Modern Stochastics: Theory and Applications 9 (3) (2022) 339–355
https://doi.org/10.15559/22-VMSTA207

Notes on spherical bifractional Brownian motion

Mohamed El Omari

Chouaïb Doukkali University, Polydisciplinary Faculty of Sidi Bennour, B.P. 299, Jabrane Khalil Jabrane Street, 24000 El Jadida, Morocco

mopzer.med@gmail.com (M. El Omari)

Received: 30 September 2021, Revised: 3 March 2022, Accepted: 17 May 2022,
Published online: 7 June 2022

Abstract The existence of the bifractional Brownian motion $B_{H,K}$ indexed by a sphere when $K \in (-\infty, 1] \setminus \{0\}$ and $H \in (0, 1/2]$ is discussed, and the asymptotics of its excursion probability $\mathbb{P}\left\{\sup_{M\in\mathbb{S}} B_{H,K}(M) > x\right\}$ as $x \to \infty$ is studied.

Keywords Fractional brownian motion, bifractional brownian motion, excursion probability, sphere, asymptotics

2010 MSC 60G15, 60G70

1 Introduction

The frational Brownian motion (fBm) has gained a huge attention since the article [20], due to its intrinsic properties (self-similarity and stationarity of increments) appearing in many areas of applications. As an extension of the fBm among others (e.g., [5, 6] and references therein), the bifractional Brownian motion (bifBm) was firstly introduced in [10]. It is formally defined on $\mathbb{R}$ as a centered Gaussian process with covariance function

$$\mathcal{R}(t,s) = \frac{1}{2^K}\left((|t|^{2H} + |s|^{2H})^K - |t-s|^{2HK}\right), \quad \text{for all } t, s \in \mathbb{R}.$$

This process satisfies the quasi-helix property (in the sense of Kahane [14]) of index HK. A brief discussion about the motivations for introducing such a process can be found in [17], in which the authors examine also the existence of the bifBm for many cases. They provided a necessary condition for the existence of one sided bifBm

$B_{H,K}(t)$, $t \geq 0$. Some properties of the bifBm have been studied in [24, 30]. In particular, in [24] the authors show that $B_{H,K}$ behaves as a fBm with Hurst parameter HK. A stochastic calculus with respect to the process $B_{H,K}$ can be found in [29, 15].

Gaussian random fields indexed by Euclidean spaces, being another interesting extension of the fBm, are studied extensively. Fractional Brownian field is the well-known instance of them. Many applications, as texture simulation or geology, require a fractional Brownian motion indexed by a manifold. Many authors (e.g., [3, 4, 8, 9, 23]) use deformations of a field indexed by $\mathbb{R}^n$. The properties of self-similarity and stationarity of the increments are lost due to such deformations; instead, only properties of local self-similarity and local stationarity are available. To overcome this problem, Istas [11] proposed the construction of the fBm indexed by a manifold as centered Gaussian field with fractional power of the distance as variance. Unfortunately, the construction of the fBm on nonflat metric space $(\mathcal{M}, d)$ is not trivial. One often needs to prove the positive definiteness of covariance kernel, which depends on the distance d. By Schoenberg results [25], the fBm indexed by a metric space $(\mathcal{M}, d)$ exists if and only if $d^{2H}(\cdot,\cdot)$ is of negative type (as defined below). This condition may not hold for some metric spaces, as shown in [31] in the case of cylinders $\mathbb{S}^1 \times (0, \varepsilon)$, $\varepsilon > 0$. Istas [12] noticed that there exists a fractional index $\beta_{\mathcal{M}} \geq 0$ depending on $(\mathcal{M}, d)$ such that $d^{\beta}(\cdot,\cdot)$ is of negative type if and only if $\beta \leq \beta_{\mathcal{M}}$.

Recently, the study of random fields on spheres is attracting more and more attention due to vast applications in astronomy [21], spatial statistics [7, 28], geoscience [22, 13] and environmental sciences [27]. In [11], the spherical fBm is formally defined as centered Gaussian field B_H on a sphere $\mathbb{S}$ such that

$$B_H(O) = 0, \quad \text{almost surely,}$$
$$\mathbb{E}\,|B_H(M) - B_H(N)|^2 = d^{2H}(M, N), \quad \text{for all } M, N \in \mathbb{S},$$

where $d(\cdot,\cdot)$ stands for the geodesic distance on $\mathbb{S}$ and O is a fixed point on $\mathbb{S}$. In a recent work [1], Cheng and Liu considered the spherical fBm and studied the asymptotic of the excursion probability $\mathbb{P}\left\{\sup_{M\in E} B_H(M) > x\right\}$ as $x \to \infty$, when E is either the sphere itself $E = \mathbb{S}$ or a geodesic disc given by $D_a := \{M \in \mathbb{S} : d(M, O) \leq a\}$. For the importance of such excursions in both theoretical and statistical point of view, we refer to [2, 1].

The aim of this work is to show that the bifractional Browninan motion (bifBm) $B_{H,K}$ indexed by a sphere $\mathbb{S}$ exists, if and only if, $0 < H \leq 1/2$, $K \in (0, 1]$, and can be extended to θ-bifBm in order to include the range $K \in (-\infty, 0)$. Its excursion probability on a unit sphere $\mathbb{P}\left\{\sup_{M\in\mathbb{S}} B_{H,K}(M) > x\right\}$ as $x \to \infty$ is also examined when $K \in (-\infty, 1) \setminus \{0\}$. In what follows we work on probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The following notations are systematically used: $\sigma^2(M)$ and $Corr(M, N)$ denote the variance and the correlation functions of the random field $B_{H,K}$ (defined below), respectively. $\|\cdot\|$ stands for the Euclidean norm with dimension space specified by the context; while the following symbols $\mathcal{O}(\cdot)$, $o(\cdot)$ are the usual big-oh and little-oh, respectively, and the symbol $f(x) \sim g(x)$, as $x \to x^*$ is used to say that $\lim_{x\to x^*} f(x)/g(x) = 1$. We set $a \wedge b := \min(a, b)$, $\Gamma(x) := \int_0^\infty t^{x-1}e^{-t}dt$ and $Var(\xi)$ denotes the variance of the random variable ξ.

We define the bifractional Browninan motion $B_{H,K}$ on a sphere $\mathbb{S}$ (if it exists) as centered Gaussian random field with covariance

$$R(M,N) = \frac{1}{2^K}\left\{\left(d^{2H}(O,M) + d^{2H}(O,N)\right)^K - d^{2HK}(M,N)\right\}, \tag{1}$$

for all $M, N \in \mathbb{S}$. Here $d(\cdot,\cdot)$ denotes the geodesic distance on $\mathbb{S}$ and O is some fixed point on $\mathbb{S}$. It follows from (1) that $B_{H,K}(O) = 0$ a.s. Before we state our main results let us recall some preliminary results about functions of positive and negative type.

Definition 1. Let $\mathcal{M}$ be a set. A symmetric funtion $\phi : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ is said to be of positive type if for every $M_1, \ldots, M_n \in \mathcal{M}$ and every $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$,

$$\sum_{j,l=1}^{n} \lambda_j \lambda_l \phi\left(M_j, M_l\right) \geq 0.$$

The following assertions hold true (e.g., [25])

- Any finite linear combination of functions of positive type with nonnegative coefficients is again of positive type.
- **The Schur product theorem:** The product of two functions of positive type is again of positive type.
- A continuous function which is the limit of a sequence of functions of positive type is itself of positive type.

