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Note on the bi-risk discrete time risk model with income rate two

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Abstract This article provides survival probability calculation formulas for bi-risk discrete time risk model with income rate two. More precisely, the possibility for the stochastic process $u+2t-\sum_{i=1}^{t} X_i - \sum_{j=1}^{\lfloor t/2 \rfloor} Y_j$, $u \in \mathbb{N} \cup \{0\}$, to stay positive for all $t \in \{1, 2, \ldots, T\}$, when $T \in \mathbb{N}$ or $T \to \infty$, is considered, where the subtracted random part consists of the sum of random variables, which occur in time in the following order: $X_1,\ X_2+Y_1,\ X_3,\ X_4+Y_2,\ \ldots$ Here $X_i,\ i \in \mathbb{N}$, and $Y_j,\ j \in \mathbb{N}$, are independent copies of two independent, but not necessarily identically distributed, nonnegative and integer-valued random variables X and Y. Following the known survival probability formulas of the similar bi-seasonal model with income rate two, $u+2t-\sum_{i=1}^{t} X_i \mathbb{1}_{\{i \text{ is odd}\}} - \sum_{j=1}^{t} Y_i \mathbb{1}_{\{j \text{ is even}\}}$, it is demonstrated how the bi-seasonal model is used to express survival probability calculation formulas in the bi-risk case. Several numerical examples are given where the derived theoretical statements are applied.

Keywords Bi-risk model, discrete time, finite time survival probability, ultimate time survival probability, recursive calculation

MSC2020 MSC 91G05, 60G50, 60J80

1 Introduction

A recent research paper [1] studied the possibility for a random walk (r.w.) $\sum_{i=1}^{t} Z_i$ to hit the line $u+2t$, $u \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $t \in \mathbb{N}$, at least once in time, when r.w. consists

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of two interchangeably occurring discrete and nonnegative integer-valued random variables, i.e. $Z_{2i-1} \stackrel{d}{=} X$ and $Z_{2i} \stackrel{d}{=} Y$, $i \in \mathbb{N}$. Here X and Y are independent but not necessarily identically distributed. The described model is called the *bi-seasonal discrete time risk model with income rate two*.

In this article we define a slightly different model

$$W(t) = u + 2t - \sum_{i=1}^{t} X_i - \sum_{j=1}^{\lfloor t/2 \rfloor} Y_j, \tag{1}$$

where:

- $t \in \mathbb{N}$ and $u \in \mathbb{N}_0$,
- $X_i \stackrel{d}{=} X$, $Y_j \stackrel{d}{=} Y$ for all $i,\ j \in \mathbb{N}$ and X, Y are independent, integer-valued and nonnegative random variables which may be distributed differently.

The present model (1) is called the *bi-risk discrete time risk model with income rate two*. Its deterministic part $u + 2t$ consists of two components: u is deemed as initial wealth or savings in some financial context, and premium rate or income per unit of time, which is just a multiplier of t. The subtracted stochastic part $\sum_{i=1}^{t} X_i + \sum_{j=1}^{\lfloor t/2 \rfloor} Y_j$ is treated as random expenses. In particular, the setup of the random part in (1) is considered in such a way that r.v. X is present at every moment of time, while Y additionally occurs at even moments of time. Of course, there are many other different setups of such type of models as (1). One of the most general models was introduced in [2] and is known as Sparre Andersen collective risk model. Equally, Refs. [9–11, 15, 20] are known as classical works on the subject. The research variety is mainly due to that every model assumption has its impact on the possibility that stochastic part never exceeds deterministic, i.e. savings and earnings are sufficient to cover occurring expenses. There are two objects we deal with investigating the model (1):

$$\varphi(u,T) := \mathbb{P}\left(\max_{1 \leqslant t \leqslant T} \left\{\sum_{i=1}^{t}(X_i - 2) + \sum_{j=1}^{\lfloor t/2 \rfloor} Y_j\right\} < u\right),\ T \in \mathbb{N}, \tag{2}$$

$$\varphi(u) := \mathbb{P}\left(\sup_{t \geqslant 1} \left\{\sum_{i=1}^{t}(X_i - 2) + \sum_{j=1}^{\lfloor t/2 \rfloor} Y_j\right\} < u\right). \tag{3}$$

The probability (2) is called the *finite time survival probability* while the later one (3) is *the ultimate time survival probability* and they both deal with the possibility that $W(t) > 0$ for all $t \in \{1, \ldots, T\}$, when T is finite or $T \to \infty$. The ultimate time survival probability (3) heavily depends on *the net profit condition*, which for the model (1), is defined as

$$2\mathbb{E}X + \mathbb{E}Y < 4. \tag{4}$$

The ultimate time survival probability also depends directly on the minimal value of the sum $X_1 + X_2 + Y$ – see Theorems 2–6.

