

Conic intrinsic volumes of Weyl chambers

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Abstract A new, direct proof of the formulas for the conic intrinsic volumes of the Weyl chambers of types A_{n-1} , B_n and D_n is given. These formulas express the conic intrinsic volumes in terms of the Stirling numbers of the first kind and their B - and D -analogues. The proof involves an explicit determination of the internal and external angles of the faces of the Weyl chambers.

Keywords Stochastic geometry, polyhedral cones, Weyl chambers, conic intrinsic volumes, external angles, internal angles, Stirling numbers, random walks and bridges

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1 Introduction

A polyhedral cone in the Euclidean space \mathbb{R}^n is a set of solutions to a finite system of linear homogeneous inequalities. That is, a polyhedral cone $C \subseteq \mathbb{R}^n$ can be represented as

$$C = \{\beta \in \mathbb{R}^n : \langle \beta, x_i \rangle \leq 0 \text{ for all } i = 1, \dots, m\}$$

for some finite collection of vectors $x_1, \dots, x_m \in \mathbb{R}^n$, where $\langle \cdot, \cdot \rangle$ denotes the standard Euclidean scalar product. The fundamental Weyl chambers of types A_{n-1} , B_n and D_n are the polyhedral cones defined by

$$\mathcal{C}(A_{n-1}) := \{\beta \in \mathbb{R}^n : \beta_1 \geq \beta_2 \geq \dots \geq \beta_n\},$$

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$$\begin{aligned} \mathcal{C}(B_n) &:= \{\beta \in \mathbb{R}^n : \beta_1 \geq \beta_2 \geq \dots \geq \beta_n \geq 0\}, \\ \mathcal{C}(D_n) &:= \{\beta \in \mathbb{R}^n : \beta_1 \geq \beta_2 \geq \dots \geq \beta_{n-1} \geq |\beta_n|\}, \end{aligned}$$

where $\beta = (\beta_1, \dots, \beta_n)$ is the coordinate representation of $\beta \in \mathbb{R}^n$.

In this paper, we shall be interested in the conic intrinsic volumes of the Weyl chambers. The conic intrinsic volumes of cones are analogues of the classical Euclidean intrinsic volumes of convex bodies in the setting of conic or spherical geometry. A recent increase of interest to conic intrinsic volumes is due to their relevance in convex optimization [6, 4, 5]. Let us briefly define the conic intrinsic volumes, referring to Section 2.1 for more details and to [25, Section 6.5] and [6, 7] for an extensive account of the theory. Given some point $x \in \mathbb{R}^n$, the Euclidean projection $\Pi_C(x)$ of x to a polyhedral cone $C \subseteq \mathbb{R}^n$ is the unique vector $y \in C$ minimizing the Euclidean distance $\|x - y\|$. For $k \in \{0, \dots, n\}$, the k -th conic intrinsic volume $v_k(C)$ of C is defined as the probability that the Euclidean projection $\Pi_C(g)$ onto C of an n -dimensional standard Gaussian random vector g lies in the relative interior of a k -dimensional face of C (meaning that the projection is contained in this face but not in any face of smaller dimension). The conic intrinsic volumes of the Weyl chambers are given by the following theorem.

Theorem 1.1. *For all $n \in \{1, 2, \dots\}$ and all $k \in \{0, \dots, n\}$ we have*

$$v_k(\mathcal{C}(A_{n-1})) = \binom{n}{k} \frac{1}{n!}, \quad v_k(\mathcal{C}(B_n)) = \frac{B[n, k]}{2^n n!}, \quad v_k(\mathcal{C}(D_n)) = \frac{D[n, k]}{2^{n-1} n!}, \quad (1.1)$$

where the $\binom{n}{k}$'s denote the Stirling numbers of the first kind, the $B[n, k]$'s their B -analogues and the $D[n, k]$'s their D -analogues defined as the coefficients of the following polynomials:

$$\begin{aligned} t(t+1) \cdot \dots \cdot (t+n-1) &= \sum_{k=0}^n \binom{n}{k} t^k, \\ (t+1)(t+3) \cdot \dots \cdot (t+2n-1) &= \sum_{k=0}^n B[n, k] t^k, \\ (t+1)(t+3) \cdot \dots \cdot (t+2n-3)(t+n-1) &= \sum_{k=0}^n D[n, k] t^k. \end{aligned}$$

The theorem is known and appeared in various, seemingly unrelated forms. One of the approaches to prove it is to relate the conic intrinsic volumes to the coefficients of the characteristic polynomial of the hyperplane arrangement generating the Weyl chambers; see [13, Theorems 3,4], [21, Theorem 5], [7, Section 6], [24], [20, Theorems 4.1 and 4.2]. The characteristic polynomials of Weyl arrangements can be computed by the finite field method; see Section 5.1 in [31] or Section 1.7.4 in [11].

On the other hand, the A_{n-1} -case of the formula (1.1) is closely related to a result, due to Sparre Andersen [30, Theorem 5], on the least concave majorant of a random walk. Consider a random walk given by $S_k := g_1 + \dots + g_k$, $1 \leq k \leq n$, and $S_0 := 0$, where the increments g_1, \dots, g_n are independent standard Gaussian random

variables. (In fact, the result stated below holds, essentially, under the assumption of exchangeability of the increments). The graph of the random walk is the set in \mathbb{R}^2 consisting of the points (k, S_k) , for $k \in \{0, 1, \dots, n\}$. The least concave majorant of the random walk is the smallest concave function defined on $[0, n]$ that stays above this graph. This function is piecewise linear. If N_n denotes the number of linearity segments of the least concave majorant, then the formula due to Sparre Andersen [30, Theorem 5] states that

$$\mathbb{P}[N_n = k] = \binom{n}{k} \frac{1}{n!}, \quad k \in \{1, \dots, n\}. \tag{1.2}$$

For further development of this probabilistic result we refer to [1–3]. Now, the projection of the point $g = (g_1, \dots, g_n)$ on the Weyl chamber $\mathcal{C}(A_{n-1})$ can explicitly be described in terms of the least concave majorant of the corresponding random walk, and, in particular, it is possible to check that the number of linearity segments of the majorant coincides with the dimension of the unique face F such that $\Pi_{\mathcal{C}(A_{n-1})}(g)$ lies in the relative interior of F . With this knowledge, the A_{n-1} -case of the formula (1.1) can be viewed as a consequence of Sparre Andersen’s formula (1.2).

In the present paper, we shall give yet another proof of Theorem 1.1. The starting point of this proof is a formula, see (2.2) below, expressing the conic intrinsic volumes of a polyhedral cone in terms of the internal and external angles of its faces. Probably the main advantage of our proof is an explicit identification of all internal and external angles involved in this formula. The determination of the external angles is closely related to a result of Gao and Vitale [15] who computed the classical intrinsic volumes of the Schläfli orthoscheme which is defined as the simplex in \mathbb{R}^n with the vertices 0 and $e_1 + \dots + e_i$, $1 \leq i \leq n$, where e_1, \dots, e_n denotes the standard orthonormal basis in \mathbb{R}^n . Equivalently, this simplex is given by

$$\{\beta \in \mathbb{R}^n : 1 \geq \beta_1 \geq \beta_2 \geq \dots \geq \beta_n \geq 0\}. \tag{1.3}$$

Later, Gao [14] computed the intrinsic volumes of the simplex with the vertices $e_1 + \dots + e_i - \frac{i}{n}(e_1 + \dots + e_n)$, $1 \leq i \leq n$, which can be also given by

$$\{\beta \in \mathbb{R}^n : \beta_1 \geq \beta_2 \geq \dots \geq \beta_n, \beta_1 + \dots + \beta_n = 0, \beta_1 - \beta_n \leq 1\}.$$

These simplices are closely related to the Weyl chambers of types B_n and A_{n-1} , respectively. Both the computation of the external angles of the faces of the Weyl chambers (which follows the method of Gao and Vitale [15] and Gao [14]), and the computation of the internal angle sums proceed by re-arranging the solid angles under interest in such a way that they cover the whole space, from which we conclude that the sum of the angles is 1.

The rest of the paper is mostly devoted to the proof of Theorem 1.1.

