

Asymptotic properties of the parabolic equation driven by stochastic measure

Boris Manikin

*Taras Shevchenko National University of Kyiv,
Kyiv, Ukraine*

bmanikin@gmail.com (B. Manikin)

Received: 20 April 2022, Revised: 20 August 2022, Accepted: 23 August 2022,
Published online: 6 September 2022

Abstract A stochastic parabolic equation on $[0, T] \times \mathbb{R}$ driven by a general stochastic measure, for which we assume only σ -additivity in probability, is considered. The asymptotic behavior of its solution as $t \rightarrow \infty$ is studied.

Keywords Stochastic measure, mild solution, stochastic parabolic equation, asymptotic behavior

2010 MSC 60G57

1 Introduction

In this paper we consider the stochastic parabolic equation

$$\begin{cases} \mathcal{L}u(t, x)dt + f(t, x, u(t, x)) dt + \sigma(t, x) d\mu(x) = 0, \\ u(0, x) = u_0(x), \end{cases} \quad (1)$$

where $(t, x) \in [0, T] \times \mathbb{R}$, μ is a general stochastic measure on the Borel σ -algebra on \mathbb{R} (see Section 2), f, σ are measurable functions, \mathcal{L} is the operator of the kind

$$\mathcal{L}u(t, x) = a(t) \frac{\partial^2 u(t, x)}{\partial^2 x} + b(t) \frac{\partial u(t, x)}{\partial x} + c(t)u(t, x) - \frac{\partial u(t, x)}{\partial t}. \quad (2)$$

Here a, b, c are defined on $[0, T]$. We prove that under certain conditions on a, b, c, f, σ the solution of (1), considered in the mild sense, tends to 0 a.s. uniformly on x .

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Note that regularity of the solution was proved in [2], the solution's convergence in the case of integrator's convergence was proved in [18] and the averaging principle for such an equation was established in [12].

The asymptotic behavior of the moments of solutions of a stochastic differential system driven by a Brownian motion was considered in [5]. The problem of the convergence of the solution of a nonautonomous logistic differential equation to zero as time coordinate goes to infinity, with disturbance of coefficients by white noise, centered and noncentered Poisson noises, was studied in [4]. Asymptotics of the solution of the stochastic heat equation with white noise, as time variable goes to infinity for the fixed spatial coordinate, was studied in [10] while asymptotic properties of the solution of the stochastic wave equation driven by a Lévy process were given in [7]. Behavior of solutions of different equations with a general stochastic measure when spatial coordinate goes to infinity was considered in [3] and [1]. In comparison to [15], where asymptotics of the heat equation driven by a general stochastic measure when time coordinate tends to infinity was considered, we study a more general parabolic equation.

The paper is organized as follows. Section 2 contains some general facts about stochastic measures and integrals with respect to them. In Section 3 we prove some technical facts and formulate the main result, which is proved in Section 4 jointly with the auxiliary lemma.

2 Preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and \mathcal{B} be a Borel σ -algebra on \mathbb{R} . Denote by $L_0 = L_0(\Omega, \mathcal{F}, \mathbb{P})$ the set of all real-valued random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Convergence in L_0 means the convergence in probability.

Definition 1. A σ -additive mapping $\mu : \mathcal{B} \rightarrow L_0$ is called *stochastic measure* (SM).

In other words, μ is a vector measure with values in L_0 . We do not assume any martingale properties or moment existence for SM.

Consider some examples of SMs. If M_t is a square integrable martingale then $\mu(A) = \int_0^T \mathbf{1}_A(t) dM_t$ is an SM. An α -stable random measure on \mathcal{B} for $\alpha \in (0, 1) \cup (1, 2]$, as it is defined in [19, Sections 3.2–3.3], is an SM in sense of Definition 1. For a fractional Brownian motion W_t^H with Hurst index $H > 1/2$ and a bounded measurable function $f : [0, T] \rightarrow \mathbb{R}$ we can define an SM $\mu(A) = \int_0^T f(t) \mathbf{1}_A(t) dW_t^H$, see [13, Theorem 1.1]. Ref. [17] contains some other examples.

The integral $\int_A g d\mu$, where $g : \mathbb{R} \rightarrow \mathbb{R}$ is a deterministic measurable function, $A \in \mathcal{B}$ and μ is an SM, is defined and its basic properties are given in [11, Chapter 7]. In that paper the integral with respect to general stochastic measure was constructed and studied for μ defined on an arbitrary σ -algebra, but here we consider SM on Borel subsets of \mathbb{R} . Note that every bounded measurable g is integrable with respect to (w.r.t.) any μ .

In the sequel, μ denotes an SM, C and $C(\omega)$ denote positive constant and positive random constant, respectively, whose exact values are not important ($C < \infty$, $C(\omega) < \infty$ a.s.).

We use the following statement.

Lemma 1 (Lemma 3.1 in [14]). *Let $\phi_l : \mathbb{R} \rightarrow \mathbb{R}$, $l \geq 1$, be measurable functions such that $\tilde{\phi}(x) = \sum_{l=1}^{\infty} |\phi_l(x)|$ is integrable w.r.t. μ on \mathbb{R} . Then*

$$\sum_{l=1}^{\infty} \left(\int_{\mathbb{R}} \phi_l d\mu \right)^2 < \infty \quad \text{a.s.}$$

We consider the Besov spaces $B_{22}^{\alpha}([c, d])$, $0 < \alpha < 1$, with a standard norm

$$\|g\|_{B_{22}^{\alpha}([c, d])} = \|g\|_{L_2([c, d])} + \left(\int_0^{d-c} (w_2(g, r))^2 r^{-2\alpha-1} dr \right)^{1/2}, \quad (3)$$

where

$$w_2(g, r) = \sup_{0 \leq h \leq r} \left(\int_c^{d-h} |g(y+h) - g(y)|^2 dy \right)^{1/2}.$$

For any $j \in \mathbb{Z}$ and all $n \geq 0$, put

$$d_{kn}^{(j)} = j + k2^{-n}, \quad 0 \leq k \leq 2^n, \quad \Delta_{kn}^{(j)} = (d_{(k-1)n}^{(j)}, d_{kn}^{(j)}], \quad 1 \leq k \leq 2^n.$$

The following lemma is a key tool for estimates of the stochastic integral.

Lemma 2 (Lemma 3 in [16]). *Let Z be an arbitrary set, and the function $q(z, s) : Z \times [j, j+1] \rightarrow \mathbb{R}$ be such that all paths $q(z, \cdot)$ are continuous on $[j, j+1]$. Denote*

$$q_n(z, s) = \sum_{1 \leq k \leq 2^n} q(z, d_{(k-1)n}^{(j)}) \mathbf{1}_{\Delta_{kn}^{(j)}}(s).$$

Then the random function

$$\eta(z) = \int_{(j, j+1]} q(z, s) d\mu(s), \quad z \in Z,$$

has a version

$$\begin{aligned} \tilde{\eta}(z) &= \int_{(j, j+1]} q_0(z, s) d\mu(s) \\ &+ \sum_{n \geq 1} \left(\int_{(j, j+1]} q_n(z, s) d\mu(s) - \int_{(j, j+1]} q_{n-1}(z, s) d\mu(s) \right) \end{aligned} \quad (4)$$

such that for all $\beta > 0$, $\omega \in \Omega$, $z \in Z$

$$\begin{aligned} |\tilde{\eta}(z)| &\leq |q(z, j)\mu((j, j+1])| + \left\{ \sum_{n \geq 1} 2^{n\beta} \sum_{1 \leq k \leq 2^n} |q(z, d_{kn}^{(j)}) - q(z, d_{(k-1)n}^{(j)})|^2 \right\}^{1/2} \\ &\quad \times \left\{ \sum_{n \geq 1} 2^{-n\beta} \sum_{1 \leq k \leq 2^n} |\mu(\Delta_{kn}^{(j)})|^2 \right\}^{1/2}. \end{aligned} \quad (5)$$