Table 1. Map between bi-seasonal and bi-risk models

Case	$\min X$	$\min Y$	$\min(X+Y)$	$\min(2X+Y)$
1	0	0	0	0
2	0	1	1	1
3	1	0	1	2
4	1	1	2	3
5	2	0	2	$\cancel{4}$
6	0	2	2	2
7	0	3	3	3
8	1	2	3	$\cancel{4}$
9	2	1	3	$\cancel{5}$
10	3	0	3	$\cancel{6}$

As mentioned previously, this article is based on the research done in [1] where the bi-seasonal model is studied. The relationship of the ultimate time survival probability expressions between bi-seasonal and bi-risk models is given in Table 1.

Table 1 should be read as follows. The net profit condition for the bi-seasonal risk model with income rate two is $\mathbb{E}X + \mathbb{E}Y < 4$. Therefore, the third column $\min(X+Y)$ indicates where the distribution of $X+Y$ may start not violating the net profit condition. That, of course, depends on which minimal value can be attained by X and Y as depicted in the second and the third columns. Turning to the bi-risk model with income rate two, a similar question arises: which minimal value can be attained by X_1+X_2+Y so that the net profit condition remain valid? The answer is present in the last column $\min(2X+Y)$, and it shows that four cases are not valid for the ultimate time as they never satisfy $2\mathbb{E}X + \mathbb{E}Y < 4$. Moreover, comparing bi-risk model with bi-seasonal one, some of the cases get rearranged due to $\min(2X+Y) \neq \min(X+Y)$ – see cases number 3 and 4 in Table 1.

For more convenient expressions of $\varphi(u,T)$ and $\varphi(u)$, we introduce the following notations. For $u \in \mathbb{N}_0$, we denote the probability mass functions (PMFs)

$$x_u := \mathbb{P}(X=u),\ y_u := \mathbb{P}(Y=u),$$
$$s_u := \mathbb{P}(X+Y=u),\ a_u := \mathbb{P}(X_1+X_2+Y=u),$$

the cumulative distribution functions (CDFs)

$$F_X(u) := \sum_{i=0}^{u} x_i,\ F_Y(u) := \sum_{i=0}^{u} y_i,\ F_S(u) := \sum_{i=0}^{u} s_i,\ F_A(u) := \sum_{i=0}^{u} a_i,$$

and tails

$$\overline{F}_X(u) := 1 - F_X(u),\ \overline{F}_Y(u) := 1 - F_Y(u),$$
$$\overline{F}_S(u) := 1 - F_S(u),\ \overline{F}_A(u) := 1 - F_A(u).$$

The following equality, implied by [1, eq. (2)],

$$\varphi(u) = \sum_{k=1}^{u+4} \varphi(k)a_{u+4-k} - (x_{u+3}s_0 + x_{u+2}s_1)\varphi(1) - x_{u+2}s_0\varphi(2), \tag{5}$$

shows where the problem of finding $\varphi(u)$ for all $u \in \mathbb{N}_0$ stems from. In order to use the recurrence relation (5), we need to know several initial values. How many of them exactly? This depends on the smallest value of $X_1 + X_2 + Y$. For example, if $a_0 > 0$ and $u = 0$, the formula (5) implies the relation among $\varphi(0), \ldots, \varphi(4)$, if $u = 0$ and $a_0 = 0$, $a_1 > 0$, then we have the relation among $\varphi(0), \ldots, \varphi(3)$, etc. With that said, the next Section 2 is structured as follows: Theorem 1 deals with the finite time survival probability, Theorems 2–5 express the ultimate time survival probability under the net profit condition and the remaining Theorem 6 provides the values of $\varphi(u)$ under the breach of the net profit condition. In Section 3 we present the outputs of these mentioned theorems with some chosen random variables X and Y, while in Section 4 we give a summarized overview and a brief look at more generalized discrete time risk models.

It is worth mentioning that research related to the present one can also be found in papers [7] and [12]. In general, the research done in this paper might be considered as a study of a certain randomness in attaining some large values. Multiple research papers are written each year on the subject, just few of them are [4–6, 8, 13, 14] and [17]. Also, due to some model's specifics, only particular distributions of a random part are considered, see [19].

