Modern Stochastics: Theory and Applications 9 (3) (2022) 313–338
https://doi.org/10.15559/22-VMSTA205

Gaussian Volterra processes with power-type kernels. Part I

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Received: 30 November 2021, Revised: 13 February 2022, Accepted: 21 March 2022,
Published online: 27 April 2022

Abstract The stochastic process of the form

$$X_t = \int_0^t s^\alpha \left(\int_s^t u^\beta (u-s)^\gamma \, du \right) dW_s$$

is considered, where W is a standard Wiener process, $\alpha > -\frac{1}{2}$, $\gamma > -1$, and $\alpha + \beta + \gamma > -\frac{3}{2}$. It is proved that the process X is well-defined and continuous. The asymptotic properties of the variances and bounds for the variances of the increments of the process X are studied. It is also proved that the process X satisfies the single-point Hölder condition up to order $\alpha + \beta + \gamma + \frac{3}{2}$ at point 0, the "interval" Hölder condition up to order $\min\bigl(\gamma + \frac{3}{2},\ 1\bigr)$ on the interval $[t_0, T]$ (where $0 < t_0 < T$), and the Hölder condition up to order $\min\bigl(\alpha + \beta + \gamma + \frac{3}{2},\ \gamma + \frac{3}{2},\ 1\bigr)$ on the entire interval $[0, T]$.

Keywords Gaussian Volterra processes, fractional Brownian motion, Hölder continuity, quasi-helix property

2010 MSC 60G15, 60G17, 60G18, 60G22

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1 Introduction

Consider the stochastic process of the form

$$X_t = C(\alpha, \beta, \gamma) \int_0^t s^\alpha \int_s^t u^\beta (u-s)^\gamma \, du \, dW_s, \tag{1}$$

where W is a Wiener process, $C(\alpha, \beta, \gamma)$ is a constant.

Our assumptions on the values of powers ensuring the existence and smoothness of X are

$$\alpha > -\frac{1}{2}, \qquad \gamma > -1, \qquad \text{and} \qquad \alpha + \beta + \gamma > -\frac{3}{2}. \tag{2}$$

The process X from (1) is a representative of the processes of the form

$$X_t = \int_0^t a(s) \int_s^t b(u) \, c(u-s) \, du \, dW_s, \tag{3}$$

which are studied in [4]. Here $a(s)$, $b(s)$ and $c(s)$ are measurable functions $[0, T] \to [-\infty, \infty]$. Initially, in this paper we intended to apply the results of [4] to power functions. However, the results in [4] are directly applicable only if, in addition to (2), $\alpha^- + \beta^- + \gamma^- < \frac{3}{2}$. (Here we use notation $x^- = \max(-x, 0)$ and $x^+ = \max(x, 0)$. The condition above can be rewritten as $(0 \wedge \alpha) + (0 \wedge \beta) + (0 \wedge \gamma) > -\frac{3}{2}$.) It turned out that for the power kernel we can formulate more specific and weaker conditions of smoothness and other properties of X that are finer than in the general case.

Note that process (3) belongs to the class of processes with Volterra kernels, i.e., the processes of the form

$$X_t = \int_0^t K(t, s) \, dW_s.$$

Such processes are discussed in [1, 2]. They are the particular case of the processes with Fredholm kernels, which are studied in [1, 8].

As it is well known, a fractional Brownian motion B^H with Hurst index $H \in \left(\frac{1}{2}, 1\right)$ admits the Molchan representation (see [5, Theorem 1.8.3] or [7, Theorem 5.2]):

$$B_t^H = \left(H - \tfrac{1}{2}\right) c_H \int_0^t s^{\frac{1}{2}-H} \int_s^t u^{H-\frac{1}{2}} (u-s)^{H-\frac{3}{2}} \, du \, dW_s,$$

where

$$c_H = \left(\frac{2H \, \Gamma(1.5-H)}{\Gamma(H+0.5) \, \Gamma(2-2H)} \right)^{1/2}.$$

Thus, a fractional Brownian motion is an example of the process of the form (1).

Concerning the related results in this direction, Azmoodeh et al. provide [2] necessary and sufficient conditions for the Hölder continuity of Gaussian processes and, as an application, for Fredholm processes. They also provide necessary and sufficient conditions as well as sufficient-only conditions for Volterra processes and for self-similar Gaussian processes. However, the sufficient-only conditions for self-similar Gaussian process, which are stated in [2, Proposition 3], are not satisfied for the process (1), at least, for some values of parameters α, β and γ that satisfy (2). The Fredholm representations of Gaussian processes were also considered in [8].

Table 1. Self-similarity exponents, “waning memory” exponents and maximum order for the Hölder condition for some well-known Gaussian processes

The process	The self-similarity exponent	The exponent in asymptotics of $\mathsf{E}\, X_1 X_t$ λ_1	The exponent in asymptotics of $\mathsf{E}\, X_1(X_{t+1}-X_t)$ λ_2	Hölder condition, up to order
Standard fBm, B^H	H	$0 \vee (2H-1)$	$2H-2$ if $H\neq\frac{1}{2}$	H
Sub-fractional Brownian motion $(B_t^H - B_{-t}^H)/\sqrt{2}$	H	$2H-1$	$2H-2$ if $H\neq\frac{1}{2}$	H
Bifractional Brownian motion $B^{H,K}$	HK	$\max(2HK-1, 2H(K-1))$	$2HK-2H-1$ or $2HK-2$	HK if $H\in(0,1)$ and $K\in(0,1]$
Mixed fBm $B_t^{H_1}+B_t^{H_2}$, $H_1<H_2$	not self-similar	$0\vee(2H_2-1)$	$2H_2-2$	H_1
Process X defined in (1)	$\alpha+\beta+\gamma+\frac{3}{2}$	$\beta+\gamma+1$ if $\beta+\gamma\neq-1$	$\beta+\gamma$	$\min\big(1,\gamma+\frac{3}{2},\alpha+\beta+\gamma+\frac{3}{2}\big)$ on $[0,T]$; $\big(\gamma+\frac{3}{2}\big)\wedge 1$ on $[t_0,T]$.

For bifractional Brownian motion, the asymptotics is

$$\mathsf{E}\, B_1^{H,K}(B_{t+1}^{H,K}-B_t^{H,K}) \sim \frac{HK}{2^{K-1}}\Big((K-1)t^{2KH-2H-1}+(2HK-1)t^{2HK-2}\Big) \quad \text{as } t\to\infty,$$

which gives the value of λ_2.

Even though we consider the process X on the interval $[0,T]$, it can be defined by (1) on the infinite interval $[0,\infty)$. Compare X with other Gaussian process such as fractional Brownian motion B^H, sub-fractional Brownian motion $\{(B_t^H-B_{-t}^H)/\sqrt{2}, t\geq 0\}$, bifractional Brownian motion $B^{H,K}$, and mixed fractional Brownian motion $B^{H_1}+B^{H_2}$, $0<H_1<H_2<1$ (the processes of this kind are studied in [6]). Here B^H is a fractional Brownian motion on $\mathbb{R}$, B^{H_1} and B^{H_2} are two independent fractional Brownian motions with different Hurst indices. All these processes except $B^{H_1}+B^{H_2}$ are self-similar. We compare the self-similarity exponents, orders of the Hölder continuity on a finite interval, and exponents λ_1 and λ_2 in the asymptotics $\mathsf{E}\, X_1X_t \asymp t^{\lambda_1}$ and $\mathsf{E}\, X_1(X_{t+1}-X_t) \asymp t^{\lambda_2}$ as $t\to+\infty$. The results are shown in Table 1. The process X defined in (1) is a fractional Brownian motion for $\alpha=\frac{1}{2}-\alpha$, $\beta=H-\frac{1}{2}$, $\gamma=H-\frac{3}{2}$ and $C(\alpha,\beta,\gamma)=(H-\frac{1}{2})c_H$, $H\in\big(\frac{1}{2},1\big)$. Otherwise, the process X does not coincide with other processes mentioned in Table 1.

In the present paper we are going to prove that the process X has a modification that satisfies the Hölder condition, and to find the upper bound for its order. To that end, we study the asymptotics of the variances of increments of the process X, construct bounds for them, and obtain the so-called generalized quasi-helix property.

For the technical simplicity, in what follows we put $C(\alpha,\beta,\gamma)=1$ and consider a process of the form

$$X_t=\int_0^t s^{\alpha}\int_s^t u^{\beta}(u-s)^{\gamma}\,du\,dW_s. \tag{4}$$

Here is a small remark on the notation. We adopt common definitions of asymptotic equivalence and negligibility. Notation $f(t) \sim g(t)$ means that $f(t) = c_1(t)g(t)$ for some function $c_1(t) \to 1$, while $f(t) = o(g(t))$ means that $f(t) = c_0(t)g(t)$ for some function $c_0(t) \to 0$.

The paper is organized as follows. In Section 2 we prove that, under conditions (2), the process (1) is well-defined and self-similar. In Section 3 we study asymptotic properties of variances and covariances of the increments of the process X. In Section 4 we find the set of parameters for which the process X has stationary increments. Quasi-helix properties of the process X are studied in Section 5; the continuity and the Hölder condition are proved in Section 6. Auxiliary results are obtained in Appendix (Section A).

2 Existence and self-similarity of Gaussian Volterra processes with power-type kernels

2.1 Well-posedness of the process X

For the process defined in (4), the Volterra kernel equals

$$K(t,s) = s^{\alpha} \int_s^t u^{\beta}(u-s)^{\gamma}\,du.$$

Therefore,

$$\begin{aligned} K(kt,ks) &= k^{\alpha}s^{\alpha}\int_{ks}^{kt} u^{\beta}(u-ks)^{\gamma}\,du \\ &= k^{\alpha+\beta+\gamma+1}s^{\alpha}\int_s^t v^{\beta}(v-s)^{\gamma}\,dv = k^{\alpha+\beta+\gamma+1}K(t,s). \end{aligned} \tag{5}$$

Thus, the function $K(t,s)$ is homogeneous of degree $\alpha+\beta+\gamma+1$.

Theorem 1. *Let $T > 0$. Consider the process X defined by (4) with exponents α, β and γ satisfying (2). Then*

$$\sup_{t\in(0,T]} \int_0^t K(t,s)^2\,ds < \infty. \tag{6}$$

So, the process $\{X_t,\ t \in [0,T]\}$ is well-defined and has bounded variance.

Proof. For any fixed $t > 0$, function $K(t,s)$ is continuous in s on $(0,t]$. Let us apply Lemma 4 and consider three cases.

Case 1. If $\beta+\gamma < -1$, then due to Lemma 4

$$\int_s^t u^{\beta}(u-s)^{\gamma}\,du \sim Cs^{\beta+\gamma+1} \quad \text{as } s \to 0+\,,$$

whence

$$K(t,s) \sim Cs^{\alpha+\beta+\gamma+1}$$

as $s \to 0+$, where $C > 0$ is a constant. Relations (2) imply that $\alpha + \beta + \gamma + 1 > -\frac{1}{2}$ whence $\int_0^t K(t,s)^2\,ds < \infty$.

