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Reflected generalized discontinuous BSDEs with rcll barrier and an obstacle problem of IPDE with nonlinear Neumann boundary conditions

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Abstract Reflected generalized backward stochastic differential equations (BSDEs) with one discontinuous barrier are investigated when the noise is driven by a Brownian motion and an independent Poisson measure. The existence and uniqueness of the solution are derived when the generators are monotone and the barrier is right-continuous with left limits (rcll). The link is established between this solution and a viscosity solution for an obstacle problem of integral-partial differential equations with nonlinear Neumann boundary conditions.

Keywords Generalized BSDE with jumps, reflected BSDE, rcll barrier, viscosity solution, integral-partial differential equations
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1 Introduction

General nonlinear backward stochastic differential equations (BSDEs, for short) in the framework of Brownian motion were first introduced by Pardoux and Peng [18] and then extended to the case of jumps by Tang and Li [28] and Rong [26]. Since then, the theory of BSDEs has grown rapidly and has been applied to various areas

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such as mathematical finance [4, 5], stochastic control and stochastic game theory [13, 16] and the theory of partial differential equations [1, 19, 20, 22].

BSDEs have been extended to the reflected case by El Karoui et al. [6] in the case of a Brownian filtration. In their setting, one of the components of the solution is forced to stay above a given barrier which is a continuous adapted stochastic process. Hamadène and Ouknine [11] treat the case where the filtration is generated by the Brownian motion and a Poisson random measure when the square-integrable obstacle has only inaccessible jump times. Later on, Essaky [9] and Hamadène and Ouknine [12] have extended these results to a right-continuous left-limited (*rcll*) obstacle with predictable and inaccessible jumps. In [8], one of the authors studied the reflected BSDE driven by a Lévy process in both cited cases.

In another context, in order to provide a probabilistic representation for a solution of a system of parabolic or elliptic semilinear PDEs with nonlinear Neumann boundary condition, Pardoux and Zhang in [21] initiated a new class of BSDEs which involves the integral with respect to a continuously increasing process, which is the local times of a diffusion process on the boundary. This kind of equations is called generalized BSDEs. Following this way, Pardoux in [17] considered the generalized BSDE with jumps, El Otmani in [7] studied the generalized BSDE driven by Lévy process, Ren and Xia [25] have introduced the notion of reflected generalized BSDEs and Ren and El Otmani [24] have treated the reflected generalized BSDE driven by Lévy process.

In this paper, we focus on the following reflected generalized BSDEs with jumps and rcll barrier (reflected generalized BSDEs, for short)

$$\begin{cases} Y_t = \xi + \int_t^T f(s, Y_s, Z_s, V_s)ds + \int_t^T g(s, Y_s)dA_s + K_T - K_t - \int_t^T Z_s dW_s \\ \qquad\qquad\qquad\qquad\qquad\qquad - \int_t^T \int_U V_s(e)\tilde{N}(ds, de), \\ Y_t \geq L_t, \ t \leq T \quad \text{and} \quad \int_0^T (Y_{t^-} - L_{t^-})dK_t = 0, \end{cases} \tag{1}$$

where W is a standard Brownian motion and $\tilde{N}$ is a compensated Poisson random measure. The second condition in (1) indicates that the first component Y of the solution is forced to stay above the barrier L and the role of K is to push Y upwards in order to keep it above L in a minimal way according to the second part of this condition, i.e. K increases only when $Y = L$. Note that in this case, the jumps can be inaccessible (which stem from the stochastic integral with respect to $\tilde{N}$) or predictable (which comes from the predictable negative jumps of the barrier L) (see, e.g., [3]).

Due to the presence of the process $(K_t)_{t\leq T}$, the integral with respect to the continuous increasing process $(A_t)_{t\leq T}$, which is the local time of a solution for a reflected stochastic differential equation with jumps, and the following special form of the coefficient:

$$f(t, x, y, z, r) = f\left(t, x, y, z, \int_U r(e)\gamma(x, e)\lambda(de)\right),$$

we will establish that equation (1) provides, in a Markovian case, a probabilistic interpretation of the solution, in the viscosity sense, of the following obstacle problem of parabolic integral-partial differential equation (IPDE) with nonlinear Neumann boundary conditions

$$\begin{cases} (u-\ell)\wedge(-\dfrac{\partial u}{\partial t}-\mathscr{L}u-f(t,x,u,(\nabla_x u\sigma),\mathscr{B}u))=0, \quad \forall (t,x)\in[0,T]\times G;\\ u(T,x)=H(x), \quad \forall x\in G,\\ \dfrac{\partial u}{\partial n}+g(t,x,u)=0, \qquad \forall x\in\partial G, \end{cases} \tag{2}$$

where:

- G is an open connected bounded domain of $\mathbb{R}^l$ ($l\geq 1$) which is such that for a function $\Phi\in C_b^2(\mathbb{R})$, G and its boundary ∂G are characterized by $G=\{\Phi>0\}$, $\partial G=\{\Phi=0\}$ and for any $x\in\partial G$, $\nabla\Phi(x)$ is the unit normal vector pointing toward the interior of G.
- $\mathscr{L}$ is the second-order integral-differential operator

$$\mathscr{L}=R+S$$

  with

$$R\phi=\frac{1}{2}Tr\big[\sigma\sigma^T(x)\big]D_x^2\phi(t,x)+\big\langle b(x),\nabla_x\phi(t,x)\big\rangle,$$
$$S\phi=\int_U\big(\phi\big(t,x+c(x,e)\big)-\phi(t,x)-\big\langle\nabla_x\phi(t,x),c(x,e)\big\rangle\big)\lambda(de).$$

- $\mathscr{B}$ is an integral operator defined as

$$\mathscr{B}\phi=\int_U\big(\phi\big(t,x+c(x,e)\big)-\phi(t,x)\big)\gamma(x,e)\lambda(de).$$

- For every $x\in\partial G$,

$$\frac{\partial\phi}{\partial n}=\big\langle\nabla_x\phi,\nabla\Phi(x)\big\rangle.$$

- f, g, H, ℓ, b, σ, c and γ are supposed to satisfy suitable assumptions.

Therefore, in the first part of this paper, the main objective is to prove the uniqueness and existence of the solution of (1) when the coefficients f and g are only monotone w.r.t. y and satisfy a linear growth condition. We first consider the case when the coefficients f and g are Lipschitz, then we solve the problem when f and g depend only on (t,y) and we generalize the result using the fixed point theorem.

The second main aim of this paper is to deal with the obstacle problem of the IPDE with nonlinear Neumann boundary condition (2). By using the results obtained in the first part, i.e. related to the existence and uniqueness of the solution of (1) we prove that equation (2) has a unique viscosity solution.

The paper is organized as follows. Section 2 is devoted to the study of the reflected generalized BSDE: first, we establish a priori estimate of the solution, and then we prove the uniqueness and existence of the solution. A comparison theorem will be presented. Section 3 focuses on the link between this reflected generalized BSDE and the obstacle problem of IPDE with nonlinear Neumann boundary conditions.

Preliminaries and notations

Let $T > 0$ be a fixed time and consider a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ carrying a standard d-dimensional Brownian motion $(W_t)_{t\leq T}$ and an independent martingale measure $(\tilde{N}_t)_{t\leq T}$ corresponding to a standard Poisson random measure N on $\mathbb{R}^+ \times U$ where $U := \mathbb{R}^k \setminus \{0\}$ $(k \geq 1)$ is equipped with its Borel σ-algebra $\mathscr{U}$. Namely, for any Borel measurable subset $\Lambda \in \mathscr{U}$ such that $\lambda(\Lambda) < \infty$, it holds $\tilde{N}_t(\Lambda) := N_t(\Lambda) - t\lambda(\Lambda)$ where λ is assumed to be a σ-finite measure on $(U, \mathscr{U})$ and satisfying $\int_U (1 \wedge |e|^2)\lambda(de) < \infty$.

We assume that $\mathscr{F}_t := \mathscr{F}_t^W \vee \mathscr{F}_t^{\tilde{N}}$ where $\mathscr{F}_t^{\tilde{N}} = \{\int_{[0,s]\times\Lambda} N(du, de), s \leq t, \Lambda \in \mathscr{U}\}$.

We will denote by $|.|$ the Euclidian norm on $\mathbb{R}^d$ and for a given *rcll* process $(X_t)_{t\leq T}$,

$$X_{t-} = \lim_{s\nearrow t} X_s \quad \text{and} \quad \Delta X_t = X_t - X_{t-}, \quad t \leq T.$$

Let $(A_t)_{t\in[0,T]}$ be a continuous one-dimensional increasing $\mathscr{F}_t$-progressively measurable real valued process satisfying $A_0 = 0$.

For every $\mu > 0$, we denote:

- $\mathscr{P}$ (resp. $\mathscr{P}^d$) is the σ-algebra of $\mathscr{F}_t$-progressively measurable (resp. predictable) sets on $[0, T] \times \Omega$.
- $\mathscr{S}_\mu^2$ is the space of $\mathbb{R}$-valued rcll $\mathscr{F}_t$-adapted processes $(Y_t)_{t\leq T}$ such that
$$\|Y\|^2_{\mathscr{S}_\mu^2} = \mathbb{E}\Big[\sup_{0\leq t\leq T} e^{\mu A_t}|Y_t|^2\Big] < \infty.$$
- $\mathscr{H}_{\mu,A}^2$ is the space of $\mathbb{R}$-valued rcll $\mathscr{F}_t$-adapted processes $(Y_t)_{t\leq T}$ such that
$$\|Y\|^2_{\mathscr{H}_{\mu,A}^2} = \mathbb{E}\int_0^T e^{\mu A_t}|Y_t|^2 dA_t < \infty.$$
- $\mathscr{H}_\mu^2$ is the space of $\mathbb{R}^d$-valued $\mathscr{P}$-measurable processes $(Z_t)_{t\leq T}$ such that
$$\|Z\|^2_{\mathscr{H}_\mu^2} = \mathbb{E}\int_0^T e^{\mu A_t}|Z_t|^2 dt < \infty.$$
- $\mathscr{L}_\lambda^2$ is the space of $\mathbb{R}$-valued and $\mathscr{P}^d \otimes \mathscr{U}$-measurable mapping $V : \Omega \times [0, T] \times U \to \mathbb{R}$ such that
$$\|V\|^2_\lambda = \int_U |V(e)|^2 \lambda(de) < \infty.$$
- $\mathscr{L}_{\mu,\lambda}^2$ is the subspace of $\mathscr{L}_\lambda^2$ which contains the mapping $V(t, \omega, e)$ such that
$$\|V\|^2_{\mathscr{L}_{\mu,\lambda}^2} = \mathbb{E}\int_0^T e^{\mu A_t}\|V_t\|^2_\lambda dt < \infty.$$

- $\mathscr{K}^2$ is the subspace of the $\mathscr{F}_t$-predictable, *rcll* and nondecreasing processes $(K_t)_{t\leq T}$ such that $K_0 = 0$ and $\mathbb{E}|K_T|^2 < \infty$.

- $\mathfrak{L}^2_\mu := (\mathscr{S}^2_\mu \cap \mathscr{H}^2_{\mu,A}) \times \mathscr{H}^2_\mu \times \mathscr{L}^2_{\mu,\lambda}$ and $\mathfrak{D}^2_\mu = \mathfrak{L}^2_\mu \times \mathscr{S}^2_\mu$.

We consider the data (ξ, f, g, L) composed by:

(**H1**) A terminal value ξ which is a $\mathscr{F}_T$-measurable variable such that

$$\mathbb{E}\big[e^{\mu A_T}|\xi|^2\big] < \infty.$$

(**H2**) Two functions $f : [0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}^d\times\mathscr{L}^2_\lambda \longrightarrow \mathbb{R}$ and $g : [0,T]\times\Omega\times\mathbb{R} \longrightarrow \mathbb{R}$ such that, for some $\kappa > 0$, $\alpha \in \mathbb{R}$ and $\beta < 0$, for all $t \in [0,T]$ and $(y,z,v), (y',z',v') \in \mathbb{R}\times\mathbb{R}\times\mathscr{L}^2_\lambda$:

(i) $(y-y')(f(t,y,z,v) - f(t,y',z,v)) \leq \alpha|y-y'|^2$,

(ii) $|f(t,y,z,v) - f(t,y,z',v')| \leq \kappa(|z-z'| + \|v-v'\|_\lambda)$,

(iii) $(y-y')(g(t,y) - g(t,y')) \leq \beta|y-y'|^2$,

(iv) $|f(t,y,0,0)| \leq \varphi_t + \kappa|y|$ and $|g(t,y)| \leq \psi_t + \kappa|y|$ where φ and ψ are two adapted processes with values in $[1,+\infty[$ such that

$$\mathbb{E}\int_0^T e^{\mu A_t}|\varphi_t|^2 dt + \mathbb{E}\int_0^T e^{\mu A_t}|\psi_t|^2 dA_t < +\infty,$$

(v) $y \mapsto (f(t,y,z,v), g(t,y))$ is continuous for all (z,v), (t,ω) a.s.

(**H3**) $(L_t)_{t\leq T}$ is an *obstacle* which is an $\mathscr{F}_t$-progressively measurable *rcll* real-valued process satisfying

$$\mathbb{E}\Big[\sup_{0\leq t\leq T}\big|e^{\mu A_t}(L_t)^+\big|^2\Big] < \infty \quad \text{and} \quad L_T \leq \xi,\ \mathbb{P}\text{-a.s.}$$

Remark 1.

1. Hypothesis (**H2**)(iv) implies that

$$\mathbb{E}\big[e^{\mu A_T}\big] \leq 1 + \mu\mathbb{E}\int_0^T e^{\mu A_t}|\psi_t|^2 dA_t < +\infty.$$

2. As in [2, p. 1137] we can suppose w.l.o.g. that $\alpha = 0$ and also from [21] we can show that $\beta < 0$ is not a severe restriction.

