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Asymptotic properties of the parabolic equation driven by stochastic measure

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Abstract A stochastic parabolic equation on $[0, T] \times \mathbb{R}$ driven by a general stochastic measure, for which we assume only σ-additivity in probability, is considered. The asymptotic behavior of its solution as $t \to \infty$ is studied.

Keywords Stochastic measure, mild solution, stochastic parabolic equation, asymptotic behavior

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1 Introduction

In this paper we consider the stochastic parabolic equation

$$\begin{cases} \mathcal{L}u(t,x)dt + f(t,x,u(t,x))\,dt + \sigma(t,x)\,d\mu(x) = 0\,, \\ u(0,x) = u_0(x)\,, \end{cases} \tag{1}$$

where $(t, x) \in [0, T] \times \mathbb{R}$, μ is a general stochastic measure on the Borel σ-algebra on $\mathbb{R}$ (see Section 2), f, σ are measurable functions, $\mathcal{L}$ is the operator of the kind

$$\mathcal{L}u(t,x) = a(t)\frac{\partial^2 u(t,x)}{\partial^2 x} + b(t)\frac{\partial u(t,x)}{\partial x} + c(t)u(t,x) - \frac{\partial u(t,x)}{\partial t}. \tag{2}$$

Here a, b, c are defined on $[0, T]$. We prove that under certain conditions on a, b, c, f, σ the solution of (1), considered in the mild sense, tends to 0 a.s. uniformly on x.

Note that regularity of the solution was proved in [2], the solution's convergence in the case of integrator's convergence was proved in [18] and the averaging principle for such an equation was established in [12].

The asymptotic behavior of the moments of solutions of a stochastic differential system driven by a Brownian motion was considered in [5]. The problem of the convergence of the solution of a nonautonomous logistic differential equation to zero as time coordinate goes to infinity, with disturbance of coefficients by white noise, centered and noncentered Poisson noises, was studied in [4]. Asymptotics of the solution of the stochastic heat equation with white noise, as time variable goes to infinity for the fixed spatial coordinate, was studied in [10] while asymptotic properties of the solution of the stochastic wave equation driven by a Lévy process were given in [7]. Behavior of solutions of different equations with a general stochastic measure when spatial coordinate goes to infinity was considered in [3] and [1]. In comparison to [15], where asymptotics of the heat equation driven by a general stochastic measure when time coordinate tends to infinity was considered, we study a more general parabolic equation.

The paper is organized as follows. Section 2 contains some general facts about stochastic measures and integrals with respect to them. In Section 3 we prove some technical facts and formulate the main result, which is proved in Section 4 jointly with the auxillary lemma.

2 Preliminaries

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a complete probability space and $\mathcal{B}$ be a Borel σ-algebra on $\mathbb{R}$. Denote by $\mathsf{L}_0 = \mathsf{L}_0(\Omega, \mathcal{F}, \mathsf{P})$ the set of all real-valued random variables defined on $(\Omega, \mathcal{F}, \mathsf{P})$. Convergence in L_0 means the convergence in probability.

Definition 1. A σ-additive mapping $\mu : \mathcal{B} \to \mathsf{L}_0$ is called *stochastic measure* (SM).

In other words, μ is a vector measure with values in L_0. We do not assume any martingale properties or moment existence for SM.

Consider some examples of SMs. If M_t is a square integrable martingal then $\mu(A) = \int_0^T \mathbf{1}_A(t)\, dM_t$ is an SM. An α-stable random measure on $\mathcal{B}$ for $\alpha \in (0, 1) \cup (1, 2]$, as it is defined in [19, Sections 3.2–3.3], is an SM in sense of Definition 1. For a fractional Brownian motion W_t^H with Hurst index $H > 1/2$ and a bounded measurable function $f : [0, T] \to \mathbb{R}$ we can define an SM $\mu(A) = \int_0^T f(t)\mathbf{1}_A(t)\, dW_t^H$, see [13, Theorem 1.1]. Ref. [17] contains some other examples.

The integral $\int_A g\, d\mu$, where $g : \mathbb{R} \to \mathbb{R}$ is a deterministic measurable function, $A \in \mathcal{B}$ and μ is an SM, is defined and its basic properties are given in [11, Chapter 7]. In that paper the integral with respect to general stochastic measure was constructed and studied for μ defined on an arbitrary σ-algebra, but here we consider SM on Borel subsets of $\mathbb{R}$. Note that every bounded measurable g is integrable with respect to (w.r.t.) any μ.

In the sequel, μ denotes an SM, C and $C(\omega)$ denote positive constant and positive random constant, respectively, whose exact values are not important ($C < \infty$, $C(\omega) < \infty$ a.s.).

We use the following statement.

Lemma 1 (Lemma 3.1 in [14])**.** *Let $\phi_l : \mathbb{R} \to \mathbb{R}$, $l \geq 1$, be measurable functions such that $\tilde{\phi}(x) = \sum_{l=1}^{\infty} |\phi_l(x)|$ is integrable w.r.t. μ on $\mathbb{R}$. Then*

$$\sum_{l=1}^{\infty}\Big(\int_{\mathbb{R}} \phi_l \, d\mu\Big)^2 < \infty \quad a.s.$$

We consider the *Besov spaces* $B_{22}^{\alpha}([c,d])$, $0 < \alpha < 1$, with a standard norm

$$\|g\|_{B_{22}^{\alpha}([c,d])} = \|g\|_{\mathsf{L}_2([c,d])} + \Big(\int_0^{d-c} (w_2(g,r))^2 r^{-2\alpha-1}\, dr\Big)^{1/2}, \tag{3}$$

where

$$w_2(g,r) = \sup_{0\le h\le r}\Big(\int_c^{d-h} |g(y+h) - g(y)|^2\, dy\Big)^{1/2}.$$

For any $j \in \mathbb{Z}$ and all $n \geq 0$, put

$$d_{kn}^{(j)} = j + k2^{-n}, \quad 0 \le k \le 2^n, \quad \Delta_{kn}^{(j)} = (d_{(k-1)n}^{(j)}, d_{kn}^{(j)}], \quad 1 \le k \le 2^n .$$

The following lemma is a key tool for estimates of the stochastic integral.

Lemma 2 (Lemma 3 in [16])**.** *Let Z be an arbitrary set, and the function $q(z,s) : Z \times [j, j+1] \to \mathbb{R}$ be such that all paths $q(z,\cdot)$ are continuous on $[j, j+1]$. Denote*

$$q_n(z,s) = \sum_{1\le k\le 2^n} q\big(z, d_{(k-1)n}^{(j)}\big)\mathbf{1}_{\Delta_{kn}^{(j)}}(s).$$

Then the random function

$$\eta(z) = \int_{(j,j+1]} q(z,s)\, d\mu(s), \ z \in Z,$$

has a version

$$\begin{aligned}\widetilde{\eta}(z) &= \int_{(j,j+1]} q_0(z,s)\, d\mu(s)\\ &\quad + \sum_{n\ge 1}\Big(\int_{(j,j+1]} q_n(z,s)\, d\mu(s) - \int_{(j,j+1]} q_{n-1}(z,s)\, d\mu(s)\Big)\end{aligned} \tag{4}$$

such that for all $\beta > 0$, $\omega \in \Omega$, $z \in Z$

$$\begin{aligned}|\widetilde{\eta}(z)| \le |q(z,j)\mu((j,j+1])| &+ \Big\{\sum_{n\ge 1} 2^{n\beta} \sum_{1\le k\le 2^n} |q(z,d_{kn}^{(j)}) - q(z,d_{(k-1)n}^{(j)})|^2\Big\}^{1/2}\\ &\times \Big\{\sum_{n\ge 1} 2^{-n\beta} \sum_{1\le k\le 2^n} |\mu(\Delta_{kn}^{(j)})|^2\Big\}^{1/2}.\end{aligned} \tag{5}$$

