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A modified Φ-Sobolev inequality for canonical Lévy processes and its applications

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Abstract A new modified Φ-Sobolev inequality for canonical L^2-Lévy processes, which are hybrid cases of the Brownian motion and pure jump-Lévy processes, is developed. Existing results included only a part of the Brownian motion process and pure jump processes. A generalized version of the Φ-Sobolev inequality for the Poisson and Wiener spaces is derived. Furthermore, the theorem can be applied to obtain concentration inequalities for canonical Lévy processes. In contrast to the measure concentration inequalities for the Brownian motion alone or pure jump Lévy processes alone, the measure concentration inequalities for canonical Lévy processes involve Lambert's W-function. Examples of inequalities are also presented, such as the supremum of Lévy processes in the case of mixed Brownian motion and Poisson processes.

Keywords Malliavin calculus, Lévy processes, logarithmic Sobolev inequalities, Φ-Sobolev inequalities, deviation inequalities, concentration inequalities
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1 Introduction

For smooth convex function Φ on an interval of $\mathbb{R}$, the Φ-entropy of a random variable F is defined as

$$\mathrm{Ent}_\Phi(F) = \mathbb{E}[\Phi(F)] - \Phi(\mathbb{E}[F]).$$

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In this paper, we deal with a (modified) Φ-Sobolev inequality and related concentration inequalities for canonical Lévy processes. Such inequalities related to Φ-entropies can be seen as an inclusive interpolation between Poincaré and Gross logarithmic Sobolev inequalities.

The logarithmic Sobolev inequalities (LSI) give an infinite-dimensional analog of Sobolev inequalities on finite-dimensional space such as Euclidean space. In a seminal paper [14], the following LSI on the Wiener space was introduced:

$$\mathbb{E}[F^2 \log F^2] - \mathbb{E}[F^2] \log \mathbb{E}[F^2] \leq 2\mathbb{E}[|D^W F|_{\mathbb{H}}^2],$$

for any F which is in the stochastic Sobolev space based on the Wiener process, where $|\cdot|_{\mathbb{H}}$ is the norm of the Cameron–Martin Hilbert space and D^W is the Malliavin gradient operator. However, it should be noted that this is a modern formulation and not the original one. The stochastic Sobolev space with respect to the Wiener process is the domain of the Malliavin gradient operator D^W. The left term in the above inequality means the entropy of a random variable F^2. For more details of the Malliavin calculus concerning the Wiener processes, see [20]. This inequality does not depend on the dimension of underlying space and is equivalent to the hyper-contractivity, and it implies Poincaré inequalities (or spectral gap inequalities) such as

$$\mathbb{E}[F^2] - (\mathbb{E}[F])^2 \leq \mathbb{E}[|D^W F|_{\mathbb{H}}^2].$$

We also remark that the Gaussian logarithmic Sobolev inequality goes back to [26].

The authors of [6] proved the logarithmic Sobolev inequalities on path space. Martingale representation has become the standard method for proving logarithmic Sobolev inequalities. Capitaine et al. (see [6]) gave a simple proof of the log-Sobolev inequality for functionals of the Brownian motion by using the Clark–Ocone–Haussmann formula. The Clark–Ocone–Haussmann formula leads to an explicit martingale representation for random variables in terms of the Malliavin derivatives. Moreover, they proved the logarithmic Sobolev inequalities for the Brownian motion on a manifold by the Clark–Ocone–Haussmann formula. Furthermore, they obtained some isoperimetric inequalities on path spaces.

We now review studies for the extension of LSI from the view of Poisson space. Surgailis (see [27]) proved that the logarithmic Sobolev inequalities failed for the Poisson space compared to the Wiener process case. Let π_θ be the Poisson measure on $\mathbb{N}$ with parameter $\theta > 0$ and consider the Dirichlet form on $L^2(\pi_\theta)$ given by

$$\mathcal{E}_{Poi}(f, g) = \int_N (D^\pi f \cdot D^\pi g) d\pi_\theta,$$

where

$$D^\pi f(x) = f(x+1) - f(x)$$

for all $x \in \mathbb{N}$. He proved that

$$\exists C(2) > 0, \forall F \in L^2(\pi_\theta) : \mathbb{E}_{\pi_\theta}[F^2 \log F^2] - \mathbb{E}_{\pi_\theta}[F^2] \log \mathbb{E}_{\pi_\theta}[F^2] \leq C(2)\mathcal{E}_{Poi}(F, F)$$

does not hold. Alternatively, Wu ([31]) proved a modified logarithmic Sobolev inequality for a Poisson space by using the Clark–Ocone type formula on the Poisson space. From the modified logarithmic Sobolev inequality, several previously known inequalities were derived similar to the logarithmic Sobolev inequality for Gaussian random variables. The author also proved a deviation inequality for a Poisson space as an application.

In the paper [7], Chafaï provides a synthesis for LSI. For a smooth convex function Φ, he introduced Φ-Sobolev functional inequality for the Wiener and Poisson spaces.

There are also LSI for discrete settings. The LSI for the Poisson space leads to deeper geometric extension. Privault ([22]) proved the log-Sobolev inequalities and deviation inequalities for discrete-time random walks (see also [21]). Moreover, the author introduced log-Sobolev and deviation inequalities for normal martingales on the Wiener and Poisson spaces. These inequalities were proved by using Clark–Ocone type formulas.

We now review a significant recent development for LSI. Bourguin and Peccati ([5]), by using Mehler's formula, provided direct, intrinsic proof of a modified logarithmic Sobolev inequality proved by Wu in [31]. Note that they did not use the Clark–Ocone type formula. As an application of the inequality, Bachmann and Peccati ([3]) deal with Poisson functionals and provide general concentration inequalities by combining logarithmic Sobolev inequalities for Poisson random measures with a Herbst-type argument. Moreover, Nourdin, Peccati and Yang ([19]) proved restricted hyper-contractivity on the Poisson space by using the modified logarithmic Sobolev inequality.

We also review a significant recent development for Φ-Sobolev inequality. By using the same sort argument as [6], Chafaï derived a Φ-Sobolev inequality for the Wiener measure (see Theorem 4.2 in [7]). Moreover, a modified Φ-Sobolev inequality for Poisson processes was derived by Chafaï in [7]. By using a modified Φ-Sobolev inequality derived by Chafaï in [7], Gusakova et al. ([15]) established a recursion scheme for moments. They applied its scheme to derive moment and concentration inequalities for functionals on abstract Poisson spaces. Moreover, they also applied the general results to stochastic geometry, namely Poisson cylinder models and Poisson random polytopes. On the other hand, Hariya ([16]) derived a family of inequalities that unifies the exponential and original hyper-contractivities; a generalization of the Gaussian logarithmic Sobolev inequality was obtained as a result. He also discussed a connection of those results with Φ-entropy inequalities in a general framework of Markov semi-groups. Unification of the exponential hyper-contractivity and the reverse hyper-contractivity of the Ornstein–Uhlenbeck semi-group Q was also provided.

The results mentioned above dealt with only the Gaussian part or the pure jump part. In this paper, in contrast, we treat them at the same time. The following modified Φ-Sobolev type inequality on canonical space $(\Omega, \mathcal{F}, \mathbb{P})$ are obtained:

$$\begin{aligned}\mathrm{Ent}_\Phi(F) \leq{}& \frac{1}{2}\sigma^2\mathbb{E}\Big[\int_0^T \Phi''(F)|D_{t,0}F|^2 dt\Big] \\ &+ \mathbb{E}\Big[\int_0^T \int_{\mathbb{R}_0} (\Psi_{t,z}\Phi(F) - \Phi'(F)\Psi_{t,z}F) z\nu(dz)dt\Big], \end{aligned} \tag{1.1}$$

for $F \in \mathbb{D}^{1,2}$ satisfying $F > 0$ with probability one and convex functions with some properties (we defined below in Section 3), where $\mathbb{R}_0 = \mathbb{R}\backslash\{0\}$, σ is constant number, ν the Lévy measure for the canonical Lévy process, $\mathbb{D}^{1,2}$ the stochastic Sobolev space for canonical Lévy processes and $\Psi_{t,z}$ the Malliavin increment quotient operator for functionals of canonical Lévy processes defined in [24] (see also [13]). To show the main theorem, we adopt the Malliavin calculus for canonical Lévy processes, based on [24, 13, 11, 29] and [28]. We provide a simple proof of a modified logarithmic Sobolev inequality (1.1) by using a Clark–Ocone type formula for Lévy processes. This modified logarithmic Sobolev inequality (1.1) for the canonical Lévy process unifies the logarithmic Sobolev inequalities for the Wiener, Poisson, and Lévy processes and derives the Poincaré inequalities for each process. In addition, as its application, we derive a concentration inequality for canonical Lévy processes:

$$\begin{aligned}
&\mathbb{P}(F - \mathbb{E}[F] > r) \\
&\leq \exp\Big[-\frac{r^2}{2\alpha^2} - \frac{\gamma^2}{2\alpha^2\beta^2}\Big(2\alpha^2 + 2\beta r + \gamma^2\Big) \\
&\qquad + \frac{\alpha^2}{2\beta^2}\Big(\mathcal{W}(\frac{\gamma^2}{\alpha^2}\exp\{\frac{\beta r + \gamma^2}{\alpha^2}\})\Big)^2 + \frac{\alpha^2}{\beta^2}\mathcal{W}(\frac{\gamma^2}{\alpha^2}\exp\{\frac{\beta r + \gamma^2}{\alpha^2}\})\Big]
\end{aligned}$$

for all $r > 0$, where $F \in \mathbb{D}^{1,2}$ is such that

$$\begin{aligned}
&zD_{t,z}F \leq \beta \in (0,\infty),\ q \otimes \mathbb{P}\text{-a.e.}, \\
&\sigma^2 \int_0^T |D_{t,0}F|^2 dt \leq \alpha^2 < \infty,\ \mathbb{P}\text{-a.e.}, \\
&\text{and} \int_{[0,T]\times\mathbb{R}_0} |D_{t,z}F|^2 z^2 \nu(dz)dt \leq \gamma^2 < \infty,\ \mathbb{P}\text{-a.e.},
\end{aligned}$$

and $\mathcal{W}$ is the principal branch of the Lambert W-function on $(-e^{-1}, \infty)$. Note that there are many seminal pieces of research for W-function (see, e.g., [4, 8], and [30]).

This paper is organized as follows. In Section 2, we review the Malliavin calculus for canonical Lévy processes, based on [24, 11], and [29], and [28]. In Section 3, by using the results of Section 2, we prove a modified logarithmic Sobolev inequality for canonical Lévy processes. In Section 4, we discuss applications of the modified logarithmic Sobolev inequality for canonical Lévy processes. Especially logarithmic Sobolev inequalities for the Wiener and Poisson processes, Poincaré type inequalities, and stochastic exponent are considered. In Section 5, a concentration inequality for canonical Lévy processes is proved as an application of the main theorem.

2 Malliavin calculus for canonical Lévy processes

There are various ways to develop the Malliavin calculus for Lévy processes. In this paper, we adopt the approach from [24], based on a chaos representation and increment quotient operator (see also, [11, 29] and [28]). The approach is suitable for our problems, especially since we can use some helpful calculation formulas and a Clark–Ocone type formula (see Propositions 2.1 and 2.4). In this setting, we shall construct a suitable canonical space on which a variational derivative with respect to the pure jump part of a Lévy process can be computed in a pathwise sense. The canonical space was constructed by [24].

In this section, we will give an overview of the approach of the Malliavin calculus on the canonical Lévy space according to [24].

2.1 Chaotic representation

We now recall chaotic representation based on Lévy processes. Itô first established the chaos representation for multiple Brownian integrals in [17] and generalized it to the Lévy case in [18].

We construct a probability space from the Wiener and canonical Lévy space. Let $T > 0$ be a finite time horizon, $(\Omega_W, \mathcal{F}_W, \mathbb{P}_W)$ the Wiener space, that is, the usual canonical space for a one-dimensional standard Brownian motion, with the space of continuous functions on $[0, T]$, the σ-algebra generated by the topology of uniform convergence and Wiener measure; W its coordinate mapping process, that is, a one-dimensional standard Brownian motion with $W_0 = 0$. Consider $(\Omega_J, \mathcal{F}_J, \mathbb{P}_J)$ to be the canonical Lévy space for a pure jump Lévy process J on $[0, T]$ with Lévy measure ν, and for its proper definition we refer to [24]. Now, we assume that $\int_{\mathbb{R}_0} z^2 \nu(dz) < \infty$. We denote $(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega_W \times \Omega_J, \mathcal{F}_W \otimes \mathcal{F}_J, \mathbb{P}_W \otimes \mathbb{P}_J)$ and call it the canonical space. Furthermore, we regard $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$ as the canonical filtration completed for $\mathbb{P}$. In the canonical space $(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{F})$, we can study a two-parameter Malliavin derivative.