2 Statements and proofs

We start with the statement for finite time survival probability $\varphi(u, T)$ for the bi-risk discrete time risk model with premium two. The model is defined in (1).

Theorem 1. *For any $u \in \mathbb{N}_0$, the finite time survival probability of the bi-risk discrete time risk model with income rate two, satisfies*

$$\begin{aligned}
\varphi(u,1) &= F_X(u+1),\\
\varphi(u,2) &= \sum_{k=0}^{u+1} x_k F_S(u+3-k),\\
\varphi(u,T) &= \sum_{k=0}^{u+3} \varphi(u+4-k, T-2) a_k\\
&\quad - (x_{u+2}s_1 + x_{u+3}s_0)\varphi(1, T-2) - x_{u+2}s_0\varphi(2, T-2),\ T \geqslant 3.
\end{aligned}$$

Proof. The proof follows by replacing $Y \mapsto X + Y$ and $X + Y \mapsto X_1 + X_2 + Y$ in Theorem 2.1 in [1]. □

Let us observe that Theorem 1 is independent of the net profit condition. We note that [3, Thm. 1] may be adopted for the considered finite time survival probability calculation as well.

Lets turn to the ultimate time. To express the ultimate time survival probability for model (1), when the sum $X_1 + X_2 + Y$ can attain zero with some positive probability, we define four recurrent sequences α_n, β_n, γ_n and δ_n. For $n = 0,\ 1,\ 2,\ 3$ we define

n	α_n	β_n	γ_n	δ_n
0	1	0	0	0
1	0	1	0	0
2	0	0	1	0
3	$-\frac{1}{a_0}$	$-\frac{F_A(2)+\overline{F}_X(2)s_0+\overline{F}_X(1)s_1}{a_0}$	$-\frac{F_A(1)+\overline{F}_X(1)s_0}{a_0}$	$\frac{1}{a_0}$

and, for $n = 4,\ 5,\ \ldots$,

$$\alpha_n = \frac{1}{a_0}\left(\alpha_{n-4} - \sum_{k=1}^{n-1} a_{n-k}\alpha_k\right),$$

$$\beta_n = \frac{1}{a_0}\left(\beta_{n-4} - \sum_{k=1}^{n-1} a_{n-k}\beta_k + x_{n-1}s_0 + x_{n-2}s_1\right),$$

$$\gamma_n = \frac{1}{a_0}\left(\gamma_{n-4} - \sum_{k=1}^{n-1} a_{n-k}\gamma_k + x_{n-2}s_0\right),\ \delta_n = \frac{1}{a_0}\left(\delta_{n-4} - \sum_{k=1}^{n-1} a_{n-k}\delta_k\right).$$

Theorem 2. *Suppose that $a_0 > 0$ and $2\mathbb{E}X + \mathbb{E}Y < 4$. Then, the ultimate time survival probability of the bi-risk discrete time risk model with income rate two satisfies*

$$\begin{pmatrix} \alpha_{n+1}-\alpha_n & \beta_{n+1}-\beta_n & \gamma_{n+1}-\gamma_n \\ \alpha_{n+2}-\alpha_n & \beta_{n+2}-\beta_n & \gamma_{n+2}-\gamma_n \\ \alpha_{n+3}-\alpha_n & \beta_{n+3}-\beta_n & \gamma_{n+3}-\gamma_n \end{pmatrix} \times \begin{pmatrix} \varphi(0) \\ \varphi(1) \\ \varphi(2) \end{pmatrix} + \begin{pmatrix} \delta_{n+1}-\delta_n \\ \delta_{n+2}-\delta_n \\ \delta_{n+3}-\delta_n \end{pmatrix} \times (4-2\mathbb{E}X-\mathbb{E}Y) = \begin{pmatrix} \varphi(n+1)-\varphi(n) \\ \varphi(n+2)-\varphi(n) \\ \varphi(n+3)-\varphi(n) \end{pmatrix},\ n \in \mathbb{N}_0, \tag{6}$$

$$\varphi(3) = \frac{-\varphi(0) - (F_A(2) + \overline{F}_X(2)s_0 + \overline{F}_X(1)s_1)\varphi(1)}{a_0} - \frac{(F_A(1) + \overline{F}_X(1)s_0)\varphi(2) - 4 + 2\mathbb{E}X + \mathbb{E}Y}{a_0},$$

$$\varphi(u) = \frac{1}{a_0}\left(\varphi(u-4) + (x_{u-1}s_0 + x_{u-2}s_1)\varphi(1) + x_{u-2}s_0\varphi(2) - \sum_{k=1}^{u-1} a_{u-k}\varphi(k)\right),\quad u = 4,\ 5,\ \ldots.$$