From Theorem 1.1 [9] it follows that, for $\alpha = (\beta+1)/2$,

$$\Big\{\sum_{n\ge 1} 2^{n\beta} \sum_{1\le k\le 2^n} |q\big(z,d_{kn}^{(j)}\big) - q\big(z,d_{(k-1)n}^{(j)}\big)|^2\Big\}^{1/2} \le C\|q(z,\cdot)\|_{B_{22}^{\alpha}([j,j+1])}. \tag{6}$$

Lemma 1 implies that for each $\beta > 0$, $j \in \mathbb{Z}$

$$\sum_{n\ge 1} 2^{-n\beta} \sum_{1\le k\le 2^n} |\mu(\Delta_{kn}^{(j)})|^2 < +\infty \quad \text{a.s.}$$

3 Formulation of the problem and the main result

We refer to the mild solution to (1), i.e. the measurable random function $u(t, x) = u(t, x, \omega) : [0, T] \times \mathbb{R} \times \Omega \to \mathbb{R}$ that satisfies

$$u(t, x) = \int_{\mathbb{R}} p(t, x; 0, y) u_0(y)\, dy + \int_0^t ds \int_{\mathbb{R}} p(t, x; s, y) f(s, y, u(s, y))\, dy + \int_{\mathbb{R}} d\mu(y) \int_0^t p(t, x; s, y) \sigma(s, y)\, ds, \tag{7}$$

for each $(t, x) \in [0, +\infty) \times \mathbb{R}$ a.s. The properties of such solutions are considered in [2]. For example, solution of (7) exists, is unique and can be built as

$$u(t, x) = \lim_{n \to \infty} u^{(n)}(t, x), \tag{8}$$

where $u^{(0)}(t, x) = 0$ and

$$u^{(n)}(t, x) = \int_{\mathbb{R}} p(t, x; 0, y) u_0(y)\, dy + \int_0^t ds \int_{\mathbb{R}} p(t, x; s, y) f(s, y, u^{(n-1)}(s, y))\, dy + \int_{\mathbb{R}} d\mu(y) \int_0^t p(t, x; s, y) \sigma(s, y)\, ds. \tag{9}$$

The analogous iteration process for the stochastic heat equation is considered in more detail in [14].

Let the coefficients of operator (2) satisfy the following assumptions.

Assumption 1. Functions $a(t)$, $b(t)$, $c(t)$ are continuous and bounded on $[0, +\infty)$, and

$$|a(t_1) - a(t_2)| \le L\, |t_1 - t_2|^{\lambda}, \quad a(t) \ge \delta,$$

where $t, t_1, t_2 \in [0, +\infty)$, L, λ, δ are positive constants.

Assumption 2. There exists a constant $c_0 > 0$ such that $c(t) \le -c_0\ \forall t \ge 0$.

Assumption 1 implies that $p(t, x; s, y) = p(t, x-y; s, 0)$ for each $t, s \in [0, +\infty)$, $x, y \in \mathbb{R}$. We consider u_0, f, σ in (7) under the following conditions.

Assumption 3. $u_0 : \mathbb{R} \times \Omega \to \mathbb{R}$ is measurable and for all $y, y_1, y_2 \in \mathbb{R}$

$$|u_0(y, \omega)| \le C(\omega), \quad |u_0(y_1, \omega) - u_0(y_2, \omega)| \le L_{u_0}(\omega) |y_1 - y_2|^{\beta(u_0)},$$

where $C(\omega)$, $L_{u_0}(\omega)$ are random constants, $\beta(u_0) \ge 1/2$.

Assumption 4. $f : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous, bounded, and

$$|f(s, y_1, v_1) - f(s, y_2, v_2)| \le L_f \big(|y_1 - y_2| + |v_1 - v_2|\big)$$

for some constant L_f and all $s \in \mathbb{R}_+$, $y_1, y_2, v_1, v_2 \in \mathbb{R}$.

Assumption 5. $\sigma : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is measurable and

$$|\sigma(s, y)| \le C_\sigma(s), \quad |\sigma(s, y_1) - \sigma(s, y_2)| \le L_\sigma(s) |y_1 - y_2|^{\beta(\sigma)},$$

for some constant $1/2 < \beta(\sigma) < 1$, all $s \in \mathbb{R}_+$, $y_1, y_2 \in \mathbb{R}$ and bounded functions $C_\sigma, L_\sigma : \mathbb{R}_+ \to \mathbb{R}$ such that

$$C_\sigma(s) \to 0,\ s \to \infty; \quad L_\sigma(s) \to 0,\ s \to \infty.$$

To proceed further, we need some statements about $\mathcal{L}$ and p. The following lemma [6, Theorem 10 §1] is formulated for our specific $\mathcal{L}$.

Lemma 3. *Assume that the function $v(t, x) : [0, T] \times \mathbb{R} \to \mathbb{R}$ is bounded, Assumptions 1, 2 hold, $|v(0, x)| \le M_1$, $|\mathcal{L}v(t, x)| \le M_2$. Then*

$$|v(t, x)| \le e^{-c_0 t}(M_1 + M_2 t).$$

Lemma 4. *There exist positive constants ν, η, C such that for each $x, y \in \mathbb{R}$, $t > s > 0$ the following estimates hold:*

$$|p(t, x; s, y)| \le C(t-s)^{-1/2} e^{-\frac{\nu(x-y)^2}{t-s}} e^{\eta(t-s)} E_{\lambda,\lambda+1}(\tilde{C}(t-s)^{\lambda}), \tag{10}$$

$$\Big|\frac{\partial p(t, x; s, y)}{\partial x}\Big| \le C(t-s)^{-1} e^{-\frac{\nu(x-y)^2}{t-s}} e^{\eta(t-s)} E_{\lambda,\lambda+1/2}(\tilde{C}(t-s)^{\lambda}), \tag{11}$$

$$\begin{aligned}\Big|\frac{\partial p(t, x; s, y)}{\partial x} - \frac{\partial p(t, x'; s, y)}{\partial x'}\Big| &\le C(\phi)|x - x'|^{\phi}(t-s)^{-3/2}\times \\ &\times \max\Big\{e^{-\frac{\nu(x-y)^2}{t-s}}, e^{-\frac{\nu(x'-y)^2}{t-s}}\Big\} e^{\eta(t-s)} E_{\lambda,\lambda}(\tilde{C}(t-s)^{\lambda}),\end{aligned} \tag{12}$$

where $E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}$ is the Mittag-Leffler function, $\phi < 1$.