Case 2. Let $\beta + \gamma = -1$, then due to Lemma 4

$$\int_s^t u^\beta (u-s)^\gamma\,du \sim \ln(t/s) \quad \text{as } s \to 0+\,,$$

whence

$$K(t,s) \sim s^\alpha \ln(t/s) = o\big(s^{(\alpha-1)/3}\big)$$

as $s \to 0+$ because $\frac{\alpha-1}{3} < \alpha$. Taking into account that $\frac{\alpha-1}{3} > -\frac{1}{2}$, we get that $\int_0^t K(t,s)^2\,ds < \infty$.

Case 3. If $\beta + \gamma > -1$, then due to Lemma 4

$$\int_s^t u^\beta (u-s)^\gamma\,du \to C t^{\beta+\gamma+1} \quad \text{as } s \to 0+,$$

whence

$$K(t,s) \sim C_2 s^\alpha t^{\beta+\gamma+1}$$

as $s \to 0+$. Since $\alpha > -\frac{1}{2}$, we get that $\int_0^t K(t,s)^2\,ds < \infty$.

In either case, (6) holds true. Indeed, due to (5),

$$\begin{aligned}\int_0^t K(t,s)^2\,ds &= \frac{t^{2\alpha+2\beta+2\gamma+2}}{T^{2\alpha+2\beta+2\gamma+2}} \int_0^t K\left(T, \frac{Ts}{t}\right)^2 ds \\ &= \frac{t^{2\alpha+2\beta+2\gamma+3}}{T^{2\alpha+2\beta+2\gamma+3}} \int_0^T K(T,u)^2\,du\end{aligned}$$

with $2\alpha + 2\beta + 2\gamma + 3 > 0$. Hence, the supremum in (6) is attained for $t = T$ and the inequality in (6) holds true. Due to (6), the process X in (1) is well-defined and has the bounded variance. □

2.2 *Self-similarity of the process X*

Proposition 1. *Process X defined by* (4) *with exponents α, β and γ satisfying* (2) *is self-similar with exponent $H = \alpha + \beta + \gamma + \frac{3}{2}$.*

Proof. According to (5), the covariance function of X is self-similar in the sense that

$$\begin{aligned}\operatorname{cov}(X_{ks}, X_{kt}) &= \int_0^{\min(ks,kt)} K(kt,u)K(ks,u)\,du \\ &= k \int_0^{\min(s,t)} K(kt,tv)K(ks,tv)\,dv \\ &= k^{2H} \int_0^{\min(s,t)} K(t,v)K(s,v)\,dv = k^{2H} \operatorname{cov}(X_s, X_t). \qquad (7)\end{aligned}$$

Notice that the process X is zero-mean and Gaussian. Together with (7), it implies that the process X is self-similar with exponent H. □

3 Asymptotic properties of incremental variances

Let X be a process defined by (4) with α, β and γ satisfying (2).

Then its increments can be represented as

$$\begin{aligned}
X_{t_2} - X_{t_1} &= \int_0^{t_1} K(t_2, s)\, dW_s + \int_{t_1}^{t_2} K(t_2, s)\, dW_s - \int_0^{t_1} K(t_1, s)\, dW_s \\
&= \int_0^{t_1} (K(t_2, s) - K(t_1, s))\, dW_s + \int_{t_1}^{t_2} K(t_2, s)\, dW_s \\
&= \int_0^{t_1} s^\alpha \int_{t_1}^{t_2} u^\beta (u-s)^\gamma\, du\, dW_s + \int_{t_1}^{t_2} s^\alpha \int_s^{t_2} u^\beta (u-s)^\gamma\, du\, dW_s \\
&= \int_0^{t_2} s^\alpha \int_{\max(s,t_1)}^{t_2} u^\beta (u-s)^\gamma\, du\, dW_s, \qquad 0 \le t_1 < t_2. \qquad (8)
\end{aligned}$$

Thus, the variance of the increment is equal to

$$\begin{aligned}
\operatorname{var}(X_{t_2} - X_{t_1}) &= \int_0^{t_2} s^{2\alpha} \left(\int_{\max(s,t_1)}^{t_2} u^\beta (u-s)^\gamma\, du \right)^2 ds \\
&= \iiint\limits_D s^{2\alpha} u^\beta (u-s)^\gamma v^\beta (v-s)^\gamma\, ds\, du\, dv > 0, \qquad (9)
\end{aligned}$$

where $D = \{(s, u, v) \in \mathbb{R}^3 : 0 < s \le \max(t_1, s) < \min(u, v) \le \max(u, v) \le t_2\}$. The set D has a mirror symmetry, and the integrand on the right-hand side of (9) is a symmetric function under the permutation of u and v. Therefore, the integrals over two symmetric to each other halves of the set D are equal:

$$\begin{aligned}
&\iiint\limits_{\{(s,u,v)\in D \,:\, u \le v\}} s^{2\alpha} u^\beta (u-s)^\gamma v^\beta (v-s)^\gamma\, ds\, du\, dv \\
&\qquad = \iiint\limits_{\{(s,u,v)\in D \,:\, u \ge v\}} s^{2\alpha} u^\beta (u-s)^\gamma v^\beta (v-s)^\gamma\, ds\, du\, dv.
\end{aligned}$$

Hence,

$$\begin{aligned}
\operatorname{var}(X_{t_2} - X_{t_1}) &= 2 \iiint\limits_{\{(s,u,v)\in D \,:\, u \le v\}} s^{2\alpha} u^\beta (u-s)^\gamma v^\beta (v-s)^\gamma\, ds\, du\, dv \\
&= 2 \int_{t_1}^{t_2} u^\beta \int_u^{t_2} v^\beta \int_0^u s^{2\alpha} (u-s)^\gamma (v-s)^\gamma\, ds\, dv\, du. \qquad (10)
\end{aligned}$$

Proposition 2. *Let the process X admit representation (4), where α, β and γ satisfy relations (2). Then for $t_0 > 0$ the asymptotic behavior of* $\operatorname{var}(X_{t_2} - X_{t_1})$ *as $(t_1, t_2) \to (t_0, t_0)$ is as follows:*

$$\operatorname{var}(X_{t_2} - X_{t_1}) \sim \frac{t_0^{2\alpha+2\beta}\, |t_2 - t_1|^{2\gamma+3}\, \mathrm{B}(\gamma+1,\, -2\gamma-1)}{(\gamma+1)\,(2\gamma+3)} \qquad \text{if } \gamma < -\frac{1}{2}, \qquad (11)$$

$$\operatorname{var}(X_{t_2}-X_{t_1}) \sim t_0^{2\alpha+2\beta}\,(t_2-t_1)^2\,\ln\!\left(\frac{t_0}{|t_2-t_1|}\right) \qquad \textit{if } \gamma=-\frac{1}{2},$$

$$\operatorname{var}(X_{t_2}-X_{t_1}) \sim t_0^{2\alpha+2\beta+2\gamma+1}\,(t_2-t_1)^2\,\mathrm{B}(2\alpha{+}1,\ 2\gamma{+}1) \qquad \textit{if } \gamma>-\frac{1}{2}.$$

Proof. Without loss of generality, assume that $0<t_1<t_2$. Consider three cases.

Case 1. Let $\gamma<-\frac{1}{2}$. Due to (10),

$$\begin{aligned}\operatorname{var}(X_{t_2}-X_{t_1}) &= 2\int_{t_1}^{t_2} u^\beta \int_u^{t_2} v^\beta \int_0^u s^{2\alpha}(u-s)^\gamma(v-s)^\gamma\,ds\,dv\,du\\ &= 2\int_{t_1}^{t_2}\int_u^{t_2} u^\beta v^\beta \int_0^u (s^{2\alpha}-u^{2\alpha})\,(u-s)^\gamma(v-s)^\gamma\,ds\,dv\,du\\ &\quad+2\int_{t_1}^{t_2} u^{2\alpha+\beta}\int_u^{t_2} v^\beta\int_0^u (u-s)^\gamma(v-s)^\gamma\,ds\,dv\,du. \end{aligned} \tag{12}$$

According to Lemma 5,

$$\begin{aligned}&\lim_{\substack{(u,v)\to(t_0,t_0)\\ u<v}} u^\beta v^\beta \int_0^u (s^{2\alpha}-u^{2\alpha})\,(u-s)^\gamma(v-s)^\gamma\,ds\\ &\qquad= t_0^{2\beta}\int_0^{t_0}(s^{2\alpha}-t_0^{2\alpha})\,(t_0-s)^{2\gamma}\,ds,\end{aligned}$$

where the integral on the right-hand side is finite.

Therefore,

$$\begin{aligned}&2\int_{t_1}^{t_2}\int_u^{t_2} u^\beta v^\beta\int_0^u (s^{2\alpha}-u^{2\alpha})\,(u-s)^\gamma(v-s)^\gamma\,ds\,dv\,du\\ &\quad\sim 2\int_{t_1}^{t_2}\int_u^{t_2} dv\,du\, t_0^{2\beta}\int_0^{t_0}(s^{2\alpha}-t_0^{2\alpha})\,(t_0-s)^{2\gamma}\,ds\\ &\quad=(t_2-t_1)^2\,t_0^{2\beta}\int_0^{t_0}(s^{2\alpha}-t_0^{2\alpha})\,(t_0-s)^{2\gamma}\,ds \end{aligned}\tag{13}$$

as $(t_1,t_2)\to(t_0,t_0)$.

With Lemma 2.2, (ii) from [7], we come to

$$\begin{aligned}&2\int_{t_1}^{t_2} u^{2\alpha+\beta}\int_u^{t_2} v^\beta\int_0^u (u-s)^\gamma(v-s)^\gamma\,ds\,dv\,du\\ &\quad=2\int_{t_1}^{t_2} u^{2\alpha+\beta}\int_u^{t_2} v^\beta (v-u)^{2\gamma+1}\int_1^{v/(v-u)}(t-1)^\gamma t^\gamma\,dt\,dv\,du\\ &\quad=2\int_{t_1}^{t_2} u^{2\alpha+\beta}\int_u^{t_2} v^\beta (v-u)^{2\gamma+1}\int_0^{u/v} s^\gamma(1-s)^{-2\gamma-2}\,ds\,dv\,du.\end{aligned}$$

Since

$$\lim_{\substack{(u,v)\to(t_0,t_0)\\ t_1<t_2}} u^{2\alpha+\beta}v^\beta\int_0^{u/v} s^\gamma(1-s)^{-2\gamma-2}\,ds = t_0^{2\alpha+2\beta}\mathrm{B}(\gamma{+}1,\ -2\gamma{-}1),$$

(here we use the condition $\gamma < -\frac{1}{2}$),

$$\begin{aligned} 2\int_{t_1}^{t_2} u^{2\alpha+\beta} \int_u^{t_2} v^\beta \int_0^u (u-s)^\gamma (v-s)^\gamma \, ds\, dv\, du \\ \sim 2\int_{t_1}^{t_2}\int_u^{t_2} (v-u)^{2\gamma+1}\, dv\, du\, t_0^{2\alpha+2\beta} \mathrm{B}(\gamma{+}1,\ -2\gamma{-}1) \\ = \frac{(t_2-t_1)^{2\gamma+3}\, t_0^{2\alpha+2\beta}\, \mathrm{B}(\gamma{+}1,\ -2\gamma{-}1)}{(\gamma+1)\,(2\gamma+3)} \end{aligned} \tag{14}$$

as $(t_1, t_2) \to (t_0, t_0)$.