Proof. The proof is implied by Theorem 2.2 in [1] replacing $Y \mapsto X + Y$ and $X + Y \mapsto X_1 + X_2 + Y$ there. □

We now turn to the case when the lowest possible value of $X_1 + X_2 + Y$ (with positive probability) is one. Note that there is just one underlying case satisfying the mentioned condition – case number two in Table 1. Let's define three recurrent sequences $\overline{\alpha}_n$, $\overline{\beta}_n$ and $\overline{\delta}_n$. For $n = 0,\ 1,\ 2$,

n	$\overline{\alpha}_n$	$\overline{\beta}_n$	$\overline{\delta}_n$
0	1	0	0
1	0	1	0
2	$-\frac{1}{a_1}$	$-\frac{F_A(2)+\overline{F}_X(1)s_1}{a_1}$	$\frac{1}{a_1}$

and, for $n = 3, 4, \ldots$,

$$\overline{\alpha}_n = \frac{1}{a_1}\left(\overline{\alpha}_{n-3} - \sum_{k=1}^{n-1} a_{n+1-k}\overline{\alpha}_k\right),$$
$$\overline{\beta}_n = \frac{1}{a_1}\left(\overline{\beta}_{n-3} - \sum_{k=1}^{n-1} a_{n+1-k}\overline{\beta}_k + x_{n-1}s_1\right),$$
$$\overline{\delta}_n = \frac{1}{a_1}\left(\overline{\delta}_{n-3} - \sum_{k=1}^{n-1} a_{n+1-k}\overline{\delta}_k\right).$$

Theorem 3. *Suppose that $a_0 = 0$, $a_1 > 0$ and $2\mathbb{E}X + \mathbb{E}Y < 4$. Then, the ultimate time survival probability of the bi-risk discrete time risk model with income rate two satisfies*

$$\begin{pmatrix} \overline{\alpha}_{n+1} - \overline{\alpha}_n & \overline{\beta}_{n+1} - \overline{\beta}_n \\ \overline{\alpha}_{n+2} - \overline{\alpha}_n & \overline{\beta}_{n+2} - \overline{\beta}_n \end{pmatrix} \times \begin{pmatrix} \varphi(0) \\ \varphi(1) \end{pmatrix} + \begin{pmatrix} \overline{\delta}_{n+1} - \overline{\delta}_n \\ \overline{\delta}_{n+2} - \overline{\delta}_n \end{pmatrix} \times (4 - \mathbb{E}A)$$
$$= \begin{pmatrix} \varphi(n+1) - \varphi(n) \\ \varphi(n+2) - \varphi(n) \end{pmatrix}, \ n \in \mathbb{N}_0, \tag{7}$$

$$\varphi(2) = \frac{-\varphi(0) - (F_A(2) + \overline{F}_X(1)s_1)\varphi(1) + 4 - 2\mathbb{E}X - \mathbb{E}Y}{a_1},$$
$$\varphi(u) = \frac{1}{a_1}\left(\varphi(u-3) + x_{u-1}s_1\varphi(1) - \sum_{k=1}^{u-1} a_{u+1-k}\varphi(k)\right), \ u = 3, 4, \ldots$$

Proof. The proof follows from Theorem 2.3 in [1] by replacing $Y \mapsto X + Y$, $X + Y \mapsto X_1 + X_2 + Y$ there and observing that $s_0 = 0$ because of $y_0 = 0$ under the current assumptions. □

We now ask when does the sum $X_1 + X_2 + Y$ attains its minimum value of two with some positive probability? That happens in cases three and six in Table 1. The next theorem requires two recurrent sequences to be defined:

$$\hat{\alpha}_0 = 1, \ \hat{\alpha}_1 = -\frac{1}{\overline{F}_X(1)s_1 + a_2},$$
$$\hat{\alpha}_n = \frac{1}{a_2}\left(\hat{\alpha}_{n-2} - \sum_{k=1}^{n-1} a_{n+2-k}\hat{\alpha}_k + x_n s_1\hat{\alpha}_1\right), \ n = 2, 3, \ldots,$$
$$\hat{\delta}_0 = 0, \ \hat{\delta}_1 = \frac{1}{\overline{F}_X(1)s_1 + a_2},$$

$$\hat{\delta}_n = \frac{1}{a_2}\left(\hat{\delta}_{n-2} - \sum_{k=1}^{n-1} a_{n+2-k}\hat{\delta}_k + x_n s_1 \hat{\delta}_1\right),\ n = 2,\ 3,\ \ldots.$$