Proof. We represent p as

$$p(t, x; s, y) = W(t, x; s, y) + \int_s^t d\theta \int_{\mathbb{R}} W(t, x; \theta, \zeta)\Phi(\theta, \zeta; s, y)\, d\zeta$$

(see, for example, [6, (4.18)]). Here

$$W(t, x; s, y) = \frac{1}{\sqrt{4\pi(t-s)a(s)}} e^{-\frac{(x-y)^2}{4(t-s)a(s)}},$$

the function $\Phi(t, x; s, y)$ is a solution of the integral equation

$$\Phi(t, x; s, y) = \mathcal{L}W(t, x; s, y) + \int_s^t d\theta \int_{\mathbb{R}} \mathcal{L}W(t, x; \theta, \zeta)\Phi(\theta, \zeta; s, y)\, d\zeta. \tag{13}$$

It is easy to calculate that

$$\begin{aligned}\frac{\partial W(t, x; s, y)}{\partial x} &= \frac{1}{\sqrt{4\pi(t-s)a(s)}} e^{-\frac{(x-y)^2}{4(t-s)a(s)}} \frac{y-x}{2(t-s)a(s)}, \\ \frac{\partial^2 W(t, x; s, y)}{\partial x^2} &= \frac{1}{\sqrt{4\pi(t-s)a(s)}} e^{-\frac{(x-y)^2}{4(t-s)a(s)}} \Big(\frac{(x-y)^2}{4(t-s)^2 a^2(s)} - \frac{1}{2(t-s)a(s)}\Big), \\ \frac{\partial W(t, x; s, y)}{\partial t} &= \frac{1}{\sqrt{4\pi(t-s)a(s)}} e^{-\frac{(x-y)^2}{4(t-s)a(s)}} \Big(\frac{(x-y)^2}{4(t-s)^2 a(s)} - \frac{1}{2(t-s)}\Big).\end{aligned}$$

Using Assumption 1 and boundedness of the function $x^{\alpha}e^{-x}$ on $[0, +\infty)$ for arbitrary $\alpha > 0$, we easily obtain that

$$a(s)\frac{\partial^2 W}{\partial x^2} - \frac{\partial W}{\partial t} = 0, \tag{14}$$

$$|W(t,x;s,y)| \le C e^{-\frac{\nu(x-y)^2}{t-s}} (t-s)^{-1/2}, \tag{15}$$

$$\left|\frac{\partial W(t,x;s,y)}{\partial x}\right| \le C e^{-\frac{\nu(x-y)^2}{t-s}} (t-s)^{-1}, \tag{16}$$

$$\left|\frac{\partial^2 W(t,x;s,y)}{\partial x^2}\right| \le C e^{-\frac{\nu(x-y)^2}{t-s}} (t-s)^{-3/2}, \tag{17}$$

where $0 < \nu < (4 \sup_{t\in\mathbb{R}} a(t))^{-1}$. The solution of (13) can be rewritten as

$$\Phi(t,x;s,y) = \sum_{k=1}^{\infty} \Phi_k(t,x;s,y),$$

where

$$\Phi_1(t,x;s,y) = \mathcal{L}W(t,x;s,y), \quad \Phi_{k+1}(t,x;s,y) = \int_s^t d\theta \int_{\mathbb{R}} \mathcal{L}W(t,x;\theta,\zeta)\Phi_k(\theta,\zeta;s,y)\,d\zeta.$$

Using (14)–(17), we obtain that

$$\begin{aligned} &|\Phi_1(t,x;s,y)| \\ &\le |a(t)-a(s)|\left|\frac{\partial^2 W(t,x;s,y)}{\partial x^2}\right| + |b(t)|\left|\frac{\partial W(t,x;s,y)}{\partial x}\right| + |c(t)||W(t,x;s,y)| \\ &\le C e^{-\frac{\nu(x-y)^2}{t-s}} \left((t-s)^{-3/2+\lambda} + (t-s)^{-1} + (t-s)^{-1/2}\right) \\ &\le C e^{-\frac{\nu(x-y)^2}{t-s}} (t-s)^{-3/2+\lambda} e^{\eta_1(t-s)}. \end{aligned}$$

Analogously to [6, (4.58)] we show that

$$|\Phi_k(t,x;s,y)| \le \frac{C}{\Gamma((k-1)\lambda+\lambda)} (\tilde{C}(t-s)^{\lambda})^{k-1} e^{-\frac{\nu(x-y)^2}{t-s}} e^{\eta_1(t-s)} (t-s)^{-3/2+\lambda}, \tag{18}$$

where constants C, $\tilde{C}$ depend on λ. Taking the sum of (18) for each $k \ge 1$, we get the inequality

$$|\Phi(t,x;s,y)| \le C e^{-\frac{\nu(x-y)^2}{t-s}} e^{\eta_1(t-s)} (t-s)^{-3/2+\lambda} E_{\lambda,\lambda}(\tilde{C}(t-s)^{\lambda}). \tag{19}$$

The following inequality plays an important role in further estimates

$$\int_{\mathbb{R}} (t-\theta)^{-1/2}(\theta-s)^{-1/2} e^{-\nu\left(\frac{(\zeta-y)^2}{\theta-s}+\frac{(x-\zeta)^2}{t-\theta}\right)}\,d\zeta = \sqrt{\frac{\pi}{\nu}}(t-s)^{-1/2} e^{-\frac{\nu(x-y)^2}{t-s}}. \tag{20}$$

Now we use (19) and (20) to obtain (10).

$$\begin{aligned} &\left|\int_s^t d\theta \int_{\mathbb{R}} W(t,x;\theta,\zeta)\Phi(\theta,\zeta;s,y)\,d\zeta\right| \\ &\le C \int_s^t d\theta \int_{\mathbb{R}} (t-\theta)^{-\frac{1}{2}} e^{-\frac{\nu(x-\zeta)^2}{t-\theta}} \frac{E_{\lambda,\lambda}(\tilde{C}(\theta-s)^{\lambda})}{(\theta-s)^{3/2-\lambda}} e^{\eta_1(\theta-s)} e^{-\frac{\nu(\zeta-y)^2}{\theta-s}}\,d\zeta \end{aligned}$$

$$\le Ce^{\eta_1(t-s)}\int_s^t \frac{E_{\lambda,\lambda}(\tilde C(\theta-s)^\lambda)}{(\theta-s)^{1-\lambda}}d\theta\int_{\mathbb{R}}(t-\theta)^{-\frac12}(\theta-s)^{-\frac12}e^{-\nu\left(\frac{(\zeta-y)^2}{\theta-s}+\frac{(x-\zeta)^2}{t-\theta}\right)}d\zeta$$
$$= Ce^{\eta_1(t-s)}(t-s)^{-1/2}e^{-\frac{\nu(x-y)^2}{t-s}}\int_s^t(\theta-s)^{-1+\lambda}E_{\lambda,\lambda}(\tilde C(\theta-s)^\lambda)d\theta$$
$$= C(t-s)^{-1/2+\lambda}e^{-\frac{\nu(x-y)^2}{t-s}}e^{\eta_1(t-s)}E_{\lambda,\lambda+1}(\tilde C(t-s)^\lambda), \tag{21}$$

where the last equality is a consequence of

$$\int_0^z E_{\rho,\mu}(\lambda t^\rho)t^{\mu-1}dt = z^\mu E_{\rho,\mu+1}(\lambda z^\rho) \tag{22}$$

[8, chapter III, (1.15)]. From (15), (21) and the inequalities $t \ge s$ $e^{\eta(t-s)} \ge 1$, $E_{\lambda,\lambda+1}(\tilde C(t-s)^\lambda) \ge \frac{1}{\Gamma(\lambda+1)}$ we obtain (10). We prove (11) analogously, using

$$\frac{1}{\sqrt{\pi}}\int_0^z E_{\rho,\mu}(\lambda t^\rho)(z-t)^{-1/2}t^{\mu-1}dt = z^{\mu-1/2}E_{\rho,\mu+1/2}(\lambda z^\rho)$$

[8, chapter III, (1.16) with $\alpha = 1/2$] instead of (22).