Functions of positive type are the covariances of random fields indexed by $\mathcal{M}$. In particular, a centred Gaussian random field indexed by $\mathcal{M}$ with covariance ϕ exists if and only if ϕ is of a positive type (see, for instance, [16]). Furthermore they are of great importance for "kernel method" in machine learning of nonlinear data (e.g., [26]). Functions of positive type are strongly related to functions of negative type (defined below).

Definition 2. Let $\mathcal{M}$ be a set. A symmetric funtion $\psi : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ is said to be (conditionally) of negative type if $\psi(M, M) = 0$ for all $M \in \mathcal{M}$, and for every $M_1, \ldots, M_n \in \mathcal{M}$ and every $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ such that $\sum_{j=1}^{n} \lambda_j = 0$ we have

$$\sum_{j,l=1}^{n} \lambda_j \lambda_l \phi\left(M_j, M_l\right) \leq 0.$$

Note that a function of negative type may not have an opposite of positive type, but the converse is true if the function vanishes on the diagonal. This is why the word "conditionally "is used to avoid any ambiguity. This is shown by the example of a finite metric space $\mathcal{M} = \{M_1, M_2, M_3\}$ with metric d, in which $d\left(M_j, M_l\right) = \rho > 0$ (fixed) for all $j \neq l$. Clearly, the function d is of negative type, but its opposite is not of positive type. The following lemma is an immediate result of the Schur product theorem.

Lemma 1. *Let ϕ and ψ be two symmetric functions on $\mathcal{M} \times \mathcal{M}$ of positive and negative type, respectively. Then $\phi \times \psi$ is symmetric and of negative type.*

2 Main Results

Theorem 1. *The spherical bifractional Browninan motion $B_{H,K}$ exists, if and only if,* $0 < H \le 1/2$ *and* $K \in (0, 1]$.

Proof. The case $K = 1$ reduces $B_{H,K}$ to the spherical fBm, which is treated in [11]. We shall treat the case $H \le 1/2$, $K < 1$. Let $M, N \in \mathbb{S}$. We need to show that $R(\cdot,\cdot)$ given in (1) is of positive type. Using the fact that for every $K \in (0, 1)$ and $\lambda \ge 0$,

$$\lambda^K = \frac{K}{\Gamma(1-K)} \int_0^\infty \frac{1-e^{-\lambda u}}{u^{K+1}} du, \tag{2}$$

we obtain

$$R(M, N) = \rho_K \int_0^\infty \frac{du}{u^{K+1}} \left(e^{-ud^{2H}(M,N)} - e^{-u(d^{2H}(O,M)+d^{2H}(O,N))}\right), \tag{3}$$

where $\rho_K = \frac{K}{2^K\Gamma(1-K)}$. Let $M_1, \ldots, M_n \in \mathbb{S}$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$. From (3) it follows that

$$\begin{aligned}
&\rho_K^{-1} \sum_{j,l=1}^n \lambda_j \lambda_l R\left(M_j, M_l\right) \\
&= \sum_{j,l=1}^n \lambda_j \lambda_l \int_0^\infty \frac{du}{u^{K+1}} \left(e^{-ud^{2H}(M_j,M_l)} - e^{-u(d^{2H}(O,M_j)+d^{2H}(O,M_l))}\right) \\
&= \int_0^\infty \frac{du}{u^{K+1}} \sum_{j,l=1}^n \lambda_j \lambda_l \left(1 + e^{-ud^{2H}(M_j,M_l)} - e^{-ud^{2H}(O,M_j)} - e^{-ud^{2H}(O,M_l)}\right) \\
&\quad - \int_0^\infty \frac{du}{u^{K+1}} \sum_{j,l=1}^n \lambda_j \lambda_l \left(1 - e^{-ud^{2H}(O,M_j)}\right)\left(1 - e^{-ud^{2H}(O,M_l)}\right) \\
&= \int_0^\infty \frac{du}{u^{K+1}} \sum_{j,l=1}^n \lambda_j \lambda_l \left(1 + e^{-ud^{2H}(M_j,M_l)} - e^{-ud^{2H}(O,M_j)} - e^{-ud^{2H}(O,M_l)}\right) \\
&\quad - \int_0^\infty \frac{du}{u^{K+1}} \left|\sum_{j=1}^n \lambda_j \left(1 - e^{-ud^{2H}(O,M_j)}\right)\right|^2 .
\end{aligned} \tag{4}$$

Let $\xi(u) = \sum_{j=1}^n \lambda_j \left(1 - e^{-i\sqrt{2u}B_H(M_j)}\right)$, where $i = \sqrt{-1}$ and B_H is the spherical fBm vanishing at O (which exists because $H \le 1/2$ (see [11, Theorem 3.1])). We have

$$\begin{aligned}
\mathbb{E}\left|\xi(u)\right|^2 &= \sum_{j,l=1}^n \lambda_j \lambda_l \mathbb{E}\left\{\left(1 - e^{-i\sqrt{2u}B_H(M_j)}\right)\overline{\left(1 - e^{-i\sqrt{2u}B_H(M_l)}\right)}\right\} \\
&= \sum_{j,l=1}^n \lambda_j \lambda_l \left\{1 - \mathbb{E}\left(e^{-i\sqrt{2u}B_H(M_j)}\right) - \mathbb{E}\left(e^{i\sqrt{2u}B_H(M_l)}\right)\right.
\end{aligned}$$

$$+\,\mathbb{E}\left(e^{i\sqrt{2u}(B_H(M_l)-B_H(M_j))}\right)\Big\}$$
$$=\sum_{j,l=1}^{n}\lambda_j\lambda_l\left\{1+e^{-ud^{2H}(M_j,M_l)}-e^{-ud^{2H}(O,M_j)}-e^{-ud^{2H}(O,M_l)}\right\}. \quad (5)$$

On the other hand,

$$|\mathbb{E}(\xi(u))|^2=\left|\sum_{j=1}^{n}\lambda_j\mathbb{E}\left(1-e^{-i\sqrt{2u}B_H(M_j)}\right)\right|^2$$
$$=\left|\sum_{j=1}^{n}\lambda_j\left(1-e^{-ud^{2H}(O,M_j)}\right)\right|^2. \quad (6)$$

Combining (5)–(6) with (4) yields

$$\sum_{j,l=1}^{n}\lambda_j\lambda_l R\left(M_j,M_l\right)=\rho_K\int_0^\infty\frac{du}{u^{K+1}}\left(\mathbb{E}\,|\xi(u)|^2-|\mathbb{E}(\xi(u))|^2\right)$$
$$=\rho_K\int_0^\infty\frac{Var\,(\xi(u))}{u^{K+1}}du\ \geq\ 0,$$

and we conclude that $B_{H,K}$ exists.