From Theorem 1.1 [9] it follows that, for $\alpha = (\beta + 1)/2$,

$$\left\{ \sum_{n \geq 1} 2^{n\beta} \sum_{1 \leq k \leq 2^n} |q(z, d_{kn}^{(j)}) - q(z, d_{(k-1)n}^{(j)})|^2 \right\}^{1/2} \leq C \|q(z, \cdot)\|_{B_{22}^{\alpha}([j, j+1])}. \quad (6)$$

Lemma 1 implies that for each $\beta > 0$, $j \in \mathbb{Z}$

$$\sum_{n \geq 1} 2^{-n\beta} \sum_{1 \leq k \leq 2^n} |\mu(\Delta_{kn}^{(j)})|^2 < +\infty \quad \text{a.s.}$$

3 Formulation of the problem and the main result

We refer to the mild solution to (1), i.e. the measurable random function $u(t, x) = u(t, x, \omega) : [0, T] \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}$ that satisfies

$$\begin{aligned} u(t, x) = & \int_{\mathbb{R}} p(t, x; 0, y) u_0(y) dy + \int_0^t ds \int_{\mathbb{R}} p(t, x; s, y) f(s, y, u(s, y)) dy \\ & + \int_{\mathbb{R}} d\mu(y) \int_0^t p(t, x; s, y) \sigma(s, y) ds, \end{aligned} \quad (7)$$

for each $(t, x) \in [0, +\infty) \times \mathbb{R}$ a.s. The properties of such solutions are considered in [2]. For example, solution of (7) exists, is unique and can be built as

$$u(t, x) = \lim_{n \rightarrow \infty} u^{(n)}(t, x), \quad (8)$$

where $u^{(0)}(t, x) = 0$ and

$$\begin{aligned} u^{(n)}(t, x) = & \int_{\mathbb{R}} p(t, x; 0, y) u_0(y) dy + \int_0^t ds \int_{\mathbb{R}} p(t, x; s, y) f(s, y, u^{(n-1)}(s, y)) dy \\ & + \int_{\mathbb{R}} d\mu(y) \int_0^t p(t, x; s, y) \sigma(s, y) ds. \end{aligned} \quad (9)$$

The analogous iteration process for the stochastic heat equation is considered in more detail in [14].

Let the coefficients of operator (2) satisfy the following assumptions.

Assumption 1. Functions $a(t)$, $b(t)$, $c(t)$ are continuous and bounded on $[0, +\infty)$, and

$$|a(t_1) - a(t_2)| \leq L |t_1 - t_2|^\lambda, \quad a(t) \geq \delta,$$

where $t, t_1, t_2 \in [0, +\infty)$, L, λ, δ are positive constants.

Assumption 2. There exists a constant $c_0 > 0$ such that $c(t) \leq -c_0 \forall t \geq 0$.

Assumption 1 implies that $p(t, x; s, y) = p(t, x - y; s, 0)$ for each $t, s \in [0, +\infty)$, $x, y \in \mathbb{R}$. We consider u_0, f, σ in (7) under the following conditions.

Assumption 3. $u_0 : \mathbb{R} \times \Omega \rightarrow \mathbb{R}$ is measurable and for all $y, y_1, y_2 \in \mathbb{R}$

$$|u_0(y, \omega)| \leq C(\omega), \quad |u_0(y_1, \omega) - u_0(y_2, \omega)| \leq L_{u_0}(\omega) |y_1 - y_2|^{\beta(u_0)},$$

where $C(\omega), L_{u_0}(\omega)$ are random constants, $\beta(u_0) \geq 1/2$.

Assumption 4. $f : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous, bounded, and

$$|f(s, y_1, v_1) - f(s, y_2, v_2)| \leq L_f (|y_1 - y_2| + |v_1 - v_2|)$$

for some constant L_f and all $s \in \mathbb{R}_+, y_1, y_2, v_1, v_2 \in \mathbb{R}$.

Assumption 5. $\sigma : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ is measurable and

$$|\sigma(s, y)| \leq C_\sigma(s), \quad |\sigma(s, y_1) - \sigma(s, y_2)| \leq L_\sigma(s) |y_1 - y_2|^{\beta(\sigma)},$$

for some constant $1/2 < \beta(\sigma) < 1$, all $s \in \mathbb{R}_+, y_1, y_2 \in \mathbb{R}$ and bounded functions $C_\sigma, L_\sigma : \mathbb{R}_+ \rightarrow \mathbb{R}$ such that

$$C_\sigma(s) \rightarrow 0, \quad s \rightarrow \infty; \quad L_\sigma(s) \rightarrow 0, \quad s \rightarrow \infty.$$

To proceed further, we need some statements about \mathcal{L} and p . The following lemma [6, Theorem 10 §1] is formulated for our specific \mathcal{L} .

Lemma 3. *Assume that the function $v(t, x) : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is bounded, Assumptions 1, 2 hold, $|v(0, x)| \leq M_1$, $|\mathcal{L}v(t, x)| \leq M_2$. Then*

$$|v(t, x)| \leq e^{-c_0 t} (M_1 + M_2 t).$$

Lemma 4. *There exist positive constants ν, η, C such that for each $x, y \in \mathbb{R}$, $t > s > 0$ the following estimates hold:*

$$|p(t, x; s, y)| \leq C(t-s)^{-1/2} e^{-\frac{\nu(x-y)^2}{t-s}} e^{\eta(t-s)} E_{\lambda, \lambda+1}(\tilde{C}(t-s)^\lambda), \quad (10)$$

$$\left| \frac{\partial p(t, x; s, y)}{\partial x} \right| \leq C(t-s)^{-1} e^{-\frac{\nu(x-y)^2}{t-s}} e^{\eta(t-s)} E_{\lambda, \lambda+1/2}(\tilde{C}(t-s)^\lambda), \quad (11)$$

$$\begin{aligned} \left| \frac{\partial p(t, x; s, y)}{\partial x} - \frac{\partial p(t, x'; s, y)}{\partial x'} \right| &\leq C(\phi) |x - x'|^\phi (t-s)^{-3/2} \times \\ &\times \max \left\{ e^{-\frac{\nu(x-y)^2}{t-s}}, e^{-\frac{\nu(x'-y)^2}{t-s}} \right\} e^{\eta(t-s)} E_{\lambda, \lambda}(\tilde{C}(t-s)^\lambda), \end{aligned} \quad (12)$$

where $E_{\alpha, \beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}$ is the Mittag-Leffler function, $\phi < 1$.

Proof. We represent p as

$$p(t, x; s, y) = W(t, x; s, y) + \int_s^t d\theta \int_{\mathbb{R}} W(t, x; \theta, \zeta) \Phi(\theta, \zeta; s, y) d\zeta$$

(see, for example, [6, (4.18)]). Here

$$W(t, x; s, y) = \frac{1}{\sqrt{4\pi(t-s)a(s)}} e^{-\frac{(x-y)^2}{4(t-s)a(s)}},$$

the function $\Phi(t, x; s, y)$ is a solution of the integral equation

$$\Phi(t, x; s, y) = \mathcal{L}W(t, x; s, y) + \int_s^t d\theta \int_{\mathbb{R}} \mathcal{L}W(t, x; \theta, \zeta) \Phi(\theta, \zeta; s, y) d\zeta. \quad (13)$$

It is easy to calculate that

$$\begin{aligned} \frac{\partial W(t, x; s, y)}{\partial x} &= \frac{1}{\sqrt{4\pi(t-s)a(s)}} e^{-\frac{(x-y)^2}{4(t-s)a(s)}} \frac{y-x}{2(t-s)a(s)}, \\ \frac{\partial^2 W(t, x; s, y)}{\partial x^2} &= \frac{1}{\sqrt{4\pi(t-s)a(s)}} e^{-\frac{(x-y)^2}{4(t-s)a(s)}} \left(\frac{(x-y)^2}{4(t-s)^2 a^2(s)} - \frac{1}{2(t-s)a(s)} \right), \\ \frac{\partial W(t, x; s, y)}{\partial t} &= \frac{1}{\sqrt{4\pi(t-s)a(s)}} e^{-\frac{(x-y)^2}{4(t-s)a(s)}} \left(\frac{(x-y)^2}{4(t-s)^2 a(s)} - \frac{1}{2(t-s)} \right). \end{aligned}$$