To formulate the Malliavin calculus on the canonical space, we also need to prepare the Lévy–Itô decomposition for the Lévy process on $(\Omega, \mathcal{F}, \mathbb{P})$. It is well known that a square integrable centered Lévy process $X = (X_t)_{t\in[0,T]}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is given by

$$X_t = \sigma W_t + J_t - t \int_{\mathbb{R}_0} z\nu(dz), \tag{2.1}$$

where $\sigma \geq 0$ (see, e.g., pp. 162 in [12]). Note that J_t is given by

$$J_t = \int_0^t \int_{\mathbb{R}_0} zN(ds, dz),$$

where N is the Poisson random measure defined as

$$N(t, A) = \sum_{s\leq t} \mathbb{1}_A(\Delta X_s),$$

for $A \in \mathcal{B}(\mathbb{R}_0)$, $t \in [0, T]$, and $\Delta X_s = X_s - X_{s-}$.

Moreover, we give another representation. Denoting the compensated measure of the Poisson random measure N by

$$\widetilde{N}(dt, dz) = N(dt, dz) - \nu(dz)dt,$$

we have the following Lévy–Itô representation for X_t with respect to the compensated measure $\widetilde{N}$ as

$$X_t = \sigma W_t + \int_0^t \int_{\mathbb{R}_0} z\widetilde{N}(ds, dz). \tag{2.2}$$

We shall give measures to establish the chaotic representation, which follows the exposition from [24]. With the preceding notations, define a finite measure q on $[0, T] \times \mathbb{R}$ by

$$q(E) = \sigma^2 \int_{E(0)} dt\delta_0(dz) + \int_{E'} z^2 dt\nu(dz), \quad E \in \mathcal{B}([0, T] \times \mathbb{R}),$$

where $E(0) = \{(t, 0) \in [0, T] \times \mathbb{R}; (t, 0) \in E\}$, $E' = E - E(0)$ and δ_0 denotes the Dirac measure located at the origin, and the random measure Q on $[0, T] \times \mathbb{R}$ by

$$Q(E) = \sigma \int_{E(0)} dW_t\delta_0(dz) + \int_{E'} z\tilde{N}(dt, dz), \quad E \in \mathcal{B}([0, T] \times \mathbb{R}).$$

For $n \in \mathbb{N}$ and a simple function $h_n = \mathbb{1}_{E_1 \times \cdots \times E_n}$, with pairwise disjoints sets $E_1, \dots, E_n \in \mathcal{B}([0, T] \times \mathbb{R})$, a multiple two-parameter integral with respect to the random measure Q

$$I_n(h_n) = \int_{([0,T]\times\mathbb{R})^n} h_n((t_1, z_1), \dots, (t_n, z_n)) Q(dt_1, dz_1) \cdots Q(dt_n, dz_n)$$

can be defined as $I_n(h_n) = Q(E_1) \cdots Q(E_n)$. It is well known that the integral can be extended to the space $L^2_{T,q,n}$ of product measurable, deterministic functions $h_n : ([0, T] \times \mathbb{R})^n \to \mathbb{R}$ satisfying

$$\|h_n\|^2_{L^2_{T,q,n}} = \int_{([0,T]\times\mathbb{R})^n} |h_n((t_1, z_1), \dots, (t_n, z_n))|^2 q(dt_1, dz_1) \cdots q(dt_n, dz_n) < \infty.$$

Theorem 2 of [18] yields that any $\mathcal{F}$-measurable square-integrable random variable F on the canonical space has a unique chaotic representation

$$F = \sum_{n=0}^{\infty} I_n(h_n), \ \mathbb{P}\text{-a.s.},$$

with functions $h_n \in L^2_{T,q,n}$ that are symmetric in the n pairs (t_i, z_i), $1 \le i \le n$, and we have the isometry

$$\mathbb{E}[F^2] = \sum_{n=0}^{\infty} n! \|h_n\|^2_{L^2_{T,q,n}}.$$

See also Section 2 of [24] and Section 3 of [11].

2.2 *Stochastic Sobolev spaces and Malliavin derivatives*

Thanks to the chaotic representation, we can introduce some classes of stochastic Sobolev spaces $\mathbb{D}^{1,2}$, $\mathbb{D}^{1,2}_0$, $\mathbb{D}^{1,2}_1$, $\text{Dom}(D^W)$, $\mathbb{D}^W$, $\mathbb{D}^J$ and $\mathbb{D}^{1,2}_J$, and the Malliavin derivatives.

Denote by $\mathbb{D}^{1,2}$ the set of $\mathcal{F}$-measurable random variables $F \in L^2(\mathbb{P})$ with the representation $F = \sum_{n=0}^{\infty} I_n(h_n)$ satisfying

$$\sum_{n=1}^{\infty} nn!\|h_n\|^2_{L^2_{T,q,n}} < \infty.$$

The Malliavin derivative $DF : \Omega \times [0, T] \times \mathbb{R} \to \mathbb{R}$ of a random variable $F \in \mathbb{D}^{1,2}$ is a stochastic process defined by

$$D_{t,z}F = \sum_{n=1}^{\infty} nI_{n-1}(h_n((t, z), \cdot)), \quad \text{valid for } q\text{-a.e. } (t, z) \in [0, T] \times \mathbb{R}, \mathbb{P}\text{-a.s.}$$

To establish useful calculation formulas for the Mallivin calculus, we shall divide it into two cases; the case of $z = 0$ and the case of $z \neq 0$.

If the Brownian motion part exists, that is, $\sigma \neq 0$, we can consider the Malliavin derivative to the Brownian part. The class $\mathbb{D}^{1,2}_0$ means the set of $\mathcal{F}$-measurable random variables $F \in L^2(\mathbb{P})$ with the representation $F = \sum_{n=0}^{\infty} I_n(f_n)$ satisfying

$$\sum_{n=1}^{\infty} nn! \int_0^T \|f_n(\cdot, (t, 0))\|^2_{L^2_{T,q,n-1}} \sigma^2 dt < \infty,$$

and we can define

$$D_{t,0}F = \sum_{n=1}^{\infty} nI_{n-1}(f_n((t, 0), \cdot)), \quad \text{valid for } q\text{-a.e. } (t, 0) \in [0, T] \times \{0\}, \mathbb{P}\text{-a.s.},$$

for $F \in \mathbb{D}^{1,2}_0$.

If there exists the jump part, that is, $\nu \neq 0$, $\mathbb{D}^{1,2}_1$ is the set of $\mathcal{F}$-measurable random variables $F \in L^2(\mathbb{P})$ with the representation $F = \sum_{n=0}^{\infty} I_n(f_n)$ satisfying

$$\sum_{n=1}^{\infty} nn! \int_0^T \int_{\mathbb{R}_0} \|f_n(\cdot, (t, z))\|^2_{L^2_{T,q,n-1}} z^2 \nu(dz)dt < \infty.$$

Then, we can define, for $F \in \mathbb{D}^{1,2}_1$,

$$D_{t,z}F = \sum_{n=1}^{\infty} nI_{n-1}(f_n((t, z), \cdot)), \quad \text{valid for } q\text{-a.e. } (t, z) \in [0, T] \times \mathbb{R}_0, \mathbb{P}\text{-a.s.}$$

In the next two subsubsections, we shall summarize the Malliavin calculus on the Wiener spaces and pure jump Lévy spaces, and the mixtures.

2.2.1 Malliavin calculus on the Wiener spaces

We summarize the Malliavin calculus on the Wiener spaces. For $n \in \mathbb{N}$, let $L^2_{T,\lambda,n}$ denote the set of product measurable, deterministic functions $h^W_n : ([0, T])^n \to \mathbb{R}$ satisfying

$$\|h^W_n\|^2_{L^2_{T,\lambda,n}} = \int_{([0,T])^n} |h^W_n(t_1, \ldots, t_n)|^2 dt_1 \cdots dt_n < \infty.$$

For $n \in \mathbb{N}$ and $h_n^W \in L^2_{T,\lambda,n}$, we denote

$$I_n^W(h_n) = \int_{([0,T])^n} h_n^W(t_1, \ldots, t_n) dW_{t_1} \cdots dW_{t_n}.$$

In this setting, the Malliavin differential operator for the Wiener functionals is defined by

$$D_t^W F = \sum_{n=1}^{\infty} n I_{n-1}^W (f_n^W(t, \cdot)),$$

λ-a.e. $t \in [0, T]$, $\mathbb{P}$-a.s., for

$$F \in \mathrm{Dom}(D^W) = \{F = \sum_{n=0}^{\infty} I_n^W(f_n^W) \in L^2(\mathbb{P}^W); \sum_{n=1}^{\infty} n n! \| f_n^W \|^2_{L^2_{T,\lambda,n}} < \infty\}.$$

2.2.2 *Malliavin calculus for pure jump Lévy processes*

We summarize the Malliavin calculus for pure jump Lévy processes. For $n \in \mathbb{N}$, let $L^2_{T,\lambda\times\nu,n}$ denote the set of product measurable, deterministic functions $h_n^J : ([0, T] \times \mathbb{R}_0)^n \to \mathbb{R}$ satisfying

$$\|h_n^J\|^2_{L^2_{T,\lambda\times\nu,n}} = \int_{([0,T]\times\mathbb{R}_0)^n} |h_n^J((t_1, z_1), \ldots, (t_n, z_n))|^2 dt_1 \nu(dz_1) \cdots dt_n \nu(dz_n) < \infty.$$

Moreover, for $n \in \mathbb{N}$ and $h_n^J \in L^2_{T,\lambda\times\nu,n}$, we denote

$$I_n^J(h_n) = \int_{([0,T]\times\mathbb{R}_0)^n} h_n^J((t_1, z_1), \ldots, (t_n, z_n)) \tilde{N}(dt_1, dz_1) \cdots \tilde{N}(dt_n, dz_n).$$

Then, the Malliavin difference operator for pure jump Lévy functionals is defined by

$$D^J_{(t,z)} F = \sum_{n=1}^{\infty} n I_{n-1}^J (f_n^J((t, z), \cdot)),$$

$\lambda \times \nu$-a.e. $(t, z) \in [0, T] \times \mathbb{R}_0$, $\mathbb{P}$-a.s., for

$$F \in \mathbb{D}_J^{1,2} = \{F = \sum_{n=0}^{\infty} I_n^J(f_n^J) \in L^2(\mathbb{P}^J); \sum_{n=1}^{\infty} n n! \| f_n^J \|^2_{L^2_{T,\lambda\times\nu,n}} < \infty\}.$$

2.2.3 *The increment quotient operator for Lévy functionals*

We next consider the increment quotient operator for Lévy functionals defined by [24]. Let F be a random variable on $\Omega_W \times \Omega_J$. Then we define the increment quotient operator

$$\Psi_{t,z} F = \frac{F(\omega_W, \omega_J^{t,z}) - F(\omega_W, \omega_J)}{z}, \quad z \neq 0,$$

where $\omega_J^{t,z}$ transforms a family $\omega_J = ((t_1, z_1), (t_2, z_2), \ldots) \in \Omega_J$ into a new family $\omega_J^{t,z} = ((t, z), (t_1, z_1), (t_2, z_2), \ldots) \in \Omega_J$, by adding a jump of size z at time t into the trajectory.

2.3 Related formulas

In this subsection, we introduce calculation formulas for the Malliavin calculus for the Lévy processes. To this end, we first define the following classes:

$$\mathbb{D}^W = \Big\{F \in L^2(\mathbb{P});\ F(\cdot,\omega_J) \in \mathrm{Dom}(D^W) \text{ for } \mathbb{P}^N\text{-a.e. } \omega_J \in \Omega_J\Big\}$$

and

$$\mathbb{D}^J = \Big\{F \in L^2(\mathbb{P});\ \mathbb{E}\Big[\int_0^T \int_{\mathbb{R}_0} |\Psi_{t,z}F|^2 z^2 \nu(dz)dt\Big] < \infty\Big\}.$$

With the preceding settings, thanks to Propositions 2.6.1–2.6.2 in [10], the result of Section 3.3 in [1], Proposition 5.5, Remark 2.2 in [24], and the definitions of the Malliavin operators for the Wiener and pure jump Lévy functionals, we can derive the following formulas.

Proposition 2.1. *$\mathbb{D}^{1,2}$ is equal to $\mathbb{D}^W \cap \mathbb{D}^J$, and the following formulas hold.*

(1) *Let $F \in \mathbb{D}^W$. Then $F \in \mathbb{D}^{1,2}_0$ and*

$$D_{t,0}F = \mathbb{1}_{\{\sigma>0\}}\sigma^{-1}D^W_t F(\cdot,\omega_J)(\omega_W)$$

holds for q -a.e. $(t,z) \in [0,T] \times \{0\}$, $\mathbb{P}$-a.s.