Now let $H>1/2$. We shall prove that $R(\cdot,\cdot)$ is not of positive type. Set $2H=1+2H'$ with $H'<1/2$. After a change of variables $v=ud(M,N)$, formula (3) can be rewritten as

$$R(M,N)=\rho_K\int_0^\infty\frac{dv}{v^{K+1}}\left(d^K(M,N)\,e^{-vd^{2H'}(M,N)}-e^{-v(d^{2H}(O,M)+d^{2H}(O,N))}\right)$$
$$=\rho_K\left\{\int_0^\infty\frac{dv}{v^{K+1}}\Phi_1(M,N)+\int_0^\infty\frac{dv}{v^{K+1}}\left(\Phi_2(M,N)+\Phi_3(M,N)\right)\right\},$$

where

$$\Phi_1(M,N)=d^K(M,N)\left(e^{-vd^{2H'}(M,N)}-e^{-v(d^{2H'}(O,M)+d^{2H'}(O,N))}\right),$$
$$\Phi_2(M,N)=d^K(M,N)\,e^{-v(d^{2H'}(O,M)+d^{2H'}(O,N))},$$
$$\Phi_3(M,N)=-e^{-v(d^{2H}(O,M)+d^{2H}(O,N))}.$$

It is easy to see that Φ_j, $j=1,2$, are symmetric and of negative type by Lemma 1 and the condition $H'<1/2$. Note d^K is of negative type since $\beta_{\mathbb{S}}=1$ and $K\leq 1$ (see [12, Proposition 2.6]). The function Φ_3 is not of negative type because it does not vanish on the diagonal. However, we have $\sum_{j,l}\lambda_j\lambda_l\Phi_3(M_j,M_l)\leq 0$, for all $M_1,\ldots,M_n\in\mathbb{S}$ and $\lambda_1,\ldots,\lambda_n\in\mathbb{R}$. As a result, $\sum_{j,l=1}^{n}\lambda_j\lambda_l R\left(M_j,M_l\right)\leq 0$, for all $M_1,\ldots,M_n\in\mathbb{S}$ and $\lambda_1,\ldots,\lambda_n\in\mathbb{R}$ such that $\sum_{j=1}^{n}\lambda_j=0$. Suppose that $R(\cdot,\cdot)$ is of positive type, then the last inequality becomes equality. By choosing $\lambda_1=-\lambda_2=1$ and $M_1,M_2\in\mathbb{S}$ so that $d(O,M)=d(O,N)=d(M,N)/2$ we obtain a contradiction. $\square$

Theorem 2. *Let $\theta > 0$. The symmetric function*

$$R_\theta(M,N) = 2^{-K} \left| \left(\theta + d^{2H}(O,M) + d^{2H}(O,N)\right)^K - \left(\theta + d^{2H}(M,N)\right)^K \right|, \tag{7}$$

for all $M, N \in \mathbb{S}$, defines a covariance function if $K \in (-\infty, 1] \setminus \{0\}$ and $H \in (0, 1/2]$.

Remark. Our motivation for introducing $\theta > 0$ in the covariance structures in Theorem 2 and propositions below is threefold. First, we try to give covariance functions more general as in [19, Theorem 1], but we consider negative exponents (as parameters $K < 0$ in Theorem 2 or functions $\nu_j(M) < 0$ in propositions below). Second, if negative exponents are considered, one has to face the problem that the covariance may explode. This is what makes the condition $\theta > 0$ very crucial and why it cannot be dropped. Third, the necessary condition for the existence of bifBm on $\mathbb{R}_+$ (see [17]) to exist is trivial in our case $K < 0$.

Definition 3. A process indexed by a sphere $\mathbb{S}$ is said to be spherical θ-bifBm with parameters $(\theta, H, K,)$ if it is centered Gaussian process with covariance function given by (7).

Proof of Theorem 2. First, note that the necessary condition for the bifractional Brownian motion on $\mathbb{R}_+$ to exist is that $K \le 2$ and $HK \le 1$ (see [17]). This condition is trivial if $K < 0$ and $H \ge 0$. Observe also that for all $\beta, \lambda > 0$

$$\lambda^{-\beta} = \frac{1}{\Gamma(\beta)} \int_0^\infty u^{\beta-1} e^{-\lambda u} du. \tag{8}$$

The case $\theta = 0$, $K \in (0, 1]$ is already treated in Theorem 1, and incorporating the parameter $\theta > 0$ does not affect the proof. Let $K < 0 < \theta$, $H \in (0, 1/2]$ and set $\beta = -K > 0$. By using (8) we have, for all $M, N \in \mathbb{S}$,

$$\begin{aligned}
R_\theta(M,N) &= 2^\beta \left[\left(\theta + d^{2H}(M,N)\right)^{-\beta} - \left(\theta + d^{2H}(O,M) + d^{2H}(O,N)\right)^{-\beta} \right] \\
&= \frac{2^\beta}{\Gamma(\beta)} \int_0^\infty u^{\beta-1} \left[e^{-u\left(\theta + d^{2H}(M,N)\right)} - e^{-u\left(\theta + d^{2H}(O,M) + d^{2H}(O,N)\right)} \right] du \\
&= \frac{2^\beta}{\Gamma(\beta)} \int_0^\infty u^{\beta-1} e^{-\theta u} \left[e^{-u d^{2H}(M,N)} - e^{-u\left(d^{2H}(O,M) + d^{2H}(O,N)\right)} \right] du.
\end{aligned}$$

Now let $M_1, \ldots, M_n \in \mathbb{S}$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$. We have

$$\begin{aligned}
&\sum_{j,l=1}^n \lambda_j \lambda_l R_\theta\left(M_j, M_l\right) \\
&\quad = \frac{2^\beta}{\Gamma(\beta)} \int_0^\infty u^{\beta-1} e^{-\theta u} \sum_{j,l=1}^n \lambda_j \lambda_l \left[e^{-u d^{2H}(M_j, M_l)} - e^{-u\left(d^{2H}(O,M_j) + d^{2H}(O,M_l)\right)} \right] du \\
&\quad = \frac{2^\beta}{\Gamma(\beta)} \int_0^\infty u^{\beta-1} e^{-\theta u} Var\left(\xi(u)\right) du \ge 0,
\end{aligned} \tag{9}$$

where $\xi(u) = \sum_{j=1}^{n} \lambda_j \left(1 - e^{-i\sqrt{2u}B_H(M_j)}\right)$ with $i = \sqrt{-1}$. For the equality (9), see the proof of Theorem 1. □

Based on a scalar variogram $\gamma(M, N)$ defined on an index set $\mathbb{D}$ (in general), we can extend Theorem 2 to include multivariate multifractional Brownian motions. The following propositions can be established by the procedures used in [19] and the fact (8). Thus, the proofs are omitted.

Proposition 1. *Let $\theta > 0$. If $\gamma(M, N)$, $M, N \in \mathbb{D}$, is a scalar variogram, and $\nu_j(M), \ldots, \nu_p(M)$ are negative functions with $\nu_j(M) < 0$, $j = 1, \ldots, p$, for all $M \in \mathbb{D}$, then the symmetric functions*

$$\begin{aligned} R_\theta^{j,l}(M,N) = \Gamma\left(\left|\nu_j(M)\right| + |\nu_l(M)|\right) \Big\{&\theta^{\nu_j(M)+\nu_l(N)} + \left[\theta + \gamma(M,N)\right]^{\nu_j(M)+\nu_l(N)} \\ &- \left[\theta + \gamma(M,O)\right]^{\nu_j(M)+\nu_l(N)} - \left[\theta + \gamma(N,O)\right]^{\nu_j(M)+\nu_l(N)}\Big\}, \\ &M, N \in \mathbb{D}, \ j, l = 1, \ldots, p, \end{aligned}$$

define a covariance matrix function.

Proposition 2. *Let $\theta > 0$. If $\gamma(M, N)$, $M, N \in \mathbb{D}$, is a scalar variogram, and $\nu_j(M), \ldots, \nu_p(M)$ are negative functions with $\nu_j(M) < 0$, $j = 1, \ldots, p$, for all $M \in \mathbb{D}$, then the symmetric functions*

$$\begin{aligned} R_\theta^{j,l}(M,N) = \Gamma\left(\left|\nu_j(M)\right| + |\nu_l(M)|\right) \Big\{&\left[\theta + \gamma(M,N)\right]^{\nu_j(M)+\nu_l(N)} \\ &- \left[\theta + \gamma(M,O) + \gamma(N,O)\right]^{\nu_j(M)+\nu_l(N)}\Big\}, \\ &M, N \in \mathbb{D}, \ j, l = 1, \ldots, p, \end{aligned}$$

define a covariance matrix function.