2 Reflected generalized BSDEs with *rcll* barrier

A solution of the reflected generalized BSDE is a quadruplet (Y, Z, V, K) satisfying

$$\begin{cases} \text{(i)} & \mathbb{E}[\sup_{0\leq t\leq T} |Y_t|^2 + \int_0^T |Y_t|^2 dA_t + \int_0^T (|Z_t|^2 + \|V_t\|_\lambda^2) dt] < \infty, \\ \text{(ii)} & Y_t = \xi + \int_t^T f(s, Y_s, Z_s, V_s) ds + \int_t^T g(s, Y_s) dA_s + K_T - K_t \\ & \qquad - \int_t^T Z_s dW_s - \int_t^T \int_U V_s(e) \tilde{N}(ds, de), \\ \text{(iii)} & Y_t \geq L_t, \quad t \leq T, \\ \text{(iv)} & K \text{ is a nondecreasing rcll process with } K_0 = 0 \text{ and } \mathbb{E}|K_T|^2 < \infty \text{ and} \\ & K = K^c + K^d \text{ where } K^c (\text{resp.} K^d) \text{ is the continuous} \\ & \text{(resp. purely discontinuous) part of } K \text{ and almost surely} \\ & \int_0^T (Y_t - L_t) dK_t^c = 0 \text{ and } \Delta K_t^d = (Y_t - L_{t-})^- \mathbb{1}_{\{Y_{t-} = L_{t-}\}}. \end{cases} \tag{3}$$

Remark 2.

1. The jump of the process Y can be inaccessible or predictable: the inaccessible jumps come from the martingales $(\int_0^t \int_U V_s(e)\tilde{N}(ds, de))_t$ and the predictable jumps are derived from the negative jumps of the process L.

2. The Skorokhod condition $\int_0^T (Y_{t-} - L_{t-}) dK_t = 0$ in (1) and the characterization (iv) in (3) are equivalent. Indeed, if the Skorokhod condition is satisfied, then
$$\int_0^T (Y_s - L_s) dK_s^c = 0 \quad \text{and} \quad \int_0^T (Y_{s-} - L_{s-}) dK_s^d = 0.$$
The process K^d does act only when the process Y has a predictable jump. In this case, the role of K^d is to make the necessary jump to Y in order to bring it above L. Therefore, for every predictable stopping time $\tau \leq T$, we have
$$\begin{aligned} \Delta K_\tau^d &= -\Delta Y_\tau = -(Y_\tau - Y_{\tau-}) = (L_{\tau-} - Y_\tau)^+ \mathbb{I}_{(L_{\tau-} = Y_{\tau-}) \cap (\Delta L_\tau < 0)} \\ &= (Y_\tau - L_{\tau-})^- \mathbb{I}_{(L_{\tau-} = Y_{\tau-})}. \end{aligned}$$
Conversely,
$$\begin{aligned} \int_0^T (Y_{t-} - L_{t-}) dK_t &= \int_0^T (Y_t - L_t) dK_t^c + \sum_{0<t\leq T} (Y_{t-} - L_{t-}) \Delta K_t^d \\ &= \int_0^T (Y_t - L_t) dK_t^c + \sum_{0<t\leq T} (Y_{t-} - L_{t-})(Y_t - L_{t-})^- \mathbb{1}_{\{Y_{t-} = L_{t-}\}} \\ &= 0. \end{aligned}$$

2.1 A priori estimate

Proposition 1. *Assume that* (***H1***)*–*(***H3***) *hold and let* (Y, Z, V, K) *and* (Y', Z', V', K') *be the solutions of reflected generalized BSDE* (3) *with data* (ξ, f, g, L) *and* (ξ', f', g', L')*, respectively. Then there exists a constant* $C = C(\alpha, \beta, \mu, T, \kappa)$ *such that*

$$\begin{aligned}
&\sup_{0\le t\le T} \mathbb{E}\big[e^{\mu A_t}|Y_t - Y'_t|^2\big] \\
&\quad + \mathbb{E}\int_0^T e^{\mu A_s}\big[|Y_s - Y'_s|^2 dA_s + \big[|Z_s - Z'_s|^2 + \|V_s - V'_s\|_\lambda^2\big]ds\big] \\
&\le C\bigg\{\mathbb{E}\big[e^{\mu A_T}|\xi - \xi'|^2\big] + \mathbb{E}\int_0^T e^{\mu A_s}\big|f\big(s, Y'_s, Z'_s, V'_s\big) - f'\big(s, Y'_s, Z'_s, V'_s\big)\big|^2 ds \\
&\qquad + \mathbb{E}\int_0^T e^{\mu A_s}\big|g\big(s, Y'_s\big) - g'\big(s, Y'_s\big)\big|^2 dA_s \\
&\qquad + \mathbb{E}\int_0^T e^{\mu A_s}\big[\big(L_{s^-} - Y'_{s^-}\big)dK_s - \big(Y_{s^-} - L'_{s^-}\big)dK'_s\big]\bigg\}.
\end{aligned}$$

Proof. Denote $\overline{\Re} = \Re - \Re'$ for $\Re = Y, Z, V, K$. Using Itô's formula (see [23, Theorem 33, p. 81]), we can write, for some $\gamma > 0$ and for all $t \le T$,

$$\begin{aligned}
&e^{\gamma T+\mu A_T}|\overline{\xi}|^2 \\
&= e^{\gamma t+\mu A_t}|\overline{Y}_t|^2 + \gamma\int_t^T e^{\gamma s+\mu A_s}|\overline{Y}_s|^2 ds + \mu\int_t^T e^{\gamma s+\mu A_s}|\overline{Y}_s|^2 dA_s \\
&\quad - 2\int_t^T e^{\gamma s+\mu A_s}\overline{Y}_s\big[f(s, Y_s, Z_s, V_s) - f'\big(s, Y'_s, Z'_s, V'_s\big)\big]ds \\
&\quad - 2\int_t^T e^{\gamma s+\mu A_s}\overline{Y}_s\big[g(s, Y_s) - g'\big(s, Y'_s\big)\big]dA_s - 2\mathbb{E}\int_t^T e^{\gamma s+\mu A_s}\overline{Y}_{s^-}d\overline{K}_s \\
&\quad + 2\int_t^T e^{\gamma s+\mu A_s}\overline{Y}_s\overline{Z}_s dW_s + 2\int_t^T\int_U e^{\gamma s+\mu A_s}\overline{Y}_{s^-}\overline{V}_s(e)\tilde{N}(ds, de) \\
&\quad + \int_t^T e^{\gamma s+\mu A_s}|\overline{Z}_s|^2 ds + \int_t^T\int_U e^{\gamma s+\mu A_s}\big|\overline{V}_s(e)\big|^2 N(ds, de).
\end{aligned}$$

Taking expectation and using assumption (**H2**), the Skorokhod condition and the inequality $2|ab| \le \varepsilon a^2 + \frac{b^2}{\varepsilon}$, we obtain

$$\begin{aligned}
&\mathbb{E}\big[e^{\gamma t+\mu A_t}|\overline{Y}_t|^2\big] + \mathbb{E}\int_t^T e^{\gamma s+\mu A_s}|\overline{Y}_s|^2(\gamma ds + \mu dA_s) \\
&\quad + \mathbb{E}\int_t^T e^{\gamma s+\mu A_s}\big(|\overline{Z}_s|^2 + \|\overline{V}_s\|_\lambda^2\big)ds \\
&\le \mathbb{E}\big[e^{\gamma T+\mu A_T}|\overline{\xi}|^2\big] + \big(1 + 2\alpha^+ + 4\kappa^2\big)\mathbb{E}\int_t^T e^{\gamma s+\mu A_s}|\overline{Y}_s|^2 ds \\
&\quad + \frac{1}{2}\mathbb{E}\int_t^T e^{\gamma s+\mu A_s}\big[|\overline{Z}_s|^2 + \|\overline{V}_s\|_\lambda^2\big]ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2|\beta|}\mathbb{E}\int_t^T e^{\gamma s+\mu A_s}\big|g\big(s,Y'_s\big)-g'\big(s,Y'_s\big)\big|^2 dA_s\\
&+\mathbb{E}\int_t^T e^{\gamma s+\mu A_s}\big|f\big(s,Y'_s,Z'_s,V'_s\big)-f'\big(s,Y'_s,Z'_s,V'_s\big)\big|^2 ds\\
&+2\mathbb{E}\int_t^T e^{\gamma s+\mu A_s}\big(L_{s^-}-Y'_{s-}\big)dK_s-2\mathbb{E}\int_t^T e^{\gamma s+\mu A_s}\big(Y_{s^-}-L'_{s-}\big)dK'_s.
\end{aligned}
$$

Choosing $\gamma = 1 + 2\alpha^+ + 4\kappa^2$, we get the desired result. □

Corollary 1. *Let (Y, Z, V, K) be the solution of (3). Then there exists a constant $\mathscr{C} > 0$ such that*

$$
\begin{aligned}
&\mathbb{E}\Big[\sup_{0\le t\le T} e^{\mu A_t}|Y_t|^2\Big]+\mathbb{E}\int_0^T e^{\mu A_s}|Y_s|^2 dA_s+\mathbb{E}\int_0^T e^{\mu A_s}\big[|Z_s|^2+\|V_s\|_\lambda^2\big]ds+\mathbb{E}|K_T|^2\\
&\le \mathscr{C}\bigg\{\mathbb{E}\big[e^{\mu A_T}|\xi|^2\big]+\mathbb{E}\int_0^T e^{\mu A_s}\big[|\varphi_s|^2 ds+|\psi_s|^2 dA_s\big]+\mathbb{E}\sup_{0\le t\le T}\big|e^{\mu A_t}(L_t)^+\big|^2\bigg\}.
\end{aligned}
$$

Proof. Remark that $(Y', Z', V', K') = (0, 0, 0, 0)$ is the unique solution of (3) with data $(\xi', f', g', L') = (0, 0, 0, 0)$. Then, directly by Proposition 1 we have, for all $t \le T$,

$$
\begin{aligned}
&\sup_{0\le t\le T}\mathbb{E}\big[e^{\mu A_t}|Y_t|^2\big]+\mathbb{E}\int_0^T e^{\mu A_s}|Y_s|^2 dA_s+\mathbb{E}\int_0^T e^{\mu A_s}\big[|Z_s|^2+\|V_s\|_\lambda^2\big]ds\\
&\qquad\le C\mathbb{E}\bigg[e^{\mu A_T}|\xi|^2+\int_0^T e^{\mu A_s}|f(s,0,0,0)|^2 ds+\int_0^T e^{\mu A_s}|g(s,0)|^2 dA_s\\
&\qquad\qquad+\int_0^T e^{\mu A_s}L_{s^-}dK_s\bigg]\\
&\qquad\le C\mathbb{E}\bigg[e^{\mu A_T}|\xi|^2+\int_0^T e^{\mu A_s}\big[|\varphi_s|^2 ds+|\psi_s|^2 dA_s\big]\bigg]\\
&\qquad\qquad+\rho C^2\mathbb{E}\sup_{0\le t\le T}\big|e^{\mu A_t}(L_t)^+\big|^2+\frac{1}{\rho}\mathbb{E}|K_T|^2.
\end{aligned}
$$

However, by (3)(ii) and (**H2**) we get by using the Hölder inequality

$$
\begin{aligned}
\mathbb{E}|K_T|^2 &\le 6\bigg\{|Y_0|^2+\mathbb{E}|\xi|^2+\mathbb{E}\Big(\int_0^T|f(s,Y_s,Z_s,V_s)|ds\Big)^2\\
&\quad+\mathbb{E}\Big(\int_0^T|g(s,Y_s)|dA_s\Big)^2+\mathbb{E}\Big|\int_0^T Z_s dW_s\Big|^2+\mathbb{E}\Big|\int_0^T\int_U V_s(e)\tilde{N}(ds,de)\Big|^2\bigg\}\\
&\le 6|Y_0|^2+6\mathbb{E}|\xi|^2+24T\mathbb{E}\int_0^T|\varphi_s|^2 ds+\frac{12}{\mu}\mathbb{E}\int_0^T e^{\mu A_s}|\psi_s|^2 dA_s\\
&\quad+24\kappa^2 T\mathbb{E}\int_0^T|Y_s|^2 ds+6\big(1+4\kappa^2 T\big)\mathbb{E}\int_0^T\big[|Z_s|^2+\|V_s\|_\lambda^2\big]ds
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{12\kappa^2}{\mu}\mathbb{E}\int_0^T e^{\mu A_s}|Y_s|^2 dA_s \\
&\le 6\mathbb{E}\big[e^{\mu A_T}|\xi|^2\big] + 24T\mathbb{E}\int_0^T e^{\mu A_s}|\varphi_s|^2 ds + \frac{12}{\mu}\mathbb{E}\int_0^T e^{\mu A_s}|\psi_s|^2 dA_s \\
&\quad + 6\big(1+4\kappa^2T^2\big)\sup_{0\le t\le T}\mathbb{E}\big[e^{\mu A_t}|Y_t|^2\big] + \frac{12\kappa^2}{\mu}\mathbb{E}\int_0^T e^{\mu A_s}|Y_s|^2 dA_s \\
&\quad + 6\big(1+4\kappa^2T\big)\mathbb{E}\int_0^T e^{\mu A_s}\big[|Z_s|^2 + \|V_s\|_\lambda^2\big] ds.
\end{aligned}
$$

Choosing $\rho > \max\{6(1+4\kappa^2T^2);\ 6(1+4\kappa^2T);\ \frac{12\kappa^2}{\mu}\}$, we obtain

$$
\begin{aligned}
&\sup_{0\le t\le T}\mathbb{E}\big[e^{\mu A_t}|Y_t|^2\big] + \mathbb{E}\int_0^T e^{\mu A_s}|Y_s|^2 dA_s + \mathbb{E}\int_0^T e^{\mu A_s}\big[|Z_s|^2 + \|V_s\|_\lambda^2\big] ds \\
&\le \mathscr{C}_1\left\{\mathbb{E}\big[e^{\mu A_T}|\xi|^2\big] + \mathbb{E}\int_0^T e^{\mu A_s}\big[|\varphi_s|^2 ds + |\psi_s|^2 dA_s\big] + \mathbb{E}\sup_{0\le t\le T}\big|e^{\mu A_t}(L_t)^+\big|^2\right\},
\end{aligned}
$$

implying that

$$
\begin{aligned}
\mathbb{E}|K_T|^2 \le \mathscr{C}_2 \Bigg\{&\mathbb{E}\big[e^{\mu A_T}|\xi|^2\big] + \mathbb{E}\int_0^T e^{\mu A_s}\big[|\varphi_s|^2 ds + |\psi_s|^2 dA_s\big] \\
&+ \mathbb{E}\sup_{0\le t\le T}\big|e^{\mu A_t}(L_t)^+\big|^2\Bigg\}.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\mathbb{E}\Big[\sup_{0\le t\le T} e^{\mu A_t}|Y_t|^2\Big] &\le \mathbb{E}\big[e^{\mu A_T}|\xi|^2\big] + \big(1+2\kappa+2\kappa^2\big)\mathbb{E}\int_0^T e^{\mu A_s}|Y_s|^2 ds \\
&\quad + \mathbb{E}\int_0^T e^{\mu A_s}\left[|\varphi_s|^2 ds + \frac{|\psi_s|^2}{\mu} dA_s\right] + \mathbb{E}|K_T|^2 + \mathbb{E}\sup_{0\le t\le T}\big|e^{\mu A_t}(L_t)^+\big|^2 \\
&\quad + 2\mathbb{E}\sup_{0\le t\le T}\left|\int_t^T e^{\mu A_s} Y_s Z_s dW_s\right| + 2\mathbb{E}\sup_{0\le t\le T}\left|\int_t^T\int_U e^{\mu A_s} Y_{s^-} V_s(e)\tilde{N}(ds,de)\right|.
\end{aligned}
\tag{4}
$$