(2) *If $F \in \mathbb{D}^J$, then $F \in \mathbb{D}^{1,2}_1$ and $D_{t,z}F = \Psi_{t,z}F$ for q-a.e. $(t,z) \in [0,T] \times \mathbb{R}_0$, $\mathbb{P}$-a.s.*

(3) *In the case $\sigma = 0$, $\nu \neq 0$, we have $\mathbb{D}^{1,2} = \mathbb{D}^{1,2}_J$ and $D^J_{(t,z)}F = zD_{t,z}F$ for q-a.e. $(t,z) \in [0,T] \times \mathbb{R}_0$, $\mathbb{P}$-a.s.*

(4) *When $\sigma \neq 0$, $\nu = 0$, one has $\mathbb{D}^{1,2} = \mathrm{Dom}(D^W)$ and $D^W_t F = \sigma D_{t,0}F$ for q-a.e. $(t,z) \in [0,T] \times \{0\}$, $\mathbb{P}$-a.s.*

Remark 2.2. Let $F \in \mathbb{D}^{1,2}$ be such that $F > 0$ with probability one. Then, $F + zD_{t,z}F > 0$ for $(t,z) \in [0,T] \times \mathbb{R}_0$, $q \times \mathbb{P}$-a.e. Indeed, Proposition 2.1 (2) implies that

$$\begin{aligned} F + zD_{t,z}F &= F + z\Psi_{t,z}F \\ &= F + z\frac{F(\omega_W,\omega_J^{t,z}) - F(\omega_W,\omega_J)}{z} \\ &= F + F(\omega_W,\omega_J^{t,z}) - F(\omega_W,\omega_J) = F(\omega_W,\omega_J^{t,z}) > 0, \quad z \neq 0 \end{aligned}$$

because $F > 0$ a.s.

The Malliavin derivative satisfies the following chain rule (see [29] and [28]).

Proposition 2.3 (Chain rule)**.** *Let $\varphi \in C^1(\mathbb{R}^n;\mathbb{R})$ and $F = (F_1,\ldots,F_n)$, where $F_1,\ldots,F_n \in \mathbb{D}^{1,2}$. Assume further that $\varphi(F) \in L^2(\mathbb{P})$ and*

$$\sum_{k=1}^n \frac{\partial}{\partial x_k}\varphi(F)D_{t,0}F_k\mathbb{1}_{\{0\}}(z)$$

$$+\frac{\varphi(F_1+zD_{t,z}F_1,\dots,F_n+zD_{t,z}F_n)-\varphi(F_1,\dots,F_n)}{z}\mathbb{1}_{\mathbb{R}_0}(z)\in L^2(q\times\mathbb{P}).$$

Then, we obtain $\varphi(F)\in\mathbb{D}^{1,2}$ *and*

$$\begin{aligned}D_{t,z}\varphi(F)=&\sum_{k=1}^n\frac{\partial\varphi}{\partial x_k}(F)D_{t,0}F_k\mathbb{1}_{\{0\}}(z)\\&+\frac{\varphi(F_1+zD_{t,z}F_1,\dots,F_n+zD_{t,z}F_n)-\varphi(F_1,\dots,F_n)}{z}\mathbb{1}_{\mathbb{R}_0}(z).\end{aligned}$$

We next present an explicit form of the martingale representation formula based on the Malliavin calculus (see, e.g., Theorem 3.5.2 in [10] and Theorem 10 in [25]). This is a key formula for the main theorem.

Proposition 2.4 (Clark–Ocone type formula for canonical Lévy functionals)**.**

$$\begin{aligned}F&=\mathbb{E}[F]+\int_{[0,T]\times\mathbb{R}}\mathbb{E}[D_{t,z}F|\mathcal{F}_{t-}]Q(dt,dz)\\&=\mathbb{E}[F]+\sigma\int_0^T\mathbb{E}[D_{t,0}F|\mathcal{F}_{t-}]dW_t+\int_0^T\int_{\mathbb{R}_0}\mathbb{E}[D_{t,z}F|\mathcal{F}_{t-}]z\tilde{N}(dt,dz)\end{aligned}$$

holds for all $F\in\mathbb{D}^{1,2}$.

By Proposition 2.4, we can derive the following Poincaré inequality.

Proposition 2.5 (Poincaré inequality)**.** *For* $F\in\mathbb{D}^{1,2}$*, one has*

$$\mathbb{E}[(F-\mathbb{E}[F])^2]\le\mathbb{E}\Big[\int_{[0,T]\times\mathbb{R}}|D_{t,z}F|^2q(dt,dz)\Big].$$

Proof. Proposition 2.4 implies that

$$\begin{aligned}\mathbb{E}[(F-\mathbb{E}[F])^2]&=\mathbb{E}\Big[\Big(\int_{[0,T]\times\mathbb{R}}\mathbb{E}[D_{t,z}F|\mathcal{F}_{t-}]Q(dt,dz)\Big)^2\Big]\\&=\mathbb{E}\Big[\int_{[0,T]\times\mathbb{R}}(\mathbb{E}[D_{t,z}F|\mathcal{F}_{t-}])^2q(dt,dz)\Big]\\&\le\mathbb{E}\Big[\int_{[0,T]\times\mathbb{R}}|D_{t,z}F|^2q(dt,dz)\Big]\end{aligned}$$

by the Itô isometry and Jensen's inequality. □

3 Φ-entropy and a modified Φ-Sobolev inequality for canonical Lévy processes

To derive a modified Φ-Sobolev inequality for canonical Lévy processes, we first introduce the following class of functions, as introduced in the work by Chafaï in [7].

Definition 3.1. We denote by $\mathscr{C}$ the space of functions $\Phi:\mathbb{R}_+\to\mathbb{R}$ satisfying the following three properties:

(1) Φ is convex and continuous,

(2) Φ is twice differentiable on $(0, \infty)$,

(3) Φ is either affine or Φ'' is strictly positive, and $1/\Phi''$ is concave.

Typical examples of functions belonging to $\mathscr{C}$ are $\Phi_{\log}(x) = x \log x$, $x \in \mathbb{R}_+$, and $\Phi_r(x) = x^{2/r}$, $r \in (1, 2)$, $x \in \mathbb{R}_+$. We next define the Φ-entropy.

Definition 3.2 (Φ-entropy)**.** For $\Phi \in \mathscr{C}$, the Φ-entropy of a random variable F is defined as

$$\mathrm{Ent}_\Phi(F) = \mathbb{E}[\Phi(F)] - \Phi(\mathbb{E}[F]).$$

In particular, the classical entropy

$$\mathrm{Ent}_{\Phi_{\log}}[F] = \mathbb{E}[F \log F] - \mathbb{E}[F] \log \mathbb{E}[F]$$

of F is recovered by taking $\Phi(x) = \Phi_{\log}(x)$, $x \in \mathbb{R}_+$.

Based on the previous preparation, we derive the following theorem.

Theorem 3.3. *Let $F \in \mathbb{D}^{1,2}$ be such that $F > 0$ with probability one. Then,*

$$\begin{aligned}\mathrm{Ent}_\Phi(F) \leq \frac{1}{2}\sigma^2 \mathbb{E}\Big[\int_0^T \Phi''(F)|D_{t,0}F|^2 dt\Big] \\ + \mathbb{E}\Big[\int_0^T \int_{\mathbb{R}_0} (\Psi_{t,z}\Phi(F) - \Phi'(F)\Psi_{t,z}F) z\nu(dz)dt\Big]\end{aligned}$$

holds for $\Phi \in \mathscr{C}$.

Proof. First, note that by a standard approximation argument, we can assume that there exist finite constants ε, η such that $0 < \varepsilon < F < \eta$ with probability one. In this way, classical measure-theoretical results justify all computations appearing below, involving, particularly, exchanging derivations and expectations. Moreover, note that F is $\mathcal{F}_{T-}$-measurable.

Propositions 2.1 and 2.4 imply that

$$\begin{aligned}F &= \mathbb{E}[F] + \sigma \int_0^T \mathbb{E}[D_{t,0}F|\mathcal{F}_{t-}]dW_t + \int_0^T \int_{\mathbb{R}_0} \mathbb{E}[D_{t,z}F|\mathcal{F}_{t-}]z\tilde{N}(dt, dz) \\ &= \mathbb{E}[F] + \sigma \int_0^T \mathbb{E}[D_{t,0}F|\mathcal{F}_{t-}]dW_t + \int_0^T \int_{\mathbb{R}_0} \mathbb{E}[\Psi_{t,z}F|\mathcal{F}_{t-}]z\tilde{N}(dt, dz). \quad (3.1)\end{aligned}$$

Denoting $U_t = \mathbb{E}[F|\mathcal{F}_{t-}]$, $\zeta_t = \mathbb{E}[D_{t,0}F|\mathcal{F}_{t-}]$ and $\xi_{t,z} = \mathbb{E}[z\Psi_{t,z}F|\mathcal{F}_{t-}]$, we have

$$\begin{aligned}\Phi(F) - \Phi(\mathbb{E}[F]) = \sigma \int_0^T \Phi'(U_t)\zeta_t dW_t + \frac{1}{2}\sigma^2 \int_0^T \Phi''(U_t)\zeta_t^2 dt \\ + \int_0^T \int_{\mathbb{R}_0} \{\Phi(U_t + \xi_{t,z}) - \Phi(U_t) - \Phi'(U_t)\xi_{t,z}\}\nu(dz)dt \\ + \int_0^T \int_{\mathbb{R}_0} \{\Phi(U_t + \xi_{t,z}) - \Phi(U_t)\}\tilde{N}(dt, dz)\end{aligned}$$

by the Itô formula (see, e.g., Theorem 9.4 in [12]) and (3.1). Hence, we obtain

$$\begin{aligned}
&\mathbb{E}[\Phi(F)] - \Phi(\mathbb{E}[F]) \\
&= \frac{1}{2}\sigma^2 \int_0^T \mathbb{E}[\Phi''(U_t)\zeta_t^2]dt \\
&\quad + \int_0^T \int_{\mathbb{R}_0} \mathbb{E}[\{\Phi(U_t + \xi_{t,z}) - \Phi(U_t) - \Phi'(U_t)\xi_{t,z}\}]\nu(dz)dt \\
&= \frac{1}{2}\sigma^2 \int_0^T \mathbb{E}[\widetilde{\Phi}_1(U_t, \zeta_t)]dt + \int_0^T \int_{\mathbb{R}_0} \mathbb{E}[\widetilde{\Phi}_2(U_t, \xi_{t,z})]\nu(dz)dt,
\end{aligned}$$

where

$$\widetilde{\Phi}_1(x, y) = \Phi''(x)y^2, \qquad x \in \mathbb{R}_+,\ y \in \mathbb{R},$$

and

$$\widetilde{\Phi}_2(x, y) = \Phi(x + y) - \Phi(x) - \Phi'(x)y, \qquad x, y \in \mathbb{R}_+.$$

(1) Since $\Phi \in \mathscr{C}$, $\widetilde{\Phi}_1$ is convex on $\{(x, y) \in \mathbb{R}_+ \times \mathbb{R}\}$ from [7]. Thus, using Jensen's inequality, we see that $\mathbb{P}$-almost surely and for dt-almost all $t \in [0, T]$,

$$\widetilde{\Phi}_1(U_t, \zeta_t) = \widetilde{\Phi}_1(\mathbb{E}[F|\mathcal{F}_{t-}], \mathbb{E}[D_{t,0}F|\mathcal{F}_{t-}]) \le \mathbb{E}[\widetilde{\Phi}_1(F, D_{t,0}F)|\mathcal{F}_{t-}].$$

(2) From [7], for $\Phi \in \mathscr{C}$, the function $\widetilde{\Phi}_2$ is convex on $\{(x, y) \in \mathbb{R}_+ \times \mathbb{R}_+ : x + y > 0\}$. Hence, using Jensen's inequality, we see that $\mathbb{P}$-almost surely and for $\nu(dz)dt$-almost all $(t, z) \in [0, T] \times \mathbb{R}_0$,

$$\widetilde{\Phi}_2(U_t, \xi_{t,z}) = \widetilde{\Phi}_2(\mathbb{E}[F|\mathcal{F}_{t-}], \mathbb{E}[z\Psi_{t,z}F|\mathcal{F}_{t-}]) \le \mathbb{E}[\widetilde{\Phi}_2(F, z\Psi_{t,z}F)|\mathcal{F}_{t-}].$$

As a consequence, we conclude that

$$\begin{aligned}
\mathrm{Ent}_\Phi(F) &= \mathbb{E}[\Phi(F)] - \Phi(\mathbb{E}[F]) \\
&\le \frac{1}{2}\sigma^2 \int_0^T \mathbb{E}[\mathbb{E}[\widetilde{\Phi}_1(F, D_{t,0}F)|\mathcal{F}_{t-}]dt \\
&\quad + \int_0^T \int_{\mathbb{R}_0} \mathbb{E}[\mathbb{E}[\widetilde{\Phi}_2(F, z\Psi_{t,z}F)|\mathcal{F}_{t-}]]\nu(dz)dt \\
&= \frac{1}{2}\sigma^2 \int_0^T \mathbb{E}[\widetilde{\Phi}_1(F, D_{t,0}F)]dt + \int_0^T \int_{\mathbb{R}_0} \mathbb{E}[\widetilde{\Phi}_2(F, z\Psi_{t,z}F)]\nu(dz)dt \\
&= \frac{1}{2}\sigma^2 \mathbb{E}\Big[\int_0^T \Phi''(F)|D_{t,0}F|^2 dt\Big] \\
&\quad + \mathbb{E}\Big[\int_0^T \int_{\mathbb{R}_0} (\Phi(F + z\Psi_{t,z}F) - \Phi(F) - z\Phi'(F)\Psi_{t,z}F)\nu(dz)dt\Big] \\
&= \frac{1}{2}\sigma^2 \mathbb{E}\Big[\int_0^T \Phi''(F)|D_{t,0}F|^2 dt\Big] \\
&\quad + \mathbb{E}\Big[\int_0^T \int_{\mathbb{R}_0} (\Psi_{t,z}\Phi(F) - \Phi'(F)\Psi_{t,z}F)z\nu(dz)dt\Big],
\end{aligned}$$

where we use

$$\Psi_{t,z}\Phi(F) = \frac{\Phi(F + z\Psi_{t,z}F) - \Phi(F)}{z}, \qquad (t, z) \in [0, T] \times \mathbb{R}_0. \qquad \square$$

Remark 3.4. By using Theorem 3.3, we shall derive a modified log-Sobolev inequality and a Poincaré type inequality for L^2-canonical Lévy processes.