Proposition 3. *Let $\theta > 0$. If $\psi_1(M), \ldots, \psi_p(M)$, $M \in \mathbb{D}$, are nonnegative functions, and $\nu_j(M), \ldots, \nu_p(M)$ are negative functions with $\nu_j(M) < 0$, $j = 1, \ldots, p$, for all $M \in \mathbb{D}$, then the symmetric functions*

$$\begin{aligned} R_\theta^{j,l}(M,N) = \Gamma\left(\left|\nu_j(M)\right| + |\nu_l(M)|\right) \Big\{&\theta^{\nu_j(M)+\nu_l(N)} \\ &+ \left[\theta + \psi_j(M) + \psi_l(N)\right]^{\nu_j(M)+\nu_l(N)} \\ &- \left[\theta + \psi_j(M)\right]^{\nu_j(M)+\nu_l(N)} - \left[\theta + \psi_l(N)\right]^{\nu_j(M)+\nu_l(N)}\Big\}, \\ &M, N \in \mathbb{D}, \ j, l = 1, \ldots, p, \end{aligned}$$

define a covariance matrix function.

For the excursion probability problem, we follow Cheng and Liu [1]. The idea is to consider $B_{H,K}$ defined on $\mathbb{S}$ as Gaussian random field on the Euclidean space by using the spherical coordinate transformation. In such way, the local behaviour of standard deviation and correlation functions of $B_{H,K}$ can be studied under spherical coordinates. As a result, relevant existing results in Euclidean space can be applied to derive the desired asymptotics of the excursion probabilities. Without loss of generality we consider the unit sphere $\mathbb{S} = \mathbb{S}^n \subset \mathbb{R}^{n+1}$ with geodesic distance

$$d(M, N) = \arccos\left(\sum_{j=1}^{n+1} x_j y_j\right), \tag{10}$$

where $M = (x_1, \ldots, x_{n+1})$ and $N = (y_1, \ldots, y_{n+1})$ are two points on $\mathbb{S}^n$ identified by their Cartesian coordinates. If the point $M \in \mathbb{S}^n$ is identified by spherical coordinates, it will be denoted as $\widetilde{M} = (\theta_1, \ldots, \theta_n) \in [0, \pi]^{n-1} \times [0, 2\pi) = \Theta$, and one has

$$\begin{aligned}
x_1 &= \cos(\theta_1),\\
x_2 &= \sin(\theta_1)\cos(\theta_2),\\
x_3 &= \sin(\theta_1)\sin(\theta_2)\cos(\theta_3),\\
&\vdots\\
x_n &= \sin(\theta_1)\sin(\theta_2)\cdots\sin(\theta_{n-1})\cos(\theta_n),\\
x_{n+1} &= \sin(\theta_1)\sin(\theta_2)\cdots\sin(\theta_{n-1})\sin(\theta_n).
\end{aligned}$$

Let $O = (0, \ldots, 0, 1, 0) \in \mathbb{S}^n$. Using Cartesian coordinates we have $d(O, M) = \arccos(x_n)$ and the standard deviation of $B_{H,K}(M)$ is $\sigma(M) = \arccos^{HK}(x_n)$. Clearly, $\sigma(M)$ attains its maximum π^{HK} at $M^* = (0, \ldots, 0, -1, 0)$ (or equivalently $\widetilde{M^*} = (\pi/2, \ldots, \pi/2, \pi)$ which is an interior point in Θ). Set

$$\Psi(x) := (2\pi)^{-1/2} \int_x^\infty e^{-u^2/2} du. \tag{11}$$

We have the following result.

Theorem 3. *If $0 < H \leq 1/2$ and $K \in (0, 1)$, then as $x \to \infty$ we have*

$$\mathbb{P}\left\{\sup_{M \in \mathbb{S}^n} B_{H,K}(M) > x\right\} \sim \mathcal{C}_{n,H,K} \Psi\left(\pi^{-HK} x\right) x^{\frac{(1-2HK)n}{HK}}, \tag{12}$$

where $\mathcal{C}_{n,H,K} = \pi^{(2HK-1)n} \mathcal{H}^n_{2HK} \int_{\mathbb{R}^n} e^{-2^{1/(2H)} HK \|u\|} du$ and $\mathcal{H}^n_{2HK}$ is the Pickands constant (see [1]).

In the case $K = 1$ the process $B_{H,K}$ reduces to the spherical fBm. The asymptotics of its excursion probability is already examined in [1, Theorem 3.4]. Before we prove Theorem 3, we state a technical lemma for which the proof can be found in [2].

Lemma 2. *Let $M, N \in \mathbb{S}^n$ be two points with spherical coordinates $\widetilde{M} = (\theta_1, \ldots, \theta_n)$ and $\widetilde{N} = (\varphi_1, \ldots, \varphi_n)$, respectively. Let N be fixed. Then as $d(M, N) \to 0$,*

$$d^2(M, N) \sim (\theta_1 - \varphi_1)^2 + (\theta_2 - \varphi_2)^2 \sin^2(\theta_1) + \cdots + (\theta_n - \varphi_n)^2 \prod_{j=1}^{n-1} \sin^2(\theta_j).$$

Proof of Theorem 3. We consider the process $B_{H,K}$ as Gaussian random field defined on the compact set $\overline{\Theta} \subset \mathbb{R}^n$. The variance function $\sigma^2(\widetilde{M})$ attains its maximum at interior point $\widetilde{M^*} = (\pi/2, \ldots, \pi/2, \pi)$. The proof is based on Lemma 3.3 in [1]. Thus, we shall verify the conditions

$$\sigma(\widetilde{M}) = \pi^{HK} - HK\pi^{HK-1} \left\| \widetilde{M} - \widetilde{M^*} \right\| (1 + o(1)), \text{ as } \widetilde{M} \to \widetilde{M^*}, \tag{13}$$

$$1 - Corr\big(\widetilde{M}, \widetilde{N}\big) \sim \frac{\left\|\widetilde{M} - \widetilde{N}\right\|^{2HK}}{2^K \pi^{2HK}}, \text{ as } \widetilde{M}, \widetilde{N} \to \widetilde{M^*}, \tag{14}$$

$$\mathbb{E}\left|B_{H,K}(\widetilde{M}) - B_{H,K}(\widetilde{N})\right|^2 \le C \left\|\widetilde{M} - \widetilde{N}\right\|^{2HK}, \text{ for all } \widetilde{M}, \widetilde{N} \in \overline{\Theta}, \tag{15}$$

where C is some nonnegative constant. Since $\sigma(\widetilde{M}) = \sigma(M) = \arccos^{HK}(x_n)$, the condition (13) can be established in a similar fashion as it is done in [1, Lemma 3.2] (just replace β by HK). Let $\widetilde{M}, \widetilde{N} \in \overline{\Theta}$ and set $\eta_K = K/(\Gamma(1-K))$. We have

$$\begin{aligned}
1 - Corr\big(\widetilde{M}, \widetilde{N}\big) &= 1 - Corr(M, N) \\
&= 1 - \frac{\left(d^{2H}(O, M) + d^{2H}(O, N)\right)^K - d^{2HK}(M, N)}{2^K d^{HK}(O, M) d^{HK}(O, N)} \\
&= \frac{d^{2HK}(M, N) + \mathcal{A}_{H,K}(M, N)}{2^K d^{HK}(O, M) d^{HK}(O, N)},
\end{aligned} \tag{16}$$

where

$$\begin{aligned}
\mathcal{A}_{H,K}(M, N) &= \left(2d^H(O, M) d^H(O, N)\right)^K - \left(d^{2H}(O, M) + d^{2H}(O, N)\right)^K \\
&= \eta_K \int_0^\infty \frac{du}{u^{K+1}} \left(e^{-u(d^{2H}(O,M)+d^{2H}(O,N))} - e^{-u(2d^H(O,M)d^H(O,N))}\right).
\end{aligned}$$