Using Assumption 1 and boundedness of the function $x^\alpha e^{-x}$ on $[0, +\infty)$ for arbitrary $\alpha > 0$, we easily obtain that

$$a(s) \frac{\partial^2 W}{\partial x^2} - \frac{\partial W}{\partial t} = 0, \quad (14)$$

$$|W(t, x; s, y)| \leq C e^{-\frac{\nu(x-y)^2}{t-s}} (t-s)^{-1/2}, \quad (15)$$

$$\left| \frac{\partial W(t, x; s, y)}{\partial x} \right| \leq C e^{-\frac{\nu(x-y)^2}{t-s}} (t-s)^{-1}, \quad (16)$$

$$\left| \frac{\partial^2 W(t, x; s, y)}{\partial x^2} \right| \leq C e^{-\frac{\nu(x-y)^2}{t-s}} (t-s)^{-3/2}, \quad (17)$$

where $0 < \nu < (4 \sup_{t \in \mathbb{R}} a(t))^{-1}$. The solution of (13) can be rewritten as

$$\Phi(t, x; s, y) = \sum_{k=1}^{\infty} \Phi_k(t, x; s, y),$$

where

$$\begin{aligned} \Phi_1(t, x; s, y) &= \mathcal{L}W(t, x; s, y), \quad \Phi_{k+1}(t, x; s, y) \\ &= \int_s^t d\theta \int_{\mathbb{R}} \mathcal{L}W(t, x; \theta, \zeta) \Phi_k(\theta, \zeta; s, y) d\zeta. \end{aligned}$$

Using (14)–(17), we obtain that

$$\begin{aligned} &|\Phi_1(t, x; s, y)| \\ &\leq |a(t) - a(s)| \left| \frac{\partial^2 W(t, x; s, y)}{\partial x^2} \right| + |b(t)| \left| \frac{\partial W(t, x; s, y)}{\partial x} \right| + |c(t)| |W(t, x; s, y)| \\ &\leq C e^{-\frac{\nu(x-y)^2}{t-s}} \left((t-s)^{-3/2+\lambda} + (t-s)^{-1} + (t-s)^{-1/2} \right) \\ &\leq C e^{-\frac{\nu(x-y)^2}{t-s}} (t-s)^{-3/2+\lambda} e^{\eta_1(t-s)}. \end{aligned}$$

Analogously to [6, (4.58)] we show that

$$|\Phi_k(t, x; s, y)| \leq \frac{C}{\Gamma((k-1)\lambda + \lambda)} (\tilde{C}(t-s)^\lambda)^{k-1} e^{-\frac{\nu(x-y)^2}{t-s}} e^{\eta_1(t-s)} (t-s)^{-3/2+\lambda}, \quad (18)$$

where constants C, \tilde{C} depend on λ . Taking the sum of (18) for each $k \geq 1$, we get the inequality

$$|\Phi(t, x; s, y)| \leq C e^{-\frac{\nu(x-y)^2}{t-s}} e^{\eta_1(t-s)} (t-s)^{-3/2+\lambda} E_{\lambda, \lambda}(\tilde{C}(t-s)^\lambda). \quad (19)$$

The following inequality plays an important role in further estimates

$$\int_{\mathbb{R}} (t-\theta)^{-1/2} (\theta-s)^{-1/2} e^{-\nu \left(\frac{(\zeta-y)^2}{\theta-s} + \frac{(x-\zeta)^2}{t-\theta} \right)} d\zeta = \sqrt{\frac{\pi}{\nu}} (t-s)^{-1/2} e^{-\frac{\nu(x-y)^2}{t-s}}. \quad (20)$$

Now we use (19) and (20) to obtain (10).

$$\begin{aligned} &\left| \int_s^t d\theta \int_{\mathbb{R}} W(t, x; \theta, \zeta) \Phi(\theta, \zeta; s, y) d\zeta \right| \\ &\leq C \int_s^t d\theta \int_{\mathbb{R}} (t-\theta)^{-\frac{1}{2}} e^{-\frac{\nu(x-\zeta)^2}{t-\theta}} \frac{E_{\lambda, \lambda}(\tilde{C}(\theta-s)^\lambda)}{(\theta-s)^{3/2-\lambda}} e^{\eta_1(\theta-s)} e^{-\frac{\nu(\zeta-y)^2}{\theta-s}} d\zeta \end{aligned}$$

$$\begin{aligned}
&\leq C e^{\eta_1(t-s)} \int_s^t \frac{E_{\lambda,\lambda}(\tilde{C}(\theta-s)^\lambda)}{(\theta-s)^{1-\lambda}} d\theta \int_{\mathbb{R}} (t-\theta)^{-\frac{1}{2}} (\theta-s)^{-\frac{1}{2}} e^{-v\left(\frac{(\xi-y)^2}{\theta-s} + \frac{(x-\xi)^2}{t-\theta}\right)} d\xi \\
&= C e^{\eta_1(t-s)} (t-s)^{-1/2} e^{-\frac{v(x-y)^2}{t-s}} \int_s^t (\theta-s)^{-1+\lambda} E_{\lambda,\lambda}(\tilde{C}(\theta-s)^\lambda) d\theta \\
&= C (t-s)^{-1/2+\lambda} e^{-\frac{v(x-y)^2}{t-s}} e^{\eta_1(t-s)} E_{\lambda,\lambda+1}(\tilde{C}(t-s)^\lambda), \tag{21}
\end{aligned}$$

where the last equality is a consequence of

$$\int_0^z E_{\rho,\mu}(\lambda t^\rho) t^{\mu-1} dt = z^\mu E_{\rho,\mu+1}(\lambda z^\rho) \tag{22}$$

[8, chapter III, (1.15)]. From (15), (21) and the inequalities $t \geq s$ $e^{\eta(t-s)} \geq 1$, $E_{\lambda,\lambda+1}(\tilde{C}(t-s)^\lambda) \geq \frac{1}{\Gamma(\lambda+1)}$ we obtain (10). We prove (11) analogously, using

$$\frac{1}{\sqrt{\pi}} \int_0^z E_{\rho,\mu}(\lambda t^\rho) (z-t)^{-1/2} t^{\mu-1} dt = z^{\mu-1/2} E_{\rho,\mu+1/2}(\lambda z^\rho)$$

[8, chapter III, (1.16) with $\alpha = 1/2$] instead of (22).

In the proof of (12) we use

$$\begin{aligned}
&\left| \frac{\partial p(t, x; s, y)}{\partial x} - \frac{\partial p(t, x'; s, y)}{\partial x'} \right| \\
&= \left| \frac{\partial p(t, x; s, y)}{\partial x} - \frac{\partial p(t, x'; s, y)}{\partial x'} \right| \mathbf{1}_{(x'-x)^2 < A(t-s)} \\
&\quad + \left| \frac{\partial p(t, x; s, y)}{\partial x} - \frac{\partial p(t, x'; s, y)}{\partial x'} \right| \mathbf{1}_{(x'-x)^2 \geq A(t-s)}, \tag{23}
\end{aligned}$$

where $A > 0$. Firstly assume that $(x' - x)^2 < A(t - s)$; we prove that for such t, x, s

$$\begin{aligned}
\left| \frac{\partial p(t, x; s, y)}{\partial x} - \frac{\partial p(t, x'; s, y)}{\partial x'} \right| &\leq C(\phi, A) |x - x'|^\phi (t-s)^{-3/2} \times \\
&\quad \times e^{-\frac{v(x-y)^2}{t-s}} e^{\eta(t-s)} E_{\lambda,\lambda}(\tilde{C}(t-s)^\lambda). \tag{24}
\end{aligned}$$