Applying the Burkholder–Davis–Gundy inequality (see, e.g., [23, Theorem 48, p. 195]), there is a universal positive constant c such that

$$
2\mathbb{E}\sup_{0\le t\le T}\left|\int_t^T e^{\mu A_s} Y_s Z_s dW_s\right| \le \frac{1}{4}\mathbb{E}\Big[\sup_{0\le t\le T} e^{\mu A_t}|Y_t|^2\Big] + 4c^2\mathbb{E}\int_0^T e^{\mu A_s}|Z_s|^2 ds
$$

and

$$
2\mathbb{E}\sup_{0\le t\le T}\left|\int_t^T\int_U e^{\mu A_s} Y_{s^-} V_s(e)\tilde{N}(ds,de)\right|
$$

$$\leq 2c\mathbb{E}\left(\int_0^T \int_U e^{2\mu A_s}|Y_{s^-}|^2|V_s(e)|^2 N(ds,de)\right)^{\frac{1}{2}}$$
$$\leq \frac{1}{4}\mathbb{E}\Big[\sup_{0\leq t\leq T} e^{\mu A_t}|Y_t|^2\Big] + 4c^2\mathbb{E}\int_0^T e^{\mu A_s}|V_s(e)|^2 N(ds,de)$$
$$= \frac{1}{4}\mathbb{E}\Big[\sup_{0\leq t\leq T} e^{\mu A_t}|Y_t|^2\Big] + 4c^2\mathbb{E}\int_0^T e^{\mu A_s}\|V_s\|_\lambda^2 ds.$$

Plugging those inequalities in (4), we conclude that

$$\mathbb{E}\Big[\sup_{0\leq t\leq T} e^{\mu A_t}|Y_t|^2\Big]$$
$$\leq \mathscr{C}_3\left\{\mathbb{E}\big[e^{\mu A_T}|\xi|^2\big] + \mathbb{E}\int_0^T e^{\mu A_s}\big[|\varphi_s|^2 ds + |\psi_s|^2 dA_s\big] + \mathbb{E}\sup_{0\leq t\leq T}\left|e^{\mu A_t}(L_t)^+\right|^2\right\}.$$

The result is therefore verified. □

2.2 Existence and uniqueness results

Proposition 2. *Under the hypothesis (**H1**)–(**H3**), the reflected generalized BSDE* (3) *associated with the data* (ξ, f, g, L) *has at most one solution.*

Proof. A direct consequence of Proposition 1. □

Our approach to prove existence is based on the following strategy: first, we establish uniqueness and existence when the coefficients f and g are Lipschitz, then we solve the problem when f does not depend on (z, v) and then we generalize the result using the fixed point theorem. The following proposition gives the first step.

Proposition 3. *Suppose that the assumptions (**H1**)–(**H3**) hold for* $\mu > 1$. *We suppose in addition that, for all* $t \leq T$ *and* $y, y' \in \mathbb{R}$,

$$|f(t,y,z,v) - f\big(t,y',z,v\big)| + |g(t,y) - g\big(t,y'\big)| \leq \kappa'|y-y'|.$$

Then the reflected generalized BSDE (3) *has a unique solution.*

We can use two methods to prove this claim. We have omitted the first technique, which is identical to that used in [9] and is based on a monotonic limit theorem and penalization method. The fixed point argument is utilized in the second one. Here is the proof of that.

Proof. First, let us endow the space $\mathfrak{L}^2_\mu$ with the norm

$$\|(Y,Z,V)\|^2_{\gamma,\mu} = \mathbb{E}\left[\int_0^T e^{\gamma t+\mu A_t}\big[|Y_t|^2+|Z_t|^2+\|V_t\|^2_\lambda\big]dt + \int_0^T e^{\gamma t+\mu A_t}|Y_t|^2 dA_t\right].$$

We define the map Ψ of $(\mathfrak{L}^2_\mu, \|.\|_{\gamma,\mu})$ into itself as follows: for every $(Y,Z,V) \in \mathfrak{L}^2_\mu$, we put

$$\Psi(Y,Z,V) = (\tilde{Y}, \tilde{Z}, \tilde{V}),$$

where $(\tilde{Y}, \tilde{Z}, \tilde{V}, \tilde{K})$ is the solution of the reflected generalized BSDE associated with $(\xi, f(t, Y, Z, V), g(t, Y), L)$.

The process $(\tilde{Y}, \tilde{Z}, \tilde{V}, \tilde{K})$ is constructed as follows. Let η be the process defined by

$$\eta_t = \xi \mathbb{I}_{\{t=T\}} + L_t \mathbb{I}_{\{t<T\}} + \int_0^t f(s, Y_s, Z_s, V_s)ds + \int_0^t g(s, Y_s)dA_s.$$

Note that η is *rcll* and $\mathbb{E}(\sup_{0\le t\le T} |\eta_t|^2) < \infty$. Let $\mathscr{S}(\eta)$ be the Snell envelope of η given by

$$\mathscr{S}_t(\eta) = \operatorname{ess\,sup}_{\nu\in\mathscr{T}_t} \mathbb{E}[\eta_\nu|\mathscr{F}_t].$$

However, it is the smallest *rcll* supermartingale dominating the process η which verifies

$$\mathbb{E} \sup_{0\le t\le T} |\mathscr{S}_t(\eta)|^2 < \infty.$$

Then, $\mathscr{S}(\eta)$ is of class [D]. Henceforth, it has the following Doob–Meyer decomposition (see [23, Theorem 8, p. 111]):

$$\mathscr{S}_t(\eta) = \mathbb{E}\left[\xi + \int_0^T f(s, Y_s, Z_s, V_s)ds + \int_0^T g(s, Y_s)dA_s + \tilde{K}_T|\mathscr{F}_t\right] - \tilde{K}_t$$

where $\tilde{K}$ is an $\mathscr{F}_t$-adapted *rcll* nondecreasing process ($\tilde{K}_0 = 0$) and $\mathbb{E}|\tilde{K}_T|^2 < \infty$. Through the martingale representation theorem, there exists two processes $\tilde{Z}$ and $\tilde{V}$ such that

$$\begin{aligned}\xi + \int_0^T f(s, Y_s, Z_s, V_s)ds + \int_0^T g(s, Y_s)dA_s + \tilde{K}_T \\ = \mathbb{E}\left[\xi + \int_0^T f(s, Y_s, Z_s, V_s)ds + \int_0^T g(s, Y_s)dA_s + \tilde{K}_T|\mathscr{F}_t\right] \\ + \int_0^T \tilde{Z}_s dW_s + \int_0^T \int_U \tilde{V}_s(e)\tilde{N}(ds, de),\end{aligned}$$

where $\mathbb{E}\int_0^T |\tilde{Z}_s|^2 ds < \infty$ and $\mathbb{E}\int_0^T \|\tilde{V}_s\|_\lambda^2 ds < \infty$.

Then, if we denote

$$\tilde{Y}_t = \operatorname{ess\,sup}_{\tau\in\mathscr{T}_t} \mathbb{E}\left[\xi\mathbb{I}_{(\tau=T)} + L_\tau \mathbb{I}_{(\tau<T)} + \int_t^\tau f(s, Y_s, Z_s, V_s)ds + \int_t^\tau g(s, Y_s)dA_s|\mathscr{F}_t\right],$$

we get

$$\begin{aligned}\tilde{Y}_t + \int_0^t f(s, Y_s, Z_s, V_s)ds + \int_0^t g(s, Y_s)dA_s + \tilde{K}_t \\ = \mathscr{S}_t(\eta) + \tilde{K}_t \\ = \xi + \int_0^T f(s, Y_s, Z_s, V_s)ds + \int_0^T g(s, Y_s)dA_s + \tilde{K}_T \\ - \int_t^T \tilde{Z}_s dW_s - \int_t^T \int_U \tilde{V}_s(e)\tilde{N}(ds, de).\end{aligned}$$

Hence

$$\tilde{Y}_t = \xi + \int_t^T f(s, Y_s, Z_s, V_s)ds + \int_t^T g(s, Y_s)dA_s + \tilde{K}_T - \tilde{K}_t \\ - \int_t^T \tilde{Z}_s dW_s - \int_t^T \int_U \tilde{V}_s(e)\tilde{N}(ds, de).$$

To finish this construction it remains to show that

$$\int_0^T (\tilde{Y}_{t-} - L_{t-})d\tilde{K}_t^d = \int_0^T (\tilde{Y}_t - L_t)d\tilde{K}_t^c = 0.$$

First, recall that $\{\Delta \tilde{K}^d > 0\} \subset \{\mathscr{S}_-(\eta) = \underline{\eta}\}$ where $\underline{\eta}_s = \limsup_{t \nearrow s} \eta_t$ (see, e.g., [6, Proposition 2.34, p. 131]). So, we can write

$$\int_0^T (\tilde{Y}_{s-} - L_{s-})d\tilde{K}_s^d = \sum_{0<s\leq T} (\tilde{Y}_{s-} - L_{s-})\mathbb{I}_{\{\Delta \tilde{K}_s^d > 0\}} \Delta \tilde{K}_s^d \\ = \sum_{0<s\leq T} (\tilde{Y}_{s-} - L_{s-})\big(\eta_{s-} - \mathscr{S}_s(\eta)\big)^+ \mathbb{I}_{\{\eta_{s-} = \mathscr{S}_{s-}(\eta)\}} = 0.$$

By the property of the Snell envelope, we know that $\int_0^T (S_{t^-}(\eta) - \eta_{t^-})d\tilde{K}_t = 0$ (see Lemma A.4 in [15]), i.e.

$$0 = \int_0^T \big(S_{t^-}(\eta) - \eta_{t^-}\big)d\tilde{K}_t = \int_0^T (Y_{t^-} - L_{t^-})d\tilde{K}_t.$$

Therefore, we have $\int_0^T (\tilde{Y}_t - L_t)d\tilde{K}_t^c = 0$.

The construction is now complete. Additionally, the fact that $(\tilde{Y}, \tilde{Z}, \tilde{V}, \tilde{K})$ is in $\mathfrak{D}_\mu^2$ results from estimations similar to those of Corollary 1.

Now let (Y, Z, V) and $(Y', Z', V') \in \mathfrak{L}_\mu^2$ be such that

$$\Psi(Y, Z, V) = (\tilde{Y}, \tilde{Z}, \tilde{V}), \quad \Psi\big(Y', Z', V'\big) = \big(\tilde{Y}', \tilde{Z}', \tilde{V}'\big).$$

Applying Itô's formula, for $\gamma > 0$ one has

$$\mathbb{E}\big[e^{\gamma t + \mu A_t}\big(\tilde{Y}_t - \tilde{Y}_t'\big)^2\big] + \gamma \mathbb{E}\int_t^T e^{\gamma s + \mu A_s}\big|\tilde{Y}_s - \tilde{Y}_s'\big|^2 ds \\ + \mu \mathbb{E}\int_t^T e^{\gamma s + \mu A_s}\big|\tilde{Y}_s - \tilde{Y}_s'\big|^2 dA_s \\ + \mathbb{E}\int_t^T e^{\gamma s + \mu A_s}|\tilde{Z}_s - \tilde{Z}_s'|^2 ds + \mathbb{E}\int_t^T e^{\gamma s + \mu A_s}\|\tilde{V}_s - \tilde{V}_s'\|_\lambda^2 ds \\ \leq 2\mathbb{E}\int_t^T e^{\gamma s + \mu A_s}\big(\tilde{Y}_s - \tilde{Y}_s'\big)\big(f(s, Y_s, Z_s, V_s) - f\big(s, Y_s', Z_s', V_s'\big)\big)ds \\ + 2\mathbb{E}\int_t^T e^{\gamma s + \mu A_s}\big(\tilde{Y}_s - \tilde{Y}_s'\big)\big(g(s, Y_s) - g\big(s, Y_s'\big)\big)dA_s$$

$$+ 2\mathbb{E}\int_t^T e^{\gamma s+\mu A_s}\big(\tilde{Y}_{s^-} - \tilde{Y}'_{s^-}\big)\big(d\tilde{K}_s - d\tilde{K}'_s\big). \tag{5}$$

First, let us show that

$$\mathbb{E}\int_t^T e^{\gamma s+\mu A_s}\big(\tilde{Y}_{s^-} - \tilde{Y}'_{s^-}\big)\big(d\tilde{K}_s - d\tilde{K}'_s\big) \leq 0.$$

Since $\tilde{Y}$ and $\tilde{Y}'$ belong to $\mathscr{S}^2_\mu$ and their jumps are nonpositive, the sets

$$\delta(\omega) := \big\{t \in [0, T], \Delta_t\tilde{Y} \neq 0\big\} \quad \text{and} \quad \delta'(\omega) := \big\{t \in [0, T], \Delta_t\tilde{Y}' \neq 0\big\}$$

are at most countable. Using the Skorokhod condition, it yields

$$\begin{aligned}
&\int_t^T e^{\gamma s+\mu A_s}\big(\tilde{Y}_{s^-} - \tilde{Y}'_{s^-}\big)\big(d\tilde{K}^c_s - d\tilde{K}'^c_s\big)\\
&\quad = -\int_t^T e^{\gamma s+\mu A_s}(\tilde{Y}_s - L_s)d\tilde{K}'^c_s + \int_t^T e^{\gamma s+\mu A_s}\big(L_s - \tilde{Y}'_s\big)d\tilde{K}^c_s\\
&\quad \leq 0.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\int_t^T e^{\gamma s+\mu A_s}\big(\tilde{Y}_{s^-} - \tilde{Y}'_{s^-}\big)\big(d\tilde{K}^d_s - d\tilde{K}'^d_s\big) &= \int_t^T e^{\gamma s+\mu A_s}\big(\tilde{Y}_{s^-} - \tilde{Y}'_{s^-}\big)d\tilde{K}^d_s\\
&\quad - \int_t^T e^{\gamma s+\mu A_s}\big(\tilde{Y}_{s^-} - \tilde{Y}'_{s^-}\big)d\tilde{K}'^d_s.
\end{aligned} \tag{6}$$