Let $\varepsilon > 0$. Using the fact that $\left|e^{-y} - e^{-x}\right| \le |y - x|$, for all $x, y \in \mathbb{R}_+$, we obtain

$$\begin{aligned}
&\left|\mathcal{A}_{H,K}(M, N)\right| \\
&\le \eta_K \int_0^\infty \frac{e^{-\varepsilon u} du}{u^{K+1}} \left|e^{-u(d^{2H}(O,M)+d^{2H}(O,N)-\varepsilon)} - e^{-u(2d^H(O,M)d^H(O,N)-\varepsilon)}\right| \\
&\le \eta_K \int_0^\infty \frac{e^{-\varepsilon u} du}{u^{K+1}} \cdot u \left|d^{2H}(O, M) + d^{2H}(O, N) - 2d^H(O, M) d^H(O, N)\right| \\
&\le K\varepsilon^{K-1} \left|d^H(O, M) - d^H(O, N)\right|^2 \\
&\le K\varepsilon^{K-1} \frac{H^2}{\Gamma(1-H)^2} \left[\int_0^\infty \frac{du}{u^{H+1}} \left(e^{-ud(O,N)} - e^{-ud(O,M)}\right)\right]^2 \\
&\le KH^2 \varepsilon^{2H+K-3} \left|d(O, M) - d(O, N)\right|^2 \\
&\le KH^2 \varepsilon^{2H+K-3} d^2(M, N).
\end{aligned}$$

In the last inequality we used the fact that $|d(O, M) - d(O, N)| \le d(M, N)$, for all M, N. It follows that $\mathcal{A}_{H,K}(M, N) = o\left(d^{2HK}(M, N)\right)$ (because $2HK \le K < 1$), and (16) becomes

$$1 - Corr\big(\widetilde{M}, \widetilde{N}\big) = \frac{d^{2HK}(M,N)}{2^K d^{HK}(O,M) d^{HK}(O,N)} (1 + o(1))$$
$$\sim \frac{d^{2HK}(M,N)}{2^K \pi^{2HK}}, \quad \text{as } M, N \to M^*. \tag{17}$$

Let $\widetilde{M} = (\theta_1, \ldots, \theta_n)$ and $\widetilde{N} = (\varphi_1, \ldots, \varphi_n)$. First, observe that $\widetilde{M}, \widetilde{N} \to \widetilde{M^*}$ implies $d(M,N) \to 0$. By Lemma 2 we have $d(M,N) \sim \|\widetilde{M} - \widetilde{N}\|$ as $\widetilde{M}, \widetilde{N} \to \widetilde{M^*}$. Thus substituting $d(M,N)$ by $\|\widetilde{M} - \widetilde{N}\|$ in (17) completes the proof of (14). For the condition (15), we use the concavity of the function $x \mapsto x^K$ to get

$$\mathcal{I} := d^{2HK}(O,M) + d^{2HK}(O,N) - 2^{1-K}\left(d^{2H}(O,M) + d^{2H}(O,N)\right)^K \leq 0$$

and

$$\begin{aligned}
&\mathbb{E}\left|B_{H,K}(\widetilde{M}) - B_{H,K}(\widetilde{N})\right|^2 \\
&\quad = \mathbb{E}\left|B_{H,K}(M) - B_{H,K}(N)\right|^2 \\
&\quad = \sigma^2(M) + \sigma^2(N) - 2R(M,N) \\
&\quad = d^{2HK}(O,M) + d^{2HK}(O,N) \\
&\qquad - 2^{1-K}\left[\left(d^{2H}(O,M) + d^{2H}(O,N)\right)^K - d^{2HK}(M,N)\right] \\
&\quad = 2^{1-K} d^{2HK}(M,N) + \mathcal{I} \leq 2^{1-K} d^{2HK}(M,N).
\end{aligned}$$

To complete the proof we must show that there is some $\mathcal{C} > 0$ such that

$$d(M,N) \leq \mathcal{C}\left\|\widetilde{M} - \widetilde{N}\right\|. \tag{18}$$

Set $M = (x_1, \ldots, x_{n+1})$ and $N = (y_1, \ldots, y_{n+1})$. Considering the Cartesian coordinates x_j as functions of (spherical coordinates) θ_k, $k \leq j \wedge n$, and using the mean value theorem one has $\left|x_j - y_j\right| \leq c_j \left\|\widetilde{M} - \widetilde{N}\right\|$, for each j, where c_j are some non-negative constants depending only on n. We conclude that there exists $C_1 > 0$ such that

$$\|M - N\| \leq C_1 \left\|\widetilde{M} - \widetilde{N}\right\|. \tag{19}$$

Set $X = d(M,N)/2$ and observe that $X \leq \pi/2$. By using the identity (e.g., [18])

$$\|M - N\| = 2\sin\left(\frac{d(M,N)}{2}\right), \quad \text{for all } M, N \in S^n,$$

and the fact that $0 \leq x\cos(x) \leq \sin(x)$, for all $x \in [0, \pi/2]$, we obtain

$$\begin{aligned}
\frac{d^2(M,N)}{4} &= X^2\cos^2(X) + X^2\sin^2(X) \\
&\leq \left(1 + X^2\right)\sin^2(X) \leq \frac{1}{4}\left(1 + \frac{\pi^2}{4}\right)\|M - N\|^2.
\end{aligned}$$

Hence, $d(M,N) \le \sqrt{1+\pi^2/4}\,\|M-N\|$. Combining this result with (19) yields $d(M,N) \le \mathcal{C}\,\|\widetilde{M}-\widetilde{N}\|$, where $\mathcal{C} = C_1\sqrt{1+\pi^2/4}$. Since we have

$$\mathbb{P}\left\{\sup_{M\in\mathbb{S}^n} B_{H,K}(M) > x\right\} = \mathbb{P}\left\{\sup_{\widetilde{M}\in\overline{\Theta}} \frac{B_{H,K}(\widetilde{M})}{\pi^{HK}} > \frac{x}{\pi^{HK}}\right\},$$

we apply [1, Lemma 3.3] to the process $X = B_{H,K}/\pi^{HK}$ with $\eta = 1$, $\alpha = 2HK$, $A = HK\pi^{-1}I$ and $C = 2^{-1/(2H)}\pi^{-1}I$, where I is the identity matrix, and get the desired result. □

The following result provides the asymptotics of the excursion probability for the spherical θ-bifBm $B^{\theta}_{H,K}$ given in terms of Definition 3. It is an immediate result of Lemma 3 below and [1, Lemma 3.3]. Let $\sigma_\theta(M)$ be the standard deviation of the spherical θ-bifBm with $M^* = \arg\max_{M\in\mathbb{S}} \sigma_\theta(M)$. Set

$$\mathcal{D}^{\theta}_{H,K} = 2H\pi^{2H-1}\left(\frac{(\theta+2\pi^{2H})^K-\theta^K}{K}\right)^{1/2H-1}\left(\frac{\theta+2\pi^{2H}}{\theta^{1/2H}}\right)^{K-1}.$$

Theorem 4. *Consider the spherical θ-bifBm $B^{\theta}_{H,K}$ with $K < 0 < \theta$ and $H \in (0,1/2)$. Then as $x \to \infty$ we have*

$$\mathbb{P}\left\{\sup_{M\in\mathbb{S}^n} B^{\theta}_{H,K}(M) > x\right\} \sim \mathcal{B}^{\theta}_{n,H,K}\Psi\left(x/\sigma_\theta(M^*)\right)x^{\frac{(1-2H)n}{H}},$$

where $\mathcal{B}^{\theta}_{n,H,K} = \mathcal{H}^n_{2H}\sigma_\theta(M^*)^{\frac{(2H-1)n}{H}}\int_{\mathbb{R}^n} e^{-\mathcal{D}^{\theta}_{H,K}\|u\|}du$, $\Psi(x)$ is given in (11) and $\mathcal{H}^n_{2HK}$ is the Pickands constant (see [1]).