(1) Let $F \in \mathbb{D}^{1,2}$ be such that $F > 0$ with probability one and take $\Phi(x) = x \log x$, $x > 0$. Then, we obtain that

$$\begin{aligned}\mathrm{Ent}_{\Phi_{\log}}[F] \leq{}& \frac{1}{2}\sigma^2\mathbb{E}\Big[\int_0^T \frac{1}{F}|D_{t,0}F|^2 dt\Big] \\ &+ \mathbb{E}\Big[\int_0^T \int_{\mathbb{R}_0} (\Psi_{t,z}(F \log F) - (\log F + 1)\Psi_{t,z}F) z\nu(dz)dt\Big]\end{aligned}$$

holds. This is a modified log-Sobolev inequality for L^2-canonical Lévy processes.

(2) Taking $F \in \mathbb{D}^{1,2}$ such that $F > 0$ with probability one and $\Phi(x) = x^2$, $x \in \mathbb{R}$, we have

$$\begin{aligned}\mathrm{Var}[F] &\leq \frac{1}{2}\sigma^2\mathbb{E}\Big[\int_0^T 2|D_{t,0}F|^2 dt\Big] \\ &\quad + \mathbb{E}\Big[\int_0^T \int_{\mathbb{R}_0} (\Psi_{t,z}F^2 - 2F\Psi_{t,z}F) z\nu(dz)dt\Big] \\ &= \sigma^2\mathbb{E}\Big[\int_0^T |D_{t,0}F|^2 dt\Big] \\ &\quad + \mathbb{E}\Big[\int_0^T \int_{\mathbb{R}_0} (2F\Psi_{t,z}F + z(\Psi_{t,z}F)^2 - 2F\Psi_{t,z}F) z\nu(dz)dt\Big] \\ &= \mathbb{E}\Big[\int_0^T |D_{t,0}F|^2\sigma^2 dt + \int_0^T \int_{\mathbb{R}_0} (\Psi_{t,z}F)^2 z^2\nu(dz)dt\Big] \\ &= \mathbb{E}\Big[\int_0^T |D_{t,0}F|^2\sigma^2 dt + \int_0^T \int_{\mathbb{R}_0} (D_{t,z}F)^2 z^2\nu(dz)dt\Big] \\ &= \mathbb{E}\Big[\int_0^T \int_{\mathbb{R}_0} (D_{t,z}F)^2 q(dt, dz)\Big],\end{aligned}$$

since $\Phi'(x) = 2x$, $\Phi''(x) = 2$, $\Psi_{t,z}F^2 = 2F\Psi_{t,z}F + z(\Psi_{t,z}F)^2$ and $\Psi_{t,z}F = D_{t,z}F$ for $F \in \mathbb{D}^{1,2}$ hold. This is a Poincaré type inequality for L^2-canonical Lévy processes.

Theorem 3.3 and Proposition 2.1 imply the following result immediately.

Corollary 3.5. *Let $F \in \mathbb{D}^{1,2}$ be such that $F > 0$ with probability one. Then, under the assumption, we obtain the following:*

(1) *If* $\sigma \neq 0$ *and* $\nu = 0$, *then*

$$\mathrm{Ent}_{\Phi}(F) \leq \frac{1}{2}\mathbb{E}\Big[\int_0^T \Phi''(F)|D_t^W F|^2 dt\Big].$$

This is a Φ*-Sobolev inequality for the Wiener functionals.*

(2) *If* $\sigma = 0$ *and* $\nu \neq 0$, *then*

$$\mathrm{Ent}_{\Phi}(F) \leq \mathbb{E}\Big[\int_0^T \int_{\mathbb{R}_0} (D^J_{(t,z)}\Phi(F) - \Phi'(F)D^J_{(t,z)}F)\nu(dz)dt\Big].$$

This is a Chafaï type Φ*-Sobolev inequality for pure jump Lévy functionals.*

We can also derive the following.

Corollary 3.6. *Fix* $r \in (1,2)$ *and let* $F \in \mathbb{D}^{1,2}$ *be such that* $F > 0$ *with probability one. Then*

$$\begin{aligned}\mathrm{Ent}_{\Phi_r}(F) \leq{}& \frac{1}{2}\sigma^2\mathbb{E}\Big[\int_0^T \frac{2}{r}\Big(\frac{2}{r}-1\Big)F^{\frac{2}{r}-2}|D_{t,0}F|^2 dt\Big] \\ &+ \mathbb{E}\Big[\int_0^T \int_{\mathbb{R}_0} (\Psi_{t,z}(F)^{\frac{2}{r}} - \frac{2}{r}F^{\frac{2}{r}-1}\Psi_{t,z}F)z\nu(dz)dt\Big].\end{aligned}$$

Proof. This is a direct consequence of Theorem 3.3 with $\Phi = \Phi_r$, since $\Phi_r'(x) = \frac{2}{r}x^{\frac{2}{r}-1}$ and $\Phi_r''(x) = \frac{2}{r}\Big(\frac{2}{r}-1\Big)x^{\frac{2}{r}-2}$. □

From the same sort of arguments as in [15], we obtain the following:

(1) As $r \to 1$, we have

$$\mathrm{Var}[F] \leq \mathbb{E}\Big[\int_{[0,T]\times\mathbb{R}} (D_{t,z}F)^2 q(dt,dz)\Big].$$

(2) As $r \to 2$, we have

$$\begin{aligned}\mathrm{Ent}_{\Phi_{\log}}(F) \leq{}& \frac{1}{2}\sigma^2\mathbb{E}\Big[\int_0^T \frac{1}{F}|D_{t,0}F|^2 dt\Big] \\ &+ \mathbb{E}\Big[\int_0^T \int_{\mathbb{R}_0} (\Psi_{t,z}(F\log F) - (\log F + 1)\Psi_{t,z}F)z\nu(dz)dt\Big].\end{aligned}$$

To show Theorem 4.1, we also derive the following.

Corollary 3.7. *If* $F \in \mathbb{D}^{1,2}$ *satisfies:*

(1) $e^F \in L^2(\mathbb{P})$,

(2)

$$e^F D_{t,z}F\mathbb{1}_{\{0\}}(z) + \frac{e^F(e^{zD_{t,z}F}-1)}{z}\mathbb{1}_{\mathbb{R}_0}(z) \in L^2(q\times\mathbb{P}),$$

(3) $Fe^F \in L^2(\mathbb{P})$,

(4)
$$\frac{e^F(Fe^{zD_{t,z}F} + zD_{t,z}F \cdot Fe^{zD_{t,z}F} - F)}{z}\mathbb{1}_{\mathbb{R}_0}(z) \in L^2(q \times \mathbb{P}),$$

then,

$$\begin{aligned}\mathrm{Ent}_{\Phi_{\log}}[e^F] \le \frac{1}{2}\sigma^2\mathbb{E}\Big[\int_0^T e^F|D_{t,0}F|^2dt\Big]\\ + \mathbb{E}\Big[\int_0^T\int_{\mathbb{R}_0} e^F(zD_{t,z}F \cdot e^{zD_{t,z}F} - e^{zD_{t,z}F} + 1)\nu(dz)dt\Big]\end{aligned} \quad (3.2)$$

holds.

Proof. From the assumptions (1), (2) and $F \in \mathbb{D}^{1,2}$, Proposition 2.3 implies that $e^F \in \mathbb{D}^{1,2}$ and

$$D_{t,z}e^F = e^F D_{t,0}F\mathbb{1}_{\{0\}}(z) + \frac{e^F(e^{zD_{t,z}F} - 1)}{z}\mathbb{1}_{\mathbb{R}_0}(z). \quad (3.3)$$

Since $e^F > 0$, Theorem 3.3 shows that

$$\begin{aligned}&\mathrm{Ent}_{\Phi_{\log}}[e^F] \quad (3.4)\\ &\le \frac{1}{2}\sigma^2\mathbb{E}\Big[\int_0^T \frac{1}{e^F}|D_{t,0}e^F|^2dt\Big]\\ &\quad + \mathbb{E}\Big[\int_0^T\int_{\mathbb{R}_0}\{D_{t,z}(Fe^F) - (F+1)D_{t,z}e^F\}z\nu(dz)dt\Big]\\ &= \frac{1}{2}\sigma^2\mathbb{E}\Big[\int_0^T e^F|D_{t,0}F|^2dt\Big]\\ &\quad + \mathbb{E}\Big[\int_0^T\int_{\mathbb{R}_0}\{zD_{t,z}(Fe^F) - (F+1)e^F(e^{zD_{t,z}F} - 1)\}\nu(dz)dt\Big]\\ &= \frac{1}{2}\sigma^2\mathbb{E}\Big[\int_0^T e^F|D_{t,0}F|^2dt\Big]\\ &\quad + \mathbb{E}\Big[\int_0^T\int_{\mathbb{R}_0}\{zD_{t,z}(Fe^F) - e^F[Fe^{zD_{t,z}F} - F + e^{zD_{t,z}F} - 1]\}\nu(dz)dt\Big] \quad (3.5)\end{aligned}$$

where we use (3.3). We next calculate $zD_{t,z}(Fe^F)$. From assumptions (3) and (4), Proposition 2.3 implies that

$$\begin{aligned}zD_{t,z}(Fe^F) &= (F + zD_{t,z}F)e^{F+zD_{t,z}F} - Fe^F\\ &= e^F\{(F + zD_{t,z}F)e^{zD_{t,z}F} - F\}\\ &= e^F\{Fe^{zD_{t,z}F} + zD_{t,z}Fe^{zD_{t,z}F} - F\}. \quad (3.6)\end{aligned}$$

Hence, combining (3.4) with (3.6), we have

$$\mathrm{Ent}_{\Phi_{\log}}[e^F] \le \frac{1}{2}\sigma^2\mathbb{E}\Big[\int_0^T e^F|D_{t,0}F|^2 dt\Big] + \mathbb{E}\Big[\int_0^T \int_{\mathbb{R}_0} e^F(zD_{t,z}F \cdot e^{zD_{t,z}F} - e^{zD_{t,z}F} + 1)\nu(dz)dt\Big]. \quad \square$$

4 Some applications to concentration inequalities

In this section, from Corollary 3.7, we shall derive some concentration inequalities by following the Herbst method.

Theorem 4.1. *Let $F \in \mathbb{D}^{1,2}$ be such that*

$$\begin{aligned} &zD_{t,z}F \le \beta \in (0,\infty),\ q \otimes \mathbb{P}\text{-a.e.},\\ &\sigma^2 \int_0^T |D_{t,0}F|^2 dt \le \alpha^2 < \infty,\ \mathbb{P}\text{-a.e.},\\ &\text{and } \int_{[0,T]\times\mathbb{R}_0} |D_{t,z}F|^2 z^2 \nu(dz)dt \le \gamma^2 < \infty,\ \mathbb{P}\text{-a.e.} \end{aligned} \tag{4.1}$$

Then, we have

$$\mathbb{P}(F - \mathbb{E}[F] > r) \le \exp\Big[-\sup_{\lambda>0}\Big\{\lambda r - \frac{\alpha^2\lambda^2}{2} - \frac{\gamma^2}{\beta^2}(e^{\lambda\beta} - \lambda\beta - 1)\Big\}\Big] \tag{4.2}$$

$$= \exp\Big[-\frac{r^2}{2\alpha^2} - \frac{\gamma^2}{2\alpha^2\beta^2}\Big(2\alpha^2 + 2\beta r + \gamma^2\Big) + \frac{\alpha^2}{2\beta^2}\Big(\mathcal{W}(\frac{\gamma^2}{\alpha^2}\exp\{\frac{\beta r + \gamma^2}{\alpha^2}\})\Big)^2 + \frac{\alpha^2}{\beta^2}\mathcal{W}(\frac{\gamma^2}{\alpha^2}\exp\{\frac{\beta r + \gamma^2}{\alpha^2}\})\Big] \tag{4.3}$$

for all $r > 0$, where $\mathcal{W}$ is the principal branch of the Lambert W-function on $(-e^{-1}, \infty)$. The Lambert W-function $\mathcal{W}(x)$ represents the solutions y of the equation $ye^y = x$ for any complex number x (see [8]).