In the proof of (12) we use

$$\begin{aligned}&\left|\frac{\partial p(t,x;s,y)}{\partial x}-\frac{\partial p(t,x';s,y)}{\partial x'}\right|\\ &\quad=\left|\frac{\partial p(t,x;s,y)}{\partial x}-\frac{\partial p(t,x';s,y)}{\partial x'}\right|\mathbf{1}_{(x'-x)^2<A(t-s)}\\ &\quad\quad+\left|\frac{\partial p(t,x;s,y)}{\partial x}-\frac{\partial p(t,x';s,y)}{\partial x'}\right|\mathbf{1}_{(x'-x)^2\ge A(t-s)},\end{aligned} \tag{23}$$

where $A > 0$. Firstly assume that $(x'-x)^2 < A(t-s)$; we prove that for such t, x, s

$$\begin{aligned}\left|\frac{\partial p(t,x;s,y)}{\partial x}-\frac{\partial p(t,x';s,y)}{\partial x'}\right| &\le C(\phi,A)|x-x'|^\phi(t-s)^{-3/2}\times\\ &\times e^{-\frac{\nu(x-y)^2}{t-s}}e^{\eta(t-s)}E_{\lambda,\lambda}(\tilde C(t-s)^\lambda).\end{aligned} \tag{24}$$

Notice that

$$\begin{aligned}\left|\frac{\partial W(t,x;s,y)}{\partial x}-\frac{\partial W(t,x';s,y)}{\partial x'}\right| &\le C\left|\int_x^{x'}\frac{1}{(t-s)^{3/2}}e^{-\frac{\nu(v-y)^2}{t-s}}dv\right|\\ &\le C\frac{1}{(t-s)^{3/2}}e^{-\frac{\nu_1(x^*-y)^2}{t-s}}\int_0^{\frac{|x-x'|}{2}}e^{-\frac{(\nu-\nu_1)w^2}{t-s}}dw\\ &\le C\frac{1}{(t-s)^{3/2}}e^{-\frac{\nu_1(x^*-y)^2}{t-s}}\int_0^{\frac{|x-x'|}{2}}\left(\frac{t-s}{w^2}\right)^l dw\\ &\le C\frac{1}{(t-s)^{3/2-l}}e^{-\frac{\nu_1(x^*-y)^2}{t-s}}|x-x'|^{1-2l},\end{aligned}$$

where $0 \le l < 1/2$, $x^* = \epsilon x + (1-\epsilon)x'$, $0 \le \epsilon \le 1$, $0 < \nu_1 < \nu$; here we used the fact that the function $x^\alpha e^{-x}$ is bounded on $[0, +\infty)$ for arbitrary $\alpha > 0$. Now we show that $(x'-x)^2 < A(t-s)$ implies

$$e^{-\frac{\nu_1 (x^*-y)^2}{t-s}} \le C e^{-\frac{\nu_2 (x-y)^2}{t-s}}, \tag{25}$$

where $\nu_1 > \nu_2 > 0$, C, ν_2 do not depend on y and x^*, but C depends on A. We consider two cases.

1. $|x-y| \le 3|x-x'|$. We have the inequalities:

$$e^{-\frac{\nu_1 (x^*-y)^2}{t-s}} \le 1 \le e^A e^{-\frac{(x'-x)^2}{t-s}} \le e^A e^{-\frac{(x-y)^2}{9(t-s)}} = C_1 e^{-\frac{\nu_{21}(x-y)^2}{t-s}}.$$

2. $|x-y| > 3|x-x'|$. In this case we have the estimates:

$$\begin{aligned} e^{-\frac{\nu_1 (x^*-y)^2}{t-s}} &= e^{-\frac{\nu_1 (x-y)^2}{t-s}} e^{\frac{\nu_1 (x-x^*)(x+x^*-2y)}{t-s}} \le e^{-\frac{\nu_1 (x-y)^2}{t-s}} e^{\frac{\nu_1 |x-x'|(2|x-y|+|x-x^*|)}{t-s}} \\ &< e^{-\frac{\nu_1 (x-y)^2}{t-s}} e^{\frac{7\nu_1 (x-y)^2}{9(t-s)}} = e^{-\frac{2\nu_1 (x-y)^2}{9(t-s)}} = C_2 e^{-\frac{\nu_{22}(x-y)^2}{t-s}}. \end{aligned}$$

Now we set $C = \max\{C_1, C_2\}$, $\nu_2 = \min\{\nu_{21}, \nu_{22}\}$ and obtain (25). Therefore, the following estimate holds:

$$\left| \frac{\partial W(t,x;s,y)}{\partial x} - \frac{\partial W(t,x';s,y)}{\partial x'} \right| \le C \frac{1}{(t-s)^{3/2-l}} e^{-\frac{\nu_2 (x-y)^2}{t-s}} |x-x'|^{1-2l}. \tag{26}$$

Consider the expression

$$\begin{aligned} \frac{\partial p(t,x;s,y)}{\partial x} - \frac{\partial p(t,x';s,y)}{\partial x'} &= \left(\frac{\partial W(t,x;s,y)}{\partial x} - \frac{\partial W(t,x';s,y)}{\partial x'} \right) \\ &+ \int_{t-\frac{|x'-x|^2}{2A}}^{t} d\theta \int_{\mathbb{R}} \frac{\partial W(t,x;\theta,\zeta)}{\partial x} \Phi(\theta,\zeta;s,y)\, d\zeta \\ &- \int_{t-\frac{|x'-x|^2}{2A}}^{t} d\theta \int_{\mathbb{R}} \frac{\partial W(t,x';\theta,\zeta)}{\partial x'} \Phi(\theta,\zeta;s,y)\, d\zeta \\ &+ \int_{s}^{t-\frac{|x'-x|^2}{2A}} d\theta \int_{\mathbb{R}} \left(\frac{\partial W(t,x;\theta,\zeta)}{\partial x} - \frac{\partial W(t,x';\theta,\zeta)}{\partial x'} \right) \Phi(\theta,\zeta;s,y)\, d\zeta \\ &= J_0 + J_1 + J_2 + J_3. \end{aligned}$$