Lemma 3. *Consider the spherical θ-bifBm $B^{\theta}_{H,K}$ with $K < 0 < \theta$ and $H \in (0,1/2)$. Then there exist $C^{(l)} > 0$, $l = 1,2,3$, such that*

$$\sigma_\theta(\widetilde{M}) = 2^{-K/2}\left(\theta^K - \left(\theta+2\pi^{2H}\right)^K\right)^{1/2} - C^{(1)}\left\|\widetilde{M}-\widetilde{M^*}\right\|(1+o(1)),\ \textit{as}\ \widetilde{M}\to\widetilde{M^*}, \tag{20}$$

$$\mathbb{E}\left|B^{\theta}_{H,K}(\widetilde{M}) - B^{\theta}_{H,K}(\widetilde{N})\right|^2 \le C^{(2)}\left\|\widetilde{M}-\widetilde{N}\right\|^{2H},\quad \textit{for all}\ \widetilde{M},\widetilde{N}\in\overline{\Theta}, \tag{21}$$

$$1 - Corr_\theta\left(\widetilde{M},\widetilde{N}\right) = C^{(3)}\left\|\widetilde{M}-\widetilde{N}\right\|^{2H}(1+o(1)),\ \textit{as}\ \widetilde{M},\widetilde{N}\to\widetilde{M^*}. \tag{22}$$

Here $\widetilde{M}$ denotes the spherical coordinate of the point $M \in \mathbb{S}^n$ and $Corr_\theta(\cdot,\cdot)$ stands for the correlation function of the process $B^{\theta}_{H,K}$.

Before we give the proof of our statements (20)–(22), we state the following results.

- **(a)** The point M^* maximizes also the function $M \mapsto d^{2H}(M,O)$, and one has $d(M^*,O) = \pi$; $\sigma_\theta(M)^2 = 2^{-K}\left(\theta^K - (\theta+2d^{2H}(M,O))^K\right)$.
- **(b)** $d^{2H}(M,O) - d^{2H}(M^*,O) = -2H\pi^{2H-1}\left\|\widetilde{M}-\widetilde{M^*}\right\|(1+o(1))$, $\widetilde{M}\to\widetilde{M^*}$.

(c) $d^{2H}(M,O) - d^{2H}(N,O) = -2Hd^{2H}(N,O)^{2H-1}(d(N,O)-d(M,O))(1+o(1))$, as $M \to N$, provided that N is fixed and $N \neq O$. In particular, $d^{2H}(M,O) - d^{2H}(N,O) = \|\widetilde{M} - \widetilde{N}\| \mathcal{O}(1)$, as $\widetilde{M} \to \widetilde{N}$ (or equivalently $d(M,N) \to 0$).

The proof of **(b)** can be found in [1, Lemma 3.2] (just replace β by $2H$); while the statement **(a)** follows by direct computations and the fact that $\mathbb{S}^n$ is a unit sphere.

Let us establish the statement **(c)**. Let $H \in (0, 1/2)$ and $M, N \in \mathbb{S}^n$. By virtue of (2) we have

$$d^{2H}(N,O) - d^{2H}(M,O) = \frac{2H}{\Gamma(1-2H)} \int_0^\infty u^{-2H-1}\left(e^{-ud(M,O)} - e^{-ud(N,O)}\right) du.$$

By applying Taylor's theorem with the integral remainder to the function $x \mapsto e^{-x}$, the above equality becomes

$$\begin{aligned} &d^{2H}(N,O) - d^{2H}(M,O) \\ &= \frac{-2H}{\Gamma(1-2H)} \int_0^\infty u^{-2H}(d(M,O) - d(N,O))\, e^{-ud(N,O)} du \\ &\quad + \frac{2H}{\Gamma(1-2H)} \int_0^\infty u^{-2H-1} r_u(M,N) du \\ &= 2Hd(N,O)^{2H-1}(d(N,O) - d(M,O)) \\ &\quad + \frac{2H}{\Gamma(1-2H)} \int_0^\infty u^{-2H-1} r_u(M,N) du, \end{aligned}$$

where $r_u(M,N) = \int_{ud(N,O)}^{ud(M,O)} e^{-t}(ud(M,O) - t)\, dt$ and

$$\begin{aligned} |r_u(M,N)| &\le u\,|d(N,O) - d(M,O)| \left| e^{-ud(M,O)} - e^{-ud(N,O)} \right| \quad (\text{for all } u \in \mathbb{R}_+) \\ &= u\,|d(N,O) - d(M,O)|\, o(1), \quad \text{as } d(M,N) \to 0. \end{aligned}$$

We conclude that the first part of **(c)** holds true. Since

$$|d(N,O) - d(M,O)| \le d(M,N) \ \text{ for all } M, N,$$

the second part of **(c)** follows from the previous one and (18). Now we are ready to prove Lemma 3.

Proof of Lemma 3. Set $\Delta(M, M^*) := \sigma_\theta(M^*)^2 - \sigma_\theta(M)^2$ and let $K < 0 < \theta$, $H \in (0, 1/2)$. Once again, we apply Taylor's theorem with the integral remainder and **(a)** to get

$$\begin{aligned} &\Delta(M, M^*) \\ &= 2^{-K}\left[\left(\theta^K - (\theta + 2d(M^*,O)^{2H})^K\right) - \left(\theta^K - (\theta + 2d(M,O)^{2H})^K\right)\right] \\ &= 2^{-K}\left[\left(\theta + 2d(M,O)^{2H}\right)^K - \left(\theta + 2d(M^*,O)^{2H}\right)^K\right] \end{aligned}$$

$$
\begin{aligned}
&= \frac{2^{-K}}{\Gamma(-K)} \int_0^\infty u^{-K-1} e^{-\theta u} \left(e^{-2ud^{2H}(M,O)} - e^{-2ud^{2H}(M^*,O)} \right) du \text{ (by (8))} \\
&= -\frac{2^{1-K}}{\Gamma(-K)} \left(d^{2H}(M,O) - d^{2H}(M^*,O) \right) \int_0^\infty u^{-K} e^{-u(\theta + 2d^{2H}(M^*,O))} du \\
&\quad + \left(d^{2H}(M,O) - d^{2H}(M^*,O) \right) o(1) \\
&= K 2^{1-K} \left(\theta + 2d^{2H}(M^*,O) \right)^{K-1} \left(d^{2H}(M,O) - d^{2H}(M^*,O) \right) (1 + o(1)) \\
&= -2^{2-K} K H \pi^{2H-1} \left(\theta + 2\pi^{2H} \right)^{K-1} \left\| \widetilde{M} - \widetilde{M^*} \right\| (1 + o(1)), \text{ as } \widetilde{M} \to \widetilde{M^*}.
\end{aligned}
$$

In the last equality we used **(b)**. As a result we have

$$
\begin{aligned}
\sigma_\theta(\widetilde{M}) - \sigma_\theta(\widetilde{M^*}) = \sigma_\theta(M) - \sigma_\theta(M^*) &= \frac{-\Delta(M, M^*)}{\sigma_\theta(M) + \sigma_\theta(M^*)} \\
&= -C^{(1)} \left\| \widetilde{M} - \widetilde{M^*} \right\| (1 + o(1)), \text{ as } \widetilde{M} \to \widetilde{M^*},
\end{aligned}
$$

and (20) holds true with $C^{(1)} := -2^{1-K} K H \pi^{2H-1} \left(\theta + 2\pi^{2H} \right)^{K-1} / \sigma_\theta(M^*)$. Note that $K < 0$.