Further, assuming $zD_{t,z}F \le 0$, $q \otimes \mathbb{P}$-a.e., we have

$$\mathbb{P}(F - \mathbb{E}[F] > r) \le \exp\Big[-\frac{r^2}{2(\alpha^2+\gamma^2)}\Big] \tag{4.4}$$

for all $r > 0$.

Proof. When β decreases to zero, the inequality (4.3) becomes

$$\begin{aligned} \mathbb{P}(F - \mathbb{E}[F] > r) &\le \exp\Big[-\sup_{\lambda>0}\Big\{\lambda r - \frac{\alpha^2+\gamma^2}{2}\lambda^2\Big\}\Big]\\ &= \exp\Big[-\frac{r^2}{2(\alpha^2+\gamma^2)}\Big] \end{aligned}$$

for all $r > 0$. Hence, we obtain (4.4).

Assume at first that F is bounded. Applying (3.2) to $e^{\lambda F}$, $\lambda > 0$, we have

$$\begin{aligned}
\mathrm{Ent}_{\Phi_{\log}}[e^{\lambda F}] &\le \frac{1}{2}\sigma^2\lambda^2\mathbb{E}\Big[e^{\lambda F}\int_0^T |D_{t,0}F|^2 dt\Big] \\
&\quad + \mathbb{E}\Big[\int_0^T\int_{\mathbb{R}_0} e^{\lambda F}(\lambda z D_{t,z}F\cdot e^{\lambda z D_{t,z}F} - e^{\lambda z D_{t,z}F} + 1)\nu(dz)dt\Big] \\
&\le \frac{1}{2}\lambda^2\alpha^2\mathbb{E}[e^{\lambda F}] \\
&\quad + \mathbb{E}\Big[\int_0^T\int_{\mathbb{R}_0} e^{\lambda F}(\lambda z D_{t,z}F\cdot e^{\lambda z D_{t,z}F} - e^{\lambda z D_{t,z}F} + 1)\nu(dz)dt\Big].
\end{aligned} \tag{4.5}$$

Let $g(x) = xe^x - e^x + 1$. Then, $g(-x) \le \frac{1}{2}x^2$ for all $x \ge 0$, $\frac{g(x)}{x^2}$ is increasing for $x > 0$ and $\lim_{x\to 0+}\frac{g(x)}{x^2} = \frac{1}{2}$. Hence, we have $g(x) \le (g(\lambda\beta)/\lambda^2\beta^2)x^2$ for all $-\infty < x \le \lambda\beta(> 0)$ and

$$g(\lambda z D_{t,z}F) \le g(\lambda\beta)\frac{1}{\beta^2}(zD_{t,z}F)^2. \tag{4.6}$$

Thus, combining (4.5) with (4.6), we get

$$\begin{aligned}
\mathrm{Ent}_{\Phi_{\log}}[e^{\lambda F}] &\le \frac{1}{2}\lambda^2\alpha^2\mathbb{E}[e^{\lambda F}] + \frac{g(\lambda\beta)}{\beta^2}\mathbb{E}\Big[e^{\lambda F}\int_0^T\int_{\mathbb{R}_0}(zD_{t,z}F)^2\nu(dz)dt\Big] \\
&\le \Big[\frac{1}{2}\alpha^2\lambda^2 + \frac{g(\lambda\beta)}{\beta^2}\gamma^2\Big]\mathbb{E}[e^{\lambda F}].
\end{aligned} \tag{4.7}$$

Let $H(\lambda) = \lambda^{-1}\log\mathbb{E}[e^{\lambda F}]$. Then, we have $H(0+) = \mathbb{E}[F]$ and

$$H'(\lambda) = \frac{\mathrm{Ent}_{\Phi_{\log}}[e^{\lambda F}]}{\lambda^2\mathbb{E}[e^{\lambda F}]} \le \frac{\alpha^2}{2} + \frac{g(\lambda\beta)}{\beta^2\lambda^2}\gamma^2.$$

Whence

$$\begin{aligned}
H(\lambda) &\le \mathbb{E}[F] + \frac{\alpha^2\lambda}{2} + \frac{\gamma^2}{\beta^2}\int_0^\lambda \frac{u\beta e^{u\beta} - e^{u\beta} + 1}{u^2}du \\
&= \mathbb{E}[F] + \frac{\alpha^2\lambda}{2} + \frac{\gamma^2}{\beta^2}\frac{e^{\lambda\beta} - \lambda\beta - 1}{\lambda}.
\end{aligned}$$

In other words, we obtain

$$\mathbb{E}[e^{\lambda F}] \le \exp\Big[\lambda\mathbb{E}[F] + \frac{\alpha^2\lambda^2}{2} + \frac{\gamma^2}{\beta^2}(e^{\lambda\beta} - \lambda\beta - 1)\Big]. \tag{4.8}$$

Hence, Chebychev's inequality implies that

$$
\begin{aligned}
&\mathbb{P}[F-\mathbb{E}[F]>r]\\
&\le \inf_{\lambda>0} e^{-\lambda r}\mathbb{E}[e^{\lambda(F-\mathbb{E}[F])}]\\
&\le \exp\Big[-\sup_{\lambda>0}\Big\{\lambda r-\frac{\alpha^2\lambda^2}{2}-\frac{\gamma^2}{\beta^2}(e^{\lambda\beta}-\lambda\beta-1)\Big\}\Big]\\
&=\exp\Big[-\Big\{r\frac{\gamma^2+\beta r-\alpha^2\mathcal{W}(\frac{\gamma^2}{\alpha^2}\exp\{\frac{\beta r+\gamma^2}{\alpha^2}\})}{\alpha^2\beta}\\
&\qquad-\frac{\Big(\gamma^2+\beta r-\alpha^2\mathcal{W}(\frac{\gamma^2}{\alpha^2}\exp\{\frac{\beta r+\gamma^2}{\alpha^2}\})\Big)^2}{2\alpha^2\beta^2}\Big\}\Big]\\
&\times\exp\Big[\frac{\gamma^2}{\beta^2}\Big(\exp\Big\{\frac{\gamma^2+\beta r-\alpha^2\mathcal{W}(\frac{\gamma^2}{\alpha^2}\exp\{\frac{\beta r+\gamma^2}{\alpha^2}\})}{\alpha^2}\Big\}\\
&\qquad-\frac{\gamma^2+\beta r-\alpha^2\mathcal{W}(\frac{\gamma^2}{\alpha^2}\exp\{\frac{\beta r+\gamma^2}{\alpha^2}\})}{\alpha^2}-1\Big)\Big]\\
&=\exp\Big[-\frac{r^2}{2\alpha^2}-\frac{\gamma^2}{2\alpha^2\beta^2}\Big(2\alpha^2+2\beta r+\gamma^2\Big)+\frac{\alpha^2}{2\beta^2}\Big(\mathcal{W}(\frac{\gamma^2}{\alpha^2}\exp\{\frac{\beta r+\gamma^2}{\alpha^2}\})\Big)^2\\
&\qquad+\frac{\alpha^2}{\beta^2}\mathcal{W}(\frac{\gamma^2}{\alpha^2}\exp\{\frac{\beta r+\gamma^2}{\alpha^2}\})\Big]\\
&=: d(\alpha,\beta,\gamma,r), \qquad (4.9)
\end{aligned}
$$

where $\mathcal{W}$ is the Lambert W-Function. Hence, (4.2) is shown in the bounded case.

In the unbounded case, we can apply a similar argument as in the proof of Proposition 3.1 in [31]. Let $F_N=[(-N)\vee F]\wedge N$. It satisfies (4.1) again. By Proposition 2.5, if $\lim_{k\to\infty}\mathbb{E}[F_{N_k}]=\pm\infty$ for some subsequence (N_k) tending to infinity, then $F_{N_k}\to\pm\infty$ in probability $\mathbb{P}$. This is in contradiction to $F_N\to F\in L^0(\mathbb{P})$ in probability. Consequently, $\{\mathbb{E}[F_N];N\ge 1\}$ is bounded. From condition (4.2) and Proposition 4.1 again, $\{\mathbb{E}[F_N^2];N\ge 1\}$ is bounded, too. Thus $\{F_n;N\ge 1\}$ is uniformly integrable, and then $\mathbb{E}[|F|]<+\infty$ and $\mathbb{E}[|F_N-F|]\to 0$. Therefore,

$$\mathbb{P}(F-\mathbb{E}[F]>r)\le \liminf_{N\to\infty}\mathbb{P}(F_N-\mathbb{E}[F_N]>r)\le d(\alpha,\beta,\gamma,r).$$

The proof is concluded. □

Remark 4.2. For $A,B,C,r>0$ and the Lambert W-function $\mathcal{W}$, we can show that

$\mathcal{W}(Ce^{A+Br})=Br-\log(Br)+A+\log C+o(r^{-1+\varepsilon})$ as $r\to+\infty$ for some $\varepsilon>0$

holds (note that in the case $r=\frac{e^A}{B}>0$, we obtain $\mathcal{W}(Ce^{A+Br})=Br-\log(Br)+A+\log C$) since we know the following asymptotic behavior of $\mathcal{W}(r)$ (pp. 25–26 in [9]):

$$\mathcal{W}(r)=\log(r)-\log\log(r)+O\left(\frac{\log\log r}{r}\right)\quad \text{as } r\to\infty.$$

Therefore, (4.3) in Theorem 4.1 can be rewritten as

$$
\begin{aligned}
&\mathbb{P}[F-\mathbb{E}[F]>r]\\
&\le \exp\Big[-\frac{r^2}{2\alpha^2}-\frac{\gamma^2}{2\alpha^2\beta^2}\Big(2\alpha^2+2\beta r+\gamma^2\Big)\\
&\qquad+\frac{\alpha^2}{2\beta^2}\Big(\mathcal{W}(\frac{\gamma^2}{\alpha^2}\exp\{\frac{\beta r+\gamma^2}{\alpha^2}\})\Big)^2+\frac{\alpha^2}{\beta^2}\mathcal{W}(\frac{\gamma^2}{\alpha^2}\exp\{\frac{\beta r+\gamma^2}{\alpha^2}\})\Big]\\
&=\exp\Big[-\frac{r^2}{2\alpha^2}-\frac{\gamma^2}{2\alpha^2\beta^2}\Big(2\alpha^2+2\beta r+\gamma^2\Big)\\
&\qquad+\frac{\alpha^2}{2\beta^2}\Big(\frac{\beta}{\alpha^2}r-\log\Big(\frac{\beta}{\alpha^2}r\Big)+\frac{\gamma^2}{\alpha^2}+\log\Big(\frac{\gamma^2}{\alpha^2}\Big)\Big)^2\\
&\qquad+\frac{\alpha^2}{\beta^2}\Big(\frac{\beta}{\alpha^2}r-\log\Big(\frac{\beta}{\alpha^2}r\Big)+\frac{\gamma^2}{\alpha^2}+\log\Big(\frac{\gamma^2}{\alpha^2}\Big)\Big)+o(r^{-1+\varepsilon})\Big]\\
&=\exp\Big[-\frac{r^2}{2\alpha^2}-\frac{\gamma^2}{2\alpha^2\beta^2}\Big(2\alpha^2+2\beta r+\gamma^2\Big)\\
&\qquad+\frac{\alpha^2}{2\beta^2}\Big(\frac{\beta^2}{\alpha^4}r^2-2\frac{\beta}{\alpha^2}\Big(\frac{\gamma^2}{\alpha^2}+\log\Big(\frac{\gamma^2}{\alpha^2}\Big)\Big)r+2\Big(\frac{\gamma^2}{\alpha^2}+\log\Big(\frac{\gamma^2}{\alpha^2}\Big)\Big)\log\Big(\frac{\beta}{\alpha^2}r\Big)\\
&\qquad+\Big(\log\Big(\frac{\beta}{\alpha^2}r\Big)\Big)^2-2\frac{\beta}{\alpha^2}r\log\Big(\frac{\beta}{\alpha^2}r\Big)+\Big(\frac{\gamma^2}{\alpha^2}+\log\Big(\frac{\gamma^2}{\alpha^2}\Big)\Big)^2\Big)\\
&\qquad+\frac{\alpha^2}{\beta^2}\Big(\frac{\beta}{\alpha^2}r-\log\Big(\frac{\beta}{\alpha^2}r\Big)+\frac{\gamma^2}{\alpha^2}+\log\Big(\frac{\gamma^2}{\alpha^2}\Big)\Big)+o(r^{-1+\varepsilon})\Big]\\
&=\exp\Big[-\Big\{\frac{2\gamma^2}{\alpha^2\beta}+\frac{1}{\beta}\Big(\log\Big(\frac{\gamma^2}{\alpha^2}\Big)-1\Big)\Big\}r\\
&\qquad+\frac{\alpha^2}{\beta^2}\Big\{\frac{\gamma^2}{\alpha^2}+\log\Big(\frac{\gamma^2}{\alpha^2}\Big)-\frac{\beta}{\alpha^2}r-1+\frac{1}{2}\log\Big(\frac{\beta}{\alpha^2}r\Big)\Big\}\log\Big(\frac{\beta}{\alpha^2}r\Big)\\
&\qquad+\frac{\alpha^2}{\beta^2}\Big\{\log\Big(\frac{\gamma^2}{\alpha^2}\Big)+\frac{1}{2}\Big(\frac{\gamma^2}{\alpha^2}+\log\Big(\frac{\gamma^2}{\alpha^2}\Big)\Big)^2\Big\}-\frac{\gamma^4}{2\alpha^2\beta^2}+o(r^{-1+\varepsilon})\Big]\\
&=C\exp\Big[-\frac{\beta}{\alpha^2}r\log\Big(\frac{\beta}{\alpha^2}r\Big)+o(r\log r)\Big], \qquad (4.10)
\end{aligned}
$$

where C is a constant, as $r\to\infty$ for some $\varepsilon>0$.