Let us deal with the second part of the right-hand side of (6). We have

$$\begin{aligned}
&\int_t^T e^{\gamma s+\mu A_s}\big(\tilde{Y}_{s^-} - \tilde{Y}'_{s^-}\big)d\tilde{K}'^d_s\\
&= \sum_{t<s\leq T} e^{\gamma s+\mu A_s}\Delta_s\tilde{K}^d\Delta_s\tilde{K}'^d + \int_t^T e^{\gamma s+\mu A_s}\big(\tilde{Y}_s - \tilde{Y}'_s\big)d\tilde{K}'^d_s\\
&\quad - \sum_{t<s\leq T} e^{\gamma s+\mu A_s}\big(\Delta_s\tilde{Y}'\big)^2
\end{aligned}$$

and

$$\int_t^T e^{\gamma s+\mu A_s}\big(\tilde{Y}_s - \tilde{Y}'_s\big)d\tilde{K}'^d_s = \int_t^T e^{\gamma s+\mu A_s}(\tilde{Y}_s - L_{s^-})d\tilde{K}'^d_s + \sum_{t<s\leq T} e^{\gamma s+\mu A_s}\big(\Delta_s\tilde{Y}'\big)^2.$$

Afterwards, we have

$$\int_t^T e^{\gamma s+\mu A_s}(\tilde{Y}_s - L_{s^-})d\tilde{K}'^d_s$$

$$
\begin{aligned}
&= -\sum_{t<s\le T} e^{\gamma s+\mu A_s}\Delta_s\tilde{K}^d\Delta_s\tilde{K}'^d + \int_t^T e^{\gamma s+\mu A_s}(\tilde{Y}_s - L_{s^-})\mathbb{1}_{\{\Delta_s\tilde{Y}=0\}}d\tilde{K}'^d_s\\
&\ge -\sum_{t<s\le T} e^{\gamma s+\mu A_s}\Delta_s\tilde{K}^d\Delta_s\tilde{K}'^d,
\end{aligned}
$$

since $\int_t^T e^{\gamma s+\mu A_s}(\tilde{Y}_s - L_{s^-})\mathbb{1}_{\{\Delta_s\tilde{Y}=0\}}d\tilde{K}'^d_s \ge 0$. In conclusion,

$$
\int_t^T e^{\gamma s+\mu A_s}\big(\tilde{Y}_{s^-} - \tilde{Y}'_{s^-}\big)d\tilde{K}'^d_s \ge 0.
$$

In the same way, we can prove that

$$
\int_t^T e^{\gamma s+\mu A_s}\big(\tilde{Y}_{s^-} - \tilde{Y}'_{s^-}\big)d\tilde{K}^d_s \le 0.
$$

Now coming back to (5), one has

$$
\begin{aligned}
&\gamma\mathbb{E}\int_0^T e^{\gamma s+\mu A_s}\big|\tilde{Y}_s - \tilde{Y}'_s\big|^2 ds + \mu\mathbb{E}\int_0^T e^{\gamma s+\mu A_s}\big|\tilde{Y}_s - \tilde{Y}'_s\big|^2 dA_s\\
&\qquad\qquad + \mathbb{E}\int_0^T e^{\gamma s+\mu A_s}\big[|\tilde{Z}_s - \tilde{Z}'_s|^2 + \|\tilde{V}_s - \tilde{V}'_s\|_\lambda^2\big]ds\\
&\quad\le \varepsilon\kappa'\mathbb{E}\int_0^T e^{\gamma s+\mu A_s}\big|\tilde{Y}_s - \tilde{Y}'_s\big|^2 ds\\
&\qquad + \frac{3\kappa'}{\varepsilon}\mathbb{E}\int_0^T e^{\gamma s+\mu A_s}\big[\big|Y_s - Y'_s\big|^2 + |Z_s - Z'_s|^2 + \|V_s - V'_s\|_\lambda^2\big]ds\\
&\qquad + \varepsilon\kappa'\mathbb{E}\int_0^T e^{\gamma s+\mu A_s}\big|\tilde{Y}_s - \tilde{Y}'_s\big|^2 dA_s + \frac{\kappa'}{\varepsilon}\mathbb{E}\int_0^T e^{\gamma s+\mu A_s}\big|Y_s - Y'_s\big|^2 dA_s.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\big(\gamma - \varepsilon\kappa'\big)\mathbb{E}\int_0^T e^{\gamma s+\mu A_s}\big|\tilde{Y}_s - \tilde{Y}'_s\big|^2 ds + \big(\mu - \varepsilon\kappa'\big)\mathbb{E}\int_0^T e^{\gamma s+\mu A_s}\big|\tilde{Y}_s - \tilde{Y}'_s\big|^2 dA_s\\
&\qquad\qquad + \mathbb{E}\int_0^T e^{\gamma s+\mu A_s}\big[|\tilde{Z}_s - \tilde{Z}'_s|^2 + \|\tilde{V}_s - \tilde{V}'_s\|_\lambda^2\big]ds\\
&\quad\le \frac{3\kappa'}{\varepsilon}\Big[\mathbb{E}\int_0^T e^{\gamma s+\mu A_s}\big[\big(Y_s - Y'_s\big)^2 + |Z_s - Z'_s|^2 + \|V_s - V'_s\|_\lambda^2\big]ds\\
&\qquad\qquad + \mathbb{E}\int_0^T e^{\gamma s+\mu A_s}\big(Y_s - Y'_s\big)^2 dA_s\Big].
\end{aligned}
$$

Now, let $\gamma, \mu > 1$ and ε be such that $3\kappa' < \varepsilon < \kappa'^{-1}(\max\{\gamma, \mu\} - 1)$. Then Ψ is a contraction mapping on $\mathfrak{L}^2_\mu$. Henceforth, there exists a triple of processes (Y, Z, V) that is a fixed point of Ψ which, with K, is the unique solution of the reflected generalized BSDE (3). □

The next proposition gives the second step. This result is the key of our proof. We assume that the coefficient f does not depend on the variables (z, v).

Proposition 4. *Suppose that (**H1**)–(**H3**) hold. Then for any* $(\mathscr{Z}, \mathscr{V}) \in \mathscr{H}^2_\mu \times \mathscr{L}^2_{\mu,\lambda}$, *the reflected generalized BSDE*

$$\begin{cases} Y_t = \xi + \int_t^T f(s, Y_s, \mathscr{Z}_s, \mathscr{V}_s)ds + \int_t^T g(s, Y_s)dA_s + (K_T - K_t) \\ \qquad - \int_t^T Z_s dW_s - \int_t^T \int_U V_s(e)\tilde{N}(ds, de), \quad 0 \le t \le T, \\ Y_t \ge L_t \quad \text{and} \quad \int_0^T (Y_{t-} - L_{t-})dK_t = 0, \quad 0 \le t \le T, \end{cases} \tag{7}$$

has a unique solution.

Proof. Let $f(t, y, \mathscr{Z}_t, \mathscr{V}_t) = h(t, y)$. Considering Remark 1-(2), we shall assume that $\alpha \equiv 0$ in the remaining part of this section. Then some assumptions in **(H2)** on the function h will be change as follows:

(i)$'$ $(y - y')(h(t, y) - h(t, y')) \le 0$,

(iv)$'$ $|h(t, y)| \le \tilde{\varphi}_t + \kappa|y|$ and $|g(t, y)| \le \psi_t + \kappa|y|$ such that

$$\tilde{\varphi}_t = \varphi_t + \kappa|\mathscr{Z}_s| + \kappa\|\mathscr{V}_s\|_\lambda \quad \text{and}$$
$$\mathbb{E}\int_0^T e^{\mu A_t}|\tilde{\varphi}_t|^2 dt + \mathbb{E}\int_0^T e^{\mu A_t}|\psi_t|^2 dA_t < +\infty,$$

(v)$'$ $y \mapsto (h(t, y), g(t, y))$ is continuous $d\mathbb{P} \times dt$ a.e.

We split the proof in two parts.

Part 1. We suppose that

$$|\xi| + \sup_{0\le t\le T} |\tilde{\varphi}_t| + \sup_{0\le t\le T} |\psi_t| + \sup_{0\le t\le T} |L_t^+| \le M. \tag{8}$$

What we would like to do is to construct a sequences of Lipschitz (globally in y, uniformly w.r.t. (ω, s)) functions h_n and g_n which approximate h and g and which are monotone. However, we only manage to construct a sequence for which each h_n (resp. g_n) is monotone in a given ball (the radius depends on n). As we will see later in the proof, this "local" monotonicity is sufficient to obtain the result. We shall use an approximate identity.

Let a function $\rho : \mathbb{R} \to \mathbb{R}^+$ be in C^∞ and with a compact support in the unit ball such that $\int \rho(u)du = 1$. For any $n \ge 1$, we put $\rho_n(u) = n\rho(nu)$.

Moreover, let $\theta_q : \mathbb{R} \to [0, 1]$ be in C^∞ such that $\theta_q(y) = 1$ if $|y| \le q$ and $\theta_q(y) = 0$ if $|y| \ge q + 1$, where the value of $q(n)$ will be fixed later. For $n \ge 1$, we set:

$$\begin{cases} \tilde{\xi} = \mathbb{1}_{\{e^{\mu A_T} \le n\}}\xi, \\ h_n(t, y) = \mathbb{1}_{\{e^{\mu A_t} \le n\}}(\rho_n * \theta_{q(n)+2}h(t, .))(y), \\ g_n(t, y) = \mathbb{1}_{\{e^{\mu A_t} \le n\}}(\rho_n * \theta_{q(n)+2}g(t, .))(y), \\ \tilde{L}_t = \mathbb{1}_{\{e^{\mu A_T} \le n\}}L_t. \end{cases} \tag{9}$$

Clearly,

(a) $h_n(t,y) \to h(t,y)$ and $g_n(t,y) \to g(t,y)$ as $n \to \infty$,

(b) $\tilde{\xi}$ and $\tilde{L}_t$ satisfies respectively (**H1**) and (**H3**),

(c) h_n satisfies (**H2**)-(i)′-(ii)-(iv)′-(v)′ and g_n satisfies (**H2**)-(iii)-(iv)′-(v)′ and also they are Lipschitz in y uniformly w.r.t. (t,ω). In fact, taking into consideration that ρ is with a compact support in the unit ball, we have

$$|\nabla h_n(t,y)| \le C'_n \quad \text{and} \quad |\nabla g_n(t,y)| \le C'_n,$$

where $C'_n = n(M + \kappa q(n) + 3\kappa)$.

Then, for any $n \ge 1$, from Proposition 3 there exists a unique process (Y^n, Z^n, V^n, K^n) satisfying (3) and

$$\begin{cases} Y_t^n = \tilde{\xi} + \int_t^T h_n(s,Y_s^n)ds + \int_t^T g_n(s,Y_s^n)dA_s + (K_T^n - K_t^n) - \int_t^T Z_s^n dW_s \\ \qquad\qquad - \int_t^T \int_U V_s^n(e)\tilde{N}(ds,de), \quad 0 \le t \le T, \\ Y_t^n \ge \tilde{L}_t \quad \text{and} \quad \int_0^T (Y_{t-}^n - \tilde{L}_{t-})dK_t^n = 0, \quad 0 \le t \le T. \end{cases} \tag{10}$$

Remark 3. Under assumptions (**H1**)–(**H3**) and with a computations similar to those in Corollary 1, there exists a constant C independ of n, such that

$$\sup_n \mathbb{E}\Bigg[\sup_{0\le t\le T} e^{\mu A_t}|Y_t^n|^2 + \int_0^T e^{\mu A_s}|Y_s^n|^2 dA_s + \int_0^T e^{\mu A_s}\big[|Z_s^n|^2 + \|V_s^n\|_\lambda^2\big]ds + |K_T^n|^2\Bigg] \le C. \tag{11}$$

Now the rest of this part is based on the following lemmas.

Lemma 1. *Under (**H1**)–(**H3**),* (8)*,* (9) *and with* $\gamma, \mu > 1$*, we have*

$$|Y_t^n|^2 \le \mathscr{K}(n), \tag{12}$$

s.t. $\mathscr{K}(n) = c_0 + c_1 n + c_2 n^2$ *and* $q(n) = [\mathscr{K}^{\frac{1}{2}}]$*, where* $[r]$ *is the integer part of* r.

This justifies the choice of the integer $q(n)$ above.