The right-hand side of (13) is negligible comparing to the right-hand side of (14). Hence, according to (12), (13), (14),

$$\operatorname{var}(X_{t_2} - X_{t_2}) \sim \frac{(t_2-t_1)^{2\gamma+3}\, t_0^{2\alpha+2\beta}\, \mathrm{B}(\gamma{+}1,\ -2\gamma{-}1)}{(\gamma+1)\,(2\gamma+3)}$$

as $(t_1, t_2) \to (t_0, t_0)$.

Case 2. Let $\gamma = -\frac{1}{2}$. Relations (12) and (13) still hold true:

$$\begin{aligned} &\operatorname{var}(X_{t_2} - X_{t_1}) \\ &\quad = 2\int_{t_1}^{t_2}\int_u^{t_2} u^\beta v^\beta \int_0^u (s^{2\alpha} - u^{2\alpha})\,(u-s)^{-1/2}(v-s)^{-1/2}\, ds\, dv\, du \\ &\quad + 2\int_{t_1}^{t_2} u^{2\alpha+\beta}\int_u^{t_2} v^\beta \int_0^u (u-s)^{-1/2}(v-s)^{-1/2}\, ds\, dv\, du, \end{aligned} \tag{15}$$

$$\begin{aligned} 2\int_{t_1}^{t_2}\int_u^{t_2} u^\beta v^\beta \int_0^u (s^{2\alpha} - u^{2\alpha})\,(u-s)^{-1/2}(v-s)^{-1/2}\, ds\, dv\, du \\ \sim 2\,(t_2-t_1)^2\, t_0^{2\beta} \int_0^{t_0} \frac{s^{2\alpha} - t_0^{2\alpha}}{t_0 - s}\, ds \end{aligned} \tag{16}$$

as $(t_1, t_2) \to (t_0, t_0)$. It is easy to see that

$$\begin{aligned} \int_{t_1}^{t_2} u^{2\alpha+\beta}\int_u^{t_2} v^\beta \int_0^u (u-s)^{-1/2}(v-s)^{-1/2}\, ds\, dv\, du \\ \sim t_0^{2\alpha+2\beta} \int_{t_1}^{t_2}\int_u^{t_2}\int_0^u (u-s)^{-1/2}(v-s)^{-1/2} ds\, dv\, du \end{aligned} \tag{17}$$

as $(t_1, t_2) \to (t_0, t_0)$, and, according to Lemma 6,

$$\begin{aligned} \int_{t_1}^{t_2}\int_u^{t_2}\int_0^u (u-s)^{-1/2}(v-s)^{-1/2}\, ds\, dv\, du \\ = t_2^2 - (t_1+t_2)t_1^{1/2}t_2^{1/2} + t_1^2 + \frac{(t_2-t_1)^2}{2} \ln\!\left(\frac{t_2^{1/2}+t_1^{1/2}}{t_2^{1/2}-t_1^{1/2}}\right) \\ \sim \frac{(t_2-t_1)^2}{2} \ln\!\left(\frac{t_0}{t_2-t_1}\right) \end{aligned} \tag{18}$$

as $(t_1, t_2) \to (t_0, t_0)$. Equations (16), (17) and (18) imply that

$$2 \int_{t_1}^{t_2} u^{2\alpha+\beta} \int_{u}^{t_2} v^{\beta} \int_{0}^{u} (u-s)^{-1/2} (v-s)^{-1/2} \, ds \, dv \, du \\ \sim t_0^{2\alpha+2\beta} (t_2-t_1)^2 \ln\left(\frac{t_0}{t_2-t_1}\right)$$

as $(t_1, t_2) \to (t_0, t_0)$.

Comparing asymptotics of the summands on the right-hand side of (15), we get that the first one is negligible. Thus,

$$\operatorname{var}(X_{t_2} - X_{t_1}) \sim t_0^{2\alpha+2\beta} (t_2-t_1)^2 \ln\left(\frac{t_0}{t_2-t_1}\right)$$

as $(t_1, t_2) \to (t_0, t_0)$.

Case 3. $\gamma > -\frac{1}{2}$. According to Lemma 7,

$$\lim_{\substack{(u,v)\to(t_0,t_0)\\ u<v}} u^{\beta} v^{\beta} \int_0^u s^{2\alpha} \, (u-s)^{\gamma} (v-s)^{\gamma} \, ds = t_0^{2\beta} \int_0^{t_0} s^{2\alpha} \, (t_0-s)^{2\gamma} \, ds \\ = t_0^{2\alpha+2\beta+2\gamma+1} \, \mathrm{B}(2\alpha+1, \; 2\gamma+1).$$

Hence,

$$\operatorname{var}(X_{t_2} - X_{t_1}) = 2 \int_{t_1}^{t_2} \int_{u}^{t_2} u^{\beta} v^{\beta} \int_0^u s^{2\alpha} (u-s)^{\gamma} (v-s)^{\gamma} \, ds \, dv \, du \\ \sim 2 \int_{t_1}^{t_2} \int_{u}^{t_2} dv \, du \, t_0^{2\alpha+2\beta+2\gamma+1} \, \mathrm{B}(2\alpha+1, \; 2\gamma+1) \\ = (t_2-t_1)^2 \, t_0^{2\alpha+2\beta+2\gamma+1} \, \mathrm{B}(2\alpha+1, \; 2\gamma+1)$$

as $(t_1, t_2) \to (t_0, t_0)$. □

Proposition 3. *Let the process X admit representation* (4) *with the values of powers satisfying relations* (2)*. Let* $0 < t_2 < t_3$*. Then the asymptotic behavior of* $\mathsf{E}[X_{t_1} \, (X_{t_3} - X_{t_2})]$ *as* $t_1 \to 0+$ *is as follows:*

$$\mathsf{E}[X_{t_1} \, (X_{t_3} - X_{t_2})] \\ \sim \frac{\mathrm{B}(2\alpha+1, \; \gamma+1)}{(2\alpha+\beta+\gamma+2)(\beta+\gamma+1)} \, (t_3^{\beta+\gamma+1} - t_2^{\beta+\gamma+1}) \, t_1^{2\alpha+\beta+\gamma+2}$$

if $\beta + \gamma \neq -1$,

$$\mathsf{E}[X_{t_1} \, (X_{t_3} - X_{t_2})] \sim \frac{\mathrm{B}(2\alpha+1, \; \gamma+1)}{2\alpha+\beta+\gamma+2} \, t_1^{2\alpha+\beta+\gamma+2} \, \ln\left(\frac{t_3}{t_2}\right)$$

if $\beta + \gamma = -1$.

Proof. According to (4) and (8), for $0 < t_1 < t_2 < t_3$

$$\mathsf{E}[X_{t_1}(X_{t_3} - X_{t_2})] = \int_0^{t_1} s^{2\alpha} \left(\int_s^{t_1} u^\beta (u-s)^\gamma \, du \right) \left(\int_{t_2}^{t_3} v^\beta (v-s)^\gamma \, dv \right) ds.$$

Obviously,

$$\lim_{s \to 0} \int_{t_2}^{t_3} v^\beta (v-s)^\gamma \, dv = \int_{t_2}^{t_3} v^{\beta+\gamma} \, dv = C(t_2, t_3),$$

where

$$C(t_2, t_3) = \begin{cases} \dfrac{t_3^{\beta+\gamma+1} - t_2^{\beta+\gamma+1}}{\beta+\gamma+1} & \text{if } \beta + \gamma \ne -1, \\ \ln(t_3/t_2) & \text{if } \beta + \gamma = -1. \end{cases}$$

Hence,

$$\mathsf{E}[X_{t_1}(X_{t_3} - X_{t_2})] \sim C(t_2, t_3) \int_0^{t_1} s^{2\alpha} \left(\int_s^t u^\beta (u-s)^\gamma \, du \right) ds \quad \text{as } t_1 \to 0. \tag{19}$$

Furthermore,

$$\begin{aligned} \int_0^{t_1} s^{2\alpha} \left(\int_s^t u^\beta (u-s)^\gamma \, du \right) ds &= \int_0^{t_1} u^\beta \left(\int_0^u s^{2\alpha} (u-s)^\gamma \, ds \right) du \\ &= \mathrm{B}(2\alpha+1,\ \gamma+1) \int_0^{t_1} u^{2\alpha+\beta+\gamma+1} \, du \\ &= \frac{\mathrm{B}(2\alpha+1,\ \gamma+1)}{2\alpha+\beta+\gamma+2} \, t_1^{2\alpha+\beta+\gamma+2}, \end{aligned} \tag{20}$$

since the assumptions (2) ensure that $2\alpha + \beta + \gamma + 2 > 0$. By (19) and (20),

$$\mathsf{E}[X_{t_1}(X_{t_3} - X_{t_2})] \sim \frac{C(t_2, t_3)\,\mathrm{B}(2\alpha+1,\ \gamma+1)}{2\alpha+\beta+\gamma+2} \, t_1^{2\alpha+\beta+\gamma+2} \quad \text{as } t_1 \to 0,$$

as desired. □

4 When does the process X have stationary increments?

Recall that fractional Brownian motion with Hurst index $H \in (0, 1)$ is a zero-mean Gaussian process with covariance function $\operatorname{cov}(X_s, X_t) = (s^{2H} + t^{2H} - |t-s|^{2H})\,/\,2$.

Theorem 2. *Let stochastic process X be defined by relations (4) and (2). Then the following three statements are equivalent:*

(a) The process X has stationary increments.

(b) Up to a constant, the process X is a fractional Brownian motion with Hurst index $H \in \left(\frac{1}{2}, 1\right)$.

(c) There exists $H \in \left(\frac{1}{2}, 1\right)$ such that $\alpha = \frac{1}{2} - H$, $\beta = H - \frac{1}{2}$ and $\gamma = H - \frac{3}{2}$.

Proof. The process X is Gaussian and according to Proposition 1, it is also self-similar with exponent $H = \alpha + \beta + \gamma + \frac{3}{2}$. Suppose (a), i.e., it has stationary increments. According to [3, Section 1.3; Theorem 1.3.1] a self-similar Gaussian process with stationary increments is a fBm, up to a constant. Moreover, $H > 0$ and

(i) if $H \in (0, 1)$, then the process X is a fractional Brownian motion with Hurst index H;

(ii) if $H = 1$, then $X(t) = tX(1)$ a.s. for all $t \ge 0$ and for some Gaussian variable $X(1)$ (see Theorem 1.3.3 in [3]);

(iii) if $H > 1$, then $X(t) = 0$ almost surely for all t (see Theorem 3.1.1(ii) in [3]).

In cases (ii) and (iii) $\operatorname{var}[X_{t_2} \mid X_{t_1}] = 0$ for $t_1 < t_2$, which contradicts (44). Thus, case (i) takes place, and up to a constant, the process X is a fBm, $X_t = mB_t^H$ with exponent $H \in (0, 1)$. Then $\operatorname{var}(X_{t_2} - X_{t_1}) = m^2 \, |t_2 - t_1|^{2H}$. On the other hand, the asymptotics of $\operatorname{var}(X_{t_2} - X_{t_1})$ is obtained in Proposition 2. Since $2H < 2$, the first case in Proposition 2 occurs, namely, $\gamma < -\frac{1}{2}$ and the asymptotics satisfies (11). It means that

$$m^2 \, |t_2 - t_1|^{2H} \sim C^2(\alpha, \beta, \gamma) \, \frac{t_0^{2\alpha+2\beta} \, |t_2 - t_1|^{2\gamma+3} \, \mathrm{B}(\gamma+1, \ -2\gamma-1)}{(\gamma + 1) \, (2\gamma + 3)}$$

as $t_1 \to t_0$ and $t_2 \to t_0$, for all $t_0 \in (0, T]$. Equating the exponents, we obtain that $2\alpha + 2\beta = 0$ and $2\gamma + 3 = 2H$. Since $\gamma \in \left(-1, -\frac{1}{2}\right)$, one has $H \in \left(\frac{1}{2}, 1\right)$, and we get that (a) implies (b).

