

Parameter estimation in mixed fractional stochastic heat equation

Diana Avetisian^a, Kostiantyn Ralchenko^{a,b,*}

^a*Department of Probability Theory, Statistics and Actuarial Mathematics, Taras Shevchenko National University of Kyiv, Volodymyrska 64, 01601 Kyiv, Ukraine*

^b*Sydney Mathematical Research Institute, The University of Sydney, Sydney NSW 2006, Australia*

diana.avetisian2017@gmail.com (D. Avetisian), kostiantynralchenko@knu.ua (K. Ralchenko)

Received: 8 August 2022, Revised: 29 November 2022, Accepted: 15 January 2023,
Published online: 24 January 2023

Abstract The paper is devoted to a stochastic heat equation with a mixed fractional Brownian noise. We investigate the covariance structure, stationarity, upper bounds and asymptotic behavior of the solution. Based on its discrete-time observations, we construct a strongly consistent estimator for the Hurst index H and prove the asymptotic normality for $H < 3/4$. Then assuming the parameter H to be known, we deal with joint estimation of the coefficients at the Wiener process and at the fractional Brownian motion. The quality of estimators is illustrated by simulation experiments.

Keywords Stochastic partial differential equation, mixed fractional Brownian motion, Hurst index estimation, strong consistency, asymptotic normality

2010 MSC 60G22, 60H15, 62F10, 62F12

1 Introduction

The paper is devoted to parameter estimation in a stochastic heat equation of the following form:

$$\left(\frac{\partial u}{\partial t} - \frac{1}{2} \cdot \frac{\partial^2 u}{\partial x^2} \right) (t, x) = \sigma \dot{B}_x^H + \kappa \dot{W}_x, \quad t > 0, x \in \mathbb{R}, \quad (1)$$

*Corresponding author.

$$u(0, x) = 0, \quad x \in \mathbb{R}. \quad (2)$$

The right-hand side of (1) is a *mixed fractional noise*. It consists of two independent stochastic processes, namely, a fractional Brownian motion $B^H = \{B_x^H, x \in \mathbb{R}\}$ with Hurst parameter $H \in (0, 1)$ and a Wiener process $W = \{W_x, x \in \mathbb{R}\}$; σ and κ are some positive coefficients. The solution to the problem (1)–(2) is understood in mild sense; the precise definition will be given in Section 2.

The mixed fractional Brownian motion was first introduced by P. Cheridito [8] in order to model financial markets that are simultaneously arbitrage-free and have a memory. The properties of this process were studied in [36]. We refer to the book [28] for a detailed presentation of the modern theory in this area. More involved mixed fractional models described by stochastic differential equations are the subject of numerous publications [13, 35, 33, 17] appeared during last decades. Such equations can be used to model processes on financial markets, where two principal random noises influence the prices. The first source of randomness is the stock exchange itself with thousands of agents. The noise coming from this source can be assumed white and is best modeled by a Wiener process. The second one has the financial and economical background and can be modeled by a fractional Brownian motion B_H . Stochastic partial differential equations with such noises can be used, in particular, for the modeling of instantaneous forward rates, where the space variable corresponds to time until maturity [9, 22]. Such equations arise also in geophysics, especially in physical oceanography [30] and in geostatistics [34]. For example, in models for sea surface temperature, noise terms can represent various heat fluxes and ocean processes [29].

Existence, uniqueness and properties of solutions for stochastic differential equations with mixed noise were studied in various papers [12, 16, 17, 25–27]. A stochastic heat equation with white and fractional noises was investigated in [24]. Several approaches to parameter identification in simple linear mixed fractional models for various observation schemes were proposed in [6, 14, 23, 20, 21]. The problem of drift parameter estimation in a mixed stochastic differential equation of a general form was studied in [19]. The statistical problems for the mixed fractional Vasicek model were investigated in the recent papers [7, 31].

Similarly to our previous papers the solution $u(t, x)$ is observed at equidistant spatial points for a several fixed time instants. On one hand, there are many practical cases where the solution is observed at some discrete space points such as temperature of a heated body or velocity of a turbulent flow. In many cases the measurements with a high space resolution are available, but the time series are short. For example, this is the case for satellite observations of sea surface temperature, see [30]. For this reason, it is suitable to assume that $u(t, x)$ is observed at a large number of space points x_k and only few different time instants t_i . On the other hand, observing the solution at three time points is enough to construct estimators for unknown parameters σ , κ and H . Nevertheless, the additional information of observing the solution discretely in time can be consolidated by taking the (weighted) average of the estimators similarly to [5] or [9].

