m}$ is a finite sequence of independent random variables, independent of \mathcal{F}_τ .

Let τ_n be the sequence of $\{\mathcal{F}_t\}_{t \geq 0}$ -stopping times, $\tau_n = \frac{k}{2^n}$ on $\{\tau \in (\frac{k-1}{2^n}, \frac{k}{2^n}]\}$, $k \in \mathbb{N}$. Then $\tau_n \downarrow \tau$ and $\tau_n - \tau \leq \frac{1}{2^n}$. The stopping times τ_n take only countably many values. Therefore the process N satisfies the strong Markov property for τ_n : the processes \bar{N}_n , defined by

$$\bar{N}_n([0, s] \times U \times \Xi) := N((\tau_n, \tau_n + s] \times U \times \theta_{\tau_n} \Xi),$$

are Poisson point processes, independent of \mathcal{F}_{τ_n} , with intensity $ds \times dx \times du \times \pi$.

To prove (i), note that $\bar{N}_n((a, b) \times U \times \Xi) \rightarrow \bar{N}((a, b) \times U \times \Xi)$ a.s. and all random variables $\bar{N}_n((a, b) \times U \times \Xi)$ have the same distribution, therefore $\bar{N}((a, b) \times U \times \Xi)$ is a Poisson random variable with mean $(b - a)l(U)\pi(\Xi)$. The random variables $\bar{N}_n((a, b) \times U \times \Xi)$ are independent of \mathcal{F}_τ , hence $\bar{N}((a, b) \times U \times \Xi)$ is independent of \mathcal{F}_τ , too. Similarly, the other part of (ii) follows. \square

Let us now show that the filtration (\mathcal{G}_t) defined below (8) is right-continuous. Indeed, as in the proof of Proposition A.2, we can check that \bar{N}_a is independent of \mathcal{G}_{a+} . Since $\mathcal{G}_\infty = \sigma(\bar{N}_a) \vee \mathcal{G}_a$, $\sigma(\bar{N}_a)$ and \mathcal{G}_a are independent and $\mathcal{G}_{a+} \subset \mathcal{G}_\infty$, we see that $\mathcal{G}_{a+} \subset \mathcal{G}_a$. Thus, $\mathcal{G}_{a+} = \mathcal{G}_a$.

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