2 Preliminaries

2.1 Conic intrinsic volumes and solid angles

In this section we collect some information on polyhedral cones (called just cones, for simplicity) that is needed in later proofs. A supporting hyperplane for a cone $C \subseteq \mathbb{R}^n$

is a linear hyperplane H with the property that C lies entirely in one of the closed half-spaces H^+ and H^- induced by H . A *face* of C is a set of the form $F = C \cap H$, for a supporting hyperplane H , or the cone C itself. We denote by $\mathcal{F}_k(C)$ the set of all k -dimensional faces of C , for $k \in \{0, \dots, n\}$. The dimension of a face F is defined as the dimension of its linear hull, i.e. $\dim F = \dim \operatorname{lin}(F)$. Equivalently, the faces of C are obtained by replacing some of the half-spaces whose intersection defines the polyhedral cone by their boundaries and taking the intersection.

The *dual cone* (or the *polar cone*) of a cone $C \subseteq \mathbb{R}^n$ is defined as

$$C^\circ = \{x \in \mathbb{R}^n : \langle x, y \rangle \leq 0 \text{ for all } y \in C\}.$$

For example, if $C = L$ is a linear subspace, then $C^\circ = L^\perp$ is its orthogonal complement. There is [7, Section 2.1] a one-to-one correspondence between the k -faces of C and the $(n - k)$ -faces of C° via the bijective mapping

$$\begin{aligned} \mathcal{F}_k(C) &\rightarrow \mathcal{F}_{n-k}(C^\circ), \\ F &\mapsto N(F, C), \end{aligned}$$

where $N(F, C) := (\operatorname{lin} F)^\perp \cap C^\circ$ is called the *normal face* (or normal cone) of F with respect to C . A version of this one-to-one correspondence for polytopes can be found in [33, Corollary 2.14].

The *positive hull* of a finite set $\{x_1, \dots, x_m\} \subseteq \mathbb{R}^n$ is defined as the smallest cone containing this set, that is

$$\operatorname{pos}\{x_1, \dots, x_m\} = \{\lambda_1 x_1 + \dots + \lambda_m x_m : \lambda_1 \geq 0, \dots, \lambda_m \geq 0\}.$$

We will repeatedly make use of the well-known duality relations

$$\operatorname{pos}\{x_1, \dots, x_m\}^\circ = \bigcap_{i=1}^m x_i^- \quad \text{and} \quad \operatorname{pos}\{x_1, \dots, x_m\} = \left(\bigcap_{i=1}^m x_i^- \right)^\circ \tag{2.1}$$

for $x_1, \dots, x_m \in \mathbb{R}^n$ and $x_i^- := \{v \in \mathbb{R}^n : \langle v, x_i \rangle \leq 0\}$, $i = 1, \dots, m$.

Now, we define the conic intrinsic volumes. The definition and further properties are taken from [7, Section 2.2] and [17, Section 2].

Let $C \subseteq \mathbb{R}^n$ be a polyhedral cone, and g be an n -dimensional standard Gaussian random vector. Then, for $k \in \{0, \dots, n\}$, the k -th *conic intrinsic volume* (or, for simplicity, just *intrinsic volume*) of C is defined by

$$v_k(C) := \sum_{F \in \mathcal{F}_k(C)} \mathbb{P}(\Pi_C(g) \in \operatorname{relint}(F)),$$

Here, Π_C denotes the Euclidean projection on C , that is $\Pi_C(x)$ is the vector in C minimizing the Euclidean distance to $x \in \mathbb{R}^n$. Also, $\operatorname{relint}(F)$ denotes the interior of F taken with respect to its linear hull $\operatorname{lin}(F)$ as an ambient space.

The *solid angle* (or just angle) of a cone $C \subseteq \mathbb{R}^n$, denoted by $\alpha(C)$, is defined as

$$\alpha(C) = \mathbb{P}(Z \in C),$$

where Z is uniformly distributed on the unit sphere in the linear hull $\text{lin } C$ of C . In fact, for the distribution of Z we can choose any rotation invariant distribution living on the linear hull of C . For a k -dimensional cone $C \subseteq \mathbb{R}^n, k \in \{1, \dots, n\}$, the k -th conic intrinsic volume coincides with the solid angle of C , that is

$$v_k(C) = \alpha(C).$$

The *external angle* of a cone $C \subseteq \mathbb{R}^n$ at a face F is defined as the solid angle of the normal face of F with respect to C , that is the external angle is $\alpha(N(F, C))$. The *internal angle* of a face F at 0 is defined as $\alpha(F)$. For the purpose of this paper, we need the well-known expression of the intrinsic volumes in terms of internal and external angles:

$$v_k(C) = \sum_{F \in \mathcal{F}_k(C)} \alpha(F)\alpha(N(F, C)), \quad k \in \{0, \dots, n\}, \tag{2.2}$$

see Section 2.2 from [7]. We shall compute the intrinsic volumes of Weyl chambers by evaluating the internal and external angles at their faces.

2.2 Stirling numbers of the first kind and their generating functions

In this section we recall some facts on the Stirling numbers and their B - and D -analogues. These numbers, well known in combinatorics, appear in the formulas for the intrinsic volumes of the Weyl chambers. The (unsigned) Stirling numbers of the first kind are denoted by $\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]$ and defined as the coefficients of the polynomial

$$t(t+1) \cdot \dots \cdot (t+n-1) = \sum_{k=1}^n \left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right] t^k. \tag{2.3}$$

By convention, $\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right] = 0$ for $n \in \mathbb{N}, k \notin \{1, \dots, n\}$. Equivalently, $\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]$ can be defined as the number of permutations of the set $\{1, \dots, n\}$ having exactly k cycles. Other representations of the Stirling numbers of the first kind are known, e.g.,

$$\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right] = \frac{n!}{k!} \sum_{\substack{i_1, \dots, i_k \in \mathbb{N} \\ i_1 + \dots + i_k = n}} \frac{1}{i_1 i_2 \cdot \dots \cdot i_k}; \tag{2.4}$$

see [23, (1.9) and (1.13)]. We shall also need the generating functions of the Stirling numbers of the first kind:

$$\sum_{n=0}^{\infty} \left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right] \frac{t^n}{n!} = \frac{(-\log(1-t))^k}{k!} \quad \text{and} \quad \sum_{n=0}^{\infty} \sum_{k=0}^n \left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right] \frac{t^n}{n!} y^k = (1-t)^{-y}, \tag{2.5}$$

for all complex t such that $|t| < 1$ and all $y \in \mathbb{C}$. By convention, $\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right] = 1$.

The B -analogues of the (signless) Stirling numbers of the first kind are denoted by $B[n, k]$ and defined as the coefficients of the polynomial

$$(t+1)(t+3) \cdot \dots \cdot (t+2n-1) = \sum_{k=0}^n B[n, k] t^k. \tag{2.6}$$

Again, by convention, we put $B[n, k] = 0$ for $k \notin \{0, \dots, n\}$. These numbers appear as entry A028338 (or A039757 for the signed version) in the On-Line Encyclopedia of Integer Sequences [27] and also in [7, 9, 8, 10, 12, 13, 16, 19, 20, 22, 32]. As follows easily from their definition, the numbers $B[n, k]$ satisfy the recurrence relation

$$B[n, k] = (2n - 1)B[n - 1, k] + B[n - 1, k - 1]; \tag{2.7}$$

see [20, Section 2.2]. The following proposition collects some useful explicit formulas for $B[n, k]$ including their exponential generating function.

Proposition 2.1. *The B-analogues $B[n, k]$ of the Stirling numbers of the first kind are explicitly given by the formulas*

$$B[n, k] = \sum_{i=k}^n 2^{n-i} \binom{i}{k} \left[\begin{matrix} n \\ i \end{matrix} \right], \tag{2.8}$$

$$B[n, k] = \sum_{r=0}^{n-k} 2^{n-k-2r} \binom{2r}{r} \left[\begin{matrix} n-r \\ k \end{matrix} \right] \frac{n!}{(n-r)!}, \tag{2.9}$$

for $k \in \{0, \dots, n\}$. The exponential generating function of $(B[n, k])_{n,k \geq 0}$ is given by

$$\sum_{n=0}^{\infty} \sum_{k=0}^n B[n, k] \frac{t^n}{n!} y^k = (1 - 2t)^{-\frac{1}{2}(y+1)} \tag{2.10}$$

for all complex $|t| < 1/2$ and $y \in \mathbb{C}$. By convention, $B[0, 0] = 1$.