Proof of Lemma 1. By virtue of Itô's formula with $\gamma, \mu > 1$, we have

$$|Y_t^n|^2 + \gamma\int_t^T e^{\gamma(s-t)+\mu(A_s-A_t)}|Y_s^n|^2 ds + \mu\int_t^T e^{\gamma(s-t)+\mu(A_s-A_t)}|Y_s^n|^2 dA_s + \int_t^T e^{\gamma(s-t)+\mu(A_s-A_t)}|Z_s^n|^2 ds + \int_t^T \int_U e^{\gamma(s-t)+\mu(A_s-A_t)}|V_s^n|^2 N(ds,de)$$

$$
\begin{aligned}
&= e^{\gamma(T-t)+\mu(A_T-A_t)}|\tilde{\xi}|^2 + 2\int_t^T e^{\gamma(s-t)+\mu(A_s-A_t)} Y_s^n h_n\big(s, Y_s^n\big)ds \\
&\quad + 2\int_t^T e^{\gamma(s-t)+\mu(A_s-A_t)} Y_s^n g_n\big(s, Y_s^n\big)dA_s \\
&\quad + 2\int_t^T e^{\gamma(s-t)+\mu(A_s-A_t)} Y_s^n Z_s^n dW_s \\
&\quad + 2\int_t^T \int_U e^{\gamma(s-t)+\mu(A_s-A_t)} Y_{s-}^n V_s^n(e)\tilde{N}(ds, de) \\
&\quad + \int_t^T e^{\gamma(s-t)+\mu(A_s-A_t)} Y_{s-}^n dK_s^n.
\end{aligned}
$$

Taking conditional expectation $\mathbb{E}(.|\mathscr{F}_t) \triangleq \mathbb{E}^{\mathscr{F}_t}(.)$, we obtain

$$
\begin{aligned}
&|Y_t^n|^2 + \gamma\mathbb{E}^{\mathscr{F}_t}\int_t^T e^{\gamma(s-t)+\mu(A_s-A_t)}|Y_s^n|^2 ds + \mu\mathbb{E}^{\mathscr{F}_t}\int_t^T e^{\gamma(s-t)+\mu(A_s-A_t)}|Y_s^n|^2 dA_s \\
&\qquad + \mathbb{E}^{\mathscr{F}_t}\int_t^T e^{\gamma(s-t)+\mu(A_s-A_t)}\big[|Z_s^n|^2 + \|V_s^n\|_\lambda^2\big]ds \\
&\leq \mathbb{E}^{\mathscr{F}_t} e^{\gamma(T-t)+\mu(A_T-A_t)}|\tilde{\xi}|^2 + 2\mathbb{E}^{\mathscr{F}_t}\int_t^T e^{\gamma(s-t)+\mu(A_s-A_t)} Y_s^n h_n\big(s, Y_s^n\big)ds \\
&\quad + \mathbb{E}^{\mathscr{F}_t}\int_t^T e^{\gamma(s-t)+\mu(A_s-A_t)} Y_s^n g_n\big(s, Y_s^n\big)dA_s \\
&\quad + \mathbb{E}^{\mathscr{F}_t}\int_t^T e^{\gamma(s-t)+\mu(A_s-A_t)} \tilde{L}_{s-} dK_s^n.
\end{aligned} \tag{13}
$$

From assumption (**H2**), for all y we have

$$
2yh_n(s, y) \leq \mathbb{1}_{\{e^{\mu A_s}\leq n\}}\bigg((\gamma-1)y^2 + \frac{1}{\gamma-1}\tilde{\varphi_s}^2\bigg),
$$

and

$$
2yg_n(s, y) \leq \mathbb{1}_{\{e^{\mu A_s}\leq n\}}\bigg((\mu-1)y^2 + \frac{1}{\mu-1}\psi_s^2\bigg).
$$

Coming back to (13), using the above inequalities and taking into account the assumption (8), one get for $\rho > 0$,

$$
\begin{aligned}
&|Y_t^n|^2 + \mathbb{E}^{\mathscr{F}_t}\int_t^T e^{\gamma(s-t)+\mu(A_s-A_t)}|Y_s^n|^2 ds + \mathbb{E}^{\mathscr{F}_t}\int_t^T e^{\gamma(s-t)+\mu(A_s-A_t)}|Y_s^n|^2 dA_s \\
&\qquad + \mathbb{E}^{\mathscr{F}_t}\int_t^T e^{\gamma(s-t)+\mu(A_s-A_t)}\big[|Z_s^n|^2 + \|V_s^n\|_\lambda^2\big]ds \\
&\leq e^{\gamma T}M^2.n + e^{\gamma T}.\frac{T.M^2.n}{\gamma-1} + e^{\gamma T}.\frac{M^2.n}{\mu(\mu-1)} + \rho.e^{\gamma T}.n^2.M^2 + \frac{1}{\rho}\mathbb{E}^{\mathscr{F}_t}|K_T^n - K_t^n|^2.
\end{aligned} \tag{14}
$$

Again by (**H2**), (8), Hölder's inequality and isometry property (see, e.g., Theorem 2.3.3, p. 23 in [4]), we end up with

$$
\begin{aligned}
\mathbb{E}^{\mathscr{F}_t}|K_T^n - K_t^n|^2 &\leq 6\Big\{|Y_t^n|^2 + \mathbb{E}^{\mathscr{F}_t}|\tilde{\xi}|^2 + \mathbb{E}^{\mathscr{F}_t}\Big(\int_t^T |h_n\big(s, Y_s^n\big)|ds\Big)^2 \\
&\quad + \mathbb{E}^{\mathscr{F}_t}\Big(\int_t^T |g_n\big(s, Y_s^n\big)|dA_s\Big)^2 + \mathbb{E}^{\mathscr{F}_t}\Big|\int_t^T Z_s^n dW_s\Big|^2 \\
&\quad + \mathbb{E}^{\mathscr{F}_t}\Big|\int_t^T \int_U V_s^n(e)\tilde{N}(ds, de)\Big|^2\Big\} \\
&\leq 6M^2 + 12T^2M^2 + \frac{12}{\mu^2}M^2.n + 6|Y_t^n|^2 + 12\kappa^2 T\mathbb{E}^{\mathscr{F}_t}\int_t^T |Y_s^n|^2 ds \\
&\quad + \frac{12\kappa^2}{\mu}\mathbb{E}^{\mathscr{F}_t}\int_t^T e^{\mu A_s}|Y_s^n|^2 dA_s + 6\mathbb{E}^{\mathscr{F}_t}\int_t^T \big[|Z_s^n|^2 + \|V_s^n\|_\lambda^2\big]ds. \qquad (15)
\end{aligned}
$$

Plugging (15) in (14), we get

$$
\begin{aligned}
&\Big(1 - \frac{6}{\rho}\Big)|Y_t^n|^2 + \Big(1 - \frac{12\kappa^2 T}{\rho}\Big)\mathbb{E}^{\mathscr{F}_t}\int_t^T e^{\gamma(s-t)+\mu(A_s-A_t)}|Y_s^n|^2 ds \\
&\qquad + \Big(1 - \frac{12\kappa^2}{\mu\rho}\Big)\mathbb{E}^{\mathscr{F}_t}\int_t^T e^{\gamma(s-t)+\mu(A_s-A_t)}|Y_s^n|^2 dA_s \\
&\qquad + \Big(1 - \frac{6}{\rho}\Big)\mathbb{E}^{\mathscr{F}_t}\int_t^T e^{\gamma(s-t)+\mu(A_s-A_t)}\big[|Z_s^n|^2 + \|V_s^n\|_\lambda^2\big]ds \\
&\leq \Big(e^{\gamma T}M^2.n + e^{\gamma T}.\frac{T.M^2.n}{\gamma - 1} + e^{\gamma T}.\frac{M^2.n}{\mu(\mu-1)} + \rho.e^{\gamma T}.n^2.M^2\Big) \\
&\qquad + \frac{1}{\rho}\Big(6M^2 + 12T^2M^2 + \frac{12}{\mu^2}M^2.n\Big).
\end{aligned}
$$

Choosing ρ such that

$$\rho > \max\Big\{6;\ 12\kappa^2 T;\ \frac{12\kappa^2}{\mu}\Big\},$$

we obtain

$$|Y_t^n|^2 \leq c_0 + c_1 n + c_2 n^2. \qquad \square$$

Lemma 2. *The processes (Y^n, Z^n, V^n, K^n) converge in $\mathfrak{D}_\mu$.*

Proof of Lemma 2. First, mention that h_n and g_n are not necessary monotone on the entire space considered first, but they are monotone in the ball with the center 0 and radius $q(n)+1$. In fact, we have for any $|y|, |y'| \leq q(n)+1$,

$$\big(y-y'\big)\big(h_n(t, y) - h_n\big(t, y'\big)\big) = \int \rho_n(u).\big(y-y'\big)\big(h(t, y-u) - h\big(t, y'-u\big)\big)du \leq 0,$$

and

$$\big(y-y'\big)\big(g_n(t, y) - g_n\big(t, y'\big)\big) = \int \rho_n(u).\big(y-y'\big)\big(g(t, y-u) - g\big(t, y'-u\big)\big)du \leq 0.$$

Let $m \geq n$. Using Itô's formula and taking expectation, we end up with

$$\begin{aligned}
\mathbb{E}\big[e^{\gamma t+\mu A_t}|Y_t^m - Y_t^n|^2\big] + \gamma\mathbb{E}\int_t^T e^{\gamma s+\mu A_s}|Y_s^m - Y_s^n|^2 ds \\
+ \mu\mathbb{E}\int_t^T e^{\gamma s+\mu A_s}|Y_s^m - Y_s^n|^2 dA_s \\
+ \mathbb{E}\int_t^T e^{\gamma s+\mu A_s}\big[|Z_s^m - Z_s^n|^2 + \|V_s^m - V_s^n\|_\lambda^2\big]ds \\
\leq 2\mathbb{E}\int_t^T e^{\gamma s+\mu A_s}\big(Y_s^m - Y_s^n\big)\big(h_m\big(s, Y_s^m\big) - h_n\big(s, Y_s^n\big)\big)ds \\
+ 2\mathbb{E}\int_t^T e^{\gamma s+\mu A_s}\big(Y_s^m - Y_s^n\big)\big(g_m\big(s, Y_s^m\big) - g_n\big(s, Y_s^n\big)\big)dA_s \\
+ 2\mathbb{E}\int_t^T e^{\gamma s+\mu A_s}\big(Y_{s-}^m - Y_{s-}^n\big)\big(dK_s^m - dK_s^n\big).
\end{aligned}$$

We cannot use the priori estimates because the functions h_m, h_n, g_m and g_n are not globally monotone. Nevertheless, h_m and g_m are monotone on the ball with radius $a = q(m) + 1$. Since $|Y_s^m| \leq q(m) + 1$ and $|Y_s^n| \leq q(n) + 1 \leq q(m) + 1$, in view of (12), Y^m and Y^n belong to this ball. As a result,

$$\big(Y_s^m - Y_s^n\big)\big(h_m\big(s, Y_s^m\big) - h_n\big(s, Y_s^n\big)\big) \leq 2a \sup_{|y|\leq a} |h_m(s, y) - h_n(s, y)|,$$

and

$$\big(Y_s^m - Y_s^n\big)\big(g_m\big(s, Y_s^m\big) - g_n\big(s, Y_s^n\big)\big) \leq 2a \sup_{|y|\leq a} |g_m(s, y) - g_n(s, y)|.$$

On the other hand, we have

$$\int_0^T e^{\gamma s+\mu A_s}\big(Y_{s-}^m - Y_{s-}^n\big)\big(dK_s^m - dK_s^n\big) \leq 0.$$

This implies that

$$\begin{aligned}
\mathbb{E}\int_0^T e^{\gamma s+\mu A_s}\big[|Z_s^m - Z_s^n|^2 + \|V_s^m - V_s^n\|_\lambda^2\big]ds \\
\leq 4a\mathbb{E}\int_0^T e^{\gamma s+\mu A_s} \sup_{|y|\leq a} |h_m(s, y) - h_n(s, y)|ds \\
+ 4a\mathbb{E}\int_0^T e^{\gamma s+\mu A_s} \sup_{|y|\leq a} |g_m(s, y) - g_n(s, y)|dA_s.
\end{aligned}$$

Since $y \to h(t, y)$ and $y \to g(t, y)$ are continuous, $h_n(t, .)$ converges towards $h(t, .)$ and $g_n(t, .)$ converges towards $g(t, .)$ uniformly on the compact $\lambda \otimes \mathbb{P}$ a.s. Moreover, it follows from assumption (**H2**)-(iv)$'$ and the dominated convergence theorem that (Z^n, V^n) is a Cauchy sequence.

Using the Burkholder–Davis–Gundy inequality, we can prove that

$$\mathbb{E}\Big[\sup_{0\le t\le T} e^{\mu A_t}|Y_t^m - Y_t^n|^2\Big] \underset{m,n\to\infty}{\to} 0.$$

This implies that Y^n is a Cauchy sequence in $\mathscr{S}_\mu^2$. Then (Y^n, Z^n, V^n) converges in $\mathfrak{L}_\mu^2$, i.e. there exists a process (Y, Z, V) such that $Y^n \to Y$ in $\mathscr{S}_\mu^2 \cap \mathscr{H}_{\mu,A}^2$, $Z^n \to Z$ in $\mathscr{H}_\mu^2$ and $V^n \to V$ in $\mathscr{L}_{\mu,\lambda}^2$.

Finally, for any $n \ge 0$, we have

$$K_t^n = Y_0^n - Y_t^n - \int_0^T h_n\big(s, Y_s^n\big)ds - \int_0^T g_n\big(s, Y_s^n\big)dA_s + \int_0^T Z_s^n dW_s + \int_0^T \int_U V_s^n \tilde{N}(ds, de).$$

Using the same argument, we get also

$$\mathbb{E}\big[\sup_{0\le t\le T} e^{\mu A_t}|K_t^m - K_t^n|^2\big] \underset{m,n\to\infty}{\to} 0.$$

Then (Y^n, Z^n, V^n, K^n) is a Cauchy sequence in $\mathfrak{D}_\mu$. □

Now we will end this part by demonstrating that the limiting process (Y, Z, V, K) of (Y^n, Z^n, V^n, K^n) in $\mathfrak{D}_\mu$ is a solution of the reflected generalized BSDE (7). Actually, passing to the limit in the reflected generalized BSDE driven by $\tilde{\xi}$, h_n, g_n and $\tilde{L}$, we have that Y_t^n converge to Y_t in $\mathscr{S}_\mu^2 \cap \mathscr{H}_{\mu,A}^2$ and we have also the following convergences in $\mathbb{L}^2$ which are resulted by the martingales representation:

$$\int_t^T Z_s^n dW_s \to \int_t^T Z_s dW_s, \quad \int_t^T \int_U V_s^n(e)\tilde{N}(ds, de) \to \int_t^T \int_U V_s(e)\tilde{N}(ds, de).$$

On the other hand,

$$\mathbb{E}\left[\sup_{0\le t\le T}\left|\int_t^T \big(h_n\big(s, Y_s^n\big) - h(s, Y_s)\big)ds\right|^2\right] \to 0,$$

and using Remark 1-(1), we get

$$\mathbb{E}\left[\sup_{0\le t\le T}\left|\int_t^T \big(g_n\big(s, Y_s^n\big) - g(s, Y_s)\big)dA_s\right|^2\right] \to 0.$$

Also, we have the convergence

$$K_t = Y_0 - Y_t - \int_0^t h(s, Y_s)ds - \int_0^t g(s, Y_s)dA_s + \int_0^t Z_s dW_s + \int_0^t \int_U V_s(e)\tilde{N}(ds, de).$$

Note that $(K_t)_t$ is an increasing process and $\mathbb{E}|K_T|^2 < +\infty$.