Having (b), find the asymptotics for $\mathsf{E}[X_{t_1} \, (X_{t_3} - X_{t_2})]$:

$$\begin{aligned}\mathsf{E}[X_{t_1} \, (X_{t_3} - X_{t_2})] &= \frac{m^2 \, (t_3^{2H} - |t_3 - t_1|^{2H} - t_2^{2H} + |t_2 - t_1|^{2H})}{2} \\ &\sim H \, m^2 \, (t_3^{2H-1} - t_2^{2H-1}) \, t_1\end{aligned}$$

as $t_1 \to 0$, for all $t_2 \in (0, T]$ and $t_3 \in (0, T]$ such that $t_2 \ne t_3$. Compare this with the result of Proposition 3. The first case, $\beta + \gamma \ne -1$, occurs in Proposition 3, and

$$H \, m^2 \, (t_3^{2H-1} - t_2^{2H-1}) \, t_1 \sim C(t_3^{\beta+\gamma+1} - t_2^{\beta+\gamma+1}) \, t_1^{2\alpha+\beta+\gamma+2}$$

as $t_1 \to 0$, where $C > 0$ is a constant. Thus, $\beta+\gamma+1 = 2H-1$ and $2\alpha+\beta+\gamma+2 = 1$. Now we can find α, β and γ from the system of linear equations:

$$2\alpha + 2\beta = 0, \quad 2\gamma + 3 = 2H, \quad \beta + \gamma + 1 = 2H - 1, \quad 2\alpha + \beta + \gamma + 2 = 1,$$

whence

$$\alpha = \frac{1}{2} - H, \qquad \beta = H - \frac{1}{2}, \qquad \gamma = H - \frac{3}{2}.$$

So, (b) implies (c). Implication $(c) \Rightarrow (a)$ is evident. $\square$

Remark 1. Note that the Volterra representation of the fractional Brownian motion with Hurst index $0 < H < \frac{1}{2}$ has a more complex formula than for $\frac{1}{2} < H < 1$, see [7, Theorem 5.2]. Particularly, the fractional Brownian motion with $0 < H < \frac{1}{2}$ cannot be represented in the form of (1).

5 Quasi-helix and generalized quasi-helix conditions

In this section we present the uniform inequalities for the incremental variances of Gaussian processes with Volterra kernels.

5.1 Definitions

Definition 1. Let $0 \le t_0 < T$. The process $\{X_t,\ t \in [t_0, T]\}$ is a quasi-helix with exponent $\lambda > 0$ if there exist two constants $C_i > 0$, $i = 1, 2$, such that for any $t_0 \le t_1 < t_2 \le T$

$$C_1(t_2 - t_1)^{2\lambda} \le \operatorname{var}(X_{t_2} - X_{t_1}) \le C_2(t_2 - t_1)^{2\lambda}. \tag{21}$$

Sometimes we can construct lower and upper bounds for the variance with different exponents. Thus, we come to the notion of the generalized quasi-helix.

Definition 2. The process $\{X_t,\ t\in[t_0, T]\}$ is a generalized quasi-helix with exponents $\lambda_i > 0$, $i = 1, 2$, if there exist two constants $C_i > 0$, $i = 1, 2$, such that for any $t_0 \le t_1 < t_2 \le T$

$$C_1(t_2 - t_1)^{2\lambda_1} \le \operatorname{var}(X_{t_2} - X_{t_1}) \le C_2(t_2 - t_1)^{2\lambda_2}.$$

Remark 2. Unless $\operatorname{var}(X_{t_2} - X_{t_1}) = 0$ for all $t_1, t_2 \in [t_0, T]$, the exponents λ_i satisfy the relation $0 < \lambda_2 \le \min(1, \lambda_1)$.

Definition 3. The process $\{X_t,\ t \in [t_0, T]\}$ is a pseudo-quasihelix with exponent $\lambda > 0$ if for any λ_1 and λ_2 such that $0 < \lambda_2 < \lambda < \lambda_1$ it is a generalized quasi-helix with exponents λ_1 and λ_2.

5.2 Quasi-helix on $[t_0, T]$

The following lemma allows to figure out when a self-similar process is a quasi-helix considering the asymptotic behavior of its small incremental variances.

Lemma 1. *Let $0 < t_0 < T$, and let the stochastic process $\{X_t,\ t \in (0, T]\}$ satisfy the following conditions:*

(i) $\{X_t,\ t \in (0, T]\}$ *is self-similar with exponent H;*

(ii) $\operatorname{var}(X_{t_2} - X_{t_1}) > 0$ *whenever $t_0 \le t_1 < t_2 \le T$;*

(iii) for some $C > 0$ and $\lambda > 0$ and for all $t \in [t_0, T]$ the variances of increments asymptotically behave as follows:

$$\operatorname{var}(X_{t_1} - X_{t_2}) \sim C\, t^{2H-2\lambda}(t_2 - t_1)^{2\lambda} \quad \text{as} \quad t_1 \to t,\ t_2 \to t,\ t_1{<}t_2. \tag{22}$$

Then the process $\{X_t,\ t \in [t_0, T]\}$ is a quasi-helix with exponent λ.

Proof. Because of (22), the process $\{X_t,\ t \in [t_0, T]\}$ is mean-square continuous. Now, consider the function

$$f(t) = \frac{\operatorname{var}(X_t - X_{t_0})}{t_0^{2H-2\lambda}(t - t_0)^{2\lambda}}, \quad t \in (t_0, T].$$

Obviously, function $f(t)$ is continuous on $(t_0, T]$, $\lim_{t\to t_0} f(t) = C > 0$ and $f(t) > 0$, $t \in (t_0, T]$, due to condition (iii). As a consequence, it is bounded on $[t_0, T]$, and there exist $0 < c_1 < c_2$ such that $c_1 \le f(t) \le c_2$ for all $t \in (t_0, T]$. Since the process X is self-similar with exponent H, we have that

$$\operatorname{var}(X_{t_2} - X_{t_1}) = \frac{t_1^{2H}}{t_0^{2H}} \operatorname{var}(X_{t_2 t_0/t_1} - X_{t_0}) = t_1^{2H-2\lambda}(t_2 - t_1)^{2\lambda} f\left(\frac{t_2 t_0}{t_1}\right)$$

for all t_1 and t_2 such that $t_0 \le t_1 < t_2 \le T$.

Thus,

$$c_1 t_1^{2H-2\lambda}(t_2 - t_1)^{2\lambda} \le \operatorname{var}(X_{t_2} - X_{t_1}) \le c_2 t_1^{2H-2\lambda}(t_2 - t_1)^{2\lambda},$$

whence

$$\begin{aligned} c_1 \min(t_0^{2H-2\lambda}, T^{2H-2\lambda})(t_2 - t_1)^{2\lambda} &\le \operatorname{var}(X_{t_2} - X_{t_1}) \\ &\le c_2 \max(t_0^{2H-2\lambda}, T^{2H-2\lambda})(t_2 - t_1)^{2\lambda}. \end{aligned}$$

It means that inequality (21) holds true for $C_1 = c_1 \min(t_0^{2H-2\lambda}, T^{2H-2\lambda})$ and $C_2 = c_2 \max(t_0^{2H-2\lambda}, T^{2H-2\lambda})$. So, the process X is a quasi-helix on the interval $[t_0, T]$ with exponent λ. □

Theorem 3. *Let $0 < t_0 < T$, and let the process X be defined by* (4) *and* (2). *Moreover, assume that $\gamma \ne -\frac{1}{2}$. Then the process $\{X_t,\ t \in [t_0, T]\}$ is a quasi-helix with exponent $\gamma + \frac{3}{2}$ if $-1 < \gamma < -\frac{1}{2}$, and with exponent 1 if $\gamma > -\frac{1}{2}$.*

Proof. The proof immediately follows from Proposition 1, inequality (9), Proposition 2, and Lemma 1. □

5.3 *Generalized quasi-helix on* $[0, T]$. *Case* $\gamma \ne -1/2$

Lemma 2. *Let $T > 0$ and let stochastic process $\{X_t,\ t \in [0, T]\}$ satisfy the following conditions:*

(i) $X_0 = 0$*;*

(ii) $\{X_t,\ t \in [0, T]\}$ *is self-similar with exponent* $H > 0$*;*

(iii) $\operatorname{var}(X_{t_2} - X_{t_1}) > 0$ *whenever* $0 \le t_1 < t_2 \le T$*;*

(iv) for some $C > 0$, $\lambda > 0$ and for all $t \in (0, T]$ the incremental variances asymptotically behave as follows:

$$\operatorname{var}(X_{t_1} - X_{t_2}) \sim C\, t^{2H-2\lambda}(t_2 - t_1)^{2\lambda} \quad \text{as} \quad t_1 \to t,\ t_2 \to t,\ t_1 < t_2. \tag{23}$$

Then the process $\{X_t,\ t \in [0, T]\}$ is a generalized quasi-helix with exponents $H \vee \lambda$ and $H \wedge \lambda$.

Proof. The process $\{X_t,\ t \in [0, T]\}$ is mean-square continuous. The continuity at point 0 follows from self-similarity with exponent $H > 0$, while the continuity on $(0, T]$ follows from (23) with $\lambda > 0$.

Consider the function

$$f(t) = \frac{\operatorname{var}(X_T - X_t)}{(T - t)^{2\lambda}}, \qquad t \in [0, T).$$

The function $f(t)$ is continuous on $[0, T)$, $f(t) > 0$ on $[0, T)$, and

$$\lim_{t \to T} f(t) = CT^{2H-2\gamma} \in (0, \infty).$$

As the consequence, there exist $c_1 > 0$ and c_2 such that $c_1 \le f(t) \le c_2$ on $[0, T)$.