Notice that

$$\begin{aligned}
\left| \frac{\partial W(t, x; s, y)}{\partial x} - \frac{\partial W(t, x'; s, y)}{\partial x'} \right| &\leq C \left| \int_x^{x'} \frac{1}{(t-s)^{3/2}} e^{-\frac{v(v-y)^2}{t-s}} dv \right| \\
&\leq C \frac{1}{(t-s)^{3/2}} e^{-\frac{v_1(x^*-y)^2}{t-s}} \int_0^{\frac{|x-x'|}{2}} e^{-\frac{(v-v_1)w^2}{t-s}} dw \\
&\leq C \frac{1}{(t-s)^{3/2}} e^{-\frac{v_1(x^*-y)^2}{t-s}} \int_0^{\frac{|x-x'|}{2}} \left(\frac{t-s}{w^2} \right)^l dw \\
&\leq C \frac{1}{(t-s)^{3/2-l}} e^{-\frac{v_1(x^*-y)^2}{t-s}} |x - x'|^{1-2l},
\end{aligned}$$

where $0 \leq l < 1/2$, $x^* = \epsilon x + (1 - \epsilon)x'$, $0 \leq \epsilon \leq 1$, $0 < \nu_1 < \nu$; here we used the fact that the function $x^\alpha e^{-x}$ is bounded on $[0, +\infty)$ for arbitrary $\alpha > 0$. Now we show that $(x' - x)^2 < A(t - s)$ implies

$$e^{-\frac{\nu_1(x^*-y)^2}{t-s}} \leq C e^{-\frac{\nu_2(x-y)^2}{t-s}}, \quad (25)$$

where $\nu_1 > \nu_2 > 0$, C , ν_2 do not depend on y and x^* , but C depends on A . We consider two cases.

1. $|x - y| \leq 3|x - x'|$. We have the inequalities:

$$e^{-\frac{\nu_1(x^*-y)^2}{t-s}} \leq 1 \leq e^A e^{-\frac{(x'-x)^2}{t-s}} \leq e^A e^{-\frac{(x-y)^2}{9(t-s)}} = C_1 e^{-\frac{\nu_{21}(x-y)^2}{t-s}}.$$

2. $|x - y| > 3|x - x'|$. In this case we have the estimates:

$$\begin{aligned} e^{-\frac{\nu_1(x^*-y)^2}{t-s}} &= e^{-\frac{\nu_1(x-y)^2}{t-s}} e^{\frac{\nu_1(x-x^*)(x+x^*-2y)}{t-s}} \leq e^{-\frac{\nu_1(x-y)^2}{t-s}} e^{\frac{\nu_1|x-x'|(|2|x-y|+|x-x^*|)}{t-s}} \\ &< e^{-\frac{\nu_1(x-y)^2}{t-s}} e^{\frac{7\nu_1(x-y)^2}{9(t-s)}} = e^{-\frac{2\nu_1(x-y)^2}{9(t-s)}} = C_2 e^{-\frac{\nu_{22}(x-y)^2}{t-s}}. \end{aligned}$$

Now we set $C = \max\{C_1, C_2\}$, $\nu_2 = \min\{\nu_{21}, \nu_{22}\}$ and obtain (25). Therefore, the following estimate holds:

$$\left| \frac{\partial W(t, x; s, y)}{\partial x} - \frac{\partial W(t, x'; s, y)}{\partial x'} \right| \leq C \frac{1}{(t-s)^{3/2-l}} e^{-\frac{\nu_2(x-y)^2}{t-s}} |x - x'|^{1-2l}. \quad (26)$$

Consider the expression

$$\begin{aligned} \frac{\partial p(t, x; s, y)}{\partial x} - \frac{\partial p(t, x'; s, y)}{\partial x'} &= \left(\frac{\partial W(t, x; s, y)}{\partial x} - \frac{\partial W(t, x'; s, y)}{\partial x'} \right) \\ &+ \int_{t-\frac{|x'-x|^2}{2A}}^t d\theta \int_{\mathbb{R}} \frac{\partial W(t, x; \theta, \zeta)}{\partial x} \Phi(\theta, \zeta; s, y) d\zeta \\ &- \int_{t-\frac{|x'-x|^2}{2A}}^t d\theta \int_{\mathbb{R}} \frac{\partial W(t, x'; \theta, \zeta)}{\partial x'} \Phi(\theta, \zeta; s, y) d\zeta \\ &+ \int_s^{t-\frac{|x'-x|^2}{2A}} d\theta \int_{\mathbb{R}} \left(\frac{\partial W(t, x; \theta, \zeta)}{\partial x} - \frac{\partial W(t, x'; \theta, \zeta)}{\partial x'} \right) \Phi(\theta, \zeta; s, y) d\zeta \\ &= J_0 + J_1 + J_2 + J_3. \end{aligned}$$

We estimate J_1 in the following way:

$$\begin{aligned} |J_1| &= \left| \int_{t-\frac{|x'-x|^2}{2A}}^t d\theta \int_{\mathbb{R}} \frac{\partial W(t, x; \theta, \zeta)}{\partial x} \Phi(\theta, \zeta; s, y) d\zeta \right| \\ &\leq C \int_{t-\frac{|x'-x|^2}{2A}}^t d\theta \int_{\mathbb{R}} e^{-\nu\left(\frac{(\zeta-y)^2}{\theta-s} + \frac{(x-\zeta)^2}{t-\theta}\right)} \frac{E_{\lambda, \lambda}(\tilde{C}(\theta-s)^\lambda)}{(t-\theta)(\theta-s)^{3/2-\lambda}} e^{\eta_1(\theta-s)} d\zeta \end{aligned}$$

$$\begin{aligned}
&\leq C e^{\eta_1(t-s)} e^{-\frac{\nu(x-y)^2}{t-s}} \frac{E_{\lambda,\lambda}(\tilde{C}(t-s)^\lambda)}{(t-s)^{1/2}} \int_{t-\frac{|x'-x|^2}{2A}}^t (t-\theta)^{-1/2} (\theta-s)^{\lambda-1} d\theta \\
&\leq C e^{\eta_1(t-s)} e^{-\frac{\nu(x-y)^2}{t-s}} \frac{E_{\lambda,\lambda}(\tilde{C}(t-s)^\lambda)}{(t-s)^{1/2}} \left(t-s - \frac{|x'-x|^2}{2A}\right)^{\lambda-1} \left(\frac{|x'-x|^2}{2A}\right)^{1/2} \\
&\leq C e^{\eta_1(t-s)} e^{-\frac{\nu(x-y)^2}{t-s}} (t-s)^{-3/2+\lambda} E_{\lambda,\lambda}(\tilde{C}(t-s)^\lambda) |x-x'|. \tag{27}
\end{aligned}$$

Here we used the inequality $t-s - \frac{|x'-x|^2}{2A} > \frac{t-s}{2}$ and (20). We estimate J_2 in analogous way using (25):

$$\begin{aligned}
|J_2| &= \left| \int_{t-\frac{|x'-x|^2}{2A}}^t d\theta \int_{\mathbb{R}} \frac{\partial W(t, x'; \theta, \zeta)}{\partial x'} \Phi(\theta, \zeta; s, y) d\zeta \right| \\
&\leq C e^{\eta_1(t-s)} e^{-\frac{\nu_2(x-y)^2}{t-s}} (t-s)^{-3/2+\lambda} E_{\lambda,\lambda}(\tilde{C}(t-s)^\lambda) |x-x'|. \tag{28}
\end{aligned}$$

We apply (20) and (26) with $\tilde{A} = 2A$ to prove the estimation for J_3 :

$$\begin{aligned}
|J_3| &\leq \int_s^{t-\frac{|x'-x|^2}{2A}} d\theta \int_{\mathbb{R}} \left| \frac{\partial W(t, x; \theta, \zeta)}{\partial x} - \frac{\partial W(t, x'; \theta, \zeta)}{\partial x'} \right| |\Phi(\theta, \zeta; s, y)| d\zeta \\
&\leq C |x-x'|^{1-2l} \int_s^{t-\frac{|x'-x|^2}{2A}} d\theta \int_{\mathbb{R}} \frac{e^{-\nu_2\left(\frac{(\zeta-y)^2}{\theta-s} + \frac{(x-\zeta)^2}{t-\theta}\right)}}{(t-\theta)^{3/2-l} (\theta-s)^{3/2-\lambda}} e^{\eta_1(\theta-s)} d\zeta \\
&\leq C |x-x'|^{1-2l} e^{\eta_1(t-s)} e^{-\frac{\nu_2(x-y)^2}{t-s}} \frac{E_{\lambda,\lambda}(\tilde{C}(t-s)^\lambda)}{(t-s)^{1/2}} \int_s^t (t-\theta)^{l-1} (\theta-s)^{\lambda-1} d\theta \\
&\leq C |x-x'|^{1-2l} e^{\eta_1(t-s)} e^{-\frac{\nu_2(x-y)^2}{t-s}} (t-s)^{-3/2+l+\lambda} E_{\lambda,\lambda}(\tilde{C}(t-s)^\lambda), \tag{29}
\end{aligned}$$

for arbitrary $l \in (0, 1/2)$. Now (24) follows from (26), (27), (28) and (29).