We estimate J_1 in the following way:

$$\begin{aligned} |J_1| &= \left| \int_{t-\frac{|x'-x|^2}{2A}}^{t} d\theta \int_{\mathbb{R}} \frac{\partial W(t,x;\theta,\zeta)}{\partial x} \Phi(\theta,\zeta;s,y)\, d\zeta \right| \\ &\le C \int_{t-\frac{|x'-x|^2}{2A}}^{t} d\theta \int_{\mathbb{R}} e^{-\nu\left(\frac{(\zeta-y)^2}{\theta-s} + \frac{(x-\zeta)^2}{t-\theta}\right)} \frac{E_{\lambda,\lambda}(\tilde{C}(\theta-s)^\lambda)}{(t-\theta)(\theta-s)^{3/2-\lambda}} e^{\eta_1(\theta-s)} d\zeta \end{aligned}$$

$$\leq Ce^{\eta_1(t-s)}e^{-\frac{\nu(x-y)^2}{t-s}}\frac{E_{\lambda,\lambda}(\tilde{C}(t-s)^\lambda)}{(t-s)^{1/2}}\int_{t-\frac{|x'-x|^2}{2A}}^{t}(t-\theta)^{-1/2}(\theta-s)^{\lambda-1}\,d\theta$$

$$\leq Ce^{\eta_1(t-s)}e^{-\frac{\nu(x-y)^2}{t-s}}\frac{E_{\lambda,\lambda}(\tilde{C}(t-s)^\lambda)}{(t-s)^{1/2}}\Big(t-s-\frac{|x'-x|^2}{2A}\Big)^{\lambda-1}\Big(\frac{|x'-x|^2}{2A}\Big)^{1/2}$$

$$\leq Ce^{\eta_1(t-s)}e^{-\frac{\nu(x-y)^2}{t-s}}(t-s)^{-3/2+\lambda}E_{\lambda,\lambda}(\tilde{C}(t-s)^\lambda)|x-x'|. \tag{27}$$

Here we used the inequality $t-s-\frac{|x'-x|^2}{2A}>\frac{t-s}{2}$ and (20). We estimate J_2 in analogous way using (25):

$$|J_2|=\Big|\int_{t-\frac{|x'-x|^2}{2A}}^{t}d\theta\int_{\mathbb{R}}\frac{\partial W(t,x';\theta,\zeta)}{\partial x'}\Phi(\theta,\zeta;s,y)\,d\zeta\Big|$$

$$\leq Ce^{\eta_1(t-s)}e^{-\frac{\nu_2(x-y)^2}{t-s}}(t-s)^{-3/2+\lambda}E_{\lambda,\lambda}(\tilde{C}(t-s)^\lambda)|x-x'|. \tag{28}$$

We apply (20) and (26) with $\tilde{A}=2A$ to prove the estimation for J_3:

$$|J_3|\leq\int_{s}^{t-\frac{|x'-x|^2}{2A}}d\theta\int_{\mathbb{R}}\Big|\frac{\partial W(t,x;\theta,\zeta)}{\partial x}-\frac{\partial W(t,x';\theta,\zeta)}{\partial x'}\Big||\Phi(\theta,\zeta;s,y)|\,d\zeta$$

$$\leq C|x-x'|^{1-2l}\int_{s}^{t-\frac{|x'-x|^2}{2A}}d\theta\int_{\mathbb{R}}\frac{e^{-\nu_2\left(\frac{(\zeta-y)^2}{\theta-s}+\frac{(x-\zeta)^2}{t-\theta}\right)}E_{\lambda,\lambda}(\tilde{C}(\theta-s)^\lambda)}{(t-\theta)^{3/2-l}(\theta-s)^{3/2-\lambda}}e^{\eta_1(\theta-s)}d\zeta$$

$$\leq C|x-x'|^{1-2l}e^{\eta_1(t-s)}e^{-\frac{\nu_2(x-y)^2}{t-s}}\frac{E_{\lambda,\lambda}(\tilde{C}(t-s)^\lambda)}{(t-s)^{1/2}}\int_{s}^{t}(t-\theta)^{l-1}(\theta-s)^{\lambda-1}\,d\theta$$

$$\leq C|x-x'|^{1-2l}e^{\eta_1(t-s)}e^{-\frac{\nu_2(x-y)^2}{t-s}}(t-s)^{-3/2+l+\lambda}E_{\lambda,\lambda}(\tilde{C}(t-s)^\lambda), \tag{29}$$

for arbitrary $l\in(0,1/2)$. Now (24) follows from (26), (27), (28) and (29).

Let $(x'-x)^2\geq A(t-s)$. This implies

$$\Big|\frac{\partial W(t,x;s,y)}{\partial x}-\frac{\partial W(t,x';s,y)}{\partial x'}\Big|\leq\Big|\frac{\partial W(t,x;s,y)}{\partial x}\Big|+\Big|\frac{\partial W(t,x';s,y)}{\partial x'}\Big|$$

$$\leq C(t-s)^{-1}\max\Big\{e^{-\frac{\nu(x-y)^2}{t-s}},e^{-\frac{\nu(x'-y)^2}{t-s}}\Big\}$$

$$\leq C(t-s)^{-3/2+l}\Big(\frac{|x'-x|}{\sqrt{A}}\Big)^{1-2l}\max\Big\{e^{-\frac{\nu(x-y)^2}{t-s}},e^{-\frac{\nu(x'-y)^2}{t-s}}\Big\}, \tag{30}$$

where $l\in(0,1)$. On the other hand,

$$\Big|\int_{s}^{t}d\theta\int_{\mathbb{R}}\frac{\partial W(t,x;\theta,\zeta)}{\partial x}\Phi(\theta,\zeta;s,y)\,d\zeta\Big|$$

$$\leq C\int_{s}^{t}d\theta\int_{\mathbb{R}}e^{-\nu\left(\frac{(\zeta-y)^2}{\theta-s}+\frac{(x-\zeta)^2}{t-\theta}\right)}\frac{E_{\lambda,\lambda}(\tilde{C}(\theta-s)^\lambda)}{(t-\theta)(\theta-s)^{3/2-\lambda}}e^{\eta_1(\theta-s)}d\zeta$$

$$\leq Ce^{\eta_1(t-s)}e^{-\frac{\nu(x-y)^2}{t-s}}\frac{E_{\lambda,\lambda}(\tilde{C}(t-s)^\lambda)}{(t-s)^{1/2}}\Big(\frac{|x'-x|}{\sqrt{A}}\Big)^{1-2l}\int_{s}^{t}(\theta-s)^{-1+\lambda}(t-\theta)^{-1+l}d\theta$$

$$= Ce^{\eta_1(t-s)}(t-s)^{-3/2+l+\lambda}e^{-\frac{\nu(x-y)^2}{t-s}}\left(\frac{|x'-x|}{\sqrt{A}}\right)^{1-2l}E_{\lambda,\lambda}(\tilde{C}(t-s)^{\lambda}),$$

and the same estimates hold for $\int_s^t d\theta \int_{\mathbb{R}} \frac{\partial W(t,x';\theta,\zeta)}{\partial x'}\Phi(\theta,\zeta;s,y)\,d\zeta$. Using (30), (24) and (23), we obtain (12); in (23) we can set, for example, $A=1$. □

The main result of the paper is this theorem.