Because of self-similarity, for any $0 \le t_1 < t_2 \le T$

$$\operatorname{var}(X_{t_2} - X_{t_1}) = \frac{t_2^{2H}}{T^{2H}} \operatorname{var}(X_T - X_{t_1 T/t_2}) = t_2^{2H-2\lambda} T^{2\lambda-2H} (t_2 - t_1)^{2\lambda} f\left(\frac{t_1 T}{t_2}\right).$$

Hence,

$$c_1 t_2^{2H-2\lambda} T^{2\lambda-2H} (t_2 - t_1)^{2\lambda} \le \operatorname{var}(X_{t_2} - X_{t_1}) \le c_2 t_2^{2H-2\lambda} T^{2\lambda-2H} (t_2 - t_1)^{2\lambda}.$$

In the following inequalities, we use that $0 < t_2 - t_1 \le t_2 \le T$, however the inequalities depend on the sign of $2H - 2\lambda$. If $2H \le 2\lambda$, then

$$\begin{aligned} c_1 (t_2 - t_1)^{2\lambda} &\le \frac{c_1\, T^{2\lambda-2H} (t_2 - t_1)^{2\lambda}}{t_2^{2\lambda-2H}} \le \operatorname{var}(X_{t_2} - X_{t_1}) \\ &\le \frac{c_2\, T^{2\lambda-2H} (t_2 - t_1)^{2\lambda}}{t_2^{2\lambda-2H}} \le c_2\, T^{2\lambda-2H} (t_2 - t_1)^{2H}. \end{aligned}$$

If $2H \ge 2\lambda$, then

$$\begin{aligned} \frac{c_1 (t_2 - t_1)^{2H}}{T^{2H-2\lambda}} &\le \frac{c_1\, t_2^{2H-2\lambda} (t_2 - t_1)^{2\lambda}}{T^{2H-2\lambda}} \le \operatorname{var}(X_{t_2} - X_{t_1}) \\ &\le \frac{c_2\, t_2^{2H-2\lambda} (t_2 - t_1)^{2\lambda}}{T^{2H-2\lambda}} \le c_2\, (t_2 - t_1)^{2\lambda}, \end{aligned}$$

and the proof follows. □

Theorem 4. *Let $T > 0$, and let the process X be defined by* (4) *and* (2). *Moreover, assume that $\gamma \ne -\frac{1}{2}$. If $-1 < \gamma < -\frac{1}{2}$, then the process $\{X_t,\ t \in [0, T]\}$ is a generalized quasi-helix with exponents $(\gamma + \frac{3}{2}) \vee (\alpha + \beta + \gamma + \frac{3}{2})$ and $(\gamma + \frac{3}{2}) \wedge (\alpha + \beta + \gamma + \frac{3}{2})$. Otherwise, if $\gamma > -\frac{1}{2}$, then the process $\{X_t,\ t \in [0, T]\}$ is a generalized quasi-helix with exponents $1 \vee (\alpha + \beta + \gamma + \frac{3}{2})$ and $1 \wedge (\alpha + \beta + \gamma + \frac{3}{2})$.*

Proof. Theorem 4 follows immediately from Propositions 1 and 2, inequality (9) and Lemma 2. □

If a self-similar process is a quasi-helix on $[0, T]$, then the exponents in the self-similarity condition and in the quasi-helix condition must be the same. If for some $t_0 \in [0, T]$ the variances of a quasi-helix satisfy the relation $\operatorname{var}(X_{t_1} - X_{t_2}) \sim C(t_0)\,(t_2 - t_1)^{2\lambda}$ as $t_1 \to t_0$, $t_2 \to t_0$, $t_1 < t_2$ for some $C(t_0) > 0$ and $\lambda > 0$, then the exponent in the quasi-helix condition must be equal to λ. If the variances of a stochastic process satisfy the relation given in the second case in Proposition 2 (for $\gamma = -\frac{1}{2}$), then the process cannot be quasi-helix. This proves the necessity condition in the following corollary. The sufficiency follows from Theorems 3 and 4.

Corollary 1. *Let $0 < t_0 < T$, and let the process X be defined by (4) and (2). The process X is a quasi-helix on $[t_0, T]$ if and only if $\gamma \neq -\frac{1}{2}$. The process X is a quasi-helix on $[0, T]$ in (and only in) two cases:*

- *if $-1 < \lambda < -\frac{1}{2}$ and $\alpha + \beta = 0$, or*
- *if $\lambda > -\frac{1}{2}$ and $\alpha + \beta + \gamma = -\frac{1}{2}$.*

5.4 *Generalized quasi-helix on $[0, T]$. The bordering case $\gamma = -\frac{1}{2}$*

Theorem 5. *Let $0 < t_0 < T$, and let the process X be defined by (4) and (2) with*

$$\alpha > -\frac{1}{2}, \qquad \alpha + \beta > -1, \qquad \text{and} \qquad \gamma = -\frac{1}{2}. \tag{24}$$

Then the following holds true:

1. *For any $\lambda \in (0, 1)$ the process $\{X_t,\ t \in [t_0, T]\}$ is a generalized quasi-helix with exponents 1 and λ.*
2. *If $\alpha + \beta < 0$, then the process $\{X_t,\ t \in [0, T]\}$ is a generalized quasi-helix with exponents 1 and $\alpha + \beta + 1$.*
3. *If $\alpha + \beta \geq 0$, then for any $\lambda \in (0, 1)$ the process $\{X_t,\ t \in [0, T]\}$ is a generalized quasi-helix with exponents $\alpha + \beta + 1$ and λ.*

Proof. We divide the proof into five parts. First, we obtain the lower and upper bounds for $\operatorname{var}(X_{t_2} - X_{t_1})$. Then we subsequently prove the three statements of the theorem.

To start with, the process X is self-similar with the exponent $\alpha + \beta + \gamma + \frac{3}{2} = \alpha + \beta + 1 > 0$. It is square-continuous on $[0, T]$. Denote

$$f_\lambda(t) = \frac{\operatorname{var}(X_T - X_t)}{(T - t)^{2\lambda}}, \qquad t \in [0, T).$$

Here we assume $\lambda \in \mathbb{R}$. The function f_λ is continuous on $[0, T)$, $f_\lambda(t) > 0$ for all $t \in [0, T)$ and $f_\lambda(t) \sim T^{2\alpha+2\beta}(T - t)^{2-2\lambda}(\ln(T) - \ln(T - t))$ as $t \to T$, due to Proposition 2.

(i) Lower bound. Let's apply function f_λ with $\lambda = 1$. In this case

$$f_1(t) = \frac{\operatorname{var}(X_T - X_t)}{(T - t)^2}, \qquad t \in [0, T),$$

$$f_1(t) \sim T^{2\alpha+2\beta}(\ln(T) - \ln(T-t)) \quad \text{as} \quad t \to T,$$
$$\lim_{t\to T} f_1(t) = +\infty.$$

Together with continuity and positivity of function f_1 on $[0, T)$ this implies that for some $c_1 > 0$

$$\forall\, t \in [0, T) \,:\, f_1(t) > c_1.$$

Furthermore, self-similarity of X implies that

$$\begin{aligned}
\operatorname{var}(X_{t_2} - X_{t_1}) &= \frac{t_2^{2\alpha+2\beta+2}}{T^{2\alpha+2\beta+2}} \operatorname{var}(X_T - X_{t_1 T/t_2}) \\
&= \frac{t_2^{2\alpha+2\beta}(t_2-t_1)^2}{T^{2\alpha+2\beta}} f_1\left(\frac{t_1 T}{t_2}\right) \ge \frac{c_1\, t_2^{2\alpha+2\beta}(t_2-t_1)^2}{T^{2\alpha+2\beta}}.
\end{aligned}$$

If $\alpha + \beta < 0$, then

$$\operatorname{var}(X_{t_2} - X_{t_1}) \ge c_1\,(t_2 - t_1)^2 \tag{25}$$

for all t_1 and t_2 such that $0 \le t_1 < t_2 \le T$.

If $\alpha + \beta \ge 0$, then

$$\operatorname{var}(X_{t_2} - X_{t_1}) \ge \frac{c_1\, t_0^{2\alpha+2\beta}(t_2-t_1)^2}{T^{2\alpha+2\beta}} \tag{26}$$

for all $0 < t_0 \le t_1 < t_2 \le T$, and

$$\operatorname{var}(X_{t_2} - X_{t_1}) \ge \frac{c_1(t_2-t_1)^{2\alpha+2\beta+2}}{T^{2\alpha+2\beta}} \tag{27}$$

for all $0 \le t_1 < t_2 \le T$.

(ii) Upper bound. Let $\lambda \in (0, 1)$. Then, as $2 - 2\lambda > 0$,

$$\lim_{t\to T} f_\lambda(t) = 0.$$

With continuity of f_λ on $[0, T)$, this implies that the function f_λ is bounded on $[0, T)$. Thus, for some finite $c_2(\lambda)$,

$$\forall\, t \in [0, T) \,:\, f_\lambda(t) \le c_2(\lambda).$$

Furthermore,

$$\begin{aligned}
\operatorname{var}(X_{t_2} - X_{t_1}) &= \frac{t_2^{2\alpha+2\beta+2}}{T^{2\alpha+2\beta+2}} \operatorname{var}(X_T - X_{t_1 T/t_2}) \\
&= \frac{t_2^{2\alpha+2\beta+2-2\lambda}(t_2-t_1)^{2\lambda}}{T^{2\alpha+2\beta+2-2\lambda}} f_\lambda\left(\frac{t_1 T}{t_2}\right) \le \frac{c_2(\lambda)\, t_2^{2\alpha+2\beta+2-2\lambda}(t_2-t_1)^{2\lambda}}{T^{2\alpha+2\beta+2-2\lambda}}.
\end{aligned}$$

Obviously,

$$\max_{t_2\in[t_0,T]} t_2^{2\alpha+2\beta+2-2\lambda} = \max(t_0^{2\alpha+2\beta+2-2\lambda},\ T^{2\alpha+2\beta+2-2\lambda}).$$

Thus,

$$\operatorname{var}(X_{t_2} - X_{t_1}) \le c_2(\lambda)\, \max\left(\frac{t_0^{2\alpha+2\beta+2-2\lambda}}{T^{2\alpha+2\beta+2-2\lambda}},\ 1\right) (t_2 - t_1)^{2\lambda} \tag{28}$$

for all t_1 and t_2 such that $t_0 \le t_1 < t_2 \le T$.

If $\lambda \le \alpha + \beta + 1$ in addition to $\lambda \in (0, 1)$, then

$$\operatorname{var}(X_{t_2} - X_{t_1}) \le c_2(\lambda)\, (t_2 - t_1)^{2\lambda} \tag{29}$$

for all t_1 and t_2 such that $0 \le t_1 < t_2 \le T$.

(iii) Proof of statement 1. Let $\lambda \in (0, 1)$. If $\alpha + \beta < 0$, then, due to (25) and (28),

$$c_1\, (t_2 - t_1)^2 \le \operatorname{var}(X_{t_2} - X_{t_1}) \le c_2(\lambda)\, \max\left(\frac{t_0^{2\alpha+2\beta+2-2\lambda}}{T^{2\alpha+2\beta+2-2\lambda}},\ 1\right) (t_2 - t_1)^{2\lambda}.$$

If $\alpha + \beta \ge 0$ (and, as a consequence, $\lambda < \alpha + \beta + 1$), then, due to (26) and (29),

$$\frac{c_1\, t_0^{2\alpha+2\beta} (t_2 - t_1)^2}{T^{2\alpha+2\beta}} \le \operatorname{var}(X_{t_2} - X_{t_1}) \le c_2(\lambda)\, (t_2 - t_1)^{2\lambda}$$

for all t_1 and t_2 such that $t_0 \le t_1 < t_2 \le T$.

In either case, the process $\{X_t,\ \ t \in [t_0, T]\}$ is a generalized quasi-helix with exponents 1 and λ.

(iv) Proof of statement 2. Let $\alpha + \beta < 0$. Denote the self-similarity exponent by λ: $\lambda = \alpha + \beta + 1$. Then $\lambda \in (0, 1)$. The condition $\lambda \le \alpha + \beta + 1$ of (29) is also satisfied.