The present paper is devoted to the problem of estimating unknown parameters H , σ , κ in the equation (1), by discrete observations of its solution $u(t, x)$. The re-

sults of this paper are an extension of our previous works [2, 3], where the problems of estimating H and σ were studied for the equation (1) with the fractional Brownian motion only (that is, for $\kappa = 0$). The diffusion parameter estimator for SPDE with white noise and its properties was considered in [4]. Similarly to the mentioned articles, in the present paper we start with proving stationarity and ergodicity of the solution $u(t, x)$ as a function of the spatial variable x by analyzing the behavior of the covariance function. Based on these results we construct a strongly consistent estimator of H (assuming that the parameters σ and κ are unknown). The asymptotic normality of this estimator is proved for any $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, \frac{3}{4})$. Then we consider the problem of estimating the couple of parameters (σ, κ) when the value of H is known. We prove strong consistency of the estimator and investigate its asymptotic normality.

The paper is organized as follows. In Section 2 we introduce the definition of a mild solution to SPDE (1) and present its properties. Furthermore, we prove the limit theorem for it, needed for establishing properties of statistical estimators. The statistical problems are investigated in Section 3. In Subsection 3.1 we construct an estimator of the Hurst index H and prove its strong consistency and asymptotic normality. Subsection 3.2 is devoted to the estimators of the parameters σ and κ and their asymptotic properties. Numerical results are presented and discussed in Section 4.

2 Preliminaries

Assume that $B^H = \{B_x^H, x \in \mathbb{R}\}$ is a two-sided fractional Brownian motion with the Hurst index $H \in (0, 1)$, while $W = \{W_x, x \in \mathbb{R}\}$ is a Wiener process, independent of B^H . Let G be Green's function of the heat equation, that is

$$G(t, x) = \begin{cases} \frac{1}{\sqrt{2\pi t}} \exp\left\{-\frac{x^2}{2t}\right\}, & \text{if } t > 0, \\ \delta_0(x), & \text{if } t = 0. \end{cases}$$

Similarly to [2–4] (see also [10] and the references cited therein), we define a solution to SPDE (1) in a mild sense as follows.

Definition 1. The random field $\{u(t, x), t \geq 0, x \in \mathbb{R}\}$ defined by

$$u(t, x) = \sigma \int_0^t \int_{\mathbb{R}} G(t-s, x-y) dB_y^H ds + \kappa \int_0^t \int_{\mathbb{R}} G(t-s, x-y) dW_y ds \quad (3)$$

is called a *solution* to SPDE (1)–(2).

Remark 1. As shown in [2], both stochastic integrals in (3) exist as pathwise Riemann–Stieltjes integrals. This fact follows from the Hölder regularity of the integrands and integrators. Namely, Green's function is obviously Lipschitz continuous, while sample paths of the fractional Brownian motion are Hölder continuous up to order H . Such regularity guarantees the existence of the first integral in (3). The second integral is also well defined, since the integrand is square integrable, see [4, Theorem 2.1].

The next proposition summarizes basic properties of the solution $u(t, x)$. These properties, especially stationarity and ergodicity, enable us to construct and investigate statistical estimators for H , κ and σ .

Proposition 1. *Let $u = \{u(t, x), t \in [0, T], x \in \mathbb{R}\}$ be a solution to (1) defined by (3). Then the following properties hold.*

1. For all $t, s \in [0, T]$ and $x, z \in \mathbb{R}$,

$$\begin{aligned} \text{cov}(u(t, z), u(s, x + z)) &= \text{cov}(u(t, 0), u(s, x)) \\ &= \frac{1}{\sqrt{2\pi}} \int_0^t \int_0^s (q + r)^{-\frac{3}{2}} \int_{\mathbb{R}} \left(\sigma^2 H |y|^{2H-1} + \frac{\kappa^2}{2} \right) \\ &\quad \times (\text{sign } y)(y - x) \exp \left\{ -\frac{(y - x)^2}{2(q + r)} \right\} dy dq dr. \end{aligned} \quad (4)$$

Consequently, for fixed distinct points $t_1, \dots, t_n \in [0, T]$, the stochastic process $\begin{pmatrix} u(t_1, x) \\ \vdots \\ u(t_n, x) \end{pmatrix}$, $x \in \mathbb{R}$, is a multivariate stationary Gaussian process.