Theorem 1. *Let Assumptions 1–5 hold. Then there exists the version of the solution of* (7) *such that for each* $\omega \in \Omega$*:*

$$\sup_{x\in\mathbb{R}} |u(t,x)| \to 0,\ \ t \to \infty. \tag{31}$$

4 Proof of the auxillary lemma and the main result

To prove Theorem 1, we consider the integral

$$\int_{\mathbb{R}} d\mu(y) \int_0^t p(t,x;s,y)\sigma(s,y)\,ds. \tag{32}$$

Lemma 5. *Assume Assumptions 1, 2, 5 hold. Then there exists a version* $v_1(t,x)$ *of integral* (32) *such that for each* $\omega \in \Omega$

$$\sup_{x\in\mathbb{R}} |v_1(t,x)| \to 0,\ \ t \to \infty. \tag{33}$$

Proof. It follows from Lemma 2 and (6) that a version $v_1(t,x)$ of integral (32) exists such that for each $x \in \mathbb{R}$, $t \geq 0$, $\omega \in \Omega$

$$\begin{aligned}|v_1(t,x)| &\leq \sum_{j\in\mathbb{Z}} |q(t,x,j)\mu((j,j+1])| + C\sum_{j\in\mathbb{Z}} \|q(t,x,\cdot)\|_{B^{\alpha}_{22}([j,j+1])}\\ &\quad\times\left(\sum_{n=1}^{\infty} 2^{-n(2\alpha-1)}\sum_{k=1}^{2^n}|\mu(\Delta^{(j)}_{kn})|^2\right)^{1/2}\\ &\leq \Big(\sum_{j\in\mathbb{Z}}|q(t,x,j)|^2\Big)^{1/2}\Big(\sum_{j\in\mathbb{Z}}|\mu((j,j+1])|^2\Big)^{1/2}\\ &+C\Big(\sum_{j\in\mathbb{Z}}\|q(t,x,\cdot)\|^2_{B^{\alpha}_{22}([j,j+1])}\Big)^{1/2}\Big(\sum_{j\in\mathbb{Z}}\sum_{n=1}^{\infty}2^{-n(2\alpha-1)}\sum_{k=1}^{2^n}|\mu(\Delta^{(j)}_{kn})|^2\Big)^{1/2},\end{aligned} \tag{34}$$

where

$$q(t,x,y) = \int_0^t p(t,x;s,y)\sigma(s,y)\,ds.$$

Next we prove that $v_1(t,x)$ satisfies (33). In order to estimate the Besov norm on $[j,j+1]$, we consider

$$|q(t,x,y+h)-q(t,x,y)| \leq \int_0^t |p(t,x;s,y+h)-p(t,x;s,y)|\,|\sigma(s,y)|\,ds$$

$$+\int_0^t |p(t,x;s,y+h)|\,|\sigma(s,y+h)-\sigma(s,y)|\,ds = I_1 + I_2,$$

where $y,\ y+h \in [j, j+1]$. Denote

$$\begin{aligned}
\Omega_{MN} &= \big([s,+\infty)\times\mathbb{R}\big)\setminus\big([s,s+M)\times(y-N,y+N)\big),\\
\Omega_{MN}^{\gamma} &= \{(t,x) : d(\Omega_{MN},(t,x)) \le \gamma\},\\
\eta_1(v) &= Ce^{\frac{1}{|v|^2-1}}\mathbf{1}_{\{|v|<1\}};\quad v\in\mathbb{R}^2,\ \int_{\mathbb{R}^2}\eta_1(v)\,dv = 1,\\
\eta_\varepsilon(v) &= \varepsilon^{-2}\eta_1(v\varepsilon^{-1}),\\
\Theta_{MN}^{\gamma} &= (\Omega_{MN}^{2\gamma}\setminus\Omega_{MN})\cap\{t>s\},
\end{aligned}$$

where $M, N, \gamma > 0$. To estimate I_2 we introduce the function

$$\tilde{p}(t,x;s,y) = p(t,x;s,y)\mathbf{1}_{\{t>s\}}\Psi(t,x;s,y),$$

where

$$\Psi(t,x;s,y) = \int_{\Omega_{MN}^{\gamma}} \eta_\gamma(t-v_1, x-v_2)\,dv,\quad v=(v_1,v_2).$$

It is obvious that, for each $0<\gamma<\min\{M/2, N/2\}$ and arbitrary fixed pair (s,y), $\tilde{p}$ belongs to a class $C^{1,2}([0,+\infty)\times\mathbb{R})$ as a function of (t,x). It is easy to obtain that $\tilde{p}(t,x;s,y) = p(t,x;s,y)$ if $(t,x)\in\Omega_{MN}$ and $\tilde{p}(t,x;s,y)=0$ if $(t,x)\in \big([0,+\infty)\times\mathbb{R}\big)\setminus\Omega_{MN}^{2\gamma}\cup\{s\ge t\}$. Moreover, boundedness of $p(t,x;s,y)$ on $\Omega_{MN}^{2\gamma}\cap\{t\le T\}$ for each $T>0$ and the fact that $|\Psi|\le 1$ imply boundedness of $\tilde{p}(t,x;s,y)$ on $([0,T]\times\mathbb{R})$. Now we estimate $\mathcal{L}\tilde{p}$. Taking into consideration properties of $\tilde{p}$, it is easy to see that $\mathcal{L}\tilde{p}=0$ outside the set Θ_{MN}^{γ}. And inside Θ_{MN}^{γ},

$$\mathcal{L}\tilde{p} = \mathcal{L}(p\Psi) = \Psi\mathcal{L}p + p\mathcal{L}\Psi + 2a\frac{\partial p}{\partial x}\frac{\partial \Psi}{\partial x} \overset{\mathcal{L}p=0}{=} p\mathcal{L}\Psi + 2a\frac{\partial p}{\partial x}\frac{\partial \Psi}{\partial x}. \tag{35}$$

For the derivatives of $\eta_\varepsilon(v)$ we have the following well-known inequalities:

$$\left|\frac{\partial\eta_\varepsilon(v)}{\partial v_i}\right| \le C\varepsilon^{-3},\quad \left|\frac{\partial^2\eta_\varepsilon(v)}{\partial v_i^2}\right| \le C\varepsilon^{-4},\quad i=1,2, \tag{36}$$

where the constant C does not depend on ε. Let us prove, for example, the first inequality in (36):

$$\left|\frac{\partial\eta_\varepsilon(v)}{\partial v_i}\right| \le \varepsilon^{-3}\max_{|v|\le 1}\left|\frac{\partial\eta_1(v)}{\partial v_i}\right| = C\varepsilon^{-3},$$

where we used that $\eta_1\in C^\infty(\mathbb{R}^2)$. From (36) and the fact that $\eta_\varepsilon = 0$ outside the ball with radius ε it follows that

$$\left|\frac{\partial\Psi}{\partial x}\right| \le C\gamma^{-1},\quad \left|\frac{\partial\Psi}{\partial t}\right| \le C\gamma^{-1},\quad \left|\frac{\partial^2\Psi}{\partial x^2}\right| \le C\gamma^{-2}.$$

Using estimates (35), (10) and (11) we obtain the inequality

$$|\mathcal{L}\tilde{p}| \le Ce^{-\frac{\nu(x-y)^2}{t-s}}e^{\eta(t-s)}\left(\gamma^{-2}\frac{E_{\lambda,\lambda+1}(\tilde{C}(t-s)^\lambda)}{\sqrt{t-s}} + \gamma^{-1}\frac{E_{\lambda,\lambda+1/2}(\tilde{C}(t-s)^\lambda)}{t-s}\right) \tag{37}$$

for each $(t,x) \in \Theta^{\gamma}_{MN}$. Let $\gamma \in (0, 1/3)$ be fixed; consider $t > 3\gamma$. Thus, I_2 can be rewritten in following way:

$$I_2 = \int_0^{t-3\gamma} |p(t,x;s,y+h)|\,|\sigma(s,y+h) - \sigma(s,y)|\,ds$$
$$+ \int_{t-3\gamma}^{t} |p(t,x;s,y+h)|\,|\sigma(s,y+h) - \sigma(s,y)|\,ds = I_{21} + I_{22}.$$