In Entry A028338 of [27] the explicit formula (2.8) for the number $B[n, k]$ in terms of the Stirling numbers of the first kind was stated by F. Woodhouse without a proof.

Proof of Proposition 2.1. We start with the proof of (2.8). We want to check whether the numbers on the right-hand side of (2.8) coincide with the coefficients of the polynomial in (2.6). We have

$$\begin{aligned} \sum_{k=0}^n \sum_{i=k}^n 2^{n-i} \binom{i}{k} \left[\begin{matrix} n \\ i \end{matrix} \right] t^k &= 2^n \sum_{i=0}^n 2^{-i} \left[\begin{matrix} n \\ i \end{matrix} \right] \left(\sum_{k=0}^i \binom{i}{k} t^k \right) = 2^n \sum_{i=0}^n \left[\begin{matrix} n \\ i \end{matrix} \right] \left(\frac{t+1}{2} \right)^i \\ &= 2^n \left(\frac{t+1}{2} \right) \left(\frac{t+1}{2} + 1 \right) \cdot \dots \cdot \left(\frac{t+1}{2} + n - 1 \right) \\ &= (t+1)(t+3) \cdot \dots \cdot (t+2n-1) \end{aligned}$$

using the Binomial Theorem and (2.3), which completes the proof of (2.8).

Next, we prove (2.10). By (2.6) and (2.3) we have

$$\sum_{k=0}^n B[n, k] y^k = 2^n \left(\frac{y+1}{2} \right) \left(\frac{y+1}{2} + 1 \right) \cdot \dots \cdot \left(\frac{y+1}{2} + n - 1 \right)$$

$$= \sum_{i=0}^n 2^n \begin{bmatrix} n \\ i \end{bmatrix} \left(\frac{y+1}{2}\right)^i.$$

Thus, the generating function is given by

$$\sum_{n=0}^{\infty} \sum_{k=0}^n B[n, k] \frac{t^n}{n!} y^k = \sum_{n=0}^{\infty} \sum_{i=0}^n \begin{bmatrix} n \\ i \end{bmatrix} \left(\frac{y+1}{2}\right)^i \frac{(2t)^n}{n!} = (1-2t)^{-\frac{1}{2}(y+1)},$$

where we used (2.5) in the last step.

At last, we use both formulas (2.8) and (2.10) to prove (2.9). To this end, we compare the generating functions of the sequences on both sides of (2.9). Consider the following sum:

$$a_k(t) := \sum_{n=k}^{\infty} \left(\sum_{r=0}^{n-k} 2^{n-k-2r} \binom{2r}{r} \begin{bmatrix} n-r \\ k \end{bmatrix} \frac{n!}{(n-r)!} \right) \frac{t^n}{n!}, \quad k \in \mathbb{N}_0.$$

Then, we get

$$\begin{aligned} a_k(t) &= \sum_{n=k}^{\infty} \sum_{r=0}^{n-k} 2^{n-k-2r} \binom{2r}{r} \begin{bmatrix} n-r \\ k \end{bmatrix} \frac{t^n}{(n-r)!} \\ &= \sum_{r=0}^{\infty} \sum_{n=r+k}^{\infty} 2^{n-k-2r} \binom{2r}{r} \begin{bmatrix} n-r \\ k \end{bmatrix} \frac{t^n}{(n-r)!} \\ &= \sum_{r=0}^{\infty} 2^{-r-k} \binom{2r}{r} t^r \left(\sum_{n=r+k}^{\infty} \begin{bmatrix} n-r \\ k \end{bmatrix} \frac{(2t)^{n-r}}{(n-r)!} \right). \end{aligned} \tag{2.11}$$

By shifting the index in the inner sum and using the generating function of the Stirling numbers of the first kind stated in (2.5), we can rewrite (2.11) to obtain

$$\begin{aligned} a_k(t) &= \sum_{r=0}^{\infty} 2^{-r-k} \binom{2r}{r} t^r \left(\sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(2t)^n}{n!} \right) \\ &= \sum_{r=0}^{\infty} 2^{-r-k} \binom{2r}{r} t^r \frac{(-\log(1-2t))^k}{k!} \\ &= 2^{-k} \frac{(-\log(1-2t))^k}{k!} \sum_{r=0}^{\infty} \left(\frac{t}{2}\right)^r \binom{2r}{r} \\ &= 2^{-k} \frac{(-\log(1-2t))^k}{k!} \frac{1}{\sqrt{1-2t}} \end{aligned}$$

for $|t| < 1/2$. Thus, the generating function of the right-hand side of (2.9) is given by

$$\sum_{k=0}^{\infty} a_k(t) y^k = \frac{1}{\sqrt{1-2t}} \exp\left(-\frac{y}{2} \log(1-2t)\right) = (1-2t)^{-\frac{1}{2}(y+1)},$$

which coincides with the generating function of $(B[n, k])_{n,k \geq 0}$. This completes the proof. □

The D -analogues of the (signless) Stirling numbers of the first kind are denoted by $D[n, k]$ and defined as the coefficients of the polynomial

$$(t + 1)(t + 3) \cdot \dots \cdot (t + 2n - 3)(t + n - 1) = \sum_{k=0}^n D[n, k]t^k. \tag{2.12}$$

By convention, $D[n, k] = 0$ for $k \notin \{0, \dots, n\}$. The signed version of these numbers appears as entry A039762 in [27]. The D -analogues can be expressed through the B -analogues by

$$\begin{aligned} D[n, k] &= (n - 1)B[n - 1, k] + B[n - 1, k - 1] \\ &= B[n, k] - nB[n - 1, k]; \end{aligned} \tag{2.13}$$

see [20, Section 2.2] and use (2.7). In probabilistic terms, (2.3), (2.6) and (2.12) state that

$$\mathbb{P}(N_n = k) = \binom{n}{k} \frac{1}{n!}, \quad \mathbb{P}(N_n^B = k) = \frac{B[n, k]}{2^n n!}, \quad \mathbb{P}(N_n^D = k) = \frac{D[n, k]}{2^{n-1} n!},$$

for all $k \in \{0, \dots, n\}$, where N_n, N_n^B and N_n^D are random variables defined by

$$\begin{aligned} N_n &= \sum_{j=1}^n \text{Bern}\left(\frac{1}{j}\right), & N_n^B &= \sum_{j=1}^n \text{Bern}\left(\frac{1}{2j}\right), \\ N_n^D &= \sum_{j=1}^{n-1} \text{Bern}\left(\frac{1}{2j}\right) + \text{Bern}\left(\frac{1}{n}\right), \end{aligned}$$

and $\text{Bern}(\cdot)$ are independent Bernoulli variables with corresponding parameters. It is well known that N_n has the same distribution as the number of cycles in a uniform random permutation of n elements. This can be deduced from the Feller coupling or from the Chinese restaurant construction of the uniform random permutation, see [23, Section 3.1]. Similarly, it can be shown that N_n^B has the same distribution as the number of *even* cycles in a uniform random *signed* permutation of n elements, where a signed permutation is a pair (σ, ε) consisting of a permutation σ acting on $\{1, \dots, n\}$ together with a vector of signs $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \{-1, +1\}^n$, and a cycle $i_1 \mapsto \dots \mapsto i_k \mapsto i_1$ of σ is called even if $\varepsilon_{i_1} \cdot \dots \cdot \varepsilon_{i_k} = +1$. Observe that this gives a combinatorial interpretation of (2.8). More generally, it is known from [26, 5.3] or [18, p. 59, p. 63] that the numbers $\binom{n}{k}, B[n, k]$ and $D[n, k]$ count the elements in the reflection group of the corresponding type whose space of invariant vectors is k -dimensional. For example, N_n^D has the same distribution as the number of even cycles in a random uniform D -permutation of n elements, where a D -permutation is a signed permutation (σ, ε) such that $\varepsilon_1 \cdot \dots \cdot \varepsilon_n = 1$.