Let $(x'-x)^2 \geq A(t-s)$. This implies

$$\begin{aligned}
\left| \frac{\partial W(t, x; s, y)}{\partial x} - \frac{\partial W(t, x'; s, y)}{\partial x'} \right| &\leq \left| \frac{\partial W(t, x; s, y)}{\partial x} \right| + \left| \frac{\partial W(t, x'; s, y)}{\partial x'} \right| \\
&\leq C(t-s)^{-1} \max \left\{ e^{-\frac{\nu(x-y)^2}{t-s}}, e^{-\frac{\nu(x'-y)^2}{t-s}} \right\} \\
&\leq C(t-s)^{-3/2+l} \left(\frac{|x'-x|}{\sqrt{A}} \right)^{1-2l} \max \left\{ e^{-\frac{\nu(x-y)^2}{t-s}}, e^{-\frac{\nu(x'-y)^2}{t-s}} \right\}, \tag{30}
\end{aligned}$$

where $l \in (0, 1)$. On the other hand,

$$\begin{aligned}
&\left| \int_s^t d\theta \int_{\mathbb{R}} \frac{\partial W(t, x; \theta, \zeta)}{\partial x} \Phi(\theta, \zeta; s, y) d\zeta \right| \\
&\leq C \int_s^t d\theta \int_{\mathbb{R}} e^{-\nu\left(\frac{(\zeta-y)^2}{\theta-s} + \frac{(x-\zeta)^2}{t-\theta}\right)} \frac{E_{\lambda,\lambda}(\tilde{C}(\theta-s)^\lambda)}{(t-\theta)(\theta-s)^{3/2-\lambda}} e^{\eta_1(\theta-s)} d\zeta \\
&\leq C e^{\eta_1(t-s)} e^{-\frac{\nu(x-y)^2}{t-s}} \frac{E_{\lambda,\lambda}(\tilde{C}(t-s)^\lambda)}{(t-s)^{1/2}} \left(\frac{|x'-x|}{\sqrt{A}} \right)^{1-2l} \int_s^t (\theta-s)^{-1+\lambda} (t-\theta)^{-1+l} d\theta
\end{aligned}$$

$$= C e^{\eta_1(t-s)} (t-s)^{-3/2+l+\lambda} e^{-\frac{\nu(x-y)^2}{t-s}} \left(\frac{|x' - x|}{\sqrt{A}} \right)^{1-2l} E_{\lambda, \lambda}(\tilde{C}(t-s)^\lambda),$$

and the same estimates hold for $\int_s^t d\theta \int_{\mathbb{R}} \frac{\partial W(t, x'; \theta, \zeta)}{\partial x'} \Phi(\theta, \zeta; s, y) d\zeta$. Using (30), (24) and (23), we obtain (12); in (23) we can set, for example, $A = 1$. \square

The main result of the paper is this theorem.

Theorem 1. *Let Assumptions 1–5 hold. Then there exists the version of the solution of (7) such that for each $\omega \in \Omega$:*

$$\sup_{x \in \mathbb{R}} |u(t, x)| \rightarrow 0, \quad t \rightarrow \infty. \quad (31)$$

4 Proof of the auxillary lemma and the main result

To prove Theorem 1, we consider the integral

$$\int_{\mathbb{R}} d\mu(y) \int_0^t p(t, x; s, y) \sigma(s, y) ds. \quad (32)$$

Lemma 5. *Assume Assumptions 1, 2, 5 hold. Then there exists a version $v_1(t, x)$ of integral (32) such that for each $\omega \in \Omega$*

$$\sup_{x \in \mathbb{R}} |v_1(t, x)| \rightarrow 0, \quad t \rightarrow \infty. \quad (33)$$

Proof. It follows from Lemma 2 and (6) that a version $v_1(t, x)$ of integral (32) exists such that for each $x \in \mathbb{R}$, $t \geq 0$, $\omega \in \Omega$

$$\begin{aligned} |v_1(t, x)| &\leq \sum_{j \in \mathbb{Z}} |q(t, x, j) \mu((j, j+1])| + C \sum_{j \in \mathbb{Z}} \|q(t, x, \cdot)\|_{B_{22}^\alpha([j, j+1])} \\ &\quad \times \left(\sum_{n=1}^{\infty} 2^{-n(2\alpha-1)} \sum_{k=1}^{2^n} |\mu(\Delta_{kn}^{(j)})|^2 \right)^{1/2} \\ &\leq \left(\sum_{j \in \mathbb{Z}} |q(t, x, j)|^2 \right)^{1/2} \left(\sum_{j \in \mathbb{Z}} |\mu((j, j+1])|^2 \right)^{1/2} \\ &\quad + C \left(\sum_{j \in \mathbb{Z}} \|q(t, x, \cdot)\|_{B_{22}^\alpha([j, j+1])}^2 \right)^{1/2} \left(\sum_{j \in \mathbb{Z}} \sum_{n=1}^{\infty} 2^{-n(2\alpha-1)} \sum_{k=1}^{2^n} |\mu(\Delta_{kn}^{(j)})|^2 \right)^{1/2}, \quad (34) \end{aligned}$$

where

$$q(t, x, y) = \int_0^t p(t, x; s, y) \sigma(s, y) ds.$$

Next we prove that $v_1(t, x)$ satisfies (33). In order to estimate the Besov norm on $[j, j+1]$, we consider

$$|q(t, x, y+h) - q(t, x, y)| \leq \int_0^t |p(t, x; s, y+h) - p(t, x; s, y)| |\sigma(s, y)| ds$$

$$+ \int_0^t |p(t, x; s, y + h)| |\sigma(s, y + h) - \sigma(s, y)| ds = I_1 + I_2,$$

where $y, y + h \in [j, j + 1]$. Denote

$$\begin{aligned} \Omega_{MN} &= ([s, +\infty) \times \mathbb{R}) \setminus ([s, s + M) \times (y - N, y + N)), \\ \Omega_{MN}^\gamma &= \{(t, x) : d(\Omega_{MN}, (t, x)) \leq \gamma\}, \\ \eta_1(v) &= C e^{\frac{1}{|v|^2-1}} \mathbf{1}_{\{|v|<1\}}; \quad v \in \mathbb{R}^2, \quad \int_{\mathbb{R}^2} \eta_1(v) dv = 1, \\ \eta_\varepsilon(v) &= \varepsilon^{-2} \eta_1(v\varepsilon^{-1}), \\ \Theta_{MN}^\gamma &= (\Omega_{MN}^{2\gamma} \setminus \Omega_{MN}) \cap \{t > s\}, \end{aligned}$$

where $M, N, \gamma > 0$. To estimate I_2 we introduce the function

$$\tilde{p}(t, x; s, y) = p(t, x; s, y) \mathbf{1}_{\{t>s\}} \Psi(t, x; s, y),$$

where

$$\Psi(t, x; s, y) = \int_{\Omega_{MN}^\gamma} \eta_\gamma(t - v_1, x - v_2) dv, \quad v = (v_1, v_2).$$