Part 2 (General case). For any $p \geq 1$, we consider

$$\begin{cases} \xi^p = \begin{cases} \dfrac{(p \wedge |\xi|)}{|\xi|}\xi, & \text{if} \quad \xi \neq 0, \\ 0, & \text{if} \quad \xi = 0, \end{cases} \\ h_p(t,y) = \begin{cases} h(t,y) - h(t,0) + \dfrac{(p \wedge |h(t,0)|)}{|h(t,0)|}h(t,0), & \text{if} \quad h(t,0) \neq 0, \\ h(t,y), & \text{if} \quad h(t,0) = 0, \end{cases} \\ g_p(t,y) = \begin{cases} g(t,y) - g(t,0) + \dfrac{(p \wedge |g(t,0)|)}{|g(t,0)|}g(t,0), & \text{if} \quad g(t,0) \neq 0, \\ g(t,y), & \text{if} \quad g(t,0) = 0, \end{cases} \\ L_t^p = \begin{cases} \dfrac{(p \wedge \sup_t (L_t)^+)}{\sup_t (L_t)^+}L_t, & \text{if} \quad \sup_t (L_t)^+ \neq 0 \\ 0, & \text{if} \quad \sup_t (L_t)^+ = 0. \end{cases} \end{cases} \tag{16}$$

It is easy to see that, when $p \to \infty$,

$$\mathbb{E}\Big[e^{\mu A_T}|\xi^p - \xi|^2 + \int_0^T e^{\mu A_t}|h_p(t,0) - h(t,0)|^2 dt + \int_0^T e^{\mu A_t}|g_p(t,0) - g(t,0)|^2 dA_t\Big] \to 0.$$

Obviously, (ξ^p, h_p, g_p, L^p) satisfies the hypothesis of the previous step. Then the reflected generalized BSDE associated to the parameters (ξ^p, h_p, g_p, L^p) has a unique solution (Y^p, Z^p, V^p, K^p) in $\mathfrak{D}_\mu$.

Now we are going to show that the sequence (Y^p, Z^p, V^p, K^p) is a Cauchy sequence in $\mathfrak{D}_\mu$. Let $q \geq p$, using Itô's formula, the definition of ξ^p, h_p, g_p, L^p and the same arguments in Proposition 1 with the Burkholder–Davis–Gundy inequality, we end up with

$$\begin{aligned} &\mathbb{E}\Big[\sup_{0\leq t\leq T} e^{\mu A_t}|Y_t^q - Y_t^p|^2 + \int_0^T e^{\mu A_s}|Y_s^q - Y_s^p|^2 dA_s \\ &\qquad + \int_0^T e^{\mu A_s}\big[|Z_s^q - Z_s^p|^2 + \|V_s^q - V_s^p\|_\lambda^2\big]ds\Big] \\ &\leq \mathbf{C}\mathbb{E}\Big[e^{\mu A_T}|\xi^q - \xi^p|^2 + \int_0^T e^{\mu A_s}|h_q(s,0) - h_p(s,0)|^2 ds \\ &\qquad + \int_0^T e^{\mu A_s}|g_q(s,0) - g_p(s,0)|^2 dA_s \\ &\qquad + \int_0^T e^{\mu A_s}\big[\big(L_{s-}^q - Y_{s-}^p\big)dK_s^q - \big(Y_{s-}^q - L_{s-}^p\big)dK_s^p\big]\Big]. \end{aligned} \tag{17}$$

Remark that, since $q \geq p$, we have $L^q \geq L^p, 0 \leq t \leq T$. Then

$$\big(L^q_{s^-} - Y^p_{s^-}\big)dK^q_s \leq \big(L^q_{s^-} - L^p_{s^-}\big)dK^q_s.$$

Since $L_t - L^n_t \searrow 0$ and $(L_t - L^n_t)$ is a rcll process, by the generalized Dini's theorem (see [3, p. 202]), the convergence holds uniformly in $[0, T]$, i.e.

$$\lim_{n\to\infty} \mathbb{E}\big[\sup_{0\leq s\leq T} |e^{\mu A_s}\big(L_s - L^n_s\big)|^2\big] \to 0 \quad \text{as } n \to \infty.$$

Then using Remark 3, we get

$$\begin{aligned}\mathbb{E}\int_0^T e^{\mu A_s}\big(L^q_{s^-} - L^p_{s^-}\big)dK^q_s &\leq \mathbb{E}\big[\sup_{0\leq s\leq T} |e^{\mu A_s}\big(L^q_s - L^p_s\big)|^2\big]^{\frac{1}{2}}.\big(\mathbb{E}|K^q_T|^2\big)^{\frac{1}{2}}\\ &\to 0 \quad \text{as } p, q \to \infty.\end{aligned}$$

The right-hand side of (17) tends to 0 as $p, q \to \infty$.

Finally, for any $p \geq 1$, we have

$$\begin{aligned}K^p_t = Y^p_0 - Y^p_t - \int_0^T h_p\big(s, Y^p_s\big)ds - \int_0^T g_p\big(s, Y^p_s\big)dA_s + \int_0^T Z^p_s dW_s\\ + \int_0^T\int_U V^p_s \tilde{N}(ds, de).\end{aligned}$$

We also get

$$\mathbb{E}\big[\sup_{0\leq t\leq T} e^{\mu A_t}|K^q_t - K^p_t|^2\big] \underset{q,p\to\infty}{\to} 0.$$

Thus the sequence (Y^p, Z^p, V^p, K^p) is a Cauchy sequence in $\mathfrak{D}_\mu$. Then it converges towards a progressively measurable process (Y, Z, V, K). It remains to verify that the limiting process solves the reflected generalized BSDE (7). Hence, by the same argument as in Part 1, the process (Y, Z, V, K) is a solution of the reflected generalized BSDE (7) and the proof is complete. □

With the help of Proposition 4, we can now construct a solution (Y, Z, V, K) to the reflected generalized BSDE (3). We claim the following result.

Theorem 1. *Suppose that the assumptions (**H1**)–(**H3**) hold. Then the reflected generalized BSDE* (3) *has a unique solution.*

Proof. The uniqueness is already established in Proposition 2. For the existence, we will use a fixed point argument. We define the map Ψ of $(\mathfrak{L}^2_\mu, \|.\|_{\gamma,\mu})$ into itself as follows: for every $(Y, Z, V) \in \mathfrak{L}^2_\mu$ we put $\Psi(Y, Z, V) = (\tilde{Y}, \tilde{Z}, \tilde{V})$ where $(\tilde{Y}, \tilde{Z}, \tilde{V}, \tilde{K})$ is the solution of the reflected generalized BSDE associated with $(\xi, f(t, \tilde{Y}, Z, V), g(t, \tilde{Y}), L)$ which exists by Proposition 4.

Now $(\tilde{Y}, \tilde{Z}, \tilde{V}) \in \mathfrak{L}^2_\mu$ is a solution of the reflected generalized BSDE (3) if and only if it is a fixed point of Ψ. Let $(Y, Z, V), (Y', Z', V') \in \mathfrak{L}^2_\mu$ be such that $\Psi(Y, Z, V) = (\tilde{Y}, \tilde{Z}, \tilde{V})$ and $\Psi(Y', Z', V') = (\tilde{Y}', \tilde{Z}', \tilde{V}')$.

Applying Itô's formula and taking expectation, one has

$$\begin{aligned}
&\mathbb{E}\big[e^{\gamma t+\mu A_t}\big|\tilde{Y}_t-\tilde{Y}'_t\big|^2\big]+\gamma\mathbb{E}\int_t^T e^{\gamma s+\mu A_s}\big|\tilde{Y}_s-\tilde{Y}'_s\big|^2ds\\
&\quad+\mu\mathbb{E}\int_t^T e^{\gamma s+\mu A_s}\big|\tilde{Y}_s-\tilde{Y}'_s\big|^2dA_s\\
&\qquad\qquad\qquad\mathbb{E}\int_t^T e^{\gamma s+\mu A_s}\big[|\tilde{Z}_s-\tilde{Z}'_s|^2+\|\tilde{V}_s-\tilde{V}'_s\|_\lambda^2\big]ds\\
&\le 2\mathbb{E}\int_t^T e^{\gamma s+\mu A_s}\big(\tilde{Y}_s-\tilde{Y}'_s\big)\big(f(s,\tilde{Y}_s,Z_s,V_s)-f\big(s,\tilde{Y}'_s,Z'_s,V'_s\big)\big)ds\\
&\quad+2\mathbb{E}\int_t^T e^{\gamma s+\mu A_s}\big(\tilde{Y}_s-\tilde{Y}'_s\big)\big(g(s,\tilde{Y}_s)-g\big(s,\tilde{Y}'_s\big)\big)dA_s\\
&\qquad+2\mathbb{E}\int_t^T e^{\gamma s+\mu A_s}\big(\tilde{Y}_{s^-}-\tilde{Y}'_{s^-}\big)\big(d\tilde{K}_s-d\tilde{K}'_s\big).
\end{aligned}$$

In particular, if we use assumption (**H2**) and Remark 1-(2), we get, by choosing $\gamma\ge 1+4\kappa^2$ and $\mu\ge 1$, that

$$\begin{aligned}
&\mathbb{E}\int_0^T e^{\gamma s+\mu A_s}\big(\tilde{Y}_s-\tilde{Y}'_s\big)^2ds+\mathbb{E}\int_0^T e^{\gamma s+\mu A_s}\big(\tilde{Y}_s-\tilde{Y}'_s\big)^2dA_s\\
&\qquad\qquad\qquad+\mathbb{E}\int_0^T e^{\gamma s+\mu A_s}\big[|\tilde{Z}_s-\tilde{Z}'_s|^2+\|\tilde{V}_s-\tilde{V}'_s\|_\lambda^2\big]ds\\
&\quad\le\frac{1}{2}\mathbb{E}\int_0^T e^{\gamma s+\mu A_s}\big[|Z_s-Z'_s|^2+\|V_s-V'_s\|_\lambda^2\big]ds\\
&\quad\le\frac{1}{2}\mathbb{E}\Big[\int_0^T e^{\gamma s+\mu A_s}\big[|Y_s-Y'_s|^2+|Z_s-Z'_s|^2+\|V_s-V'_s\|_\lambda^2\big]ds\\
&\qquad\qquad\qquad\qquad+\int_0^T e^{\gamma s+\mu A_s}|Y_s-Y'_s|^2dA_s\Big].
\end{aligned}$$

Then

$$\big\|\big(\tilde{Y}-\tilde{Y}'\big),\big(\tilde{Z}-\tilde{Z}'\big),\big(\tilde{V}-\tilde{V}'\big)\big\|_{\gamma,\mu}^2\le\frac{1}{2}\big\|\big(Y-Y'\big),\big(Z-Z'\big),\big(V-V'\big)\big\|_{\gamma,\mu}^2.$$

Then Ψ is a contraction mapping on $(\mathfrak{L}_\mu^2,\|.\|_{\gamma,\mu})$. Henceforth, there exists a triple of processes (Y,Z,V) that is a fixed point of Ψ which, with K, is the unique solution of the reflected generalized BSDE (3). □

2.3 Comparison theorem

In general, we do not have a comparison result for solutions of BSDEs driven by a Brownian motion and an independent Poisson process, reflected or not (see, e.g., [1] for a counter-example). But in some specific cases, when the coefficients satisfy some properties, we have a comparison result.

We establish two results of the comparison. The first one is when the coefficient f does not depend on the variable v. But in the second one, we impose some monotonicity w.r.t. v. So assume there exists another quadruple of processes (Y', Z', V', K') being a solution for the reflected generalized BSDE with one lower *rcll* reflecting barrier associated with (ξ', f', g', L).

Theorem 2. *Assume that:*

i) f is independent of v;

ii) $\mathbb{P}$-a.s, for any $t \leq T$,

$$f\big(t, Y'_t, Z'_t\big) \leq f'\big(t, Y'_t, Z'_t, V'_t\big), \quad g\big(t, Y'_t\big) \leq g'\big(t, Y'_t\big) \quad \textit{and} \quad \xi \leq \xi'.$$

Then $\mathbb{P}$-a.s., for any $t \leq T$, $Y_t \leq Y'_t$. Additionally, if f' does not depend on v then we have also $K_t - K_s \geq K'_t - K'_s$, for any $0 \leq s \leq t \leq T$.

Remark 4.

i) Using Remark 2-(2), since $Y \leq Y'$, we obviously have $\mathbb{P}$-a.s., for any $s \leq t$,

$$K^d_t - K^d_s \geq K'^d_t - K'^d_s.$$

ii) If the barriers are not the same, as it is assumed in the previous theorem, we can still get the comparison result of Ys, but the comparison of Ks could fail.

The second result extends the comparison results in [27] to the case of reflected generalized BSDE with monotone generators. The three assumptions **(H1)**, **(H2)** and **(H3)** hold, but **(H2)**-(ii) is replaced by:

*(**H2**)-(ii)′: f is Lipschitz continuous w.r.t. z with constant κ, and for each $(y, z, v, v') \in \mathbb{R} \times \mathbb{R}^d \times (\mathcal{L}_\lambda)^2$, there exists a predictable process $\mathcal{K} = \mathcal{K}^{y,z,v,v'}$: $\Omega \times [0, T] \times U \to \mathbb{R}$ such that*

$$f(t, y, z, v) - f\big(t, y, z, v'\big) \leq \int_U \big(v(e) - v'(e)\big)\mathcal{K}^{y,z,v,v'}_t(e)\lambda(de)$$

with $\mathbb{P} \otimes m \otimes \lambda$-a.e. for any (y, z, v, v')

- $-1 \leq \mathcal{K}^{y,z,v,v'}_t(e)$,
- $|\mathcal{K}^{y,z,v,v'}_t(e)| \leq w(e)$ *where* $w \in \mathcal{L}_\lambda$.

Now, we are able to establish our comparison theorem.

Theorem 3. *Let (Y, Z, V, K) and (Y', Z', V', K') be solutions of the reflected generalized BSDE with one rcll reflecting barrier associated, respectively, with (ξ, f, g, L) and (ξ', f', g', L) which satisfy all the assumptions (**H1**)–(**H3**). Assume that:*

i) $\xi \leq \xi'$, $\mathbb{P}$-a.s.;

ii) $\mathbb{P}$-a.s, for any $t \leq T$,

$$f(t, Y_t, Z_t, V_t) \leq f'(t, Y_t, Z_t, V_t) \quad \textit{and} \quad g(t, Y_t) \leq g'(t, Y_t).$$

Then $\mathbb{P}$-a.s., for any $t \leq T$, $Y_t \leq Y'_t$. Additionally, if f and f' do not depend on v, then we have also $K_t - K_s \geq K'_t - K'_s$, for any $0 \leq s \leq t \leq T$.

3 Applications to the obstacle problem for integral-partial differential equations with Neumann boundary condition

With the help of BSDEs, the Feynman–Kac formula provides a probabilistic interpretation for semilinear second-order PDEs of elliptic or parabolic types, which has been generalized to systems of quasilinear second-order PDEs by Peng [22], Pardoux and Tang [20], see also Darling and Pardoux [2] and references therein. The case of PDE with nonlinear Neumann boundary conditions have been first treated by Pardoux and Zhang [17] and extended to several cases, see, e.g., [7, 24, 25]. Through the case of BSDEs with jumps these results have been generalized to treat a class of second-order integral-partial differential equations (IPDEs), see, e.g., [1].