3 Conic intrinsic volumes of Weyl chambers

This section is dedicated to proving the formulas for the intrinsic volumes of the Weyl chambers stated in (1.1). To this end, we want to use the formula (2.2) and evaluate

the internal and external angles of the faces of the Weyl chambers. The computation of the external angles relies on the ideas of Gao and Vitale [15] who derived the classical intrinsic volumes of the simplex (1.3).

3.1 Type A_{n-1}

We start with the simpler A_{n-1} case. Recall the definition of the fundamental Weyl chamber of type A_{n-1} :

$$C^A := \mathcal{C}(A_{n-1}) = \{\beta \in \mathbb{R}^n : \beta_1 \geq \beta_2 \geq \dots \geq \beta_n\}.$$

Then our first result is the following.

Theorem 3.1. *For $k = 1, \dots, n$, the k -th conic intrinsic volume of the Weyl chamber of type A_{n-1} is given by*

$$v_k(\mathcal{C}(A_{n-1})) = \begin{bmatrix} n \\ k \end{bmatrix} \frac{1}{n!},$$

where the Stirling numbers $\begin{bmatrix} n \\ k \end{bmatrix}$ are defined as in Section 2.2.

Before we start with the proof of Theorem 3.1, we compute the solid angle of the cone whose points correspond to “bridges staying below zero”.

Lemma 3.2. *The solid angle of the cone*

$$D_m := \{x \in \mathbb{R}^m : x_1 \leq 0, x_1 + x_2 \leq 0, \dots, x_1 + \dots + x_{m-1} \leq 0, \\ x_1 + \dots + x_m = 0\}$$

is given by $\alpha(D_m) = \frac{1}{m}$, for $m \in \mathbb{N}$.

Proof of Lemma 3.2. The idea is to define m cones in \mathbb{R}^m with the same solid angle as D_m , such that their union equals the linear subspace $H_m := \{x \in \mathbb{R}^m : x_1 + \dots + x_m = 0\}$ and their pairwise intersections are Lebesgue null sets (in the sense of the ambient linear subspace H_m). To this end, define

$$D_m^{(i)} = \{(x_1, \dots, x_m) \in \mathbb{R}^m : x_{i+1} \leq 0, x_{i+1} + x_{i+2} \leq 0, \dots, \\ x_{i+1} + \dots + x_m + x_1 + \dots + x_{i-1} \leq 0, \\ x_{i+1} + \dots + x_m + x_1 + \dots + x_i = 0\},$$

for $i \in \{0, \dots, m - 1\}$, where obviously $D_m^{(0)} = D_m$. Observe that $D_m^{(i)} = \mathcal{O}D_m$ for a suitable permutation matrix $\mathcal{O} \in \mathbb{R}^{m \times m}$, and thus, the rotation invariance of the spherical Lebesgue measure yields that the solid angle is the same for each $D_m^{(i)}$, $0 \leq i \leq m$. Furthermore, it is easy to see that $D_m^{(i)}$ can be written as follows:

$$D_m^{(i)} = \{x \in \mathbb{R}^m : x_1 + \dots + x_m = 0, \\ x_1 + \dots + x_i \geq \max\{x_1, x_1 + x_2, \dots, x_1 + \dots + x_m\}\}.$$

From this representation of $D_m^{(i)}$ we easily observe that for any point $x \in H_m$, there is a suitable index $i \in \{0, \dots, m - 1\}$, namely where $x_1 + \dots + x_i$ is maximal, such

that $x \in D_m^{(i)}$. On the other hand, if x is in the intersection of two cones $D_m^{(i)}$ and $D_m^{(j)}$, $i \neq j$, then x belongs to $\{x \in \mathbb{R}^m : x_1 + \dots + x_m = 0, x_1 + \dots + x_i = x_1 + \dots + x_j\}$, which is a subspace of dimension $m - 2$, and thus, a Lebesgue null set (in the sense of the ambient subspace H_m). This yields that $\alpha(D_m) = \frac{1}{m}$. Note that a probabilistic version of this reasoning is due to Sparre Andersen [29] who proved that the probability that a random bridge (satisfying certain minor conditions) of length m stays below zero is $1/m$. \square

Proof of Theorem 3.1. In order to prove Theorem 3.1, we want to evaluate the internal and external angles of the faces of C^A . Observe that the faces of C^A can be obtained by replacing some of the inequalities defining C^A by the corresponding equalities. It follows that for $2 \leq k \leq n$, any k -face of C^A is determined by some collection of indices $1 \leq l_1 < \dots < l_{k-1} < n$ and given by

$$C^A(l_1, \dots, l_{k-1}) = \{\beta \in \mathbb{R}^n : \beta_1 = \dots = \beta_{l_1} \geq \beta_{l_1+1} = \dots = \beta_{l_2} \geq \dots \geq \beta_{l_{k-1}+1} = \dots = \beta_n\}.$$

Then, (2.2) yields the formula

$$\begin{aligned} v_k(C^A) &= \sum_{F \in \mathcal{F}_k(C^A)} \alpha(F)\alpha(N(F, C^A)) \\ &= \sum_{1 \leq l_1 < \dots < l_{k-1} \leq n-1} \alpha(C^A(l_1, \dots, l_{k-1}))\alpha(N(C^A(l_1, \dots, l_{k-1}), C^A)). \end{aligned}$$

Note that C^A has no vertices and the only 1-face is $\{\beta \in \mathbb{R}^n : \beta_1 = \dots = \beta_n\}$, which corresponds to an empty collection of l 's.

External Angles. Let $1 \leq k \leq n$. Take some $1 \leq l_1 < \dots < l_{k-1} < n$. We start by computing the normal face $N(C^A(l_1, \dots, l_{k-1}), C^A)$. Our method of proof partially relies on the approach of Gao and Vitale [15]. By definition, we have

$$N(C^A(l_1, \dots, l_{k-1}), C^A) = (\text{lin } C^A(l_1, \dots, l_{k-1}))^\perp \cap (C^A)^\circ.$$

Using (2.1), we obtain

$$\begin{aligned} (C^A)^\circ &= \{x \in \mathbb{R}^n : x_1 \geq x_2 \geq \dots \geq x_n\}^\circ \\ &= \text{pos}\{e_1, e_1 + e_2, \dots, e_1 + \dots + e_n, -e_1 - \dots - e_n\}^\circ \\ &= \{x \in \mathbb{R}^n : x_1 \leq 0, \dots, x_1 + \dots + x_{n-1} \leq 0, x_1 + \dots + x_n = 0\}. \end{aligned}$$

Here, e_1, \dots, e_n denotes the standard Euclidean orthonormal basis of \mathbb{R}^n . Furthermore,

$$\begin{aligned} &(\text{lin } C^A(l_1, \dots, l_{k-1}))^\perp \\ &= \{x \in \mathbb{R}^n : x_1 = \dots = x_{l_1}, x_{l_1+1} = \dots = x_{l_2}, \dots, x_{l_{k-1}+1} = \dots = x_n\}^\perp \\ &= \{x \in \mathbb{R}^n : x_1 + \dots + x_{l_1} = 0, x_{l_1+1} + \dots + x_{l_2} = 0, \\ &\quad \dots, x_{l_{k-1}+1} + \dots + x_n = 0\}. \end{aligned}$$

The explicit representations of $(C^A)^\circ$ and $(\text{lin } C^A(l_1, \dots, l_{k-1}))^\perp$ imply the following representation of the normal face as the orthogonal product (or direct sum) of k cones:

$$N(C^A(l_1, \dots, l_{k-1}), C^A) = D_{l_1} \times D_{l_2-l_1} \times \dots \times D_{l_{k-1}-l_{k-2}} \times D_{n-l_{k-1}},$$

where again

$$D_m := \{x \in \mathbb{R}^m : x_1 \leq 0, \dots, x_1 + \dots + x_{m-1} \leq 0, x_1 + \dots + x_m = 0\}.$$

In Lemma 3.2 we proved that $\alpha(D_m) = 1/m$ holds for any $m \in \mathbb{N}$. Since the normal cone $N(C^A(l_1, \dots, l_{k-1}), C^A)$ is the orthogonal product of the k cones $D_{l_1}, D_{l_2-l_1}, \dots, D_{n-l_{k-1}}$, it follows that

$$\alpha(N(C^A(l_1, \dots, l_{k-1}), C^A)) = \frac{1}{l_1(l_2 - l_1) \cdot \dots \cdot (n - l_{k-1})}. \tag{3.1}$$