It is obvious that, for each $0 < \gamma < \min\{M/2, N/2\}$ and arbitrary fixed pair (s, y) , \tilde{p} belongs to a class $C^{1,2}([0, +\infty) \times \mathbb{R})$ as a function of (t, x) . It is easy to obtain that $\tilde{p}(t, x; s, y) = p(t, x; s, y)$ if $(t, x) \in \Omega_{MN}$ and $\tilde{p}(t, x; s, y) = 0$ if $(t, x) \in ([0, +\infty) \times \mathbb{R}) \setminus \Omega_{MN}^{2\gamma} \cup \{s \geq t\}$. Moreover, boundedness of $p(t, x; s, y)$ on $\Omega_{MN}^{2\gamma} \cap \{t \leq T\}$ for each $T > 0$ and the fact that $|\Psi| \leq 1$ imply boundedness of $\tilde{p}(t, x; s, y)$ on $([0, T] \times \mathbb{R})$. Now we estimate $\mathcal{L}\tilde{p}$. Taking into consideration properties of \tilde{p} , it is easy to see that $\mathcal{L}\tilde{p} = 0$ outside the set Θ_{MN}^γ . And inside Θ_{MN}^γ ,

$$\mathcal{L}\tilde{p} = \mathcal{L}(p\Psi) = \Psi\mathcal{L}p + p\mathcal{L}\Psi + 2a \frac{\partial p}{\partial x} \frac{\partial \Psi}{\partial x} \stackrel{\mathcal{L}p=0}{=} p\mathcal{L}\Psi + 2a \frac{\partial p}{\partial x} \frac{\partial \Psi}{\partial x}. \quad (35)$$

For the derivatives of $\eta_\varepsilon(v)$ we have the following well-known inequalities:

$$\left| \frac{\partial \eta_\varepsilon(v)}{\partial v_i} \right| \leq C\varepsilon^{-3}, \quad \left| \frac{\partial^2 \eta_\varepsilon(v)}{\partial v_i^2} \right| \leq C\varepsilon^{-4}, \quad i = 1, 2, \quad (36)$$

where the constant C does not depend on ε . Let us prove, for example, the first inequality in (36):

$$\left| \frac{\partial \eta_\varepsilon(v)}{\partial v_i} \right| \leq \varepsilon^{-3} \max_{|v| \leq 1} \left| \frac{\partial \eta_1(v)}{\partial v_i} \right| = C\varepsilon^{-3},$$

where we used that $\eta_1 \in C^\infty(\mathbb{R}^2)$. From (36) and the fact that $\eta_\varepsilon = 0$ outside the ball with radius ε it follows that

$$\left| \frac{\partial \Psi}{\partial x} \right| \leq C\gamma^{-1}, \quad \left| \frac{\partial \Psi}{\partial t} \right| \leq C\gamma^{-1}, \quad \left| \frac{\partial^2 \Psi}{\partial x^2} \right| \leq C\gamma^{-2}.$$

Using estimates (35), (10) and (11) we obtain the inequality

$$|\mathcal{L}\tilde{p}| \leq C e^{-\frac{\nu(x-y)^2}{t-s}} e^{\eta(t-s)} \left(\gamma^{-2} \frac{E_{\lambda, \lambda+1}(\tilde{C}(t-s)^\lambda)}{\sqrt{t-s}} + \gamma^{-1} \frac{E_{\lambda, \lambda+1/2}(\tilde{C}(t-s)^\lambda)}{t-s} \right) \quad (37)$$

for each $(t, x) \in \Theta_{MN}^\gamma$. Let $\gamma \in (0, 1/3)$ be fixed; consider $t > 3\gamma$. Thus, I_2 can be rewritten in following way:

$$I_2 = \int_0^{t-3\gamma} |p(t, x; s, y+h)| |\sigma(s, y+h) - \sigma(s, y)| ds \\ + \int_{t-3\gamma}^t |p(t, x; s, y+h)| |\sigma(s, y+h) - \sigma(s, y)| ds = I_{21} + I_{22}.$$

We estimate the first summand using the function $\tilde{p}(t, x; s, y)$ for $M = N = 3\gamma$. Note that $t - s < 3\gamma$ on the set Θ_{MN}^γ ; moreover, $t - s > \gamma$ or $|x - y| > \gamma$. In the first case we have the following consequence of (37):

$$|\mathcal{L}\tilde{p}| \leq Ce^{3\eta\gamma} \gamma^{-5/2} (E_{\lambda, \lambda+1}(\tilde{C}(3\gamma)^\lambda) + E_{\lambda, \lambda+1/2}(\tilde{C}(3\gamma)^\lambda)).$$

In the second case,

$$|\mathcal{L}\tilde{p}| \leq Ce^{3\eta\gamma} \gamma^{-7/2} (E_{\lambda, \lambda+1}(\tilde{C}(3\gamma)^\lambda) + E_{\lambda, \lambda+1/2}(\tilde{C}(3\gamma)^\lambda)).$$

Anyway,

$$|\mathcal{L}\tilde{p}| \leq C(\gamma) \forall (t, x) \in [0, +\infty) \times \mathbb{R}.$$

Using Lemma 3 for arbitrary $T > t$, we obtain

$$|\tilde{p}(t, x; s, y)| \leq C(\gamma)e^{-c_0 t} t,$$

where the constant C does not depend on T . On the other hand, taking into account that $\tilde{p}(t, x; s, y) = p(t, x; s, y)$ if $t - s > 3\gamma$ we obtain

$$|I_{21}| \leq C(\gamma)h^{\beta(\sigma)} e^{-c_0 t} t^2 \rightarrow 0, \quad t \rightarrow \infty. \quad (38)$$

Now we estimate I_{22} . We get

$$|I_{22}| \leq Ch^{\beta(\sigma)} \int_{t-3\gamma}^t \frac{1}{\sqrt{t-s}} e^{-\frac{v(x-y)^2}{t-s}} e^{\eta(t-s)} E_{\lambda, \lambda+1}(\tilde{C}(t-s)^\lambda) L_\sigma(s) ds \\ \leq Ch^{\beta(\sigma)} \sqrt{\gamma} e^{3\gamma\eta} E_{\lambda, \lambda+1}(\tilde{C}(3\gamma)^\lambda) \sup_{s \in [t-3\gamma, t]} |L_\sigma(s)| \rightarrow 0, \quad t \rightarrow \infty. \quad (39)$$

For further estimates we use the function

$$\hat{p}(t, x; s, y, h) = (p(t, x; s, y+h) - p(t, x; s, y)) \mathbf{1}_{\{t>s\}} \Psi(t, x; s, y).$$

Let $M > 2\gamma$, $N > 1 + 2\gamma$. Then function \hat{p} has properties, which are analogous to the properties of \tilde{p} . For example, $\hat{p}(t, x; s, y, h) = p(t, x; s, y+h) - p(t, x; s, y)$ when $(t, x) \in \Omega_{MN}$, \hat{p} is bounded on $([0, T] \times \mathbb{R})$, where $T > 0$, $\hat{p} = 0$ when $(t, x) \in ([0, +\infty) \times \mathbb{R}) \setminus \Omega_{MN}^{2\gamma} \cup \{s \geq t\}$. Now we estimate $\mathcal{L}\hat{p}$. Notice that $\mathcal{L}\tilde{p} = 0$ outside the set Θ_{MN}^γ , and for each $(t, x) \in \Theta_{MN}^\gamma$ the following estimates hold:

$$\mathcal{L}\hat{p} = \mathcal{L}((p_{y+h} - p_y)\Psi) \\ = \Psi \mathcal{L}(p_{y+h} - p_y) + (p_{y+h} - p_y) \mathcal{L}\Psi + 2a \frac{\partial(p_{y+h} - p_y)}{\partial x} \frac{\partial\Psi}{\partial x}$$

$$\stackrel{\mathcal{L}p=0}{=} (p_{y+h} - p_y)\mathcal{L}\Psi + 2a \frac{\partial(p_{y+h} - p_y)}{\partial x} \frac{\partial\Psi}{\partial x},$$