2. The variance of $u(t, x)$ is given by

$$\mathbb{E}[u(t, x)^2] = \sigma^2 v_t(H) + \kappa^2 v_t\left(\frac{1}{2}\right), \quad t > 0, x \in \mathbb{R}, \quad (5)$$

where

$$v_t(H) = c_H t^{H+1}, \quad c_H = \frac{2^{H+1}(2^H - 1)\Gamma(H + \frac{1}{2})}{\sqrt{\pi}(H + 1)}, \quad (6)$$

Γ denotes the gamma function.

3. For all $t, s \in [0, T]$ and $x > 0$, the covariance function admits the following upper bound:

$$|\text{cov}(u(t, 0), u(s, x))| \leq C_H t s \left(\sigma^2 x^{2H-2} + \kappa^2 x^{-1} \right), \quad (7)$$

where C_H is a positive constant depending only on H .

4. For all $t, s \in [0, T]$ and $x \in \mathbb{R}$,

$$\begin{aligned} \text{cov}(u(t, x), u(s, x)) &= \frac{\sigma^2 2^H \Gamma(H + \frac{1}{2}) \left((t + s)^{H+1} - t^{H+1} - s^{H+1} \right)}{\sqrt{\pi}(H + 1)} \\ &\quad + \frac{\kappa^2 2^{\frac{3}{2}} \left((t + s)^{\frac{3}{2}} - t^{\frac{3}{2}} - s^{\frac{3}{2}} \right)}{3\sqrt{\pi}}. \end{aligned} \quad (8)$$

5. For a fixed $t > 0$, the random process $\{u(t, x), x \in \mathbb{R}\}$ is ergodic.

Proof. The proposition follows from the corresponding results for the equation with pure fractional noise studied in [2, 3]. Indeed, all the statements are based on the properties of the covariance function of the solution. However, since B^H and W are independent, we see that this covariance function can be represented as

$$\text{cov}(u(t, x), u(s, z)) = \sigma^2 \text{cov}(u_b(t, x), u_b(s, z)) + \kappa^2 \text{cov}(u_w(t, x), u_w(s, z)), \quad (9)$$

where

$$\begin{aligned} u_b(t, x) &= \int_0^t \int_{\mathbb{R}} G(t-s, x-y) dB_y^H ds, \\ u_w(t, x) &= \int_0^t \int_{\mathbb{R}} G(t-s, x-y) dW_y ds. \end{aligned}$$

Then combining the equality (9) with statements of [3, Prop. 2.2], we immediately obtain formulas (4), (7) and (8). The equality (5) follows from (9) and [2, Prop. 3]. Finally, the last statement of the proposition holds, because the solution $\{u(t, x), x \in \mathbb{R}\}$ is a stationary Gaussian process, whose covariance function vanishes as $x \rightarrow \infty$, according to (8). Hence, the process $u(t, \cdot)$ is ergodic. \square

Let us fix some $\delta > 0$ and consider the following sequence:

$$V_N(t) = \frac{1}{N} \sum_{k=1}^N u(t, k\delta)^2, \quad t > 0, \quad N \in \mathbb{N}. \quad (10)$$

The sequence (10) will serve as a statistic for parameter estimation problems in Section 3. We introduce the following notation in addition to (6):

$$\begin{aligned} \mu(t) &:= \mathbf{E}V_N(t) = \mathbf{E}[u(t, 0)^2] = \sigma^2 v_t(H) + \kappa^2 v_t\left(\frac{1}{2}\right), \\ \rho_{ts}^H(k) &= \text{cov}(u(t, k\delta), u(s, 0)), \quad r_{ts}^H(H) = 2 \sum_{k=-\infty}^{\infty} \rho_{ts}^H(k)^2, \quad t, s > 0. \end{aligned} \quad (11)$$

The next theorem describes an asymptotic behavior of the stochastic process V_N . It gives a law of large numbers and a central limit theorem for its finite-dimensional distributions $(V_N(t_1), \dots, V_N(t_n))$ as $N \rightarrow \infty$. This result is crucial for construction of the estimators and for establishing their asymptotic properties.