Internal Angles. Now, we need to compute the internal angles at the faces of C^A . To this end, we consider the linear hull of $C^A(l_1, \dots, l_{k-1})$ which forms a k -dimensional linear subspace of \mathbb{R}^n . Recall that for $1 \leq l_1 < \dots < l_{k-1} \leq n - 1$ the linear hull of $C^A(l_1, \dots, l_{k-1})$ is given by

$$\text{lin } C^A(l_1, \dots, l_{k-1}) = \{\beta \in \mathbb{R}^n : \beta_1 = \dots = \beta_{l_1}, \dots, \beta_{l_{k-1}+1} = \dots = \beta_n\}.$$

Thus, the vectors y_i given by

$$\begin{aligned} y_1 &= \frac{1}{\sqrt{l_1}} \overbrace{(1, \dots, 1, 0, \dots, 0)}^{1, \dots, l_1}, \quad y_2 = \frac{1}{\sqrt{l_2 - l_1}} (0, \dots, 0, \overbrace{1, \dots, 1}^{l_1+1, \dots, l_2}, 0, \dots, 0), \dots, \\ y_{k-1} &= \frac{1}{\sqrt{l_{k-1} - l_{k-2}}} (0, \dots, 0, \overbrace{1, \dots, 1}^{l_{k-2}+1, \dots, l_{k-1}}, 0, \dots, 0), \\ y_k &= \frac{1}{\sqrt{n - l_{k-1}}} (0, \dots, 0, \overbrace{1, \dots, 1}^{l_{k-1}+1, \dots, n}) \end{aligned}$$

form an orthonormal basis of $\text{lin } C^A(l_1, \dots, l_{k-1})$. For independent and standard Gaussian random variables ξ_1, \dots, ξ_k , the random vector $N := \xi_1 y_1 + \dots + \xi_k y_k$ is k -dimensional standard Gaussian distributed on the linear subspace $\text{lin } C^A(l_1, \dots, l_{k-1})$. Thus, we obtain for the angle of $C^A(l_1, \dots, l_{k-1})$ the following formula:

$$\begin{aligned} \alpha(C^A(l_1, \dots, l_{k-1})) &= \mathbb{P}(N \in C^A(l_1, \dots, l_{k-1})) \\ &= \mathbb{P}\left(\frac{\xi_1}{\sqrt{l_1}} \geq \frac{\xi_2}{\sqrt{l_2 - l_1}} \geq \dots \geq \frac{\xi_{k-1}}{\sqrt{l_{k-1} - l_{k-2}}} \geq \frac{\xi_k}{\sqrt{n - l_{k-1}}}\right). \end{aligned} \tag{3.2}$$

Conic intrinsic volumes. Let $1 \leq k \leq n$. Using the formulas (2.2), (3.1) and (3.2) we obtain

$$v_k(C^A) = \sum_{1 \leq l_1 < \dots < l_{k-1} \leq n-1} \alpha(C^A(l_1, \dots, l_{k-1})) \alpha(N(C^A(l_1, \dots, l_{k-1}), C^A))$$

$$= \sum_{1 \leq l_1 < \dots < l_{k-1} \leq n-1} \frac{\mathbb{P}\left(\frac{\xi_1}{\sqrt{l_1}} \geq \frac{\xi_2}{\sqrt{l_2-l_1}} \geq \dots \geq \frac{\xi_{k-1}}{\sqrt{l_{k-1}-l_{k-2}}} \geq \frac{\xi_k}{\sqrt{n-l_{k-1}}}\right)}{l_1(l_2-l_1) \cdot \dots \cdot (l_{k-1}-l_{k-2})(n-l_{k-1})}.$$

Defining $i_1 = l_1, i_2 = l_2 - l_1, \dots, i_{k-1} = l_{k-1} - l_{k-2}, i_k = n - l_{k-1}$, we can change the summation indices in the above sum and obtain

$$v_k(C^A) = \sum_{\substack{i_1, \dots, i_k \in \mathbb{N} \\ i_1 + \dots + i_k = n}} \frac{1}{i_1 i_2 \cdot \dots \cdot i_k} \mathbb{P}\left(\frac{\xi_1}{\sqrt{i_1}} \geq \frac{\xi_2}{\sqrt{i_2}} \geq \dots \geq \frac{\xi_{k-1}}{\sqrt{i_{k-1}}} \geq \frac{\xi_k}{\sqrt{i_k}}\right). \tag{3.3}$$

Now, fix a permutation π from the symmetric group \mathcal{S}_k . For each tuple $(i_1, \dots, i_k) \in \mathbb{N}^k$ satisfying $i_1 + \dots + i_k = n$, the tuple $(i_{\pi(1)}, \dots, i_{\pi(k)})$ also satisfies $i_{\pi(1)} + \dots + i_{\pi(k)} = n$. Thus, the sum in (3.3) does not change if we replace the tuple (i_1, \dots, i_k) by $(i_{\pi(1)}, \dots, i_{\pi(k)})$ inside the sum. It follows that

$$\begin{aligned} v_k(C^A) &= \frac{1}{k!} \sum_{\pi \in \mathcal{S}_k} \sum_{\substack{i_1, \dots, i_k \in \mathbb{N} \\ i_1 + \dots + i_k = n}} \frac{\mathbb{P}\left(\frac{\xi_1}{\sqrt{i_{\pi(1)}}} \geq \frac{\xi_2}{\sqrt{i_{\pi(2)}}} \geq \dots \geq \frac{\xi_k}{\sqrt{i_{\pi(k)}}}\right)}{i_{\pi(1)} i_{\pi(2)} \cdot \dots \cdot i_{\pi(k)}} \\ &= \frac{1}{k!} \sum_{\substack{i_1, \dots, i_k \in \mathbb{N} \\ i_1 + \dots + i_k = n}} \frac{1}{i_1 i_2 \cdot \dots \cdot i_k} \sum_{\pi \in \mathcal{S}_k} \mathbb{P}\left(\frac{\xi_1}{\sqrt{i_{\pi(1)}}} \geq \frac{\xi_2}{\sqrt{i_{\pi(2)}}} \geq \dots \geq \frac{\xi_k}{\sqrt{i_{\pi(k)}}}\right) \\ &= \frac{1}{k!} \sum_{\substack{i_1, \dots, i_k \in \mathbb{N} \\ i_1 + \dots + i_k = n}} \frac{1}{i_1 i_2 \cdot \dots \cdot i_k} \sum_{\pi \in \mathcal{S}_k} \mathbb{P}\left(\frac{\xi_{\pi(1)}}{\sqrt{i_{\pi(1)}}} \geq \frac{\xi_{\pi(2)}}{\sqrt{i_{\pi(2)}}} \geq \dots \geq \frac{\xi_{\pi(k)}}{\sqrt{i_{\pi(k)}}}\right) \\ &= \frac{1}{k!} \sum_{\substack{i_1, \dots, i_k \in \mathbb{N} \\ i_1 + \dots + i_k = n}} \frac{1}{i_1 i_2 \cdot \dots \cdot i_k}, \end{aligned}$$

where we used that ξ_1, \dots, ξ_k are independent and identically distributed, hence exchangeable. Using the representation (2.4) of the Stirling numbers of the first kind, we obtain

$$v_k(C^A) = \frac{1}{k!} \frac{[n] k!}{[k] n!} = \frac{[n]}{[k]} \frac{1}{n!},$$

which completes the proof. □

3.2 Type B_n

Now, we proceed with the B_n -case. Recall the definition of the fundamental Weyl chamber of type B_n :

$$C^B := \mathcal{C}(B_n) = \{\beta \in \mathbb{R}^n : \beta_1 \geq \beta_2 \geq \dots \geq \beta_n \geq 0\}.$$

Then the analogue of Theorem 3.1 is the following.

Theorem 3.3. For $k = 0, 1, \dots, n$, the k -th conic intrinsic volume of the Weyl chamber $\mathcal{C}(B_n)$ is given by

$$v_k(\mathcal{C}(B_n)) = \frac{B[n, k]}{2^n n!}.$$

Similar to the A_{n-1} -case, we first need a formula for the solid angle of a cone whose points correspond to “walks staying below zero”.