Remark 4.3. This concentration inequality looks a little bit complex but unifies all cases. We explain it in the following items.

(1) The Wiener space case: $\nu=0$ and $\sigma\neq 0$. Taking $\gamma\to 0$ in (4.3), we obtain

$$\mathbb{P}(F-\mathbb{E}[F]>r)\le\exp\Big[-\frac{r^2}{2\alpha^2}\Big]$$

for all $r>0$.

(2) The Poisson space case: $\nu\neq 0$ and $\sigma=0$. Take $\alpha\to 0$ in (4.2). Hence, we

have

$$\mathbb{P}(F-\mathbb{E}[F]>r) \le \exp\Big[-\sup_{\lambda>0}\Big\{\lambda r - \frac{\gamma^2}{\beta^2}(e^{\lambda\beta}-\lambda\beta-1)\Big\}\Big]$$
$$= \exp\Big[-\Big(\frac{r}{\beta}+\frac{\gamma^2}{\beta^2}\Big)\log\Big(1+\frac{\beta r}{\gamma^2}\Big)+\frac{r}{\beta}\Big]$$
$$\le \exp\Big[-\frac{r}{2\beta}\log\Big(1+\frac{\beta r}{\gamma^2}\Big)\Big]. \tag{4.11}$$

Moreover, in the case $zD_{t,z}F \le 0$, $\nu\otimes\mathbb{P}$-a.e., letting $\beta\to 0$ in (4.11), we obtain

$$\mathbb{P}(F-\mathbb{E}[F]>r) \le \exp\Big[-\frac{r^2}{2\gamma^2}\Big].$$

It recovers Proposition 3.1 in [31].

Remark 4.4. Three conditions (4.1) in Theorem 4.1 are a little bit strong and difficult. In fact, our estimation cannot be applied to many cases, including the pure jump case, which Wu claimed (see [31]). In the following subsection, we show some examples to which we can apply our result.

4.1 Examples

In this subsection, we will give examples of concentration inequalities in Theorem 4.1.

Example 4.5. Let $f:\mathbb{R}\to\mathbb{R}$ be a bounded Lipschitz continuous function with Lipschitz constant L_f and $\sup_{x\in\mathbb{R}}|f(x)|>0$. We now consider the following function F: $F=f(X_T)$, where $X_T=\sigma W_T+\int_0^T\int_{\mathbb{R}_0} x\widetilde{N}(ds,dx)=I_1(1)\in\mathbb{D}^{1,2}$, $\sigma>0$. Then, Proposition 5.1 in [13] implies that $f(X_T)\in\mathbb{D}^{1,2}$ and

$$D_{t,z}F = GD_{t,0}X_T\mathbb{1}_{\{0\}}(z) + \frac{f(X_T+zD_{t,z}X_T)-f(X_T)}{z}\mathbb{1}_{\mathbb{R}_0}(z)$$
$$= G\mathbb{1}_{\{0\}}(z) + \frac{f(X_T+z)-f(X_T)}{z}\mathbb{1}_{\mathbb{R}_0}(z), \tag{4.12}$$

where G is a random variable which is a.s. bounded by L_f. By using (4.12), we have

$$\sigma^2\int_0^T |D_{t,0}F|^2dt = \sigma^2T|G|^2 \le \sigma^2TL_f^2 =: \alpha^2 < \infty,$$
$$zD_{t,z}F = f(X_T+z)-f(X_T)$$
$$\le |f(X_T+z)-f(X_T)| \le 2\sup_{x\in\mathbb{R}}|f(x)| =: \beta\in(0,\infty),$$

and

$$\int_0^T\int_{\mathbb{R}_0}|zD_{t,z}F|^2\nu(dz)dt = \int_0^T\int_{\mathbb{R}_0}|f(X_T+z)-f(X_T)|^2\nu(dz)dt$$

$$\leq L_f^2 T \int_{\mathbb{R}_0} z^2 \nu(dz) =: \gamma^2 = \frac{\alpha^2}{\sigma^2} \int_{\mathbb{R}_0} z^2 \nu(dz) < \infty.$$

Therefore, Theorem 4.1 shows that

$$\begin{aligned}
&\mathbb{P}(f(X_T) - \mathbb{E}[f(X_T)] > r) \\
&\leq \exp\Big[- \frac{r^2}{2\alpha^2} - \frac{\gamma^2}{2\alpha^2\beta^2}\Big(2\alpha^2 + 2\beta r + \gamma^2\Big) \\
&\quad + \frac{\alpha^2}{2\beta^2}\Big(\mathcal{W}(\frac{\gamma^2}{\alpha^2}\exp\{\frac{\beta r + \gamma^2}{\alpha^2}\})\Big)^2 + \frac{\alpha^2}{\beta^2}\mathcal{W}(\frac{\gamma^2}{\alpha^2}\exp\{\frac{\beta r + \gamma^2}{\alpha^2}\})\Big] \\
&= \exp\Big[- \frac{r^2}{2\alpha^2} - \frac{\sigma^{-2}\int_{\mathbb{R}_0} z^2\nu(dz)}{2\beta^2}\Big(\alpha^2(2 + \sigma^{-2}\int_{\mathbb{R}_0} z^2\nu(dz)) + 2\beta r\Big) \\
&\quad + \frac{\alpha^2}{2\beta^2}\Big(\mathcal{W}(\sigma^{-2}\int_{\mathbb{R}_0} z^2\nu(dz)\exp\{\frac{\beta}{\alpha^2}r + \sigma^{-2}\int_{\mathbb{R}_0} z^2\nu(dz)\})\Big)^2 \\
&\quad + \frac{\alpha^2}{\beta^2}\mathcal{W}(\sigma^{-2}\int_{\mathbb{R}_0} z^2\nu(dz)\exp\{\frac{\beta}{\alpha^2}r + \sigma^{-2}\int_{\mathbb{R}_0} z^2\nu(dz)\})\Big]
\end{aligned}$$

for all $r > 0$, where $\mathcal{W}$ is the Lambert W-function. Moreover, from (4.10) and taking $m_2 = \int_{\mathbb{R}_0} z^2\nu(dz)$ and $M = \sup_{x\in\mathbb{R}} |f(x)|$, we obtain the following approximation result:

$$\begin{aligned}
&\mathbb{P}(f(X_T) - \mathbb{E}[f(X_T)] > r) \\
&\leq \exp\Big[- \frac{r^2}{2\alpha^2} - \frac{\gamma^2}{2\alpha^2\beta^2}\Big(2\alpha^2 + 2\beta r + \gamma^2\Big) \\
&\quad + \frac{\alpha^2}{2\beta^2}\Big(\mathcal{W}(\frac{\gamma^2}{\alpha^2}\exp\{\frac{\beta r + \gamma^2}{\alpha^2}\})\Big)^2 + \frac{\alpha^2}{\beta^2}\mathcal{W}(\frac{\gamma^2}{\alpha^2}\exp\{\frac{\beta r + \gamma^2}{\alpha^2}\})\Big] \\
&= \exp\Big[- \Big\{\frac{m_2}{\sigma^2 M} + \frac{1}{2M}\log\Big(\frac{m_2}{\sigma^2}\Big)\Big\} r + \frac{r}{2M} \\
&\qquad + \frac{\sigma^2T^2L_f^2}{4(M)^2}\Big\{\frac{m_2}{\sigma^2} + \log\Big(\frac{m_2}{\sigma^2}\Big) - 2\frac{M}{\sigma^2TL_f^2}r - 1\Big\}\log\Big(\frac{2M}{\sigma^2TL_f^2}r\Big) \\
&\qquad + \frac{\sigma^2T^2L_f^2}{8(M)^2}\Big(\log\Big(\frac{2M}{\sigma^2TL_f^2}r\Big)\Big)^2 \\
&\qquad + \frac{\sigma^2T^2L_f^2}{4(M)^2}\Big\{\log\Big(\frac{m_2}{\sigma^2}\Big) + \frac{1}{2}\Big(\frac{m_2}{\sigma^2} + \log\Big(\frac{m_2}{\sigma^2}\Big)\Big)^2\Big\} - \frac{TL_f^2m_2^2}{8\sigma^2(M)^2} + o(r^{-1+\varepsilon})\Big] \\
&= C\exp\Big[- \frac{2M}{\sigma^2TL_f^2}r\log\Big(\frac{2M}{\sigma^2TL_f^2}r\Big) + o(r\log r)\Big],
\end{aligned} \tag{4.13}$$

where C is a constant, as $r \to \infty$ for some $\varepsilon > 0$ and $r > 0$.

Example 4.6. The put option is a typical example of contingent claims in mathematical finance. The payoff of the put option with strike price $K > 0$ is expressed by

$\max\{K - S_T, 0\}$, where S_T is a risky asset price process at maturity T; we omit the details of it. Now, as an example of Theorem 4.1, we deal with the following. Let

$$F = \max\{K - \max(X_T, 0),\ 0\}, \qquad K > 0.$$

Since $f(x) = \max\{K - \max(x, 0),\ 0\}$, $x \in \mathbb{R}$, is bounded Lipschitz continuous with Lipschitz constant 1, Proposition 5.1 in [13] implies that $F = f(X_T)$ belongs to $\mathbb{D}^{1,2}$ and

$$D_{t,0} f(X_T) = H D_{t,0} G = H,$$

where H is a random variable almost surely bounded by 1. Moreover, we obtain

$$D_{t,z} f(X_T) = \frac{f(X_T + z D_{t,z} X_T) - f(X_T)}{z} \mathbb{1}_{\mathbb{R}_0}(z).$$

Hence,

$$\sigma^2 \int_0^T |D_{t,0} F|^2 dt = \sigma^2 T |H|^2 \le \sigma^2 T =: \alpha^2 < \infty,$$

$$\begin{aligned} z D_{t,z} F &= f(X_T + z) - f(X_T) \\ &\le |f(X_T + z) - f(X_T)| \\ &\le 2 \sup_{x \in \mathbb{R}} |f(x)| \\ &= 2K =: \beta \in (0, \infty), \end{aligned}$$

and

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}_0} |z D_{t,z} F|^2 \nu(dz) dt \\ &= \int_0^T \int_{\mathbb{R}_0} |f(X_T + z) - f(X_T)|^2 \nu(dz) dt \\ &\le T \int_{\mathbb{R}_0} z^2 \nu(dz) = \frac{\alpha^2}{\sigma^2} \int_{\mathbb{R}_0} z^2 \nu(dz) =: \gamma^2 < \infty \end{aligned}$$

hold.