Due to (25) and (29),

$$c_1\, (t_2 - t_1)^2 \le \operatorname{var}(X_{t_2} - X_{t_1}) \le c_2(\lambda)\, (t_2 - t_1)^{2\lambda}$$

for all t_1 and t_2 such that $0 \le t_1 < t_2 \le T$. Thus, the process $\{X_t,\ \ t \in [0, T]\}$ is a generalized quasi-helix with exponents 1 and λ. We recall that $\lambda = \alpha + \beta + 1$.

(v) Proof of statement 3. Let $\alpha + \beta \ge 0$ and $\lambda \in (0, 1)$. Then the assumption $\lambda \le \alpha + \beta + 1$ of (29) is satisfied. Due to (27) and (29),

$$\frac{c_1 (t_2 - t_1)^{2\alpha+2\beta+2}}{T^{2\alpha+2\beta}} \le \operatorname{var}(X_{t_2} - X_{t_1}) \le c_2(\lambda)\, (t_2 - t_1)^{2\lambda}$$

for all t_1 and t_2 such that $0 \le t_1 < t_2 \le T$. Thus, the process $\{X_t,\ \ t \in [0, T]\}$ is a generalized quasi-helix with exponents $\alpha + \beta + 1$ and λ. □

The following corollary to Theorem 5 is complimentary to Corollary 1.

Corollary 2. *Let* $0 < t_0 < T$. *Process* X *defined by* (4) *and* (24) *is a pseudo-quasihelix on the interval* $[t_0, T]$ *with exponent* 1. *If, in addition,* $\alpha + \beta = 0$, *then* X *is a pseudo-quasihelix on the entire interval* $[0, T]$.

Quasi-helix, pseudo-quasihelix and generalized quasi-helix conditions for the process X defined by (1) and (2) are summarized in Table 2.

Table 2. Summary of quasi-helix properties

The process X is

$$X_t = \int_0^t s^\alpha \int_s^t u^\beta (u-s)^\gamma \, du \, dW_s,$$

$$\alpha > -\tfrac{1}{2}, \qquad \gamma > -1, \qquad \alpha + \beta + \gamma > -\tfrac{3}{2}.$$

The process X satisfies	on the interval $[0, T]$	on any interval $[t_0, T]$
quasi-helix	if and only if $\gamma < -\frac{1}{2}$ and $\alpha + \beta = 0$, or $\gamma > -\frac{1}{2}$ and $\alpha + \beta + \gamma = -\frac{1}{2}$.	$\gamma \neq -\frac{1}{2}$
pseudo-quasihelix	$\gamma \le -\frac{1}{2}$ and $\alpha + \beta = 0$, or $\gamma \ge -\frac{1}{2}$ and $\alpha + \beta + \gamma = -\frac{1}{2}$.	always
generalized quasi-helix	always	always

Here $0 < t_0 < T$. The entry “always” means “always whenever $\alpha > -\frac{1}{2}$, $\gamma > -1$, $\alpha + \beta + \gamma > -\frac{3}{2}$.”

The exponents in the generalized quasi-helix condition:

	the exponents in the generalized quasi-helix condition	
	on the interval $[0, T]$ are	on any interval $[t_0, T]$ are
If $\gamma < -\frac{1}{2}$ and $\alpha + \beta \le 0$,	$\gamma+\frac{3}{2}$ and $\alpha+\beta+\gamma+\frac{3}{2}$	$\gamma+\frac{3}{2}$
If $\gamma < -\frac{1}{2}$ and $\alpha + \beta \ge 0$,	$\alpha+\beta+\gamma+\frac{3}{2}$ and $\gamma+\frac{3}{2}$	$\gamma+\frac{3}{2}$
If $\gamma = -\frac{1}{2}$ and $\alpha + \beta < 0$,	1 and $\alpha + \beta + 1$	1 and $1-\epsilon$
If $\gamma = -\frac{1}{2}$ and $\alpha + \beta \ge 0$,	$\alpha+\beta+1$ and $1-\epsilon$	1 and $1-\epsilon$
If $\gamma > -\frac{1}{2}$ and $\alpha+\beta+\gamma \le -\frac{1}{2}$,	1 and $\alpha+\beta+\gamma+\frac{3}{2}$	1
If $\gamma > -\frac{1}{2}$ and $\alpha+\beta+\gamma \ge -\frac{1}{2}$,	$\alpha+\beta+\gamma+\frac{3}{2}$ and 1	1

If only one number is in the cell, then the process is a quasi-helix. In that case, the exponents in both sides of the generalized quasi-helix condition are equal. The number $1-\epsilon$ means that the upper bound in the generalized quasi-helix condition “$\exists C_2 \, \forall t_1 \, \forall t_2 : \operatorname{var}(X_{t_2} - X_{t_1}) \le C_2 \, (t_2 - t_1)^{2\lambda}$” holds true for all $\lambda \in (0, 1)$.

6 Hölder property

The Hölder property for stochastic processes follows from generalized quasi-helix property. Indeed, it is well-known that Gaussian process $\{X_t, \ t \in [t_0, T]\}$ satisfying for some $\lambda_0 > 0$ the assumption

$$\forall \lambda \in (0, \lambda_0) \; \exists C(\lambda) \; \forall t_1 \in [t_0, T] \; \forall t_2 \in [t_0, T] \; : \; \operatorname{var}(X_{t_2} - X_{t_1}) \le C(\lambda) \, |t_2 - t_1|^{2\lambda},$$

has a modification $\widetilde{X}$ whose paths are Hölder up to order λ_0, that is,

$$\forall \lambda \in (0, \lambda_0) \; \exists C(\lambda, \omega) \; \forall t_1 \in [t_0, T] \; \forall t_2 \in [t_0, T] \; : \; |\widetilde{X}_{t_2} - \widetilde{X}_{t_1}| \le C(\lambda, \omega) \, |t_2 - t_1|^{\lambda}.$$

As a consequence, a stochastic process satisfying quasi-helix or pseudo-quasihelix condition with exponent λ also has a modification that is Hölder up to order λ. A stochastic process that satisfies the generalized quasi-helix condition with exponents λ_1 and $\lambda_2 < \lambda_1$ also has a modification that is Hölder up to order λ_2.

Theorem 6. *Let* $0 < t_0 < T$*, and let the process* X *be defined by* (4) *and* (2)*.*

(i) The process $\{X_t,\ t \in [t_0, T]\}$ *has a continuous modification that satisfies the Hölder condition up to order* $\min\bigl(\gamma+\frac{3}{2},\ 1\bigr)$.

(ii) The process $\{X_t,\ t \in [0, T]\}$ *has a continuous modification that is Hölder up to order* $\min\bigl(\alpha+\beta+\gamma+\frac{3}{2},\ \gamma+\frac{3}{2},\ 1\bigr)$.

Proof. The Hölder condition follows from the results of Section 5 presented in Theorems 3, 4 and 5 and summarized in Table 2. □

Remark 3. According to Proposition 2, the process X defined by (4) and (2) does not admit the bound $\operatorname{var}(X_{t_2} - X_{t_1}) < C\,|t_2 - t_1|^{2\lambda}$ for any $\lambda > \min\bigl(\gamma + \frac{3}{2},\ 1\bigr)$. Hence, according to [2, Theorem 1], the process X cannot be Hölder of order greater than $\min\bigl(\gamma + \frac{3}{2},\ 1\bigr)$.

The process X is self-similar with exponent $H = \alpha+\beta+\gamma+\frac{3}{2}$, whence $\operatorname{var}(X_t - X_0) = C\,t^{2H}$ for some constant $C > 0$. Hence, according to [2, Theorem 1], the process X cannot satisfy the Hölder condition of order greater than H on the interval $[0, T]$.

Thus, the process X cannot satisfy the Hölder condition of order greater than specified in Theorem 6.

Lemma 3. *Let the process* $\{X_t,\ t \in [0, T]\}$ *satisfy conditions*

(i) X *is Gaussian with zero mean;*

(ii) X *is self-similar with exponent* $H > 0$*;*

(iii) incremental variances of X *satisfy the inequality*

$$\exists \lambda_0 > 0\ \exists C_0 < \infty\ \forall t_1, t_2 \in [0, T] :\ \operatorname{var}(X_{t_1} - X_{t_2}) \le C_0\,|t_2 - t_1|^{2\lambda_0}. \quad (30)$$

Then X *has a modification* $\widetilde{X}$ *whose paths are Hölder up to order* H *at point* 0:

$$\forall \lambda \in (0, H)\ \exists C_1 = C_1(\lambda, \omega) < \infty,\ \textit{a.s.}\ \forall t \in [0, T]\ :\ |\widetilde{X}_t - \widetilde{X}_0| \le C_1\,t^{\lambda}. \quad (31)$$

Remark 4. Note that in Lemma 3 we formulated the Hölder condition at a single point. The exponent in the Hölder condition at a single point may exceed 1, while the exponent in the Hölder condition on an interval does not exceed 1 unless the function or process is constant at that interval.

Proof of Lemma 3. The process X is mean-square continuous, and $X_0 = 0$ almost surely. The variance of X is a power function: $\operatorname{var}(X_t) = \operatorname{var}(X_t - X_0) = Ct^{2H}$ for some $C \ge 0$. Let us take the constants λ_0 and C_0 from (30). Since $Ct^{2H} = \operatorname{var}(X_t - X_0) \le C_0 t^{2\lambda_0}$ for all $t \in [0, T]$, the exponents H and λ_0 satisfy the inequality $0 < \lambda_0 \le H$. (Moreover, with view of Remark 2, $0 < \lambda_0 \le \min(1, H)$.)

Consider the stochastic process $Y = \{Y_s,\ s \in [0, T^{H/\lambda_0}]\}$ with $Y_s = X_{s^{\lambda_0/H}}$. For all s_1 and s_2 such that $0 \le s_1 < s_2 \le T^{H/\lambda_0}$, the incremental variances of Y are

$$\operatorname{var}(Y_{s_2} - Y_{s_1}) = \operatorname{var}(X_{s_2^{\lambda_0/H}} - X_{s_1^{\lambda_0/H}})$$

$$= \left(\frac{s_2^{\lambda_0/H}}{T} \right)^{2H} \operatorname{var}\!\left(X_T - X_{T s_1^{\lambda_0/H} s_2^{-\lambda_0/H}}\right) \le \frac{C_0 s_2^{2\lambda_0}}{T^{2H}} \left(T - \frac{T s_1^{\lambda_0/H}}{s_2^{\lambda_0/H}} \right)^{2\lambda_0}.$$

With $0 \le \frac{s_1}{s_2} < 1$, the inequality $0 < \frac{\lambda_0}{H} \le 1$ implies $\left(\frac{s_1}{s_2}\right)^{\lambda_0/H} \ge \frac{s_1}{s_2}$. Hence,

$$T - \frac{T s_1^{\lambda_0/H}}{s_2^{\lambda_0/H}} \le T - \frac{T s_1}{s_2},$$

and

$$\operatorname{var}(Y_{s_2} - Y_{s_1}) \le \frac{C_0 s_2^{2\lambda_0}}{T^{2H}} \left(T - \frac{T s_1}{s_2} \right)^{2\lambda_0} = \frac{C_0\,(s_2 - s_1)^{2\lambda_0}}{T^{2H-2\lambda_0}}.$$

Therefore Y has a modification $\widetilde{Y}$ whose paths are Hölder up to order λ_0:

$$\forall\theta \in (0, \lambda_0)\ \exists C_2 = C_2(\theta, \omega) < \infty \text{ a.s., } \forall s_1, s_2 \in [0, T^{H/\lambda_0}] :$$
$$|\widetilde{Y}_{s_2} - \widetilde{Y}_{s_1}| \le C_2\, |s_2 - s_1|^{\theta}.$$

The process $\widetilde{X} = \{\widetilde{X}_t,\ \ t \in [0, T]\}$ with $\widetilde{X}_t = \widetilde{Y}_{t^{H/\lambda_0}}$ is a modification of the process X. Then

$$\forall\theta \in (0, \lambda_0)\ \ \forall s_1, s_2 \in [0, T^{H/\lambda_0}] : |\widetilde{X}_{s_2^{\lambda_0/H}} - \widetilde{X}_{s_1^{\lambda_0/H}}| \le C_2\, |s_2 - s_1|^{\theta},$$

whence

$$\forall\theta \in (0, \lambda_0)\ \forall s \in [0, T^{H/\lambda_0}] \ :\ |\widetilde{X}_{s^{\lambda_0/H}} - \widetilde{X}_0| \le C_2\, s^{\theta}.$$

Substituting $s = t^{H/\lambda_0}$ and $\theta = \lambda\lambda_0/H$ for $\lambda \in (0, H)$, we obtain (31). □

The next result is an immediate consequence of Lemma 3. Self-similarity of X is established in Proposition 1.