Lemma 3.4. The solid angle of the cone

$$B_m := \{x \in \mathbb{R}^m : x_1 \leq 0, x_1 + x_2 \leq 0, \dots, x_1 + \dots + x_m \leq 0\},$$

is given by $\alpha(B_m) = \binom{2m}{m} \frac{1}{4^m}$, for $m \in \mathbb{N}$.

Proof of Lemma 3.4. Take an m -dimensional standard Gaussian random vector $N = (N_1, \dots, N_m)$. The solid angle of B_m coincides with the probability that $N \in B_m$. Using the well-known formula of Sparre Andersen [28] (which is valid in the special case when N_1, \dots, N_m are independent and standard Gaussian variables) we obtain

$$\begin{aligned} \alpha(B_m) &= \mathbb{P}[N \in B_m] = \mathbb{P}[N_1 \leq 0, N_1 + N_2 \leq 0, \dots, N_1 + \dots + N_m \leq 0] \\ &= \binom{2m}{m} \frac{1}{2^{2m}}, \end{aligned}$$

which completes the proof. □

Proof of Theorem 3.3. To obtain the faces of C^B we have to replace some of the inequalities defining C^B by the corresponding equalities. It follows that for $1 \leq k \leq n$ any k -face of C^B is determined by a collection of indices $1 \leq l_1 < \dots < l_k \leq n$ and given by

$$C^B(l_1, \dots, l_k) = \{\beta \in \mathbb{R}^n : \beta_1 = \dots = \beta_{l_1} \geq \dots \geq \beta_{l_{k-1}+1} = \dots = \beta_{l_k} \geq \beta_{l_k+1} = \dots = \beta_n = 0\}.$$

Then, (2.2) yields the formula

$$v_k(C^B) = \sum_{1 \leq l_1 < \dots < l_k \leq n} \alpha(C^B(l_1, \dots, l_k)) \alpha(N(C^B(l_1, \dots, l_k), C^B)).$$

External angles. The computation of the external angles for the B_n -case is similar to that for the A_{n-1} -case. Take some $k \in \{1, \dots, n\}$ and $1 \leq l_1 < \dots < l_k \leq n$. We start by computing the normal face $N(C^B(l_1, \dots, l_k), C^B)$ of $C^B(l_1, \dots, l_k)$. First of all, we have

$$\begin{aligned} (C^B)^\circ &= \text{pos}\{e_1, e_1 + e_2, \dots, e_1 + \dots + e_n\}^\circ \\ &= \{x \in \mathbb{R}^n : x_1 \leq 0, x_1 + x_2 \leq 0, \dots, x_1 + \dots + x_n \leq 0\}. \end{aligned}$$

Since

$$(\text{lin } C^B(l_1, \dots, l_k))^\perp$$

$$\begin{aligned} &= \{x \in \mathbb{R}^n : x_1 = \dots = x_{l_1}, \dots, x_{l_{k-1}+1} = \dots = x_{l_k}, x_{l_k+1} = \dots = x_n = 0\}^\perp \\ &= \{x \in \mathbb{R}^n : x_1 + \dots + x_{l_1} = 0, \dots, x_{l_{k-1}+1} + \dots + x_{l_k} = 0\}, \end{aligned}$$

it follows that the normal cone $N(C^B(l_1, \dots, l_k), C^B)$ can be written as the orthogonal product

$$N(C^B(l_1, \dots, l_k), C^B) = D_{l_1} \times D_{l_2-l_1} \times \dots \times D_{l_k-l_{k-1}} \times B_{n-l_k},$$

where again

$$\begin{aligned} D_m &= \{x \in \mathbb{R}^m : x_1 \leq 0, \dots, x_1 + \dots + x_{m-1} \leq 0, x_1 + \dots + x_m = 0\}, \\ B_m &= \{x \in \mathbb{R}^m : x_1 \leq 0, x_1 + x_2 \leq 0, \dots, x_1 + \dots + x_m \leq 0\}, \quad B_0 := \{0\}. \end{aligned}$$

Using Lemmas 3.2 and 3.4 and the orthogonal product structure of the normal face we arrive at

$$\alpha(N(C^B(l_1, \dots, l_k), C^B)) = \frac{\binom{2(n-l_k)}{n-l_k}}{l_1(l_2-l_1) \cdot \dots \cdot (l_k-l_{k-1})2^{2(n-l_k)}}. \tag{3.4}$$

Note that the above formula is also valid for the case $k = 0$ where we have $C^B(l_1, \dots, l_k) = \{0\}$.

Internal angles. Now, we need to compute the internal angles for the faces of C^B , that is, the solid angles $\alpha(C^B(l_1, \dots, l_k))$ for $1 \leq l_1 < \dots < l_k \leq n$. Recall that the linear hull of $C^B(l_1, \dots, l_k)$ is the k -dimensional linear subspace given by

$$\begin{aligned} \text{lin } C^B(l_1, \dots, l_k) &= \{x \in \mathbb{R}^n : x_1 = \dots = x_{l_1}, x_{l_1+1} = \dots = x_{l_2}, \dots, \\ &\quad x_{l_{k-1}+1} = \dots = x_n = 0\}. \end{aligned}$$

The following vectors y_1, \dots, y_k form an orthonormal basis of $\text{lin } C^B(l_1, \dots, l_k)$:

$$\begin{aligned} y_1 &= \frac{1}{\sqrt{l_1}} \overbrace{(1, \dots, 1, 0, \dots, 0)}^{1, \dots, l_1}, \quad y_2 = \frac{1}{\sqrt{l_2-l_1}} \overbrace{(0, \dots, 0, 1, \dots, 1, 0, \dots, 0)}^{l_1+1, \dots, l_2}, \dots, \\ y_k &= \frac{1}{\sqrt{l_k-l_{k-1}}} \overbrace{(0, \dots, 0, 1, \dots, 1, 0, \dots, 0)}^{l_{k-1}+1, \dots, l_k}. \end{aligned}$$

For independent and standard Gaussian random variables ξ_1, \dots, ξ_k , the random vector $N := \xi_1 y_1 + \dots + \xi_k y_k$ is k -dimensional standard Gaussian on the linear subspace $\text{lin } C^B(l_1, \dots, l_k)$. Thus, we obtain for the solid angle of $C^B(l_1, \dots, l_k)$ the following formula:

$$\begin{aligned} \alpha(C^B(l_1, \dots, l_k)) &= \mathbb{P}(N \in C^B(l_1, \dots, l_k)) \\ &= \mathbb{P}\left(\frac{\xi_1}{\sqrt{l_1}} \geq \frac{\xi_2}{\sqrt{l_2-l_1}} \geq \dots \geq \frac{\xi_k}{\sqrt{l_k-l_{k-1}}} \geq 0\right). \end{aligned} \tag{3.5}$$

Conic intrinsic volumes. Let $1 \leq k \leq n$. Using the formulas (3.5) and (3.4) for the internal and external angles, we have

$$\begin{aligned} v_k(C^B) &= \sum_{1 \leq l_1 < \dots < l_k \leq n} \alpha(C^B(l_1, \dots, l_k)) \alpha(N(C^B(l_1, \dots, l_k), C^B)) \\ &= \sum_{1 \leq l_1 < \dots < l_k \leq n} \binom{2(n-l_k)}{n-l_k} \frac{\mathbb{P}\left(\frac{\xi_1}{\sqrt{l_1}} \geq \frac{\xi_2}{\sqrt{l_2-l_1}} \geq \dots \geq \frac{\xi_k}{\sqrt{l_k-l_{k-1}}} \geq 0\right)}{l_1(l_2-l_1) \dots (l_k-l_{k-1}) 2^{2(n-l_k)}}. \end{aligned}$$