Theorem 4. *Suppose that $a_0 = a_1 = 0$, $a_2 > 0$ and $2\mathbb{E}X + \mathbb{E}Y < 4$. Then, the ultimate time survival probability of the bi-risk discrete time risk model with income rate two satisfies*

$$(\hat{\alpha}_{n+1} - \hat{\alpha}_n)\varphi(0) + (\hat{\delta}_{n+1} - \hat{\delta}_n)(4 - 2\mathbb{E}X - \mathbb{E}Y) = \varphi(n+1) - \varphi(n),\ n \in \mathbb{N}_0,$$
$$\varphi(1) = \hat{\alpha}_1\varphi(0) + \hat{\delta}_1(4 - 2\mathbb{E}X - \mathbb{E}Y),$$
$$\varphi(u) = \frac{1}{a_2}\left(\varphi(u-2) - \sum_{k=1}^{u-1} a_{u+2-k}\varphi(k) + x_u s_1 \varphi(1)\right),\ u = 2,\ 3,\ \ldots$$

Moreover, $\hat{\alpha}_{n+1} - \hat{\alpha}_n \neq 0$ for all $n \in \mathbb{N}_0$.

Proof. The proof is implied by Theorem 2.4 in [1] replacing $Y \mapsto X + Y$, $X + Y \mapsto X_1 + X_2 + Y$ there and observing that $s_1 = 0$ under the current assumptions. □

The last case, when the net profit condition can be satisfied, is when the sum $X_1 + X_2 + Y$ can attain its minimum value of three with some positive probability. This is illustrated in cases number four and seven in Table 1 and consequently, the expression of the ultimate time survival probability is straightforward.

Theorem 5. *Suppose that $a_0 = a_1 = a_2 = 0$, $a_3 > 0$ and $2\mathbb{E}X + \mathbb{E}Y < 4$. Then, the ultimate time survival probability of the bi-risk discrete time risk model with income rate two satisfies*

$$\varphi(0) = 4 - 2\mathbb{E}X - \mathbb{E}Y,\ \varphi(1) = \varphi(0)/a_3,$$
$$\varphi(u) = \frac{1}{a_3}\left(\varphi(u-1) - \sum_{k=1}^{u-1} a_{u+3-k}\varphi(k)\right),\ u = 2,\ 3,\ \ldots.$$

Proof. The proof follows from Theorem 2.5 in [1] by replacing $Y \mapsto X + Y$, $X + Y \mapsto X_1 + X_2 + Y$ there and observing that $s_0 = s_1 = 0$ under the current assumptions. □

The next theorem shows that survival is impossible, in all but a few trivial cases, if the net profit condition is violated.

Theorem 6. *Suppose that $2\mathbb{E}X + \mathbb{E}Y \geqslant 4$. Then, the ultimate time survival probability of the bi-risk discrete time risk model with income rate two is as follows:*

(i) $\varphi(u) = 0$ *for all* $u \in \mathbb{N}_0$, *when* $2\mathbb{E}X + \mathbb{E}Y > 4$,

(ii) $\varphi(u) = 0$ *for all* $u \in \mathbb{N}_0$, *when* $2\mathbb{E}X + \mathbb{E}Y = 4$ *and* $a_4 < 1$,

(iii) $\varphi(0) = 0$ *and* $\varphi(u) = 1$ *for all* $u \in \mathbb{N}$, *when* $2\mathbb{E}X + \mathbb{E}Y = 4$ *and* $a_4 = 1$.

Proof. For the model (1), the equality (20) in [1] implies

$$\begin{aligned}&\varphi(0)+(x_0s_2+s_1+s_0)\varphi(1)+(x_0s_1+s_0)\varphi(2)+x_0s_0\varphi(3)\\&=(4-2\mathbb{E}X-\mathbb{E}Y)\cdot\varphi(\infty),\end{aligned}\tag{8}$$

where

$$\varphi(\infty):=\lim_{u\to\infty}\varphi(u).$$

Due to (8), the condition $2\mathbb{E}X+\mathbb{E}Y>4$ implies $\varphi(\infty)=0$ and consequently $\varphi(u)=0$ for all $u\in\mathbb{N}_0$ because of $\varphi(u)\leqslant\varphi(u+1)$. Therefore (i) is correct.