Let $K < 0 < \theta$ and $M, N \in \mathbb{S}^n$ with spherical coordinates $\widetilde{M}, \widetilde{N} \in \Theta$. By definition of the covariance function of $B^\theta_{H,K}$ given in (7) we have

$$
\begin{aligned}
&\mathbb{E} \left| B^\theta_{H,K}(\widetilde{M}) - B^\theta_{H,K}(\widetilde{N}) \right|^2 = \mathbb{E} \left| B^\theta_{H,K}(M) - B^\theta_{H,K}(N) \right|^2 \\
&\quad = \sigma_\theta(M)^2 + \sigma_\theta(N)^2 - 2R_\theta(M,N) \\
&\quad = \frac{2^{-K}}{\Gamma(-K)} \int_0^\infty du u^{-K-1} e^{-\theta u} \Big[2 - e^{-2ud^{2H}(M,O)} - e^{-2ud^{2H}(N,O)} \\
&\quad\quad -2e^{-ud^{2H}(M,N)} + 2e^{-u\left(d^{2H}(M,O) + d^{2H}(N,O)\right)} \Big] \\
&\quad = \frac{2^{1-K}}{\Gamma(-K)} \int_0^\infty u^{-K-1} e^{-\theta u} \left(1 - e^{-ud^{2H}(M,N)} \right) du \\
&\quad\quad - \frac{2^{-K}}{\Gamma(-K)} \int_0^\infty u^{-K-1} e^{-\theta u} \left| e^{-ud^{2H}(M,O)} - e^{-ud^{2H}(N,O)} \right|^2 du \\
&\quad \le \frac{2^{1-K}}{\Gamma(-K)} d^{2H}(M,N) \int_0^\infty u^{-K} e^{-\theta u} du = -K 2^{1-K} \theta^{K-1} d^{2H}(M,N) \\
&\quad \le -K 2^{1-K} \theta^{K-1} \mathcal{C} \left\| \widetilde{M} - \widetilde{N} \right\|^{2H},
\end{aligned}
$$

where $\mathcal{C}$ is nonnegative constant due to (18). Hence, the proof of (21) is complete. To establish (22) we make the following notations.

$$
\begin{aligned}
A_\alpha(M) &:= \theta^K - \left(\theta + 2d^{2H}(M,O) \right)^K, \ \ \alpha = (\theta, H, K), \\
B_\alpha(M,N) &:= \theta^K - \left(\theta + d^{2H}(M,N) \right)^K,
\end{aligned}
$$

$$C_\alpha(M,N) := 2\left(\theta + d^{2H}(M,O) + d^{2H}(N,O)\right)^K - \left(\theta + 2d^{2H}(M,O)\right)^K - \left(\theta + 2d^{2H}(N,O)\right)^K.$$

Straightforward computations yield

$$\begin{aligned} 1 - Corr_\theta(\widetilde{M},\widetilde{N}) &= 1 - Corr_\theta(M,N), \\ &= \frac{C_\alpha(M,N) + 2B_\alpha(M,N) - \left[A_\alpha(M)^{1/2} - A_\alpha(N)^{1/2}\right]^2}{2A_\alpha(M)^{1/2}A_\alpha(N)^{1/2}}. \end{aligned} \tag{23}$$

We shall evaluate all the terms appearing in (23). First, as $M, N \to M^*$, we have $A_\alpha(M)^{1/2}A_\alpha(N)^{1/2} \longrightarrow A_\alpha(M^*)$. Second, we apply Taylor's theorem with the integral remainder to get

$$\begin{aligned} B_\alpha(M,N) &= \theta^K - \left(\theta + d^{2H}(M,N)\right)^K \\ &= \frac{1}{\Gamma(-K)}\int_0^\infty u^{-K-1}e^{-\theta u}\left(1 - e^{-ud^{2H}(M,N)}\right)du \\ &= -K\theta^{K-1}d^{2H}(M,N)(1+o(1)) \\ &= -K\theta^{K-1}\left\|\widetilde{M} - \widetilde{N}\right\|^{2H}(1+o(1)). \end{aligned}$$

The last equality is justified by the fact that $d(M,N) \sim \left\|\widetilde{M} - \widetilde{N}\right\|$, as $\widetilde{M}, \widetilde{N} \to \widetilde{M^*}$, which in turn follows from Lemma 2. Third, the term $C_\alpha(M,N)$ can be rewritten as

$$\begin{aligned} C_\alpha(M,N) &= \left[\left(\theta + d^{2H}(M,O) + d^{2H}(N,O)\right)^K - \left(\theta + 2d^{2H}(M,O)\right)^K\right] \\ &\quad + \left[\left(\theta + d^{2H}(M,O) + d^{2H}(N,O)\right)^K - \left(\theta + 2d^{2H}(N,O)\right)^K\right] \\ &=: I_1 + I_2. \end{aligned} \tag{24}$$

Because I_1 and I_2 are similar, it sufficies to evaluate the first term I_1 on the right hand side of (24). Using (8) and the fact $\left|e^{-x} - e^{-y}\right| \le |x - y|$, for all $x, y \in \mathbb{R}_+$, we obtain

$$\begin{aligned} |I_1| &\le \frac{1}{\Gamma(-K)}\int_0^\infty u^{-K-1}e^{-u(\theta + d^{2H}(M,O))}\left|e^{-ud^{2H}(N,O)} - e^{-ud^{2H}(M,O)}\right|du \\ &\le -K\theta^{K-1}\left|d^{2H}(M,O)) - d^{2H}(N,O))\right| \\ &\le -K\theta^{K-1}\left\|\widetilde{M} - \widetilde{N}\right\|\mathcal{O}(1) = \left\|\widetilde{M} - \widetilde{N}\right\|^{2H}o(1) \text{ (because } H \in (0,1/2)). \end{aligned}$$

The last inequality is justified by **(c)**. Finally, we know that $\left|x^{1/2} - y^{1/2}\right| \le |x - y|^{1/2}$, for all $x, y \in \mathbb{R}_+$. Thus,

$$\begin{aligned} &\left[A_\alpha(M)^{1/2} - A_\alpha(N)^{1/2}\right]^2 \\ &\quad \le |A_\alpha(M) - A_\alpha(N)| \\ &\quad \le \left|\left(\theta + 2d^{2H}(M,O)\right)^K - \left(\theta + 2d^{2H}(N,O)\right)^K\right| \end{aligned}$$

$$\le \frac{1}{\Gamma(-K)} \int_0^\infty u^{-K-1} e^{-\theta u} \left| e^{-2ud^{2H}(M,O)} - e^{-2ud^{2H}(N,O)} \right| du$$
$$\le -2K\theta^{K-1} \left| d^{2H}(M,O) - d^{2H}(N,O) \right| = \left\| \widetilde{M} - \widetilde{N} \right\|^{2H} o(1).$$

From the previous results we deduce (22). Hence, the proof of Lemma 3 is complete. □

Disclosure statement

No potential conflict of interest was reported by the author.