Defining $i_1 = l_1, i_2 = l_2 - l_1, \dots, i_k = l_k - l_{k-1}$, we can change the summation indices in the above sum and obtain

$$\begin{aligned} v_k(C^B) &= \sum_{\substack{i_1, \dots, i_k \in \mathbb{N} \\ i_1 + \dots + i_k \leq n}} \binom{2(n-i_1-\dots-i_k)}{n-i_1-\dots-i_k} \frac{\mathbb{P}\left(\frac{\xi_1}{\sqrt{i_1}} \geq \frac{\xi_2}{\sqrt{i_2}} \geq \dots \geq \frac{\xi_k}{\sqrt{i_k}} \geq 0\right)}{i_1 i_2 \dots i_k 2^{2(n-i_1-\dots-i_k)}} \\ &= \sum_{r=0}^{n-k} \sum_{\substack{i_1, \dots, i_k \in \mathbb{N} \\ i_1 + \dots + i_k = n-r}} \binom{2r}{r} \frac{\mathbb{P}\left(\frac{\xi_1}{\sqrt{i_1}} \geq \frac{\xi_2}{\sqrt{i_2}} \geq \dots \geq \frac{\xi_k}{\sqrt{i_k}} \geq 0\right)}{i_1 i_2 \dots i_k 2^{2r}}. \end{aligned} \tag{3.6}$$

Now, fix a number $r \in \{0, 1, \dots, n-k\}$, a permutation $\pi \in \mathcal{S}_k$ and a vector of signs $\varepsilon = (\varepsilon_1, \dots, \varepsilon_k) \in \{\pm 1\}^k$. For each tuple $(i_1, \dots, i_k) \in \mathbb{N}^k$ satisfying $i_1 + \dots + i_k = n-r$, the tuple $(i_{\pi(1)}, \dots, i_{\pi(k)})$ also satisfies $i_{\pi(1)} + \dots + i_{\pi(k)} = n-r$. Thus, the inner sum in (3.6) does not change if we replace the tuple (i_1, \dots, i_k) by $(i_{\pi(1)}, \dots, i_{\pi(k)})$ inside the sum. Furthermore, the sum does not change if we additionally replace ξ_1, \dots, ξ_k by $\varepsilon_1 \xi_1, \dots, \varepsilon_k \xi_k$, since ξ_1, \dots, ξ_k are independent and standard normal. Thus, we obtain

$$\begin{aligned} v_k(C^B) &= \sum_{r=0}^{n-k} \frac{1}{2^k k!} \sum_{(\varepsilon, \pi) \in \{\pm 1\}^k \times \mathcal{S}_k} \sum_{\substack{i_1, \dots, i_k \in \mathbb{N} \\ i_1 + \dots + i_k = n-r}} \binom{2r}{r} \\ &\quad \times \frac{\mathbb{P}\left(\frac{\varepsilon_1 \xi_1}{\sqrt{i_{\pi(1)}}} \geq \frac{\varepsilon_2 \xi_2}{\sqrt{i_{\pi(2)}}} \geq \dots \geq \frac{\varepsilon_k \xi_k}{\sqrt{i_{\pi(k)}}} \geq 0\right)}{i_{\pi(1)} i_{\pi(2)} \dots i_{\pi(k)} 2^{2r}} \\ &= \frac{1}{2^k k!} \sum_{r=0}^{n-k} \sum_{\substack{i_1, \dots, i_k \in \mathbb{N} \\ i_1 + \dots + i_k = n-r}} \frac{\binom{2r}{r}}{i_1 i_2 \dots i_k 2^{2r}} \\ &\quad \times \sum_{(\varepsilon, \pi) \in \{\pm 1\}^k \times \mathcal{S}_k} \mathbb{P}\left(\frac{\varepsilon_1 \xi_1}{\sqrt{i_{\pi(1)}}} \geq \dots \geq \frac{\varepsilon_k \xi_k}{\sqrt{i_{\pi(k)}}} \geq 0\right) \\ &= \frac{1}{2^k k!} \sum_{r=0}^{n-k} \sum_{\substack{i_1, \dots, i_k \in \mathbb{N} \\ i_1 + \dots + i_k = n-r}} \frac{\binom{2r}{r}}{i_1 i_2 \dots i_k 2^{2r}} \\ &\quad \times \sum_{(\varepsilon, \pi) \in \{\pm 1\}^k \times \mathcal{S}_k} \mathbb{P}\left(\frac{\varepsilon_1 \xi_{\pi(1)}}{\sqrt{i_{\pi(1)}}} \geq \dots \geq \frac{\varepsilon_k \xi_{\pi(k)}}{\sqrt{i_{\pi(k)}}} \geq 0\right) \end{aligned}$$

$$= \frac{1}{2^k k!} \sum_{r=0}^{n-k} \sum_{\substack{i_1, \dots, i_k \in \mathbb{N} \\ i_1 + \dots + i_k = n-r}} \frac{\binom{2r}{r}}{i_1 i_2 \dots i_k 2^{2r}}.$$

Using the representation (2.4) for the Stirling numbers of the first kind and the formula (2.9), we obtain

$$\begin{aligned} v_k(C^B) &= \sum_{r=0}^{n-k} \frac{1}{2^k k!} \frac{\binom{2r}{r}}{2^{2r}} \begin{bmatrix} n-r \\ k \end{bmatrix} \frac{k!}{(n-r)!} \\ &= \sum_{r=0}^{n-k} 2^{-k-2r} \binom{2r}{r} \begin{bmatrix} n-r \\ k \end{bmatrix} \frac{1}{(n-r)!} = \frac{B[n, k]}{2^n n!}. \end{aligned}$$

The case $k = 0$ is easy since the only 0-face is the origin $\{0\}$, and thus

$$\begin{aligned} v_0(C^B) &= \alpha(\{0\})\alpha(N(\{0\}, C^B)) = \alpha((C^B)^\circ) = \alpha(B_n) \\ &= \binom{2n}{n} \frac{1}{2^{2n}} = \frac{(2n-1)!!}{2^n n!} = \frac{B[n, 0]}{2^n n!}, \end{aligned}$$

where we applied the formula for $\alpha(B_n)$ stated in Lemma 3.4. □

3.3 Type D_n

At last, we can also compute the conic intrinsic volumes of the Weyl chambers of type D_n . These formulas follow from the intrinsic volumes of the type B_n chambers and additivity of the intrinsic volumes.

Theorem 3.5. *For $k = 0, 1, \dots, n$, the k -th conic intrinsic volume of the Weyl chamber of type D_n is given by*

$$v_k(C(D_n)) = \frac{D[n, k]}{2^{n-1} n!},$$

where the numbers $D[n, k]$ are given by (2.12).

Proof. Consider the Weyl chamber

$$\begin{aligned} C(D_n) &= \{\beta \in \mathbb{R}^n : \beta_1 \geq \dots \geq \beta_{n-1} \geq |\beta_n|\} \\ &= (C(D_n) \cap \{\beta_n \geq 0\}) \cup (C(D_n) \cap \{\beta_n \leq 0\}) \\ &= \{\beta \in \mathbb{R}^n : \beta_1 \geq \dots \geq \beta_{n-1} \geq \beta_n \geq 0\} \\ &\quad \cup \{\beta \in \mathbb{R}^n : \beta_1 \geq \dots \geq \beta_{n-1} \geq -\beta_n \geq 0\}, \end{aligned}$$

where the first set on the right-hand side is the Weyl chamber of type B_n , while the second set is isometric to it. Their intersection

$$\begin{aligned} &\{\beta \in \mathbb{R}^n : \beta_1 \geq \dots \geq \beta_{n-1} \geq \beta_n \geq 0\} \cap \{\beta \in \mathbb{R}^n : \beta_1 \geq \dots \geq \beta_{n-1} \geq -\beta_n \geq 0\} \\ &= \{\beta_1 \geq \dots \geq \beta_{n-1} \geq \beta_n = 0\} \end{aligned}$$

is a Weyl chamber of type B_{n-1} if we identify \mathbb{R}^{n-1} and $\mathbb{R}^{n-1} \times \{0\}$. The conic intrinsic volumes v_k are additive functionals, see [25, Theorem 6.5.2], and thus, we obtain

$$\begin{aligned} v_k(\mathcal{C}(D_n)) &= 2v_k(\{\beta \in \mathbb{R}^n : \beta_1 \geq \dots \geq \beta_{n-1} \geq \beta_n \geq 0\}) \\ &\quad - v_k(\{\beta_1 \geq \dots \geq \beta_{n-1} \geq \beta_n = 0\}) \\ &= 2 \frac{B[n, k]}{2^n n!} - \frac{B[n-1, k]}{2^{n-1} (n-1)!} = \frac{B[n, k] - nB[n-1, k]}{2^{n-1} n!} = \frac{D[n, k]}{2^{n-1} n!}. \end{aligned}$$