If $2\mathbb{E}X+\mathbb{E}Y=4$ and $a_4<1$, then, by (8),

$$\varphi(0)+(x_0s_2+s_1+s_0)\varphi(1)+(x_0s_1+s_0)\varphi(2)+x_0s_0\varphi(3)=0.\tag{9}$$

In view of (9), we have the following cases:

$$\begin{aligned}&a_0>0 \quad\Rightarrow\quad \varphi(0)=\varphi(1)=\varphi(2)=\varphi(3)=0,\\&a_0=0,\ a_1>0 \quad\Rightarrow\quad \varphi(0)=\varphi(1)=\varphi(2)=0,\\&a_0=a_1=0,\ a_2>0 \quad\Rightarrow\quad \varphi(0)=\varphi(1)=0,\\&a_0=a_1=a_2=0,\ a_3>0 \quad\Rightarrow\quad \varphi(0)=0,\end{aligned}$$

while $\varphi(u)=0$ for all $u\in\mathbb{N}_0$ is implied by (5). Thus, (ii) is correct.

Suppose that $2\mathbb{E}X+\mathbb{E}Y=4$ and $a_4=1$. Then, there are just three degenerated distributions:

$$X\equiv 0,\ Y\equiv 4 \text{ or } X\equiv 2,\ Y\equiv 0 \text{ or } X\equiv 1,\ Y\equiv 2,$$

all of which, by model definition (1), imply $\varphi(0)=0$ and $\varphi(u)=1$ for all $u\in\mathbb{N}$. □

3 Numerical examples

In this section we verify the formulated Theorems 1–6 with some chosen r.vs. That is performed by choosing particular distributions of X and Y and obtaining $\varphi(u,T)$ and $\varphi(u)$ for some $u\in\mathbb{N}_0$ and $T\in\mathbb{N}$. Initial wealth u and time T are chosen individually aiming to reflect the dynamics of survival probabilities. The presented survival probabilities are calculated with Python [16] and confirmed with Wolfram Mathematica [18]. Results are rounded up to three decimal places except when the rounding result is 0 or 1. As the initial values of the ultimate time survival probability in Theorems 2–4 depend on $n\in\mathbb{N}$, there $n=100$ is chosen as large enough when applying them. Ideally, we should set $n\to\infty$ and, for example, Theorem 4 would imply

$$\varphi(0)=(4-2\mathbb{E}X-\mathbb{E}Y)\lim_{n\to\infty}\frac{\hat{\delta}_n-\hat{\delta}_{n+1}}{\hat{\alpha}_{n+1}-\hat{\alpha}_n}.$$

However, it is not easy to find such limit, especially the corresponding one in Theorems 2 and 3. Therefore, we consider the chosen n as large enough when a relative change of the obtained initial value (or values) is negligible comparing results between n and $n+1$, i.e. $|L_{n+1}/L_n|<\varepsilon$, where $L_n=(\hat{\delta}_n-\hat{\delta}_{n+1})/(\hat{\alpha}_{n+1}-\hat{\alpha}_n)$.

The distribution

$$\mathbb{P}(X = k) = (1 - p)^k p,\ k = 0,\ 1,\ \ldots,$$

is called *geometric* with parameter $0 < p < 1$ (denoted $X \sim \mathcal{G}(p)$), while the generalized one

$$\mathbb{P}(X = k) = \binom{k-1}{r-1}(1 - p)^{k-r} p^r,\ k = r,\ r + 1,\ \ldots,$$

is known as *negative binomial* or *Pascal* with parameters $0 < p < 1$ and $r \in \mathbb{N}$ (denoted $X \sim \mathcal{NB}(r, p)$).

Example 1. *Assume that* $X \sim \mathcal{G}(3/4)$ *and* $Y \sim \mathcal{G}(1/4)$. *Then,* $2\mathbb{E}X + \mathbb{E}Y = 3\frac{2}{3} < 4$ *and, according to Theorems 1 and 2, we obtain Table 2.*

Table 2. Survival probability when $X \sim \mathcal{G}(3/4)$ and $Y \sim \mathcal{G}(1/4)$