Acknowledgments

The author would like to thank the Editor and the anonymous reviewers for their careful reading and constructive comments which have led to improvements of the article. The author is also grateful to Chunsheng Ma for pointing out his work on multifractional vector Brownian motions, and to Khalifa Es-Sebaiy for his discussions on the topic.

References

[1] Cheng, D., Liu, P.: Extremes of spherical fractional Brownian motion. Extremes **22**(3), 433–457 (2019) MR3984357. https://doi.org/10.1007/s10687-019-00344-4

[2] Cheng, D., Xiao, Y.: Excursion probability of gaussian random fields on sphere. Bernoulli **22**(2), 1113–1130 (2016) MR3449810. https://doi.org/10.3150/14-BEJ688

[3] Clerc, M., Mallat, S.: Estimating deformations of stationary processes. Ann. Stat. **31**(6), 1772–1821 (2003) MR2036390. https://doi.org/10.1214/aos/1074290327

[4] Cohen, S., Guyon, X., Perrin, O., Pontier, M.: Identification of an isometric transformation of the standard Brownian sheet. J. Stat. Plan. Inference **136**(4), 1317–1330 (2006) MR2253765. https://doi.org/10.1016/j.jspi.2004.09.012

[5] El Omari, M.: Mixtures of higher-order fractional Brownian motions. Commun. Stat., Theory Methods 1–16 (2021) https://doi.org/10.1080/03610926.2021.1986541

[6] El Omari, M.: An α-order fractional Brownian motion with Hurst index $H \in (0,1)$ and $\alpha \in \mathrm{R}_+$. Sankhya A, 1–28 (2022) https://doi.org/10.1007/s13171-021-00266-z

[7] Gelfand, A.E., Diggle, P., Guttorp, P., Fuentes, M.: Handbook of Spatial Statistics. CRC press (2010) MR2730964. https://doi.org/10.1201/9781420072884-c29

[8] Genton, M.G., Perrin, O.: On a time deformation reducing nonstationary stochastic processes to local stationarity. J. Appl. Probab. **41**(1), 236–249 (2004) MR2036285. https://doi.org/10.1017/s0021900200014170

[9] Guyon, X., Perrin, O.: Identification of space deformation using linear and superficial quadratic variations. Stat. Probab. Lett. **47**(3), 307–316 (2000) MR1747492. https://doi.org/10.1016/S0167-7152(99)00171-6

[10] Houdré, C., Villa, J.: An example of infinite dimensional quasi-helix. Contemp. Math. **336**, 195–202 (2003) MR2037165. https://doi.org/10.1090/conm/336/06034

[11] Istas, J.: Spherical and hyperbolic fractional Brownian motion. Electron. Commun. Probab. **10**, 254–262 (2005) MR2198600. https://doi.org/10.1214/ECP.v10-1166

[12] Istas, J.: Manifold indexed fractional fields. ESAIM Probab. Stat. **16**, 222–276 (2012) MR2956575. https://doi.org/10.1051/ps/2011106

[13] Jeong, J., Jun, M., Genton, M.G.: Spherical process models for global spatial statistics. Statistical science: a review journal of the Institute of Mathematical Statistics **32**(4), 501 (2017) MR3730519. https://doi.org/10.1214/17-STS620

[14] Kahane, J.-P.: Some Random Series of Functions vol. 5. Cambridge University Press (1993) MR0833073

[15] Kruk, I., Russo, F., Tudor, C.A.: Wiener integrals, Malliavin calculus and covariance measure structure. J. Funct. Anal. **249**(1), 92–142 (2007) MR2338856. https://doi.org/10.1016/j.jfa.2007.03.031

[16] Lifshits, M.: Lectures on gaussian processes. In: Lectures on Gaussian Processes, pp. 1–117. Springer (2012) MR3024389. https://doi.org/10.1007/978-3-642-24939-6

[17] Lifshits, M., Volkova, K.: Bifractional Brownian motion: existence and border cases. ESAIM Probab. Stat. **19**, 766–781 (2015) MR3438257. https://doi.org/10.1051/ps/2015015

[18] Ma, C.: Stationary and isotropic vector random fields on spheres. Math. Geosci. **44**(6), 765–778 (2012) MR2956272. https://doi.org/10.1007/s11004-012-9411-8

[19] Ma, C.: Multifractional vector Brownian motions, their decompositions, and generalizations. Stoch. Anal. Appl. **33**(3), 535–548 (2015) MR3339317. https://doi.org/10.1080/07362994.2015.1017108

[20] Mandelbrot, B.B., Van Ness, J.W.: Fractional Brownian motions, fractional noises and applications. SIAM Rev. **10**(4), 422–437 (1968) MR0242239. https://doi.org/10.1137/1010093

[21] Marinucci, D., Peccati, G.: Random Fields on the Sphere: Representation, Limit Theorems and Cosmological Applications vol. 389. Cambridge University Press (2011) MR2840154. https://doi.org/10.1017/CBO9780511751677

[22] Oh, H.-S., Li, T.-H.: Estimation of global temperature fields from scattered observations by a spherical-wavelet-based spatially adaptive method. J. R. Stat. Soc., Ser. B, Stat. Methodol. **66**(1), 221–238 (2004) MR2035768. https://doi.org/10.1046/j.1369-7412.2003.05220.x

[23] Perrin, O.: Quadratic variation for Gaussian processes and application to time deformation. Stoch. Process. Appl. **82**(2), 293–305 (1999) MR1700011. https://doi.org/10.1016/S0304-4149(99)00037-X

[24] Russo, F., Tudor, C.A.: On bifractional Brownian motion. Stoch. Process. Appl. **116**(5), 830–856 (2006) MR2218338. https://doi.org/10.1016/j.spa.2005.11.013

[25] Schoenberg, I.J.: Metric spaces and positive definite functions. Trans. Am. Math. Soc. **44**(3), 522–536 (1938) MR1501980. https://doi.org/10.2307/1989894

[26] Schölkopf, B., Smola, A.J., Bach, F., et al.: Learning with Kernels: Support Vector Machines, Regularization, Optimization, and Beyond. MIT press (2002)

[27] Stein, M.L.: Spatial variation of total column ozone on a global scale. Ann. Appl. Stat. **1**(1), 191–210 (2007) MR2393847. https://doi.org/10.1214/07-AOAS106

[28] Stein Michael, L.: Interpolation of Spatial Data: Some Theory for Kriging. Springer, New York (1999) MR1697409. https://doi.org/10.1007/978-1-4612-1494-6

[29] Tudor, C., Es-Sebaiy, K.: Multidimensional bifractional Brownian motion: Itô and Tanaka formulas. Stoch. Dyn. **07**(03), 365–388 (2007) MR2351043. https://doi.org/10.1142/S0219493707002050

[30] Tudor, C.A., Xiao, Y.: Sample path properties of bifractional Brownian motion. Bernoulli **13**(4), 1023–1052 (2007) MR2364225. https://doi.org/10.3150/07-BEJ6110

[31] Venet, N.: Nonexistence of fractional Brownian fields indexed by cylinders. Electron. J. Probab. **24**, 1–26 (2019) MR3978225. https://doi.org/10.1214/18-EJP256