We estimate the first summand using the function $\tilde{p}(t,x;s,y)$ for $M = N = 3\gamma$. Note that $t - s < 3\gamma$ on the set Θ^{γ}_{MN}; moreover, $t - s > \gamma$ or $|x - y| > \gamma$. In the first case we have the following consequence of (37):

$$|\mathcal{L}\tilde{p}| \le Ce^{3\eta\gamma}\gamma^{-5/2}\big(E_{\lambda,\lambda+1}(\tilde{C}(3\gamma)^{\lambda}) + E_{\lambda,\lambda+1/2}(\tilde{C}(3\gamma)^{\lambda})\big).$$

In the second case,

$$|\mathcal{L}\tilde{p}| \le Ce^{3\eta\gamma}\gamma^{-7/2}\big(E_{\lambda,\lambda+1}(\tilde{C}(3\gamma)^{\lambda}) + E_{\lambda,\lambda+1/2}(\tilde{C}(3\gamma)^{\lambda})\big).$$

Anyway,

$$|\mathcal{L}\tilde{p}| \le C(\gamma)\ \forall\,(t,x) \in [0,+\infty) \times \mathbb{R}.$$

Using Lemma 3 for arbitrary $T > t$, we obtain

$$|\tilde{p}(t,x;s,y)| \le C(\gamma)e^{-c_0 t}t,$$

where the constant C does not depend on T. On the other hand, taking into account that $\tilde{p}(t,x;s,y) = p(t,x;s,y)$ if $t - s > 3\gamma$ we obtain

$$|I_{21}| \le C(\gamma)h^{\beta(\sigma)}e^{-c_0 t}t^2 \to 0,\ t \to \infty. \tag{38}$$

Now we estimate I_{22}. We get

$$\begin{aligned}|I_{22}| &\le Ch^{\beta(\sigma)}\int_{t-3\gamma}^{t}\frac{1}{\sqrt{t-s}}e^{-\frac{\nu(x-y)^2}{t-s}}e^{\eta(t-s)}E_{\lambda,\lambda+1}(\tilde{C}(t-s)^{\lambda})L_{\sigma}(s)\,ds\\ &\le Ch^{\beta(\sigma)}\sqrt{\gamma}e^{3\gamma\eta}E_{\lambda,\lambda+1}(\tilde{C}(3\gamma)^{\lambda})\sup_{s\in[t-3\gamma,t]}|L_{\sigma}(s)| \to 0,\ t \to \infty.\end{aligned} \tag{39}$$

For further estimates we use the function

$$\hat{p}(t,x;s,y,h) = (p(t,x;s,y+h) - p(t,x;s,y))\mathbf{1}_{\{t>s\}}\Psi(t,x;s,y).$$

Let $M > 2\gamma$, $N > 1 + 2\gamma$. Then function $\hat{p}$ has properties, which are analogouos to the properties of $\tilde{p}$. For example, $\hat{p}(t,x;s,y,h) = p(t,x;s,y+h) - p(t,x;s,y)$ when $(t,x) \in \Omega_{MN}$, $\hat{p}$ is bounded on $([0,T] \times \mathbb{R})$, where $T > 0$, $\hat{p} = 0$ when $(t,x) \in \big([0,+\infty) \times \mathbb{R}\big) \setminus \Omega^{2\gamma}_{MN} \cup \{s \ge t\}$. Now we estimate $\mathcal{L}\hat{p}$. Notice that $\mathcal{L}\tilde{p} = 0$ outside the set Θ^{γ}_{MN}, and for each $(t,x) \in \Theta^{\gamma}_{MN}$ the following estimates hold:

$$\begin{aligned}\mathcal{L}\hat{p} &= \mathcal{L}((p_{y+h} - p_y)\Psi)\\ &= \Psi\mathcal{L}(p_{y+h} - p_y) + (p_{y+h} - p_y)\mathcal{L}\Psi + 2a\frac{\partial(p_{y+h} - p_y)}{\partial x}\frac{\partial\Psi}{\partial x}\end{aligned}$$

$$\stackrel{\mathcal{L}p=0}{=} (p_{y+h} - p_y)\mathcal{L}\Psi + 2a\frac{\partial(p_{y+h} - p_y)}{\partial x}\frac{\partial \Psi}{\partial x},$$

where for convenience we denote $p_y = p(t, x; s, y)$. (11) and (12) imply that

$$\Big|\frac{\partial p(t, x; s, y+h)}{\partial x} - \frac{\partial p(t, x; s, y)}{\partial x}\Big|$$
$$\le Ch^{\phi}(t-s)^{-3/2}\sup_{y\in[j,j+1]} e^{-\frac{\nu(x-y)^2}{t-s}} e^{\eta(t-s)} E_{\lambda,\lambda}(\tilde{C}(t-s)^{\lambda});$$
$$\big|p(t, x; s, y+h) - p(t, x; s, y)\big| = \Big|\int_{x-y-h}^{x-y}\frac{\partial p(t, v; s, 0)}{\partial v}\, dv\Big|$$
$$\le \int_{x-y-h}^{x-y}\frac{1}{t-s}e^{-\frac{\nu v^2}{t-s}}e^{\eta(t-s)}E_{\lambda,\lambda+1/2}(\tilde{C}(t-s)^{\lambda})\, dv$$
$$\le \frac{h}{t-s}\sup_{y\in[j,j+1]} e^{-\frac{\nu(x-y)^2}{t-s}} e^{\eta(t-s)} E_{\lambda,\lambda+1/2}(\tilde{C}(t-s)^{\lambda}).$$

Therefore, for each $(t, x) \in \Theta^{\gamma}_{MN}$,

$$|\mathcal{L}\hat{p}| \le$$
$$\le Ch^{\phi}\sup_{y\in[j,j+1]} e^{-\frac{\nu(x-y)^2}{t-s}} e^{\eta(t-s)}$$
$$\times\bigg(\gamma^{-2}\frac{E_{\lambda,\lambda+1/2}(\tilde{C}(t-s)^{\lambda})}{t-s} + \gamma^{-1}\frac{E_{\lambda,\lambda}(\tilde{C}(t-s)^{\lambda})}{(t-s)^{3/2}}\bigg). \tag{40}$$

We rewrite I_1 as

$$I_1 = \int_0^{t-3\gamma} |p(t, x; s, y+h) - p(t, x; s, y)|\,|\sigma(s, y)|\, ds$$
$$+ \int_{t-3\gamma}^{t} |p(t, x; s, y+h) - p(t, x; s, y)|\,|\sigma(s, y)|\, ds = I_{11} + I_{12}.$$

Consider the function $\hat{p}(t, x; s, y, h)$ for $M = 3\gamma$, $N = 3\gamma + 1$. We estimate I_{11} analogouosly to I_{21}. It follows from (40) and Lemma 3 that

$$|\hat{p}(t, x; s, y, h)| \le C(\gamma)h^{\phi}e^{-c_0 t}t\ \forall (t, x) \in [0, +\infty) \times \mathbb{R}.$$

On the other hand, the equality $\hat{p}(t, x; s, y, h) = p(t, x; s, y+h) - p(t, x; s, y)$ for $t - s > 3\gamma$ implies that

$$|I_{11}| \le C(\gamma)h^{\phi}e^{-c_0 t}t^2 \to 0,\ t \to \infty. \tag{41}$$

I_{12} is estimated in the following way:

$$|I_{12}| \le C\int_{t-3\gamma}^{t} ds\int_{x-y-h}^{x-y}\frac{C_{\sigma}(s)}{t-s}e^{-\frac{\nu v^2}{t-s}}e^{\eta(t-s)}E_{\lambda,\lambda+1/2}(\tilde{C}(t-s)^{\lambda})\, dv$$
$$\le Ce^{3\eta\gamma}E_{\lambda,\lambda+1/2}(\tilde{C}(3\gamma)^{\lambda})\sup_{s\in[t-3\gamma,t]}|C_{\sigma}(s)|\int_0^{h/2} v^{-2l}\, dv\int_{t-3\gamma}^{t}(t-s)^{1-l}\, ds$$

$$= Ce^{3\eta\gamma} E_{\lambda,\lambda+1/2}(\tilde{C}(3\gamma)^{\lambda}) \sup_{s\in[t-3\gamma,t]} |C_\sigma(s)|\gamma^{1-l} h^{1-2l} \to 0,\ t \to \infty, \tag{42}$$

where $l \in (0, 1/2)$. We can choose l, ϕ such that $1 - 2l = \beta(\sigma), \phi = \beta(\sigma)$.