T	$u = 0$	$u = 1$	$u = 2$	$u = 3$	$u = 4$	$u = 5$	$u = 10$	$u = 25$	$u = 50$
1	0.938	0.984	0.996	0.999	1	1	1	1	1
2	0.582	0.695	0.774	0.831	0.873	0.905	0.977	1	1
3	0.580	0.693	0.772	0.830	0.872	0.904	0.977	1	1
4	0.465	0.576	0.661	0.729	0.784	0.829	0.948	0.999	1
5	0.465	0.575	0.660	0.728	0.784	0.828	0.948	0.999	1
10	0.338	0.430	0.508	0.576	0.637	0.689	0.866	0.993	1
20	0.267	0.343	0.410	0.472	0.528	0.579	0.772	0.974	1
30	0.235	0.303	0.364	0.420	0.472	0.521	0.713	0.953	0.999
40	0.216	0.279	0.336	0.389	0.438	0.484	0.672	0.933	0.997
50	0.203	0.263	0.317	0.367	0.414	0.458	0.642	0.914	0.996
100	0.172	0.224	0.270	0.314	0.355	0.395	0.562	0.850	0.982
∞	0.138	0.18	0.218	0.253	0.287	0.319	0.460	0.731	0.915

Example 2. *Assume that* $X \sim \mathcal{G}(3/4)$ *and* $Y \sim \mathcal{NB}(1, 1/2)$. *Then,* $2\mathbb{E}X + \mathbb{E}Y = 2\frac{2}{3} < 4$ *and, according to Theorems 1 and 3, we obtain Table 3.*

Table 3. Survival probability when $X \sim \mathcal{G}(3/4)$ and $Y \sim \mathcal{NB}(1, 1/2)$

T	$u = 0$	$u = 1$	$u = 2$	$u = 3$	$u = 4$	$u = 5$	$u = 10$	$u = 15$
1	0.938	0.984	0.996	0.999	1	1	1	1
2	0.738	0.866	0.932	0.965	0.983	0.991	1	1
3	0.734	0.863	0.930	0.964	0.982	0.991	1	1
4	0.686	0.823	0.902	0.946	0.970	0.984	0.999	1
5	0.684	0.822	0.901	0.945	0.970	0.984	0.999	1
10	0.647	0.787	0.872	0.924	0.955	0.973	0.998	1
20	0.638	0.778	0.864	0.917	0.949	0.969	0.997	1
30	0.637	0.776	0.862	0.916	0.948	0.968	0.997	1
40	0.636	0.776	0.862	0.915	0.948	0.968	0.997	1
∞	0.636	0.776	0.862	0.915	0.948	0.968	0.997	1

Example 3. *Assume that* $X \sim \mathcal{NB}(1, 3/4)$ *and* $Y \sim \mathcal{G}(1/2)$. *Then,* $2\mathbb{E}X + \mathbb{E}Y = 3\frac{2}{3} < 4$ *and, according to Theorems 1 and 4, we obtain Table 4.*

Example 4. *Assume that* $X \sim \mathcal{NB}(1, 3/4)$ *and* $Y \sim \mathcal{NB}(1, 4/5)$. *Then,* $2\mathbb{E}X + \mathbb{E}Y = 3\frac{11}{12} < 4$ *and, according to Theorems 1 and 5, we obtain Table 5.*

Table 4. Survival probability when $X \sim \mathcal{NB}(1, 3/4)$ and $Y \sim \mathcal{G}(1/2)$

T	$u=0$	$u=1$	$u=2$	$u=3$	$u=4$	$u=5$	$u=10$	$u=25$	$u=40$
1	0.750	0.938	0.984	0.996	0.999	1	1	1	1
2	0.492	0.738	0.866	0.932	0.965	0.983	0.999	1	1
3	0.475	0.722	0.854	0.925	0.962	0.981	0.999	1	1
4	0.399	0.635	0.782	0.872	0.927	0.958	0.998	1	1
5	0.393	0.627	0.775	0.867	0.923	0.956	0.998	1	1
10	0.307	0.511	0.658	0.765	0.841	0.894	0.989	1	1
20	0.260	0.440	0.577	0.684	0.766	0.829	0.969	1	1
30	0.241	0.409	0.540	0.644	0.726	0.791	0.952	1	1
40	0.230	0.392	0.518	0.620	0.702	0.768	0.938	0.999	1
50	0.223	0.381	0.505	0.605	0.686	0.751	0.928	0.999	1
100	0.210	0.358	0.476	0.572	0.651	0.716	0.901	0.997	1
∞	0.203	0.347	0.462	0.556	0.633	0.967	0.884	0.993	1

Table 5. Survival probability when $X \sim \mathcal{NB}(1, 3/4)$ and $Y \sim \mathcal{NB}(1, 4/5)$