Theorem 7. *Let the process X be defined by* (4) *and* (2)*. Then X has a modification whose paths satisfy the Hölder condition up to order $\alpha + \beta + \gamma + \frac{3}{2}$ at point* 0:

$$\forall\lambda \in \left(0,\ \alpha + \beta + \gamma + \tfrac{3}{2}\right)\ \exists C = C(\lambda, \omega) < \infty \textit{ a.s. } \forall t \in [0, T] \ :\ |\widetilde{X}_t - \widetilde{X}_0| \le C\, t^{\lambda},$$

where C is an a.s. finite random variable.

A Appendix

A.1 Some inference for power integrals

Lemma 4. *Let $\beta \in \mathbb{R}$, $\gamma > -1$ and $t > 0$. Then the asymptotic behavior of the integral $\int_s^t u^{\beta}(u - s)^{\gamma}\, du$ as $s \to 0+$ is*

(i) $$\int_s^t u^{\beta}(u - s)^{\gamma}\, du \sim s^{\beta+\gamma+1}\, \mathrm{B}(\gamma+1,\ -\beta-\gamma-1) \qquad \textit{if} \quad \beta+\gamma < -1,$$

(ii) $$\int_s^t u^{\beta}(u - s)^{\gamma}\, du \sim \ln(t/s) \qquad \textit{if} \quad \beta+\gamma = -1,$$

(iii) $$\int_s^t u^{\beta}(u - s)^{\gamma}\, du \to \frac{t^{\beta+\gamma+1}}{\beta + \gamma + 1} \qquad \textit{if} \quad \beta+\gamma > -1.$$

Proof. By [7, Lemma 2.2(ii)],

$$\begin{aligned}\int_s^t u^\beta (u-s)^\gamma\, du &= s^{\beta+\gamma+1} \int_1^{t/s} v^\beta (v-1)^\gamma\, dv \\ &= s^{\beta+\gamma+1} \int_0^{1-\frac{s}{t}} x^\gamma (1-x)^{-\beta-\gamma-2}\, dx. \end{aligned} \tag{32}$$

Case 1. If $\beta+\gamma<-1$, then $-\beta-\gamma-2>-1$,

$$\begin{aligned}&\int_0^u x^\gamma (1-x)^{-\beta-\gamma-2}\, dx \to \mathrm{B}(\gamma{+}1,\ -\beta{-}\gamma{-}1) \quad \text{as} \quad u\to 1-, \\ &\int_s^t u^\beta (u-s)^\gamma\, du \sim s^{\beta+\gamma+1}\, \mathrm{B}(\gamma{+}1,\ -\beta{-}\gamma{-}1) \quad \text{as} \quad s\to 0+,\end{aligned}$$

as desired.

Case 2. Now suppose that $\beta+\gamma=-1$. Then (32) comes into

$$\int_s^t u^\beta (u-s)^\gamma\, du = \int_0^{1-\frac{s}{t}} \frac{x^\gamma}{1-x}\, dx.$$

By substitution $x = 1-e^{-y}$ and $y = z\ln(t/s)$,

$$\int_0^{1-\frac{s}{t}} \frac{x^\gamma}{1-x}\, dx = \int_0^{\ln(t/s)} (1-e^{-y})^\gamma\, dy = \ln(t/s) \int_0^1 \left(1-\left(\frac{s}{t}\right)^z\right)^\gamma\, dz.$$

Let us substantiate the convergence

$$\lim_{s\to 0+} \int_0^1 \left(1-\left(\frac{s}{t}\right)^z\right)^\gamma\, dz = \int_0^1 \lim_{s\to 0+} \left(1-\left(\frac{s}{t}\right)^z\right)^\gamma\, dz = 1. \tag{33}$$

The pre-limit integral $\int_0^t (1-s^z t^{-z})^\gamma\, dz$ is finite for all $s\in(0,t)$. The integral $\int_0^1 dz$ on the right-hand side of (33) is also finite. The integrand $(1-s^z t^{-z})^\gamma$ is monotone in s for all $z\in(0,1)$. Hence, the convergence (33) indeed holds true. Finally,

$$\int_s^t u^\beta (u-s)^\gamma\, du = \ln(t/s) \int_0^1 \left(1-\left(\frac{s}{t}\right)^z\right)^\gamma\, dz \sim \ln(t/s) \quad \text{as} \quad s\to 0+,$$

as desired.

Case 3. Now suppose that $\beta+\gamma>-1$. If $\gamma>0$, then the convergence

$$\lim_{s\to 0+} \int_s^t u^\beta (u-s)^\gamma\, du = \int_0^t \lim_{s\to 0+} u^\beta (u-s)^\gamma\, du = \int_0^t u^{\beta+\gamma}\, du = \frac{t^{\beta+\gamma+1}}{\beta+\gamma+1}$$

follows from the Lebesgue monotone convergence theorem. Otherwise, if $-1<\gamma\le 0$, then

$$\lim_{s\to 0+} \int_s^t u^\beta (u-s)^\gamma\, du = \lim_{s\to 0+} \int_0^{t-s} (v+s)^\beta v^\gamma\, dv = \int_0^t \lim_{s\to 0+} (v+s)^\beta v^\gamma\, dv$$

$$= \int_0^t v^{\beta+\gamma}\, dv = \frac{t^{\beta+\gamma+1}}{\beta+\gamma+1}$$

due to the dominated convergence theorem. However, the dominant used depends on β:

$$\text{if } \beta \le 0, \quad \text{then} \quad (v+s)^\beta v^\gamma \mathbf{1}_{(0,t-s]}(v) \le v^{\beta+\gamma} \quad \text{and} \quad \int_0^t v^{\beta+\gamma}\, dv < \infty;$$

$$\text{if } \beta > 0, \quad \text{then} \quad (v+s)^\beta v^\gamma \mathbf{1}_{(0,t-s]}(v) \le t^\beta v^\gamma \quad \text{and} \quad \int_0^t t^\beta v^\gamma\, dv < \infty.$$

In any case, there is the desired convergence. □

Lemma 5. *Let $t_0 > 0$, $\alpha > -\frac{1}{2}$ and $-1 < \gamma \le 0$. Then*

$$\lim_{\substack{(u,v)\to(t_0,t_0)\\ u\le v}} \int_0^u (u^{2\alpha} - s^{2\alpha})\,(u-s)^\gamma (v-s)^\gamma\, ds = \int_0^{t_0} (t_0^{2\alpha} - s^{2\alpha})\,(t_0-s)^{2\gamma}\, ds, \quad (34)$$

and this value is finite.

Proof. In what follows, assume that $\alpha \ne 0$, otherwise both integrals equal zero. First, prove that the integral on the right-hand side is finite. Indeed, integrand $(t_0^{2\alpha} - s^{2\alpha})(t_0 - s)^{2\gamma}$ is continuous on $(0, t_0)$, and its asymptotic behavior at endpoints is

$$(t_0^{2\alpha} - s^{2\alpha})(t_0 - s)^{2\gamma} \sim -s^{2\alpha} t_0^{2\gamma} \quad \text{as } s \to 0 \quad \text{if } \alpha < 0,$$

$$(t_0^{2\alpha} - s^{2\alpha})(t_0 - s)^{2\gamma} \to t_0^{2\gamma+2\gamma} \quad \text{as } s \to 0 \quad \text{if } \alpha > 0,$$

$$(t_0^{2\alpha} - s^{2\alpha})(t_0 - s)^{2\gamma} \sim 2\alpha (t_0 - s)^{2\gamma+1} \quad \text{as } s \to t_0.$$

As $2\alpha > -1$ and $2\gamma + 1 > -1$, the integral is finite. Moreover, by linear substitution,

$$\int_0^u (u^{2\alpha} - s^{2\alpha})(u-s)^\gamma (v-s)^\gamma\, ds = \frac{u^{2\alpha+2\gamma+1}}{t_0^{2\alpha+2\gamma+1}} \int_0^{t_0} (t_0^{2\alpha} - s^{2\alpha})\,(t_0-s)^\gamma \left(\frac{v t_0}{u} - s\right)^\gamma ds. \quad (35)$$

Obviously,

$$\frac{u^{2\alpha+2\gamma+1}}{t_0^{2\alpha+2\gamma+1}} \to 1 \quad \text{as} \quad u \to t_0,$$

and for $0 < s < t_0$ and $u \le v$

$$\left|(t_0^{2\alpha} - s^{2\alpha})\,(t_0-s)^\gamma \left(\frac{v t_0}{u} - s\right)^\gamma\right| \le \left|(s^{2\alpha} - t_0^{2\alpha})\,(t_0-s)^{2\gamma}\right|; \quad (36)$$

$$(s^{2\alpha} - t_0^{2\alpha})\,(t_0-s)^\gamma \left(\frac{v t_0}{u} - s\right)^\gamma \to (s^{2\alpha} - t_0^{2\alpha})\,(t_0-s)^{2\gamma}$$

as $(u, v) \to (t_0, t_0)$, $u \le v$ for all $s \in (0, t_0)$. By the Lebesgue dominated convergence theorem,

$$\int_0^{t_0} (s^{2\alpha} - t_0^{2\alpha})\,(t_0 - s)^\gamma \left(\frac{vt_0}{u} - s\right)^\gamma ds \to \int_0^{t_0} (t_0^{2\alpha} - s^{2\alpha})\,(t_0 - u)^{2\gamma}\, ds \tag{37}$$

as $(u, v) \to (t_0, t_0)$, $u \le v$. The proof follows from equality (35) together with (36) and (37). □

Remark 5. The condition $\gamma \le 0$ can be excluded from the assumptions of Lemma 5. If $t_0 > 0$, $\alpha > -\frac{1}{2}$ and $\gamma > -1$, then (34) holds true and the limit in (34) is finite.