The main result of this section is to prove that the solution of the reflected generalized BSDE (3) provides a probabilistic formula for a viscosity solution for an obstacle problem of a class of second-order integral-partial differential equations (IPDEs) of parabolic type with nonlinear Neumann boundary condition.

3.1 A class of reflected diffusion process

First of all let us recall some notions. Let G be an open connected bounded domain of $\mathbb{R}^l$ ($l \geq 1$). We suppose that G is a smooth domain, which is such that for a function $\Phi \in C_b^2(\mathbb{R})$, G and its boundary ∂G are characterized by $G = \{\Phi > 0\}$, $\partial G = \{\Phi = 0\}$ and for any $x \in \partial G$, $\nabla\Phi(x)$ is the unit normal vector pointing toward the interior of G. In addition, the interior sphere condition holds (see [21, p. 551]), i.e. there exists $m > 0$ such that for any $x \in \partial G$, $x' \in \bar{G}$,

$$|x' - x|^2 + m\langle\nabla\Phi(x), x' - x\rangle \geq 0. \tag{18}$$

Now from [14], we know that for every $(t, x) \in \mathbb{R}^+ \times \bar{G}$ there exists a unique pair of progressively measurable process $(X_s^{t,x}, A_s^{t,x})_{s\geq 0}$ being a solution to the following reflected stochastic differential equation (reflected SDE) with jumps:

$$\begin{cases} X_s^{t,x} = x + \int_t^{t\vee s} b(X_r^{t,x})dr + \int_t^{t\vee s} \sigma(X_r^{t,x})dW_r + \int_t^{t\vee s}\int_U c(X_{r-}^{t,x}, e)\tilde{N}(dr, de) \\ \qquad\qquad + \int_t^{t\vee s} \nabla\Phi(X_r^{t,x})dA_r^{t,x},\ 0 \leq s \leq T, \\ A_s^{t,x} = \int_t^{t\vee s} \mathbb{1}_{\{X_r^{t,x}\in\partial G\}} dA_r^{t,x}, \end{cases} \tag{19}$$

where $b : \mathbb{R}^l \to \mathbb{R}^l$ and $\sigma : \mathbb{R}^l \to \mathbb{R}^{l\times l}$ satisfy, for $\kappa > 0$ and for any $(x, x') \in \mathbb{R}^{l+l}$,

$$\begin{cases} \text{(i)} \quad |b(x) - b(x')| + |\sigma(x) - \sigma(x')| \leq \kappa|x - x'|, \\ \text{(ii)} \quad |b(x)| + |\sigma(x)| \leq \kappa(1 + |x|). \end{cases} \tag{20}$$

Moreover $c : \mathbb{R}^l \times U \to \mathbb{R}$ is a measurable function which satisfies, for any $e \in U$ and $x, x' \in \mathbb{R}^l$, the following conditions:

$$\begin{cases} \text{(i)} \quad |c(x, e)| \leq \kappa(1 \wedge |e|), \\ \text{(ii)} \quad |c(x, e) - c(x', e)| \leq \kappa|x - x'|(1 \wedge |e|). \end{cases} \tag{21}$$

Under assumptions (18), (20) and (21) we state some properties of the processes $(X_s^{t,x}, A_s^{t,x})_{s\geq 0}$ which can be found in [14] and [10].

Proposition 5. *For each $T \geq 0$, there exists two constants C_T and C'_T such that for all $t < t' \leq T$ and $x, x' \in \bar{G}$,*

$$\mathbb{E}\big[\sup_{t' \leq s \leq T} |X_s^{t,x} - X_s^{t',x'}|^4\big] \leq C_T\big(|x - x'|^4 + |t - t'|^2\big),$$
$$\mathbb{E}\big[\sup_{t' \leq s \leq T} |A_s^{t,x} - A_s^{t',x'}|^4\big] \leq C'_T\big(|x - x'|^4 + |t - t'|^2\big).$$

Furthermore for all $0 \leq t \leq s \leq r$, we have

$$X_r^{t,x} = X_r^{s, X_s^{t,x}}. \tag{22}$$

Characterization (22) implies the Markov property of the process $(X_s^{t,x})_{s \geq 0}$ that allows the relationship with integral-partial differential equations to be established.

3.2 Viscosity solution for the obstacle IPDE with Neumann boundary condition

For all $(t, x) \in [0, T] \times \bar{G}$, let $(X_s^{t,x}, A_s^{t,x})_{s \geq 0}$ denote the solution of the reflected SDE (19). Let us set

$$\begin{cases} \xi^{t,x} := H(X_T^{t,x}), \\ L_s(\omega) := \ell(s, X_s^{t,x}), \\ f(s, \omega, y, z, v) := f(s, X_s^{t,x}, y, z, \int_U v(e)\gamma(X_s^{t,x}, e)\lambda(de)), \\ g(s, \omega, y) := g(s, X_s^{t,x}, y), \end{cases} \tag{23}$$

where the functions $f : [0, T] \times \bar{G} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ and $g : [0, T] \times \bar{G} \times \mathbb{R} \to \mathbb{R}$, and $H : \bar{G} \to \mathbb{R}$ and $\ell : [0, T] \times \bar{G} \to \mathbb{R}$ are continuous and satisfy, for some constants, $\alpha, \beta \in \mathbb{R}$ and $\kappa, C, p > 0$:

$$\begin{cases} \text{(i)} & |f(t, x, 0, 0, 0)| + |g(t, x, 0)| + |H(x)| + |\ell(t, x)| \leq C(1 + |x|^p), \\ \text{(ii)} & (y - y')(f(t, x, y, z) - f(t, x, y', z)) \leq \alpha|y - y'|^2, \\ \text{(iii)} & |f(t, x, y, z, v) - f(t, x, y, z', v')| \leq \kappa(|z - z'| + \|v - v'\|_\lambda), \\ \text{(iv)} & \text{the mapping } r \to f(t, x, y, z, r) \text{ is nondecreasing}, \\ \text{(v)} & (y - y')(g(t, x, y) - g(t, x, y')) \leq \beta|y - y'|^2, \\ \text{(vi)} & \ell \in C^{1,2} \text{ such that } \ell(T, x) \leq H(x), \ \forall (t, x) \in [0, T] \times \bar{G}. \end{cases} \tag{24}$$

Moreover we suppose that the function $\gamma : \mathbb{R}^l \times U \to \mathbb{R}$ satisfies, for any $e \in U$ and $x, x' \in G$, the conditions

$$\begin{cases} \text{(i)} & \gamma(x, e) \leq \kappa(1 \wedge |e|), \\ \text{(ii)} & |\gamma(x, e) - \gamma(x', e)| \leq \kappa|x - x'|(1 \wedge |e|). \end{cases} \tag{25}$$

It follows from Theorem 1 that, for all $(t, x) \in [0, T] \times \bar{G}$, there exists a unique quadruple $(Y_s^{t,x}, Z_s^{t,x}, V_s^{t,x}, K_s^{t,x})_{t \leq s \leq T}$ being a solution of the following reflected generalized BSDE:

$$\begin{cases} \text{(i)} & \mathbb{E}[\sup_{t\le s\le T}|Y_s^{t,x}|^2+\int_t^T|Y_s^{t,x}|^2dA_s^{t,x}+\int_t^T(|Z_s^{t,x}|^2+\|V_s^{t,x}\|_\lambda^2)ds]<\infty, \\ \text{(ii)} & Y_s^{t,x}=H(X_T^{t,x})+\int_s^T f(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x},V_r^{t,x})dr \\ & \qquad +\int_s^T g(r,X_r^{t,x},Y_r^{t,x})dA_r+K_T^{t,x}-K_s^{t,x}-\int_s^T Z_r^{t,x}dW_r \\ & \qquad -\int_s^T\int_U V_r^{t,x}(e)\tilde{N}(dr,de), \quad t\le s\le T, \\ \text{(iii)} & Y_s^{t,x}\ge \ell(s,X_s^{t,x}), \quad t\le s\le T, \\ \text{(iv)} & \int_t^T(Y_{s-}^{t,x}-\ell(s,X_{s-}^{t,x}))dK_s^{t,x}=0, \quad \mathbb{P}\text{-a.s.} \end{cases} \tag{26}$$

The process $Y_s^{t,x}$ is $\mathscr{F}_s^t$-adapted and $(Z_s^{t,x},V_s^{t,x},K_s^{t,x})$ are $\mathscr{F}_s$-predictable where

$$\mathscr{F}_s^t=\sigma\big(W_s-W_t,N(]t,s],\Lambda),t\le s\le r,\Lambda\in\mathscr{U}\big)\vee\mathscr{N}.$$

Now, we consider the following related obstacle problem for a parabolic integral-partial differential equation with nonlinear Neumann boundary conditions

$$\begin{cases} (u(t,x)-\ell(t,x)) \\ \quad \wedge(-\frac{\partial u}{\partial t}(t,x)-\mathscr{L}u(t,x)-f(t,x,u(t,x),(\nabla_x u\sigma)(t,x),\mathscr{B}u(t,x)))=0, \\ \qquad \forall(t,x)\in[0,T]\times G, \\ u(T,x)=H(x), \quad \forall x\in G, \\ \frac{\partial u}{\partial n}(t,x)+g(t,x,u(t,x))=0, \quad \forall x\in\partial G, \end{cases} \tag{27}$$

where $\mathscr{L}$ is the second-order integral-differential operator

$$\mathscr{L}=R+S$$

with

$$R\phi=\frac{1}{2}Tr\big[\sigma\sigma^T(x)\big]D_x^2\phi(t,x)+\big\langle b(x),\nabla_x\phi(t,x)\big\rangle,$$
$$S\phi=\int_U\Big(\phi\big(t,x+c(x,e)\big)-\phi(t,x)-\big\langle\nabla_x\phi(t,x),c(x,e)\big\rangle\Big)\lambda(de),$$

and $\mathscr{B}$ is an integral operator defined as

$$\mathscr{B}\phi=\int_U\Big(\phi\big(t,x+c(x,e)\big)-\phi(t,x)\Big)\gamma(x,e)\lambda(de),$$

and for every $x\in\partial G$,

$$\frac{\partial\phi}{\partial n}=\big\langle\nabla_x\phi,\nabla\Phi(x)\big\rangle.$$

Now, according to Definition 3.1, Remark 3.2 and Lemma 3.3 in [1], we give a definition of the viscosity solution of (27).

Definition 1. Let u be a function which belongs to $\mathscr{C}([0,T]\times\bar{G},\mathbb{R})$. Then:

(a) It is called a subsolution of (27), if $u(T,x)\leq H(x)$, $\forall x\in\bar{G}$, and for any $\varphi\in C^{1,2}([0,T]\times\bar{G})$ such that whenever $(t,x)\in[0,T]\times\bar{G}$ is a local maximum of $u-\varphi$, we have, suppressing dependence on (t,x),

$$\begin{cases}(u-\ell)\wedge[-\varphi_t-\mathscr{L}\varphi-f(t,x,u,(\nabla\varphi\sigma),\mathscr{B}\varphi)]\leq 0, & x\in G,\\ [(u-\ell)\wedge[-\varphi_t-\mathscr{L}\varphi-f(t,x,u,(\nabla\varphi\sigma),\mathscr{B}\varphi)]] & \\ \qquad\wedge[-\frac{\partial\varphi}{\partial n}-g(t,x,u)]\leq 0, & x\in\partial G.\end{cases}$$

In other words, if $u(t,x)>\ell(t,x)$ then

$$\begin{cases}[-\varphi_t-\mathscr{L}\varphi-f(t,x,u,(\nabla\varphi\sigma),\mathscr{B}\varphi)]\leq 0, & x\in G,\\ [-\varphi_t-\mathscr{L}\varphi-f(t,x,u,(\nabla\varphi\sigma),\mathscr{B}\varphi)] & \\ \qquad\wedge[-\frac{\partial\varphi}{\partial n}-g(t,x,u)]\leq 0, & x\in\partial G.\end{cases}$$

(b) It is called a supersolution of (27), if $u(T,x)\geq H(x)$, $\forall x\in\bar{G}$, and for any $\varphi\in C^{1,2}([0,T]\times\bar{G})$ such that whenever $(t,x)\in[0,T]\times\bar{G}$ is a local minimum of $u-\varphi$, we have, suppressing dependence on (t,x),

$$\begin{cases}(u-\ell)\wedge[-\varphi_t-\mathscr{L}\varphi-f(t,x,u,(\nabla\varphi\sigma),\mathscr{B}\varphi)]\geq 0, & x\in G,\\ [(u-\ell)\wedge[-\varphi_t-\mathscr{L}\varphi-f(t,x,u,(\nabla\varphi\sigma),\mathscr{B}\varphi)]] & \\ \qquad\vee[-\frac{\partial\varphi}{\partial n}-g(t,x,u)]\geq 0, & x\in\partial G.\end{cases}$$

In other words, if $u(t,x)\geq\ell(t,x)$ then

$$\begin{cases}[-\varphi_t-\mathscr{L}\varphi-f(t,x,u,(\nabla\varphi\sigma),\mathscr{B}\varphi)]\geq 0, & x\in G,\\ [-\varphi_t-\mathscr{L}\varphi-f(t,x,u,(\nabla\varphi\sigma),\mathscr{B}\varphi)] & \\ \qquad\vee[-\frac{\partial\varphi}{\partial n}-g(t,x,u)]\geq 0, & x\in\partial G.\end{cases}$$

(c) $u\in C([0,T]\times\bar{G})$ is said to be a viscosity solution of (27), if it is both sub- and supersolution.

Now, we denote

$$u(t,x)=Y_t^{t,x}, \tag{28}$$

Obviously, this function is deterministic. By the uniqueness of the solution of (26), it is not hard to see that

$$Y_s^{t,x}=Y_s^{s,X_s^{t,x}}=u\big(s,X_s^{t,x}\big),\quad\forall\, t\leq s<T.$$

Next we will indicate some basic properties of this function.

Proposition 6. $u\in C([0,T]\times\bar{G},\mathbb{R})$.