Therefore, Example 4.5 shows that

$$\begin{aligned} &\mathbb{P}(f(X_T) - \mathbb{E}[f(X_T)] > r) \\ &\le \exp\Big[- \frac{r^2}{2\alpha^2} - \frac{\sigma^{-2} \int_{\mathbb{R}_0} z^2 \nu(dz)}{2\beta^2} \Big(\alpha^2 (2 + \sigma^{-2} \int_{\mathbb{R}_0} z^2 \nu(dz)) + 2\beta r \Big) \\ &\quad + \frac{\alpha^2}{2\beta^2} \Big(\mathcal{W}(\sigma^{-2} \int_{\mathbb{R}_0} z^2 \nu(dz) \exp\{ \frac{\beta}{\alpha^2} r + \sigma^{-2} \int_{\mathbb{R}_0} z^2 \nu(dz) \}) \Big)^2 \\ &\quad + \frac{\alpha^2}{\beta^2} \mathcal{W}(\sigma^{-2} \int_{\mathbb{R}_0} z^2 \nu(dz) \exp\{ \frac{\beta}{\alpha^2} r + \sigma^{-2} \int_{\mathbb{R}_0} z^2 \nu(dz) \}) \Big] \\ &= \exp\Big[- \frac{r^2}{2\sigma^2 T} - \frac{\sigma^{-2} \int_{\mathbb{R}_0} z^2 \nu(dz)}{8K^2} \Big(\sigma^2 T (2 + \sigma^{-2} \int_{\mathbb{R}_0} z^2 \nu(dz)) + 4Kr \Big) \end{aligned}$$

$$+\frac{\sigma^2 T}{8K^2}\Big(\mathcal{W}(\sigma^{-2}\int_{\mathbb{R}_0} z^2\nu(dz)\exp\{\frac{2K}{\sigma^2 T}r+\sigma^{-2}\int_{\mathbb{R}_0} z^2\nu(dz)\})\Big)^2$$
$$+\frac{\sigma^2 T}{4K^2}\mathcal{W}(\sigma^{-2}\int_{\mathbb{R}_0} z^2\nu(dz)\exp\{\frac{2K}{\sigma^2 T}r+\sigma^{-2}\int_{\mathbb{R}_0} z^2\nu(dz)\})\Big]$$

for all $r>0$, where $\mathcal{W}$ is the Lambert W-function. Moreover, from (4.10), (4.13) and taking $m_2=\int_{\mathbb{R}_0} z^2\nu(dz)$, we obtain the following approximation result:

$$\begin{aligned}
&\mathbb{P}(f(X_T)-\mathbb{E}[f(X_T)]>r)\\
&\leq \exp\Big[-\frac{r^2}{2\alpha^2}-\frac{\sigma^{-2}\int_{\mathbb{R}_0} z^2\nu(dz)}{2\beta^2}\Big(\alpha^2(2+\sigma^{-2}\int_{\mathbb{R}_0} z^2\nu(dz))+2\beta r\Big)\\
&\qquad+\frac{\alpha^2}{2\beta^2}\Big(\mathcal{W}(\sigma^{-2}\int_{\mathbb{R}_0} z^2\nu(dz)\exp\{\frac{\beta}{\alpha^2}r+\sigma^{-2}\int_{\mathbb{R}_0} z^2\nu(dz)\})\Big)^2\\
&\qquad+\frac{\alpha^2}{\beta^2}\mathcal{W}(\sigma^{-2}\int_{\mathbb{R}_0} z^2\nu(dz)\exp\{\frac{\beta}{\alpha^2}r+\sigma^{-2}\int_{\mathbb{R}_0} z^2\nu(dz)\})\Big]\\
&=\exp\Big[-\Big\{\frac{m_2}{\sigma^2 K}+\frac{1}{2K}\log\Big(\frac{m_2}{\sigma^2}\Big)\Big\}r+\frac{r}{2K}\\
&\qquad+\frac{\sigma^2T^2}{4K^2}\Big\{\frac{m_2}{\sigma^2}+\log\Big(\frac{m_2}{\sigma^2}\Big)-2\frac{K}{\sigma^2 T}r-1\Big\}\log\Big(\frac{2K}{\sigma^2 T}r\Big)\\
&\qquad+\frac{\sigma^2T^2}{8K^2}\Big(\log\Big(\frac{2K}{\sigma^2 T}r\Big)\Big)^2\\
&\qquad+\frac{\sigma^2T^2}{4K^2}\Big\{\log\Big(\frac{m_2}{\sigma^2}\Big)+\frac{1}{2}\Big(\frac{m_2}{\sigma^2}+\log\Big(\frac{m_2}{\sigma^2}\Big)\Big)^2\Big\}-\frac{Tm_2^2}{8\sigma^2K^2}+o(r^{-1+\varepsilon})\Big]\\
&=C\exp\Big[-\frac{2K}{\sigma^2 T}r\log\Big(\frac{2K}{\sigma^2 T}r\Big)+o(r\log r)\Big],
\end{aligned}$$

where C is a constant, as $r\to\infty$ for some $\varepsilon>0$.

Example 4.7. As an example of Theorem 4.1, we consider the following:

$$F=\delta\sigma W_T+(1-\delta)(J_T-cT),$$

where $\delta\in(0,1)$ and J_T is the Poisson process at time T with intensity $c>0$. The definition of the Malliavin derivative implies that

$$D_{t,0}F=\delta,\qquad t\in[0,T],\ q\otimes\mathbb{P}\text{-a.e.},$$

and

$$D_{t,z}F=(1-\delta),\qquad (t,z)\in[0,T]\times\{1\},\ q\otimes\mathbb{P}\text{-a.e.}$$

Hence, $zD_{t,z}F\leq(1-\delta)=:\beta$ for $(t,z)\in[0,T]\times\{1\}$, $q\otimes\mathbb{P}$-a.e.,

$$\sigma^2\int_0^T|D_{t,0}F|^2dt=\sigma^2\delta^2T=:\alpha^2$$

and

$$\int_{[0,T]\times\mathbb{R}_0}|D_{t,z}F|^2z^2\nu(dz)dt=c(1-\delta)^2T=:\gamma^2$$

hold.

Therefore, Theorem 4.1 shows that

$$
\begin{aligned}
&\mathbb{P}(F-\mathbb{E}[F]>r)\\
&\leq \exp\Big[-\frac{r^2}{2\alpha^2}-\frac{\gamma^2}{2\alpha^2\beta^2}\Big(2\alpha^2+2\beta r+\gamma^2\Big)\\
&\qquad +\frac{\alpha^2}{2\beta^2}\Big(\mathcal{W}(\frac{\gamma^2}{\alpha^2}\exp\{\frac{\beta r+\gamma^2}{\alpha^2}\})\Big)^2+\frac{\alpha^2}{\beta^2}\mathcal{W}(\frac{\gamma^2}{\alpha^2}\exp\{\frac{\beta r+\gamma^2}{\alpha^2}\})\Big]\\
&=\exp\Big[-\frac{r^2}{2\sigma^2\delta^2T}-\frac{c}{2\sigma^2}\Big(2\sigma^2\delta^2T+2(1-\delta)r+c(1-\delta)^2T\Big)\\
&\qquad +\frac{\sigma^2\delta^2T}{2(1-\delta)^2}\Big(\mathcal{W}(\frac{c(1-\delta)^2}{\sigma^2\delta^2}\exp\{\frac{(1-\delta)r+c(1-\delta)^2T}{\sigma^2\delta^2T}\})\Big)^2\\
&\qquad +\frac{\sigma^2\delta^2T}{(1-\delta)^2}\mathcal{W}(\frac{c(1-\delta)^2}{\sigma^2\delta^2}\exp\{\frac{(1-\delta)r+c(1-\delta)^2T}{\sigma^2\delta^2T}\})\Big]\\
&\overset{(4.10)}{=}\exp\Big[-\Big\{\frac{2c(1-\delta)}{\sigma^2\delta^2}+\frac{1}{1-\delta}\log\Big(\frac{c(1-\delta)^2}{\sigma^2\delta^2}\Big)\Big\}r+\frac{r}{1-\delta}\\
&\qquad +\frac{\sigma^2\delta^2T}{(1-\delta)^2}\Big\{\frac{c(1-\delta)^2}{\sigma^2\delta^2}+\log\Big(\frac{c(1-\delta)^2}{\sigma^2\delta^2}\Big)-\frac{(1-\delta)}{\sigma^2\delta^2T}r-1\Big\}\log\Big(\frac{1-\delta}{\sigma^2\delta^2T}r\Big)\\
&\qquad +\frac{\sigma^2\delta^2T}{2(1-\delta)^2}\Big(\log\Big(\frac{1-\delta}{\sigma^2\delta^2T}r\Big)\Big)^2\\
&\qquad +\frac{\sigma^2\delta^2T}{(1-\delta)^2}\Big\{\log\Big(\frac{c(1-\delta)^2}{\sigma^2\delta^2}\Big)+\frac{1}{2}\Big(\frac{c(1-\delta)^2}{\sigma^2\delta^2}+\log\Big(\frac{c(1-\delta)^2}{\sigma^2\delta^2}\Big)\Big)^2\Big\}\\
&\qquad -\frac{c^2(1-\delta)^2T}{2\sigma^2\delta^2}+o(r^{-1+\varepsilon})\Big]\\
&=C\exp\Big[-\frac{1-\delta}{\sigma^2\delta^2T}r\log\Big(\frac{1-\delta}{\sigma^2\delta^2T}r\Big)+o(r\log r)\Big],
\end{aligned}
$$

where C is a constant, as $r\to\infty$ for some $\varepsilon>0$ and $r>0$.

Assuming further that $\sigma=T=1$ and $c=\frac{1-\delta^2}{(1-\delta)^2}=\frac{1+\delta}{1-\delta}$, we have

$$\mathbb{E}[F]=0,\qquad \mathrm{Var}[F]=1$$

and

$$
\begin{aligned}
&\mathbb{P}(F-\mathbb{E}[F]>r)\\
&\leq \exp\Big[-\frac{r^2}{2\delta^2}-\frac{1-\delta^2}{2(1-\delta)^2}\Big(\delta^2+2(1-\delta)r+1\Big)\\
&\qquad +\frac{\delta^2}{2(1-\delta)^2}\Big(\mathcal{W}(\frac{1-\delta^2}{\delta^2}\exp\{\frac{(1-\delta)r+1-\delta^2}{\delta^2}\})\Big)^2\\
&\qquad +\frac{\delta^2}{(1-\delta)^2}\mathcal{W}(\frac{1-\delta^2}{\delta^2}\exp\{\frac{(1-\delta)r+1-\delta^2}{\delta^2}\})\Big]\\
&=\exp\Big[-\Big\{\frac{2(1+\delta)}{\delta^2}+\frac{1}{1-\delta}\log\Big(\frac{1-\delta^2}{\delta^2}\Big)\Big\}r+\frac{r}{1-\delta}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\delta^2}{(1-\delta)^2}\Big\{\frac{1-\delta^2}{\delta^2}+\log\Big(\frac{1-\delta^2}{\delta^2}\Big)-\frac{(1-\delta)}{\delta^2}r-1\Big\}\log\Big(\frac{1-\delta}{\delta^2}r\Big)\\
&+\frac{\delta^2}{2(1-\delta)^2}\Big(\log\Big(\frac{1-\delta}{\delta^2}r\Big)\Big)^2\\
&+\frac{\delta^2}{(1-\delta)^2}\Big\{\log\Big(\frac{1-\delta^2}{\delta^2}\Big)+\frac{1}{2}\Big(\frac{1-\delta^2}{\delta^2}+\log\Big(\frac{1-\delta^2}{\delta^2}\Big)\Big)^2\Big\}\\
&-\frac{(1+\delta)^2T}{2\delta^2}+o(r^{-1+\varepsilon})\Big]\\
=\;& C\exp\Big[-\frac{1-\delta}{\delta^2}r\log\Big(\frac{1-\delta}{\delta^2}r\Big)+o(r\log r)\Big],
\end{aligned}
$$

where C is a constant, as $r\to\infty$ for some $\varepsilon>0$ and $r>0$.

Example 4.8. As an example of Theorem 4.1, we consider the following:

$$F=\delta\sigma W_T-(1-\delta)J_T,$$

where $\sigma>0$, $\delta\in(0,1)$ and J_T is an L^2-canonical pure jump Lévy process at time T. We also assume that J has only positive jumps.

The definition of the Malliavin derivative implies that

$$D_{t,0}F=\delta,\qquad t\in[0,T],\ q\otimes\mathbb{P}\text{-a.e.},$$

and

$$D_{t,z}F=-(1-\delta),\qquad (t,z)\in[0,T]\times(0,\infty),\ q\otimes\mathbb{P}\text{-a.e.}$$

Hence, $zD_{t,z}F=-z(1-\delta)\le 0$ for $(t,z)\in[0,T]\times(0,\infty)$, $q\otimes\mathbb{P}$-a.e.,

$$\sigma^2\int_0^T|D_{t,0}F|^2dt=\sigma^2\delta^2T=:\alpha^2<\infty,$$

and

$$\int_{[0,T]\times\mathbb{R}_0}|D_{t,z}F|^2z^2\nu(dz)dt=c(1-\delta)^2T=:\gamma^2<\infty$$

hold. Therefore, from (4.4) in Theorem 4.1, we obtain

$$\mathbb{P}(F-\mathbb{E}[F]>r)\le\exp\Big[-\frac{r^2}{2(\sigma^2\delta^2+c(1-\delta^2))T}\Big]$$

for all $r>0$.

Example 4.9. We shall give a concentration inequality for the running maximum over $[0,T]$ of the following Lévy process: $L_t=\mu t+X_t$, $t\in[0,T]$, where X is the underlying Lévy process defined in (2.2) and $\mu\in\mathbb{R}$. Note that $L_t\in\mathbb{D}^{1,2}$ for any $t\in[0,T]$. We next denote $M^L=\sup_{t\in[0,T]}L_t$, and $\tau=\inf\{t\in[0,T]|L_t\vee L_{t-}=M^L\}$. Note that $M^L=\sup_{t\in[0,T]}(L_t\vee L_{t-})=L_\tau\vee L_{\tau-}$; and τ is a unique random time satisfying $M^L=L_\tau\vee L_{\tau-}$ by Lemma 49.4 of [23]. In this setting, Theorem 6.4 in [2] implies that $M^L\in\mathbb{D}^{1,2}$ and

$$D_{t,z}M^L=\mathbb{1}_{\{\tau\ge t\}}\mathbb{1}_{\{0\}}(z)+\frac{\sup_{s\in[0,T]}(L_s+z\mathbb{1}_{\{t\le s\}})-M^L}{z}\mathbb{1}_{\mathbb{R}_0}(z).$$

We next consider the following two cases for examples of Theorem 4.1.