where for convenience we denote $p_y = p(t, x; s, y)$. (11) and (12) imply that

$$\begin{aligned} & \left| \frac{\partial p(t, x; s, y+h)}{\partial x} - \frac{\partial p(t, x; s, y)}{\partial x} \right| \\ & \leq Ch^\phi (t-s)^{-3/2} \sup_{y \in [j, j+1]} e^{-\frac{v(x-y)^2}{t-s}} e^{\eta(t-s)} E_{\lambda, \lambda}(\tilde{C}(t-s)^\lambda); \\ |p(t, x; s, y+h) - p(t, x; s, y)| & = \left| \int_{x-y-h}^{x-y} \frac{\partial p(t, v; s, 0)}{\partial v} dv \right| \\ & \leq \int_{x-y-h}^{x-y} \frac{1}{t-s} e^{-\frac{v^2}{t-s}} e^{\eta(t-s)} E_{\lambda, \lambda+1/2}(\tilde{C}(t-s)^\lambda) dv \\ & \leq \frac{h}{t-s} \sup_{y \in [j, j+1]} e^{-\frac{v(x-y)^2}{t-s}} e^{\eta(t-s)} E_{\lambda, \lambda+1/2}(\tilde{C}(t-s)^\lambda). \end{aligned}$$

Therefore, for each $(t, x) \in \Theta_{MN}^\gamma$,

$$\begin{aligned} |\mathcal{L}\hat{p}| & \leq \\ & \leq Ch^\phi \sup_{y \in [j, j+1]} e^{-\frac{v(x-y)^2}{t-s}} e^{\eta(t-s)} \\ & \quad \times \left(\gamma^{-2} \frac{E_{\lambda, \lambda+1/2}(\tilde{C}(t-s)^\lambda)}{t-s} + \gamma^{-1} \frac{E_{\lambda, \lambda}(\tilde{C}(t-s)^\lambda)}{(t-s)^{3/2}} \right). \end{aligned} \quad (40)$$

We rewrite I_1 as

$$\begin{aligned} I_1 & = \int_0^{t-3\gamma} |p(t, x; s, y+h) - p(t, x; s, y)| |\sigma(s, y)| ds \\ & + \int_{t-3\gamma}^t |p(t, x; s, y+h) - p(t, x; s, y)| |\sigma(s, y)| ds = I_{11} + I_{12}. \end{aligned}$$

Consider the function $\hat{p}(t, x; s, y, h)$ for $M = 3\gamma$, $N = 3\gamma + 1$. We estimate I_{11} analogously to I_{21} . It follows from (40) and Lemma 3 that

$$|\hat{p}(t, x; s, y, h)| \leq C(\gamma)h^\phi e^{-c_0 t} t \quad \forall (t, x) \in [0, +\infty) \times \mathbb{R}.$$

On the other hand, the equality $\hat{p}(t, x; s, y, h) = p(t, x; s, y+h) - p(t, x; s, y)$ for $t-s > 3\gamma$ implies that

$$|I_{11}| \leq C(\gamma)h^\phi e^{-c_0 t} t^2 \rightarrow 0, \quad t \rightarrow \infty. \quad (41)$$

I_{12} is estimated in the following way:

$$\begin{aligned} |I_{12}| & \leq C \int_{t-3\gamma}^t ds \int_{x-y-h}^{x-y} \frac{C_\sigma(s)}{t-s} e^{-\frac{v^2}{t-s}} e^{\eta(t-s)} E_{\lambda, \lambda+1/2}(\tilde{C}(t-s)^\lambda) dv \\ & \leq C e^{3\eta\gamma} E_{\lambda, \lambda+1/2}(\tilde{C}(3\gamma)^\lambda) \sup_{s \in [t-3\gamma, t]} |C_\sigma(s)| \int_0^{h/2} v^{-2l} dv \int_{t-3\gamma}^t (t-s)^{1-l} ds \end{aligned}$$

$$= C e^{3\eta\gamma} E_{\lambda, \lambda+1/2}(\tilde{C}(3\gamma)^\lambda) \sup_{s \in [t-3\gamma, t]} |C_\sigma(s)| \gamma^{1-l} h^{1-2l} \rightarrow 0, \quad t \rightarrow \infty, \quad (42)$$

where $l \in (0, 1/2)$. We can choose l, ϕ such that $1 - 2l = \beta(\sigma)$, $\phi = \beta(\sigma)$.

Now assume that for each $y \in [j, j+1]$ and some $m \in \mathbb{N}$ the following inequality holds:

$$|x - y| \geq m + 1. \quad (43)$$

Then we consider the functions $\tilde{p}(t, x; s, y)$ and $\hat{p}(t, x; s, y, h)$ with $M = 3\gamma$ and $N = 3\gamma + m$. For such M and N , provided that (43) holds, $(t, x) \in \Omega_{MN}$. Moreover, using (37), (40), the fact that (43) implies

$$\sup_{y \in [j, j+1]} e^{-\frac{v(x-y)^2}{t-s}} \leq e^{-\frac{v_1 m^2}{3\gamma}} \sup_{y \in [j, j+1]} e^{-\frac{(1-v_1)(x-y)^2}{t-s}} \quad \forall (t, x) \in \Theta_{MN}^\gamma, \quad 0 < v_1 < v,$$

and Lemma 3, we obtain:

$$\begin{aligned} |p(t, x; s, y)| &\leq C(\gamma) e^{-c_0 t} t e^{-\frac{v_1 m^2}{3\gamma}}, \\ |p(t, x; s, y+h) - p(t, x; s, y)| &\leq C(\gamma) h^{\beta(\sigma)} e^{-c_0 t} t e^{-\frac{v_1 m^2}{3\gamma}}. \end{aligned}$$

Now it is easy to estimate I_1, I_2 :

$$I_1 \leq C(\gamma) h^{\beta(\sigma)} e^{-c_0 t} t^2 e^{-\frac{v_1 m^2}{3\gamma}} \leq C(\gamma) h^{\beta(\sigma)} e^{-c_0 t} t^2 m^{-1} \rightarrow 0, \quad t \rightarrow \infty, \quad (44)$$

$$I_2 \leq C(\gamma) h^{\beta(\sigma)} e^{-c_0 t} t^2 e^{-\frac{v_1 m^2}{3\gamma}} \leq C(\gamma) h^{\beta(\sigma)} e^{-c_0 t} t^2 m^{-1} \rightarrow 0, \quad t \rightarrow \infty. \quad (45)$$

Note that we estimate $|q(t, x, y)|$ analogously to I_2 . From (38), (39), (41), (42), (44), (45) it follows that there exists $G_\gamma(t) : [0, +\infty) \rightarrow [0, +\infty)$ such that $G_\gamma(t) \rightarrow 0$, $t \rightarrow \infty$, and

$$\begin{aligned} w_2(q, r) &\leq G_\gamma(t) r^{\beta(\sigma)} \quad \forall t, r \geq 0, j \in \mathbb{Z}, x \in \mathbb{R}; \\ |q(t, x, j)| &\leq G_\gamma(t) \quad \forall t \geq 0, j \in \mathbb{Z}, x \in \mathbb{R}; \\ w_2(q, r) &\leq G_\gamma(t) r^{\beta(\sigma)} m^{-1} \quad \forall t, r \geq 0, j \in \mathbb{Z}, x \in \mathbb{R} : \max_{y \in [j, j+1]} |x - y| \geq m + 1; \\ |q(t, x, j)| &\leq G_\gamma(t) m^{-1} \quad \forall t \geq 0, j \in \mathbb{Z}, x \in \mathbb{R} : \max_{y \in [j, j+1]} |x - y| \geq m + 1. \end{aligned}$$