Proof. First, we define for all (t,x) the solution $Y_s^{t,x}$ for all $s \in [0,T]$ by choosing $Y_s^{t,x} = Y_t^{t,x}$ for $0 \le s \le t$. Note that, for each sequence (t_n, x_n) which converges to (t,x), the following converges as $n \to \infty$:

$$\mathbb{E}\big[e^{\mu k_T}\big|H\big(X_T^{t,x}\big) - H\big(X_T^{t_n,x_n}\big)\big|^2\big] \underset{n\to\infty}{\longrightarrow} 0,$$

$$\mathbb{E}\int_0^T e^{\mu k_r}\big|f\big(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}, V_r^{t,x}\big) - f\big(r, X_r^{t_n,x_n}, Y_r^{t,x}, Z_r^{t,x}, V_r^{t,x}\big)\big|^2 dr \underset{n\to\infty}{\longrightarrow} 0,$$

$$\mathbb{E}\int_0^T e^{\mu k_r}\big|g\big(r, X_r^{t,x}, Y_r^{t,x}\big) - g\big(r, X_r^{t_n,x_n}, Y_r^{t,x}\big)\big|^2 dA_r^{t_n,x_n} \underset{n\to\infty}{\longrightarrow} 0,$$

$$\mathbb{E}\int_0^T e^{\mu k_r}\big|g\big(r, X_r^{t,x}, Y_r^{t,x}\big)\big|^2\big(dA_r^{t,x} - dA_r^{t_n,x_n}\big) \underset{n\to\infty}{\longrightarrow} 0,$$

$$\mathbb{E}\int_0^T e^{\mu k_r}\big(\ell\big(r, X_s^{t,x}\big) - \ell\big(r, X_s^{t_n,x_n}\big)\big)^2 \Delta K_r \underset{n\to\infty}{\longrightarrow} 0,$$

with $k \triangleq |\bar{A}| + A^{t_n,x_n}$ where $\bar{A} = A^{t,x} - A^{t_n,x_n}$ and $|\bar{A}|$ is the total variation of $\bar{A}$ and $\Delta K := K^{t,x} - K^{t_n,x_n}$. Those convergences follow from the continuity assumptions of f, g, H and ℓ and Proposition 5. As in the proof of Proposition 1, we can derive the desired result. □

We now prove that our reflected generalized BSDE provides a viscosity solution of (27).

Theorem 4. *The function u defined in* (28) *is a viscosity solution of* (27).

Proof. First let us show that u is a viscosity subsolution of (27). A similar argument would show that is a viscosity supersolution of (27). Let $\varphi \in C^{1,2}([0,T]\times\bar{G})$ and $(t_0,x_0) \in [0,T]\times\bar{G}$ such that $\varphi(t_0,x_0) = u(t_0,x_0)$ and $\varphi(t,x) \ge u(t,x)$ for all $(t,x) \in [0,T]\times\bar{G}$.

Step 1. Suppose that $u(t_0,x_0) > \ell(t_0,x_0)$ and $x_0 \in G$, and that

$$-\varphi_t(t_0,x_0) - \mathscr{L}\varphi(t_0,x_0) - f\big(t_0, x_0, u(t_0,x_0), (\nabla\varphi\sigma)(t_0,x_0), \mathscr{B}\varphi(t_0,x_0)\big) > 0,$$

and we will find a contradiction.

It follows from the continuity of f, g, b, σ, c and φ that there exist $\varepsilon > 0$ and $\eta_\varepsilon > 0$ such that for all (t,x), $t_0 \le t \le t_0 + \eta_\varepsilon$ and $\{x : |x - x_0| \le \eta_\varepsilon\} \subset G$, we have $u(t,x) \ge \ell(t,x) + \varepsilon$ and

$$-\varphi_t(t,x) - \mathscr{L}\varphi(t,x) - f\big(t, x, u(t,x), (\nabla\varphi\sigma)(t,x), \mathscr{B}\varphi(t,x)\big) \ge \varepsilon. \tag{29}$$

Define

$$\tau = \inf\big\{s \ge t_0 : |X_s^{t_0,x_0} - x_0| > \eta_\varepsilon\big\} \wedge (t_0 + \eta_\varepsilon). \tag{30}$$

Note that for all $s \in [t_0, \tau]$, we have

$$u\big(s, X_s^{t_0,x_0}\big) \ge \ell\big(s, X_s^{t_0,x_0}\big) + \varepsilon.$$

Consequently, the process (K^{t_0,x_0}) is constant in $[t_0, \tau]$ and, for all $t_0 \le s \le \tau$, we have

$$Y_s^{t_0,x_0} = Y_\tau^{t_0,x_0} + \int_s^\tau f\big(r, X_r^{t_0,x_0}, Y_r^{t_0,x_0}, Z_r^{t_0,x_0}, V_r^{t_0,x_0}\big)dr - \int_s^\tau Z_r^{t_0,x_0} dW_r \\ - \int_s^\tau \int_U V_r^{t_0,x_0}(e)\tilde{N}(dr, de).$$

On the other hand, applying Itô's formula to $\varphi(s, X_s^{t_0,x_0})$ yields that

$$\begin{aligned}
&\varphi\big(\tau, X_\tau^{t_0,x_0}\big) \\
&= \varphi\big(s, X_s^{t_0,x_0}\big) + \int_s^\tau \frac{\partial \varphi}{\partial r}\big(r, X_r^{t_0,x_0}\big)dr + \int_s^\tau \nabla\varphi\big(r, X_r^{t_0,x_0}\big)dX_r^{t_0,x_0} \\
&\quad + \frac{1}{2}\int_s^\tau D^2\varphi\big(r, X_r^{t_0,x_0}\big)\big(\sigma\sigma^T\big)\big(X_r^{t_0,x_0}\big)dr \\
&\quad + \int_s^\tau \int_U \big[\varphi\big(r, X_{r-}^{t_0,x_0} + c\big(X_{r-}^{t_0,x_0}, e\big)\big) - \varphi\big(r, X_{r-}^{t_0,x_0}\big) \\
&\quad - \nabla\varphi\big(r, X_{r-}^{t_0,x_0}\big)c\big(X_{r-}^{t_0,x_0}, e\big)\big]N(dr, de) \\
&= \varphi\big(s, X_s^{t_0,x_0}\big) \\
&\quad + \int_s^\tau \Big[\frac{\partial \varphi}{\partial r}\big(r, X_r^{t_0,x_0}\big) + \nabla\varphi\big(r, X_r^{t_0,x_0}\big)b\big(X_r^{t_0,x_0}\big) + \frac{1}{2}D^2\varphi\big(r, X_r^{t_0,x_0}\big)\big(\sigma\sigma^T\big)\big(X_r^{t_0,x_0}\big)\Big]dr \\
&\quad + \int_s^\tau \nabla\varphi\big(r, X_r^{t_0,x_0}\big)\sigma\big(X_r^{t_0,x_0}\big)dW_r \\
&\quad + \int_s^\tau \int_U \big[\varphi\big(r, X_{r-}^{t_0,x_0} + c\big(X_{r-}^{t_0,x_0}, e\big)\big) - \varphi\big(r, X_{r-}^{t_0,x_0}\big)\big]\tilde{N}(dr, de) \\
&\quad + \int_s^\tau \int_U \big[\varphi\big(r, X_{r-}^{t_0,x_0} + c\big(X_{r-}^{t_0,x_0}, e\big)\big) - \varphi\big(r, X_{r-}^{t_0,x_0}\big) \\
&\quad - \nabla\varphi\big(r, X_{r-}^{t_0,x_0}\big)c\big(X_{r-}^{t_0,x_0}, e\big)\big]\lambda(de)dr.
\end{aligned}$$

Then

$$\begin{aligned}
\varphi\big(s, X_s^{t_0,x_0}\big) &= \varphi\big(\tau, X_\tau^{t_0,x_0}\big) - \int_s^\tau \Big[\frac{\partial \varphi}{\partial r} + \mathscr{L}\varphi\Big]\big(r, X_r^{t_0,x_0}\big)dr \\
&\quad - \int_s^\tau \nabla\varphi\big(r, X_r^{t_0,x_0}\big)\sigma\big(X_r^{t_0,x_0}\big)dW_r \\
&\quad - \int_s^\tau \int_U \big[\varphi\big(r, X_{r-}^{t_0,x_0} + c\big(X_{r-}^{t_0,x_0}, e\big)\big) - \varphi\big(r, X_{r-}^{t_0,x_0}\big)\big]\tilde{N}(dr, de), \\
&\quad t_0 \le s \le \tau.
\end{aligned}$$

Now, by assumption (29), we have

$$-\Big[\frac{\partial \varphi}{\partial s} + \mathscr{L}\varphi\Big]\big(s, X_s^{t_0,x_0}\big)$$

$$- f\big(s, X_s^{t_0,x_0}, \varphi\big(s, X_s^{t_0,x_0}\big), (\nabla\varphi\sigma)\big(s, X_s^{t_0,x_0}\big), \mathscr{B}\varphi\big(s, X_s^{t_0,x_0}\big)\big) \geq \varepsilon.$$

Also,

$$\varphi\big(\tau, X_\tau^{t_0,x_0}\big) \geq u\big(\tau, X_\tau^{t_0,x_0}\big) = Y_\tau^{t_0,x_0}.$$

We deduce with the help of Theorem 3 that

$$\varphi(t_0, x_0) > \varphi\big(t_0, X_{t_0}^{t_0,x_0}\big) - \varepsilon(\tau - t_0) \geq u(t_0, x_0)$$

which contradicts our assumption.

Step 2. If we continue to assume that $u(t_0, x_0) > \ell(t_0, x_0)$ and $x_0 \in \partial G$, and that

$$\big[-\varphi_t(t_0, x_0) - \mathscr{L}\varphi(t_0, x_0) - f\big(t_0, x_0, u(t_0, x_0), (\nabla\varphi\sigma)(t_0, x_0), \mathscr{B}\varphi(t_0, x_0)\big)\big] \wedge \left[-\frac{\partial\varphi}{\partial n}(t_0, x_0) - g\big(t_0, x_0, \varphi(t_0, x_0)\big)\right] > 0,$$

we will find a contradiction.

It follows from the continuity of f, g, b, σ, c and φ that there exist $\varepsilon > 0$ and $\eta_\varepsilon > 0$ such that for all (t, x), $t_0 \leq t \leq t_0 + \eta_\varepsilon$ and $\{x : |x - x_0| \leq \eta_\varepsilon\}$, we have $u(t, x) \geq \ell(t, x) + \varepsilon$ and

$$\big[-\varphi_t(t, x) - \mathscr{L}\varphi(t, x) - f\big(t, x, u(t, x), (\nabla\varphi\sigma)(t, x), \mathscr{B}\varphi(t, x)\big)\big] \tag{31}$$

$$\wedge \left[-\frac{\partial\varphi}{\partial n}(t, x) - g\big(t, x, \varphi(t, x)\big)\right] \geq \varepsilon. \tag{32}$$

Let τ be the stopping time defined as above in (30), and let us note that for all $s \in [t_0, \tau]$ we have

$$u\big(s, X_s^{t_0,x_0}\big) \geq \ell\big(s, X_s^{t_0,x_0}\big) + \varepsilon.$$

Consequently, the process (K^{t_0,x_0}) is constant in $[t_0, \tau]$ and for all $t_0 \leq s \leq \tau$, and we have

$$\begin{aligned} Y_s^{t_0,x_0} = Y_\tau^{t_0,x_0} &+ \int_s^\tau f\big(r, X_r^{t_0,x_0}, Y_r^{t_0,x_0}, Z_r^{t_0,x_0}, V_r^{t_0,x_0}\big) dr \\ &+ \int_s^\tau g\big(r, X_r^{t_0,x_0}, Y_r^{t_0,x_0}\big) dA_r^{t_0,x_0} \\ &- \int_s^\tau Z_r^{t_0,x_0} dW_r - \int_s^\tau \int_U V_r^{t_0,x_0}(e)\tilde{N}(dr, de). \end{aligned}$$

On the other hand, applying Itô's formula to $\varphi(s, X_s^{t_0,x_0})$ we end up with

$$\begin{aligned} &\varphi\big(s, X_s^{t_0,x_0}\big) \\ &\quad = \varphi\big(\tau, X_\tau^{t_0,x_0}\big) - \int_s^\tau \left[\frac{\partial\varphi}{\partial r} + \mathscr{L}\varphi\right]\big(r, X_r^{t_0,x_0}\big) dr + \int_s^\tau \frac{\partial\varphi}{\partial n}\big(r, X_r^{t_0,x_0}\big) dA_r^{t_0,x_0} \\ &\quad\quad - \int_s^\tau \nabla\varphi\big(r, X_r^{t_0,x_0}\big)\sigma\big(X_r^{t_0,x_0}\big) dW_r \end{aligned}$$

$$- \int_s^\tau \int_U \big[\varphi\big(r, X_{r^-}^{t_0,x_0} + c\big(X_{r^-}^{t_0,x_0}, e\big)\big) - \varphi\big(r, X_{r^-}^{t_0,x_0}\big)\big]\tilde{N}(dr, de), \quad t_0 \le s \le \tau.$$

Now, by assumption (31), we have

$$\Bigg(-\bigg[\frac{\partial\varphi}{\partial s} + \mathscr{L}\varphi\bigg]\big(s, X_s^{t_0,x_0}\big) \\ - f\big(s, X_s^{t_0,x_0}, \varphi\big(s, X_s^{t_0,x_0}\big), (\nabla\varphi\sigma)\big(s, X_s^{t_0,x_0}\big), \mathscr{B}\varphi\big(s, X_s^{t_0,x_0}\big)\big)\Bigg) \\ \wedge \bigg[-\frac{\partial\varphi}{\partial n}\big(s, X_s^{t_0,x_0}\big) - g\big(s, X_s^{t_0,x_0}, \varphi\big(s, X_s^{t_0,x_0}\big)\big)\bigg] \ge \varepsilon.$$

Also,

$$\varphi\big(\tau, X_\tau^{t_0,x_0}\big) \ge u\big(\tau, X_\tau^{t_0,x_0}\big) = Y_\tau^{t_0,x_0}.$$

We deduce with the help of Theorem 3 that

$$\varphi(t_0, x_0) > \varphi\big(t_0, X_{t_0}^{t_0,x_0}\big) - \varepsilon(\tau - t_0) \ge Y_{t_0}^{t_0,x_0} = u(t_0, x_0).$$

which leads to a contradiction. □

Remark 5. The uniqueness of the viscosity solution will be obtained by the comparison of sub- and supersolutions of IPDE with nonlinear Neumann boundary conditions. It follows from an adaptation of standard techniques and the proof of Theorem 3.5 in [1], taking into account the continuity of the solution u and the obstacle ℓ which enables us to focus exclusively on the results on $[0, T] \times G$, and when $u > \ell$.

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