(1) In the case $\nu = c\delta_1$, we have

$$\begin{aligned} zD_{t,z}M^L &= \sup_{s\in[0,T]}(L_s + z\mathbb{1}_{\{t\le s\}}) - M^L \\ &\le \sup_{s\in[0,T]}(z\mathbb{1}_{\{t\le s\}}) \le 1 =: \beta,\ (t,z)\in[0,T]\times\{1\},\ q\otimes\mathbb{P}\text{-a.e.}, \\ &\qquad \sigma^2\int_0^T |D_{t,0}M^L|^2 dt \le \sigma^2 T =: \alpha^2 \end{aligned}$$

and

$$\int_0^T\int_{\mathbb{R}_0} z^2|D_{t,z}M^L|^2\nu(dz)dt \le cT =: \gamma^2.$$

Therefore, Theorem 4.1 shows that

$$\begin{aligned} &\mathbb{P}(M^L - \mathbb{E}[M^L] > r) \\ &\le \exp\Big[-\frac{r^2}{2\alpha^2} - \frac{\gamma^2}{2\alpha^2\beta^2}\Big(2\alpha^2 + 2\beta r + \gamma^2\Big) \\ &\qquad + \frac{\alpha^2}{2\beta^2}\Big(\mathcal{W}(\frac{\gamma^2}{\alpha^2}\exp\{\frac{\beta r + \gamma^2}{\alpha^2}\})\Big)^2 + \frac{\alpha^2}{\beta^2}\mathcal{W}(\frac{\gamma^2}{\alpha^2}\exp\{\frac{\beta r + \gamma^2}{\alpha^2}\})\Big] \\ &= \exp\Big[-\frac{r^2}{2\sigma^2 T} - \frac{c}{2\sigma^2}\Big(2\sigma^2 T + 2r + cT\Big) \\ &\qquad + \frac{\sigma^2 T}{2}\Big(\mathcal{W}(\frac{c}{\sigma^2}\exp\{\frac{r+cT}{\sigma^2 T}\})\Big)^2 + \sigma^2 T\mathcal{W}(\frac{c}{\sigma^2}\exp\{\frac{r+cT}{\sigma^2 T}\})\Big] \\ &\overset{(4.10)}{=} \exp\Big[-\Big\{\frac{2c}{\sigma^2} + \log\Big(\frac{c}{\sigma^2}\Big) - 1\Big\}r \\ &\qquad + \sigma^2 T\Big\{\frac{c}{\sigma^2} + \log\Big(\frac{c}{\sigma^2}\Big) - \frac{1}{\sigma^2 T}r - 1\Big\}\log\Big(\frac{1}{\sigma^2 T}r\Big) \\ &\qquad + \frac{\sigma^2 T}{2}\Big(\log\Big(\frac{1}{\sigma^2 T}r\Big)\Big)^2 \\ &\qquad + \sigma^2 T\Big\{\log\Big(\frac{c}{\sigma^2}\Big) + \frac{1}{2}\Big(\frac{c}{\sigma^2} + \log\Big(\frac{c}{\sigma^2}\Big)\Big)^2\Big\} - \frac{c^2 T}{2\sigma^2} + o(r^{-1+\varepsilon})\Big] \\ &= C\exp\Big[-\frac{1}{\sigma^2 T}r\log\Big(\frac{1}{\sigma^2 T}r\Big) + o(r\log r)\Big], \end{aligned}$$

where C is a constant, as $r \to \infty$ for some $\varepsilon > 0$ and $r > 0$. Especially, taking $\sigma = T = 1$, we have

$$\mathbb{P}(M^L - \mathbb{E}[M^L] > r) \le \exp\Big[-r\log r + o(r\log r)\Big],$$

where C is a constant, as $r \to \infty$ and $r > 0$.

(2) We consider the case when $\{J_t\}_{t\in[0,T]}$ has no positive jumps. In this case,

$$zD_{t,z}M^L = \sup_{s\in[0,T]}(L_s + z\mathbb{1}_{\{t\le s\}}) - M^L$$

$$\leq \sup_{s\in[0,T]} (z\mathbb{1}_{\{t\leq s\}}) \leq 0,\ (t,z)\in[0,T]\times\{z\in\mathbb{R}|z<0\},\ q\otimes\mathbb{P}\text{-a.e.},$$

$$\sigma^2\int_0^T |D_{t,0}M^L|^2dt \leq \sigma^2 T =: \alpha^2 < \infty$$

and

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}_0} z^2|D_{t,z}M^L|^2\nu(dz)dt \\
&= \int_0^T\int_{\mathbb{R}_0} |\sup_{s\in[0,T]}(L_s + z\mathbb{1}_{\{t\leq s\}}) - M^L|^2\nu(dz)dt \\
&\leq \int_0^T\int_{\mathbb{R}_0} \sup_{s\in[0,T]} |z\mathbb{1}_{\{t\leq s\}}|^2\nu(dz)dt \\
&\leq \int_0^T\int_{\mathbb{R}_0} |z|^2\nu(dz)dt = T\int_{\mathbb{R}_0} z^2\nu(dz) =: \gamma^2 < \infty
\end{aligned}$$

are derived. Thus, (4.4) in Theorem 4.1 implies that

$$\mathbb{P}(F-\mathbb{E}[F]>r) \leq \exp\left[-\frac{r^2}{2T\left(\sigma^2+\int_{\mathbb{R}_0} z^2\nu(dz)\right)}\right], \qquad \forall r>0.$$

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References

[1] Alòs, E., León, J.A., Pontier, M., Vives, J.: A Hull and White formula for a general stochastic volatility jump-diffusion model with applications to the study of the short-time behavior of the implied volatility. J. Appl. Math. Stoch. Anal., 359142, 17 pages (2008). MR2476729. https://doi.org/10.1155/2008/359142

[2] Arai, T., Suzuki, R.: Local risk-minimization for Lévy markets. Int. J. Financ. Eng. **2**(2), 1550015, 28 pages (2015). MR3454647. https://doi.org/10.1142/S2424786315500152

[3] Bachmann, S., Peccati, G.: Concentration bounds for geometric Poisson functionals: logarithmic Sobolev inequalities revisited. Electron. J. Probab. **21**, 6–44 (2016). MR3485348. https://doi.org/10.1214/16-EJP4235

[4] Beardon, A.F.: The principal branch of the Lambert *W* function. Comput. Methods Funct. Theory **21**(2), 307–316 (2021). MR4262124. https://doi.org/10.1007/s40315-020-00329-6

[5] Bourguin, S., Peccati, G.: The Malliavin-Stein method on the Poisson space. In: Stochastic Analysis for Poisson Point Processes. Bocconi Springer Ser., vol. 7, pp. 185–228. Bocconi Univ. Press, (2016). MR3585401. https://doi.org/10.1007/978-3-319-05233-5_6

[6] Capitaine, M., Hsu, E.P., Ledoux, M.: Martingale representation and a simple proof of logarithmic Sobolev inequalities on path spaces. Electron. Commun. Probab. **2**, 71–81 (1997). MR1484557. https://doi.org/10.1214/ECP.v2-986

[7] Chafaï, D.: Entropies, convexity, and functional inequalities: on Φ-entropies and Φ-Sobolev inequalities. J. Math. Kyoto Univ. **44**(2), 325–363 (2004). MR2081075. https://doi.org/10.1215/kjm/1250283556

[8] Corless, R.M., Gonnet, G.H., Hare, D.E.G., Jeffrey, D.J., Knuth, D.E.: On the Lambert *W* function. Adv. Comput. Math. **5**(4), 329–359 (1996). MR1414285. https://doi.org/10.1007/BF02124750

[9] de Bruijn, N.G.: Asymptotic Methods in Analysis, 3rd edn. p. 200. Dover Publications, Inc., New York (1981). MR0671583

[10] Delong, Ł.: Backward Stochastic Differential Equations with Jumps and Their Actuarial and Financial Applications. European Actuarial Academy (EAA) Series, p. 288. Springer, (2013). BSDEs with jumps, MR3089193. https://doi.org/10.1007/978-1-4471-5331-3

[11] Delong, Ł., Imkeller, P.: On Malliavin's differentiability of BSDEs with time delayed generators driven by Brownian motions and Poisson random measures. Stoch. Model. Appl. **120**(9), 1748–1775 (2010). MR2673973. https://doi.org/10.1016/j.spa.2010.05.001

[12] Di Nunno, G., Øksendal, B., Proske, F.: Malliavin Calculus for Lévy Processes with Applications to Finance. Universitext, p. 413. Springer, (2009). MR2460554. https://doi.org/10.1007/978-3-540-78572-9.

[13] Geiss, C., Laukkarinen, E.: Denseness of certain smooth Lévy functionals in $\mathbb{D}_{1,2}$. Probab. Math. Stat. **31**(1), 1–15 (2011). MR2804974

[14] Gross, L.: Logarithmic Sobolev inequalities. Am. J. Math. **97**(4), 1061–1083 (1975). MR0420249. https://doi.org/10.2307/2373688

[15] Gusakova, A., Sambale, H., Thäle, C.: Concentration on Poisson spaces via modified Φ-Sobolev inequalities. Stoch. Model. Appl. **140**, 216–235 (2021). MR4282691. https://doi.org/10.1016/j.spa.2021.06.009

[16] Hariya, Y.: A unification of hypercontractivities of the Ornstein-Uhlenbeck semigroup and its connection with Φ-entropy inequalities. J. Funct. Anal. **275**(10), 2647–2683 (2018). MR3853077. https://doi.org/10.1016/j.jfa.2018.08.009

[17] Itô, K.: Multiple Wiener integral. J. Math. Soc. Jpn. **3**, 157–169 (1951). MR0044064. https://doi.org/10.2969/jmsj/00310157

[18] Itô, K.: Spectral type of the shift transformation of differential processes with stationary increments. Trans. Am. Math. Soc. **81**, 253–263 (1956). MR0077017. https://doi.org/10.2307/1992916

[19] Nourdin, I., Peccati, G., Yang, X.: Restricted hypercontractivity on the Poisson space. Proc. Am. Math. Soc. **148**(8), 3617–3632 (2020). MR4108865. https://doi.org/10.1090/proc/14964

[20] Nualart, D.: The Malliavin Calculus and Related Topics, 2nd edn. Probability and its Applications, p. 382. Springer, New York (2006). MR2200233

[21] Privault, N.: Stochastic analysis of Bernoulli processes. Probab. Surv. **5**, 435–483 (2008). MR2476738. https://doi.org/10.1214/08-PS139

[22] Privault, N.: Stochastic Analysis in Discrete and Continuous Settings with Normal Martingales. Lecture Notes in Mathematics, vol. 1982, p. 310. Springer, (2009). MR2531026. https://doi.org/10.1007/978-3-642-02380-4

[23] Sato, K.: Lévy Processes and Infinitely Divisible Distributions. Cambridge Studies in Advanced Mathematics, vol. 68, p. 486. Cambridge University Press, Cambridge (1999). Translated from the 1990 Japanese original, Revised by the author. MR1739520

[24] Solé, J.L., Utzet, F., Vives, J.: Canonical Lévy process and Malliavin calculus. Stoch. Model. Appl. **117**(2), 165–187 (2007). MR2290191. https://doi.org/10.1016/j.spa.2006.06.006

[25] Solé, J.L., Utzet, F., Vives, J.: Chaos expansions and Malliavin calculus for Lévy processes. In: Stochastic Analysis and Applications. Abel Symp., vol. 2, pp. 595–612. Springer, (2007). MR2397807. https://doi.org/10.1007/978-3-540-70847-6_27,

[26] Stam, A.J.: Some inequalities satisfied by the quantities of information of Fisher and Shannon. Inf. Control **2**, 101–112 (1959). MR0109101. https://doi.org/10.1016/S0019-9958(59)90348-1

[27] Surgailis, D.: On multiple Poisson stochastic integrals and associated Markov semigroups. Probab. Math. Stat. **3**(2), 217–239 (1984). MR0764148

[28] Suzuki, R.: A Clark-Ocone type formula under change of measure for canonical Lévy processes. Accessed 14 March 2022. https://www.math.keio.ac.jp/wp-content/uploads/2022/03/14002.pdf

[29] Suzuki, R.: A Clark-Ocone type formula under change of measure for Lévy processes with L^2-Lévy measure. Commun. Stoch. Anal. **7**(3), 383–407 (2013). MR3167405. https://doi.org/10.31390/cosa.7.3.03

[30] Wu, B., Zhou, Y., Lim, C.W., Zhong, H.: Analytical approximations to the Lambert W function. Appl. Math. Model. **104**, 114–121 (2022). MR4350987. https://doi.org/10.1016/j.apm.2021.11.024

[31] Wu, L.: A new modified logarithmic Sobolev inequality for Poisson point processes and several applications. Probab. Theory Relat. Fields **118**(3), 427–438 (2000). MR1800540. https://doi.org/10.1007/PL00008749