T	$u=0$	$u=1$	$u=2$	$u=3$	$u=4$	$u=5$	$u=10$	$u=25$	$u=50$
1	0.750	0.938	0.984	0.996	0.999	1	1	1	1
2	0.450	0.765	0.912	0.970	0.990	0.997	1	1	1
3	0.422	0.738	0.896	0.962	0.987	0.996	1	1	1
4	0.344	0.652	0.837	0.930	0.972	0.989	1	1	1
5	0.332	0.636	0.824	0.922	0.968	0.987	1	1	1
10	0.240	0.497	0.694	0.825	0.905	0.951	0.999	1	1
20	0.185	0.398	0.579	0.717	0.817	0.885	0.993	1	1
30	0.161	0.350	0.517	0.652	0.755	0.832	0.982	1	1
40	0.146	0.320	0.477	0.607	0.711	0.792	0.969	1	1
50	0.137	0.300	0.449	0.575	0.678	0.760	0.956	1	1
100	0.113	0.250	0.378	0.490	0.586	0.666	0.900	0.999	1
∞	0.083	0.185	0.282	0.368	0.445	0.512	0.744	0.963	0.999

Example 5. *Assume that* $X \sim \mathcal{NB}(1, 3/4)$ *and* $Y \sim \mathcal{NB}(3, 3/4)$. *Then, the net profit condition is violated,* $2\mathbb{E}X + \mathbb{E}Y = 6\frac{2}{3} > 4$, *and, according to Theorems 1 and 6, we obtain Table 6.*

Table 6. Survival probability when $X \sim \mathcal{NB}(1, 3/4)$ and $Y \sim \mathcal{NB}(3, 3/4)$

T	$u=0$	$u=1$	$u=2$	$u=3$	$u=4$	$u=5$	$u=10$	$u=25$	$u=50$
1	0.750	0.938	0.984	0.996	0.999	1	1	1	1
2	0	0	0.237	0.534	0.756	0.886	0.999	1	1
3	0	0	0.222	0.512	0.737	0.873	0.999	1	1
4	0	0	0	0.056	0.197	0.391	0.960	1	1
5	0	0	0	0.053	0.187	0.376	0.956	1	1
10	0	0	0	0	0	0	0.115	0.997	1
20	0	0	0	0	0	0	0	0.340	1
30	0	0	0	0	0	0	0	0.001	0.943
40	0	0	0	0	0	0	0	0	0.292
50	0	0	0	0	0	0	0	0	0.006
55	0	0	0	0	0	0	0	0	0.001
∞	0	0	0	0	0	0	0	0	0

4 Concluding remarks

Random walks appear in many natural sciences where birth/death, gain/loss and upturn/downturn processes are studied. In this work, we investigated the possibility that the random walk

$$\sum_{i=1}^{t} X_i + \sum_{j=1}^{\lfloor t/2 \rfloor} Y_j$$

never hits the line $u + 2t$, $u \in \mathbb{N}_0$, when $X_i \stackrel{d}{=} X$, $Y_j \stackrel{d}{=} Y$ for all $i, j \in \mathbb{N}$ and X, Y are integer-valued, nonnegative and independent random variables, which may be distributed differently. Here $t \in \{1, 2, \ldots, T\}$, $T \in \mathbb{N}$ or $T \to \infty$. The finite time survival probability $\varphi(u, T)$ does not depend on the net profit condition $2\mathbb{E}X + \mathbb{E}Y < 4$ and its recurrent expressions are given in Theorem 1. Theorems 2–5 express the ultimate time survival probability $\varphi(u)$ when the net profit condition is satisfied, and show $\varphi(u)$ dependency on $\min(2X + Y)$. The last Theorem 6 states that survival is impossible, in all but few trivial cases, if the net profit condition is violated. In Section 3, there are examples given for particular values of survival probabilities calculated according to Theorems 1–6. Summarizing, the ultimate time survival probability of the bi-risk discrete time risk model with income rate two is expressible via certain recurrent sequences and limit laws, i.e. initial values of φ are determined by the limits of certain recurrent sequences.

As mentioned in the introduction, each distinct setup of random or deterministic part in any discrete time risk model influences an expression of survival probability. This work might be deemed as preparation for studying more generalized discrete time risk models with an arbitrary income rate $\kappa \in \mathbb{N}$ in deterministic part $u + \kappa t$ and/or an arbitrary number of nonidentically distributed random variables generating the random walk $\sum_{i=1}^{t} Z_i$. To avoid being buried in too many details, we just mention that such generalized models raise the level of abstraction significantly: a corresponding version of recurrence relation (5) would require more initial values and consequently much more effort in finding them.

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