Lemma 6. *If* $0 < t_1 < t_2$, *then*

$$\begin{aligned}\int_{t_1}^{t_2} \left(\int_u^{t_2} \left(\int_0^u (u-s)^{-1/2}(v-s)^{-1/2}\,ds\right) dv\right) du \\ = t_2^2 - (t_1 + t_2) t_1^{1/2} t_2^{1/2} + t_1^2 + \frac{(t_2 - t_1)^2}{2} \ln\left(\frac{t_2^{1/2} + t_1^{1/2}}{t_2^{1/2} - t_1^{1/2}}\right).\end{aligned} \tag{38}$$

Proof. We have

$$\begin{aligned}&\int_{u=t_1}^{t_2} \int_{v=u}^{t_2} \int_{s=0}^{u} (u-s)^{-1/2}(v-s)^{-1/2}\,ds\,dv\,du \\ &\quad= \int_{s=0}^{t_2} \int_{u=\max(s,t_1)}^{t_2} \int_{v=u}^{t_2} (u-s)^{-1/2}(v-s)^{-1/2}\,dv\,du\,ds \\ &\quad= \int_{s=0}^{t_2} \int_{u=\max(s,t_1)}^{t_2} 2\left(\sqrt{\frac{t_2 - s}{u - s}} - 1\right) du\,ds \\ &\quad= \int_{s=0}^{t_2} \Big(4(t_2 - s) - 4\sqrt{(t_2 - s)(\max(s, t_1) - s)} - 2(t_2 - \max(s, t_1))\Big)\, ds \\ &\quad= 2t_2^2 - 4\int_0^{t_1} \sqrt{(t_2 - s)(t_1 - s)}\,ds - (t_2 - t_1)(t_2 + t_1) \\ &\quad= t_1^2 + t_2^2 - 4\int_0^{t_1} \sqrt{(t_2 - s)(t_1 - s)}\,ds.\end{aligned} \tag{39}$$

By the linear substitution $s = \frac{1}{2}\,(t_1 + t_2 - (t_2 - t_1)x)$, $x \ge 1$, the last integral can be reduced to a well-known one:

$$\begin{aligned}\int \sqrt{(t_2 - s)(t_1 - s)}\,ds &= -\int \sqrt{\frac{(t_2 - t_1)^2(x^2 - 1)}{4}}\,\frac{t_2 - t_1}{2}\,dt \\ &= -\frac{(t_2 - t_1)^2}{4}\int \sqrt{x^2 - 1}\,dx \\ &= -\frac{(t_2 - t_1)^2}{8}\left(x\sqrt{x^2 - 1} - \ln(x + \sqrt{x^2 - 1})\right) + C \\ &= -\frac{(t_2 - t_1)^2}{8}\,\frac{t_1 + t_2 - 2s}{t_2 - t_1}\,\frac{2\sqrt{(t_2 - s)(t_1 - s)}}{t_2 - t_1} +\end{aligned}$$

$$+\frac{(t_2-t_1)^2}{8}\ln\left(\frac{t_1+t_2-2s+2\sqrt{(t_2-s)(t_1-s)}}{t_2-t_1}\right)+C$$
$$=-\frac{(t_1+t_2-2s)\sqrt{(t_2-s)(t_1-s)}}{4}+$$
$$+\frac{(t_2-t_1)^2}{8}\ln\left(\frac{\sqrt{t_2-s}+\sqrt{t_1-s}}{\sqrt{t_2-s}-\sqrt{t_1-s}}\right)+C,$$

whence

$$\int_0^{t_1}\sqrt{(t_2-s)(t_1-s)}\,ds=\frac{(t_1+t_2)\sqrt{t_2t_1}}{4}-\frac{(t_2-t_1)^2}{8}\ln\left(\frac{\sqrt{t_2}+\sqrt{t_1}}{\sqrt{t_2}-\sqrt{t_1}}\right). \quad (40)$$

Equations (39) and (40) imply that

$$\int_{t_1}^{t_2}\int_u^{t_2}\int_0^u (u-s)^{-1/2}(v-s)^{-1/2}\,ds\,dv\,du$$
$$=t_1^2+t_2^2-(t_1+t_2)\sqrt{t_2t_1}+\frac{(t_2-t_1)^2}{2}\ln\left(\frac{\sqrt{t_2}+\sqrt{t_1}}{\sqrt{t_2}-\sqrt{t_1}}\right),$$

which agrees with (38). □

Lemma 7. *Let $\alpha>-\frac{1}{2}$, $\beta\in\mathbb{R}$ and $\gamma>-\frac{1}{2}$. Then*

$$\lim_{\substack{(u,v)\to(t_0,t_0)\\u<v}} u^\beta v^\beta\int_0^u s^{2\alpha}\,(u-s)^\gamma(v-s)^\gamma\,ds=t_0^{2\alpha+2\beta+2\gamma+1}\,\mathrm{B}(2\alpha+1,\ 2\gamma+1). \quad (41)$$

Proof. By a linear substitution,

$$u^\beta v^\beta\int_0^u s^{2\alpha}\,(u-s)^\gamma\,(v-s)^\gamma\,ds$$
$$=u^{2\alpha+\beta+\gamma+1}v^{\beta+\gamma}\int_0^1 s^{2\alpha}\,(1-s)^\gamma\left(1-\frac{us}{v}\right)^\gamma\,ds.$$

Thus,

$$\lim_{\substack{(u,v)\to(t_0,t_0)\\u<v}} u^\beta v^\beta\int_0^u s^{2\alpha}\,(u-s)^\gamma(v-s)^\gamma\,ds$$
$$=\lim_{\substack{(u,v)\to(t_0,t_0)\\u<v}} u^{2\alpha+\beta+\gamma+1}v^{\beta+\gamma}\int_0^1 s^{2\alpha}\,(1-s)^\gamma\left(1-\frac{us}{v}\right)^\gamma\,ds$$
$$=\lim_{u\to t_0}u^{2\alpha+\beta+\gamma+1}\lim_{v\to t_0}v^{\beta+\gamma}\lim_{\substack{(u,v)\to(t_0,t_0)\\u<v}}\int_0^1 s^{2\alpha}\,(1-s)^\gamma\left(1-\frac{us}{v}\right)^\gamma\,ds$$
$$=t_0^{2\alpha+2\beta+2\gamma+1}\lim_{r\to1-}\int_0^1 s^{2\alpha}\,(1-s)^\gamma\,(1-rs)^\gamma\,ds \quad (42)$$

provided that the limit on the right-hand side exists.

If $-\frac{1}{2} < \gamma \le 0$, then

$$|s^{2\alpha}\,(1-s)^{\gamma}\,(1-rs)^{\gamma}| \le s^{2\alpha}\,(1-s)^{2\gamma},$$

for all $r \in (0, 1)$ and $s \in (0, 1)$, while

$$\int_0^1 s^{2\alpha}\,(1-s)^{2\gamma}\,ds = \mathrm{B}(2\alpha+1,\ 2\gamma+1) < \infty.$$

Otherwise, if $\gamma \ge 0$, then

$$|s^{2\alpha}\,(1-s)^{\gamma}\,(1-rs)^{\gamma}| \le s^{2\alpha},$$

for all $r \in (0, 1)$ and $s \in (0, 1)$, while

$$\int_0^1 s^{2\alpha} = \frac{1}{2\alpha+1} < \infty.$$

By the Lebesgue dominated convergence theorem,

$$\begin{aligned}\lim_{r\to 1-} \int_0^1 s^{2\alpha}\,(1-s)^{\gamma}\,(1-rs)^{\gamma}\,ds &= \int_0^1 \lim_{r\to 1-} s^{2\alpha}\,(1-s)^{\gamma}\,(1-rs)^{\gamma}\,ds \\ &= \int_0^1 s^{2\alpha}\,(1-s)^{2\gamma}\,ds = \mathrm{B}(2\alpha+1,\ 2\gamma+1);\end{aligned} \tag{43}$$

thus, the limit on the right-hand side of (42) exists as supposed. Equations (42) and (43) imply (41). □

Remark 6. In Lemma 7, the constraint $u < v$ can be relaxed as $u \le v$. If $\alpha > -\frac{1}{2}$, $\beta \in \mathbb{R}$ and $\gamma > -\frac{1}{2}$, then

$$\lim_{\substack{(u,v)\to(t_0,t_0)\\ u\le v}} u^{\beta} v^{\beta} \int_0^u s^{2\alpha}\,(u-s)^{\gamma}(v-s)^{\gamma}\,ds = t_0^{2\alpha+2\beta+2\gamma+1}\,\mathrm{B}(2\alpha+1,\ 2\gamma+1).$$

The generalization follows from the equality

$$u^{2\beta} \int_0^u s^{2\alpha}\,(u-s)^{2\gamma}\,ds = u^{2\alpha+2\beta+2\gamma+1}\,\mathrm{B}(2\alpha+1,\ 2\gamma+1).$$

A.2 The process X is not deterministic

Let X be a process defined by (4). According to (9), the increments of process X are nondegenerate in the sense that they have nonzero variances. Moreover, similarly to representation (8), for $0 < t_1 < t_2$,

$$\operatorname{var}[X_{t_2} \mid X_{t_1}] \ge \operatorname{var}[X_{t_2} \mid \mathcal{F}_{t_1}] = \int_{t_1}^{t_2} \left(s^{\alpha} \int_s^t u^{\beta}(u-s)^{\gamma}\,du \right)^2 ds > 0, \tag{44}$$

where $\mathcal{F}_{t_1}$ is a σ-algebra generated by W_t, $t \in [0, t_1]$, and the conditional variance is given as $\mathrm{var}[X \mid \mathcal{F}] = \mathsf{E}[(X - \mathsf{E}[X \mid \mathcal{F}])^2 \mid \mathcal{F}]$. The inequality $\mathrm{var}[X_{t_2} \mid X_{t_1}] \ge \mathrm{var}[X_{t_2} \mid \mathcal{F}_{t_1}]$ follows from fact that X_{t_1} is $\mathcal{F}_{t_1}$-measurable, the conditional variance allows a representation

$$\mathrm{var}[X_{t_2} \mid X_{t_1}] = \mathrm{var}[\mathsf{E}[X_{t_2} \mid \mathcal{F}_{t_1}] \mid X_{t_1}] + \mathsf{E}[\mathrm{var}[X_{t_2} \mid \mathcal{F}_{t_1}] \mid X_{t_1}]$$

due to the law of total variance, and the conditional variance $\mathrm{var}[X_{t_2} \mid \mathcal{F}_{t_1}]$ is nonrandom.

A.3 The meaning of the exponents

The order of the Hölder continuity on a finite interval separated from 0 is determined by γ. The self-similarity exponent equals $\alpha + \beta + \gamma + \frac{3}{2}$. In Proposition 2 the asymptotics of the incremental variance depends on all parameters, however, it can be split into three factors: $|t_2 - t_1|^{(2\gamma+3)\wedge 2}$ (or $-(t_2 - t_1)^2 \ln |t_2 - t_1|$ if $\lambda = -1/2$), which depends on γ and describes the rate of convergence to 0; $t_0^{2\alpha+2\beta}$ or $t_0^{2\alpha+2\beta+2\gamma+1}$ to achieve the homogeneity order compatible with the self-similarity; and a coefficient, which depends on α and γ. Some asymptotic properties of the covariance function of the process X are given in Proposition 3.

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