From this it follows that for each $\alpha \in (1/2, \beta(\sigma))$ the following inequalities hold:

$$\begin{aligned} \|q(t, x, \cdot)\|_{B_{22}^\alpha([j, j+1])} &\leq C G_\gamma(t) \quad \forall t \geq 0, j \in \mathbb{Z}, x \in \mathbb{R}; \\ \|q(t, x, \cdot)\|_{B_{22}^\alpha([j, j+1])} &\leq C G_\gamma(t) m^{-1} \\ \forall t \geq 0, j \in \mathbb{Z}, x \in \mathbb{R} : \max_{y \in [j, j+1]} |x - y| &\geq m + 1. \end{aligned}$$

These estimates imply

$$\sum_{j \in \mathbb{Z}} |q(t, x, j)|^2 \leq C G_\gamma^2(t) + C G_\gamma^2(t) \sum_{m \in \mathbb{N}} \frac{1}{m^2} = C G_\gamma^2(t),$$

$$\sum_{j \in \mathbb{Z}} \|q(t, x, \cdot)\|_{B_{22}^\alpha([j, j+1])}^2 \leq C G_\gamma^2(t) + C G_\gamma^2(t) \sum_{m \in \mathbb{N}} \frac{1}{m^2} = C G_\gamma^2(t).$$

for each $x \in \mathbb{R}$. On the other hand,

$$\begin{aligned} \sum_{j \in \mathbb{Z}} |\mu((j, j+1])|^2 &< \infty \text{ a.s.}, \\ \sum_{j \in \mathbb{Z}} \sum_{n=1}^{\infty} \sum_{k=1}^{2^n} 2^{-n(2\alpha-1)} |\mu(\Delta_{kn}^{(j)})|^2 &< \infty \text{ a.s.} \end{aligned}$$

Therefore, for each version that satisfies (34) we have

$$|v_1(t, x)| \leq C(\omega) G_\gamma(t).$$

Taking a supremum on x and sending t to infinity, we obtain the statement of the lemma. \square

Now we return to the proof of Theorem 1.

Proof. We use the iteration process (9). For each $n \in \mathbb{N}$ we consider the function

$$v_2^{(n)}(t, x) = \int_{\mathbb{R}} p(t, x; 0, y) u_0(y) dy + \int_0^t ds \int_{\mathbb{R}} p(t, x; s, y) f(s, y, u^{(n-1)}(s, y)) dy.$$

From [6, Theorem 2 §4] it follows that the function $v_2^{(n)}$ is a solution, bounded on $[0, T] \times \mathbb{R}$, of the Cauchy problem

$$\mathcal{L}v(t, x) = -f(t, x, u^{(n-1)}(t, x)), \quad v(0, x) = u_0(x),$$

for each $\omega \in \Omega$, $T > 0$. Using Lemma 3, we obtain

$$|v_2^{(n)}(t, x)| \leq C e^{-c_0 t} (1 + t). \quad (46)$$

Now from (8) and (46) it follows that

$$|u(t, x)| \leq C e^{-c_0 t} (1 + t) + \sup_{x \in \mathbb{R}} |v_1(t, x)|.$$

Taking a supremum on x , sending t to infinity and using Lemma 5, we obtain the statement of the theorem. \square

References

- [1] Bodnarchuk, I.M.: Asymptotic behavior of a mild solution of the stochastic heat equation. Bulletin of Taras Shevchenko National University of Kyiv Series: Physics & Mathematics **36**(2), 40–42 (2016).
- [2] Bodnarchuk, I.M.: Regularity of the mild solution of a parabolic equation with stochastic measure. Ukr. Math. J. **69**, 1–18 (2017). MR3631616. <https://doi.org/10.1007/s11253-017-1344-4>

- [3] Bodnarchuk, I.M., Radchenko, V.M.: Asymptotic behavior of solutions of the heat equation with stochastic measure. *Scientific Herald of Yuriy Fedkovych Chernivtsi National University* **2**(1) (2012).
- [4] Borysenko, O.D., Borysenko, D.O.: Asymptotic behavior of a solution of the non-autonomous logistic stochastic differential equation. *Theory Probab. Math. Stat.* **101**, 39–50 (2020). [MR4060330](https://doi.org/10.1090/tpms/1110). <https://doi.org/10.1090/tpms/1110>
- [5] Friedman, A.: *Stochastic Differential Equations and Applications*. Academic Press, New York (1975). [MR0494490](https://doi.org/10.1090/tpms/1110)
- [6] Ilyin, A.M., Kalashnikov, A.S., Oleynik, O.A.: Linear second-order partial differential equations of the parabolic type. *J. Math. Sci.* **108**(4), 435–542 (2002). <https://doi.org/10.1023/A:1013156322602>
- [7] Jiang, Y., Wang, S., Wang, X.: Asymptotics of the solutions to stochastic wave equations driven by a non-Gaussian Lévy process. *Acta Math. Sci.* **39**, 731–746 (2019). [MR4066502](https://doi.org/10.1007/s10473-019-0307-2). <https://doi.org/10.1007/s10473-019-0307-2>
- [8] Jrbashian, M.M.: *Integral Transformations and Function Representations in Complex Space*. Nauka, Moskva (1966).
- [9] Kamont, A.: A discrete characterization of Besov spaces. *Approx. Theory Appl. (N. S.)* **13**(2), 63–77 (1997). [MR1750304](https://doi.org/10.1007/s10473-019-0307-2)
- [10] Kohatsu-Higa, A., Nualart, D.: Large time asymptotic properties of the stochastic heat equation. *J. Theor. Probab.* **34**(3), 1455–1473 (2021). [MR4289891](https://doi.org/10.1007/s10959-020-01007-y). <https://doi.org/10.1007/s10959-020-01007-y>
- [11] Kwapien, S., Woyczynski, W.A.: *Random Series and Stochastic Integrals: Single and Multiple*. Birkhauser, Boston (1992). [MR1167198](https://doi.org/10.1007/978-1-4612-0425-1). <https://doi.org/10.1007/978-1-4612-0425-1>
- [12] Manikin, B.I.: Averaging principle for the one-dimensional parabolic equation driven by stochastic measure. *Mod. Stoch. Theory Appl.* **9**(2), 123–137 (2022). [MR4420680](https://doi.org/10.15559/21-vmsta195). <https://doi.org/10.15559/21-vmsta195>
- [13] Memin, T., Mishura, Y., Valkeia, E.: Inequalities for the moment of Wiener integrals with respect to a fractional Brownian motion. *Stat. Probab. Lett.* **51**, 197–206 (2001). [MR1822771](https://doi.org/10.1016/S0167-7152(00)00157-7). [https://doi.org/10.1016/S0167-7152\(00\)00157-7](https://doi.org/10.1016/S0167-7152(00)00157-7)
- [14] Radchenko, V.M.: Mild solution of the heat equation with a general stochastic measure. *Stud. Math.* **194**(3), 231–251 (2009). [MR2539554](https://doi.org/10.4064/sm194-3-2). <https://doi.org/10.4064/sm194-3-2>
- [15] Radchenko, V.M.: Asymptotic behavior of the solution of heat equation with a stochastic measure as $t \rightarrow \infty$. *Nauk. Visnyk Uzhgorod. Un-tu* **23**(1), 113–118 (2012).
- [16] Radchenko, V.M.: Evolution equations driven by general stochastic measures in Hilbert space. *Theory Probab. Appl.* **59**(2), 328–339 (2015). [MR3416054](https://doi.org/10.1137/S0040585X97T987119). <https://doi.org/10.1137/S0040585X97T987119>
- [17] Radchenko, V.M.: Averaging principle for equation driven by a stochastic measure. *Stochastics* **91**(6), 905–915 (2019). [MR3985803](https://doi.org/10.1080/17442508.2018.1559320). <https://doi.org/10.1080/17442508.2018.1559320>
- [18] Radchenko, V.M., Manikin, B.I.: Approximation of the solution to the parabolic equation driven by stochastic measure. *Theory Probab. Math. Stat.* **102**, 145–156 (2020). [MR4421340](https://doi.org/10.1090/tpms). <https://doi.org/10.1090/tpms>
- [19] Samorodnitsky, G., Taqqu, M.: *Stable Non-Gaussian Random Processes*. Chapman and Hall, London (1994). [MR1280932](https://doi.org/10.1090/tpms)