Now assume that for each $y \in [j, j+1]$ and some $m \in \mathbb{N}$ the following inequality holds:

$$|x - y| \geq m + 1. \tag{43}$$

Then we consider the functions $\tilde{p}(t, x; s, y)$ and $\hat{p}(t, x; s, y, h)$ with $M = 3\gamma$ and $N = 3\gamma + m$. For such M and N, provided that (43) holds, $(t, x) \in \Omega_{MN}$. Moreover, using (37), (40), the fact that (43) implies

$$\sup_{y\in[j,j+1]} e^{-\frac{\nu(x-y)^2}{t-s}} \leq e^{-\frac{\nu_1 m^2}{3\gamma}} \sup_{y\in[j,j+1]} e^{-\frac{(1-\nu_1)(x-y)^2}{t-s}} \quad \forall (t, x) \in \Theta^{\gamma}_{MN},\ 0 < \nu_1 < \nu,$$

and Lemma 3, we obtain:

$$|p(t, x; s, y)| \leq C(\gamma)e^{-c_0 t} t e^{-\frac{\nu_1 m^2}{3\gamma}},$$

$$|p(t, x; s, y + h) - p(t, x; s, y)| \leq C(\gamma)h^{\beta(\sigma)} e^{-c_0 t} t e^{-\frac{\nu_1 m^2}{3\gamma}}.$$

Now it is easy to estimate I_1, I_2:

$$I_1 \leq C(\gamma)h^{\beta(\sigma)} e^{-c_0 t} t^2 e^{-\frac{\nu_1 m^2}{3\gamma}} \leq C(\gamma)h^{\beta(\sigma)} e^{-c_0 t} t^2 m^{-1} \to 0,\ t \to \infty, \tag{44}$$

$$I_2 \leq C(\gamma)h^{\beta(\sigma)} e^{-c_0 t} t^2 e^{-\frac{\nu_1 m^2}{3\gamma}} \leq C(\gamma)h^{\beta(\sigma)} e^{-c_0 t} t^2 m^{-1} \to 0, t \to \infty. \tag{45}$$

Note that we estimate $|q(t, x, y)|$ analogously to I_2. From (38), (39), (41), (42), (44), (45) it follows that there exists $G_\gamma(t) : [0, +\infty) \to [0, +\infty)$ such that $G_\gamma(t) \to 0$, $t \to \infty$, and

$$w_2(q, r) \leq G_\gamma(t) r^{\beta(\sigma)}\ \forall t,\ r \geq 0,\ j \in \mathbb{Z}, x \in \mathbb{R};$$

$$|q(t, x, j)| \leq G_\gamma(t)\ \forall t \geq 0,\ j \in \mathbb{Z}, x \in \mathbb{R};$$

$$w_2(q, r) \leq G_\gamma(t) r^{\beta(\sigma)} m^{-1}\ \forall t,\ r \geq 0,\ j \in \mathbb{Z}, x \in \mathbb{R} : \max_{y\in[j,j+1]} |x - y| \geq m + 1;$$

$$|q(t, x, j)| \leq G_\gamma(t) m^{-1}\ \forall t \geq 0,\ j \in \mathbb{Z}, x \in \mathbb{R} : \max_{y\in[j,j+1]} |x - y| \geq m + 1.$$

From this it follows that for each $\alpha \in (1/2, \beta(\sigma))$ the following inequalities hold:

$$\|q(t, x, \cdot)\|_{B^\alpha_{22}([j,j+1])} \leq CG_\gamma(t)\ \forall t \geq 0,\ j \in \mathbb{Z}, x \in \mathbb{R};$$

$$\|q(t, x, \cdot)\|_{B^\alpha_{22}([j,j+1])} \leq CG_\gamma(t) m^{-1}$$

$$\forall t \geq 0,\ j \in \mathbb{Z}, x \in \mathbb{R} : \max_{y\in[j,j+1]} |x - y| \geq m + 1.$$

These estimates imply

$$\sum_{j\in\mathbb{Z}} |q(t, x, j)|^2 \leq CG^2_\gamma(t) + CG^2_\gamma(t) \sum_{m\in\mathbb{N}} \frac{1}{m^2} = CG^2_\gamma(t),$$

$$\sum_{j\in\mathbb{Z}} \|q(t,x,\cdot)\|^2_{B^{\alpha}_{22}([j,j+1])} \le CG^2_{\gamma}(t) + CG^2_{\gamma}(t)\sum_{m\in\mathbb{N}}\frac{1}{m^2} = CG^2_{\gamma}(t).$$

for each $x \in \mathbb{R}$. On the other hand,

$$\sum_{j\in\mathbb{Z}} |\mu((j, j+1])|^2 < \infty \text{ a.s.},$$

$$\sum_{j\in\mathbb{Z}}\sum_{n=1}^{\infty}\sum_{k=1}^{2^n} 2^{-n(2\alpha-1)}|\mu(\Delta^{(j)}_{kn})|^2 < \infty \text{ a.s.}$$

Therefore, for each version that satisfies (34) we have

$$|v_1(t,x)| \le C(\omega)G_{\gamma}(t).$$

Taking a supremum on x and sending t to infinity, we obtain the statement of the lemma. □

Now we return to the proof of Theorem 1.

Proof. We use the iteration process (9). For each $n \in \mathbb{N}$ we consider the function

$$v_2^{(n)}(t,x) = \int_{\mathbb{R}} p(t,x;0,y)u_0(y)\,dy + \int_0^t ds\int_{\mathbb{R}} p(t,x;s,y)f(s,y,u^{(n-1)}(s,y))\,dy.$$

From [6, Theorem 2 §4] it follows that the function $v_2^{(n)}$ is a solution, bounded on $[0,T]\times\mathbb{R}$, of the Cauchy problem

$$\mathcal{L}v(t,x) = -f(t,x,u^{(n-1)}(t,x)), \quad v(0,x) = u_0(x),$$

for each $\omega \in \Omega$, $T > 0$. Using Lemma 3, we obtain

$$|v_2^{(n)}(t,x)| \le Ce^{-c_0 t}(1+t). \tag{46}$$

Now from (8) and (46) it follows that

$$|u(t,x)| \le Ce^{-c_0 t}(1+t) + \sup_{x\in\mathbb{R}} |v_1(t,x)|.$$

Taking a supremum on x, sending t to infinity and using Lemma 5, we obtain the statement of the theorem. □

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