Here, we used the formula for the intrinsic volumes of the Weyl chambers of type B_n from Theorem 3.3, and in the last step the relation (2.13). \square

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References

- [1] Abramson, J., Pitman, J.: Concave majorants of random walks and related Poisson processes. *Combin. Probab. Comput.* **20**(5), 651–682 (2011). [MR2825583](#). <https://doi.org/10.1017/S0963548311000307>
- [2] Abramson, J., Pitman, J., Ross, N., Uribe Bravo, G.: Convex minorants of random walks and Lévy processes. *Electron. Commun. Probab.* **16**, 423–434 (2011). [MR2831081](#). <https://doi.org/10.1214/ECP.v16-1648>
- [3] Alsmeyer, G., Kabluchko, Z., Marynych, A., Vysotsky, V.: How long is the convex minorant of a one-dimensional random walk? *Electron. J. Probab.* **25**, 1–22 (2020). [MR4147518](#). <https://doi.org/10.1214/20-ejp497>
- [4] Amelunxen, D., Bürgisser, P.: Intrinsic volumes of symmetric cones and applications in convex programming. *Math. Program.* **149**(1-2, Ser. A), 105–130 (2015). [MR3300458](#). <https://doi.org/10.1007/s10107-013-0740-2>
- [5] Amelunxen, D., Bürgisser, P.: Probabilistic analysis of the Grassmann condition number. *Found. Comput. Math.* **15**(1), 3–51 (2015). [MR3303690](#). <https://doi.org/10.1007/s10208-013-9178-4>
- [6] Amelunxen, D., Lotz, M., McCoy, M.B., Tropp, J.A.: Living on the edge: phase transitions in convex programs with random data. *Information and Inference* **3**(3), 224–294 (2014). [MR3311453](#). <https://doi.org/10.1093/imaiai/iau005>
- [7] Amelunxen, D., Lotz, M.: Intrinsic volumes of polyhedral cones: A combinatorial perspective. *Discrete Comput. Geom.* **58**(2), 371–409 (2017). [MR3679941](#). <https://doi.org/10.1007/s00454-017-9904-9>
- [8] Bagno, E., Garber, D.: Signed partitions – A balls into urns approach. Preprint at [arXiv:1903.02877](https://arxiv.org/abs/1903.02877) (2019). [MR4408200](#)
- [9] Bagno, E., Biagioli, R., Garber, D.: Some identities involving second kind Stirling numbers of types B and D. *Elect. J. Combin.* **26**(3), 3–9 (2019). [MR3982318](#). <https://doi.org/10.37236/8703>

- [10] Bala, P.: A 3-parameter family of generalized Stirling numbers. Preprint at <https://oeis.org/A143395/a143395.pdf> (2015)
- [11] Bóna, M. (ed.): Handbook of Enumerative Combinatorics. Discrete Mathematics and its Applications (Boca Raton). CRC Press, Boca Raton, FL (2015). [MR3408702](#)
- [12] Dowling, T.A.: A class of geometric lattices based on finite groups. *J. Combinatorial Theory Ser. B* **14**, 61–86 (1973). [MR0307951](#). [https://doi.org/10.1016/s0095-8956\(73\)80007-3](https://doi.org/10.1016/s0095-8956(73)80007-3)
- [13] Drton, M., Klivans, C.J.: A geometric interpretation of the characteristic polynomial of reflection arrangements. *Proc. Amer. Math. Soc.* **138**(8), 2873–2887 (2010). [MR2644900](#). <https://doi.org/10.1090/S0002-9939-10-10369-4>
- [14] Gao, F.: The mean of a maximum likelihood estimator associated with the Brownian bridge. *Electron. Comm. Probab.* **8**, 1–5 (2003). [MR1961284](#). <https://doi.org/10.1214/ECP.v8-1064>
- [15] Gao, F., Vitale, R.A.: Intrinsic volumes of the Brownian motion body. *Discrete Comput. Geom.* **26**(1), 41–50 (2001). [MR1832728](#). <https://doi.org/10.1007/s00454-001-0023-1>
- [16] Henze, N.: Weitere Überraschungen im Zusammenhang mit dem Schnur-Orakel. *Stochastik in der Schule* **33**(3), 18–23 (2013)
- [17] Hug, D., Schneider, R.: Random conical tessellations. *Discrete Comput. Geom.* **56**(2), 395–426 (2016). [MR3530973](#). <https://doi.org/10.1007/s00454-016-9788-0>
- [18] Humphreys, J.E.: Reflection Groups and Coxeter Groups. Cambridge Studies in Advanced Mathematics, vol. 29. Cambridge University Press, Cambridge (1990). <https://doi.org/10.1017/CBO9780511623646>
- [19] Kabluchko, Z., Vysotsky, V., Zaporozhets, D.: Convex hulls of random walks: expected number of faces and face probabilities. *Adv. Math.* **320**, 595–629 (2017). [MR3709116](#). <https://doi.org/10.1016/j.aim.2017.09.002>
- [20] Kabluchko, Z., Vysotsky, V., Zaporozhets, D.: Convex hulls of random walks, hyperplane arrangements, and Weyl chambers. *Geom. Funct. Anal.* **27**(4), 880–918 (2017). [MR3678504](#). <https://doi.org/10.1007/s00039-017-0415-x>
- [21] Klivans, C.J., Swartz, E.: Projection volumes of hyperplane arrangements. *Discrete Comput. Geom.* **46**(3), 417–426 (2011). [MR2826960](#). <https://doi.org/10.1007/s00454-011-9363-7>
- [22] Lang, W.: On Sums of Powers of Arithmetic Progressions, and Generalized Stirling, Eulerian and Bernoulli numbers. Preprint at arXiv:[1707.04451](https://arxiv.org/abs/1707.04451) (2017)
- [23] Pitman, J.: Combinatorial Stochastic Processes. Lecture Notes in Mathematics. Springer (2006). [MR2245368](#)
- [24] Schneider, R.: Combinatorial identities for polyhedral cones. *St. Petersburg Math. J.* **29**(1), 209–221 (2018). [MR3660692](#). <https://doi.org/10.1090/spmj/1489>
- [25] Schneider, R., Weil, W.: Stochastic and Integral Geometry. Springer (2008). [MR2455326](#). <https://doi.org/10.1007/978-3-540-78859-1>
- [26] Shephard, G.C., Todd, J.A.: Finite unitary reflection groups. *Canad. J. Math.* **6**, 274–304 (1954). [MR0059914](#). <https://doi.org/10.4153/cjm-1954-028-3>
- [27] Sloane (editor), N.J.A.: The On-Line Encyclopedia of Integer Sequences. <https://oeis.org>
- [28] Sparre Andersen, E.: On the number of positive sums of random variables. *Scandinavian Actuarial Journal* **1949**(1), 27–36 (1949). [MR0032115](#). <https://doi.org/10.1080/03461238.1949.10419756>
- [29] Sparre Andersen, E.: On the fluctuations of sums of random variables. *Math. Scand.* **1**, 263–285 (1953). [MR0058893](#). <https://doi.org/10.7146/math.scand.a-10385>

- [30] Sparre Andersen, E.: On the fluctuations of sums of random variables II. *Math. Scand.* **2**, 195–223 (1954). [MR0068154](#)
- [31] Stanley, R.P.: An introduction to hyperplane arrangements. In: Miller, E., Reiner, V., Sturmfels, B. (eds.) *Geometric Combinatorics*. IAS/Park City Mathematics Series, vol. 13, pp. 389–496. AMS (2007). [MR2383131](#). <https://doi.org/10.1090/pcms/013/08>
- [32] Suter, R.: Two analogues of a classical sequence. *J. Integer Seq.* **3**(1), 1–18 (2000). [MR1750742](#)
- [33] Ziegler, G.M.: *Lectures on Polytopes*. Springer, New York (1995). [MR1311028](#). <https://doi.org/10.1007/978-1-4613-8431-1>