Theorem 1. *Let $H \in (0, 1)$.*

1. *For any $t > 0$,*

$$V_N(t) \rightarrow \mu(t) \quad a. s., \quad \text{as } N \rightarrow \infty. \quad (12)$$

2. *If additionally $H \in (0, \frac{3}{4})$, then for any $n \geq 1$ and any distinct positive t_1, \dots, t_n ,*

$$\sqrt{N} \begin{pmatrix} V_N(t_1) - \mu(t_1) \\ \vdots \\ V_N(t_n) - \mu(t_n) \end{pmatrix} \xrightarrow{d} \mathcal{N}(\mathbf{0}, \mathbf{R}) \quad \text{as } N \rightarrow \infty, \quad (13)$$

where $\mathcal{N}(\mathbf{0}, R)$ is the multivariate normal distribution with zero mean and the covariance matrix

$$R = (r_{t_i t_j}(H))_{i,j=1}^n.$$

Proof. 1. The ergodic theorem implies that for any $t > 0$

$$\frac{1}{N} \sum_{k=1}^N u(t, k\delta)^2 \rightarrow \mathbb{E}[u(t, 0)^2] \quad \text{a. s. as } N \rightarrow \infty,$$

which is equivalent to (12).

2. In order to prove the convergence (13), we shall apply the Cramér–Wold theorem. In other words, we need to show that for all $\alpha_1, \dots, \alpha_n \in \mathbb{R}$, the convergence

$$\sqrt{N} \left[\sum_{i=1}^n \alpha_i (V_N(t_i) - \mu(t_i)) \right] \xrightarrow{d} \mathcal{N}(0, s^2) \quad (14)$$

holds with the asymptotic variance

$$s^2 = \sum_{i=1}^n \alpha_i^2 r_{t_i t_i}(H) + 2 \sum_{1 \leq i < j \leq n} \alpha_i \alpha_j r_{t_i t_j}(H). \quad (15)$$

Representing V_N as the sum (10) and using (11), we rewrite (14) in the form

$$\frac{1}{\sqrt{N}} \sum_{k=1}^N \sum_{i=1}^n \left[\alpha_i (u(t_i, k\delta)^2 - \mathbb{E}u(t_i, k\delta)^2) \right] \xrightarrow{d} \mathcal{N}(0, s^2). \quad (16)$$

Further, since $\begin{pmatrix} u(t_1, k\delta) \\ \vdots \\ u(t_n, k\delta) \end{pmatrix}$, $x \in \mathbb{R}$, is a multivariate stationary Gaussian sequence, the convergence (16) can be established by application of the multivariate Breuer–Major theorem [1, Theorem 4]. In order to verify the conditions of this theorem, note that the function $F(x_1, \dots, x_n) = \sum_{i=1}^n \alpha_i x_i^2$ has Hermite rank 2, therefore we need to check the convergence of the series:

$$\sum_{k=-\infty}^{\infty} \rho_{t_i t_j}^H(k)^2 < \infty, \quad i, j = 1, \dots, n.$$

It follows immediately from the upper bound (7) that these power series converge if and only if $H \in (0, \frac{3}{4})$. Thus, the assumptions of [1, Theorem 4] are satisfied, whence the convergence (16) holds with the following asymptotic variance:

$$\begin{aligned} s^2 &= \text{Var} \left(F(u(t_1, 0), \dots, u(t_n, 0)) \right) \\ &+ 2 \sum_{k=1}^{\infty} \text{cov} \left(F(u(t_1, 0), \dots, u(t_n, 0)), F(u(t_1, k\delta), \dots, u(t_n, k\delta)) \right). \end{aligned}$$

Now we must only check that the asymptotic variance s^2 satisfies (15). By the definition of the function F , we have

$$\begin{aligned}
s^2 &= \text{Var} \left(\sum_{i=1}^n \alpha_i u(t_i, 0)^2 \right) + 2 \sum_{k=1}^{\infty} \text{cov} \left(\sum_{i=1}^n \alpha_i u(t_i, 0)^2, \sum_{j=1}^n \alpha_j u(t_j, k\delta)^2 \right) \\
&= \sum_{i=1}^n \alpha_i^2 \text{Var}(u(t_i, 0)^2) + 2 \sum_{1 \leq i < j \leq n} \alpha_i \alpha_j \text{cov} \left(u(t_i, 0)^2, u(t_j, 0)^2 \right) \\
&\quad + 2 \sum_{k=1}^{\infty} \sum_{i=1}^n \alpha_i^2 \text{cov} \left(u(t_i, 0)^2, u(t_i, k\delta)^2 \right) \\
&\quad + 2 \sum_{k=1}^{\infty} \sum_{1 \leq i < j \leq n} \alpha_i \alpha_j \left(\text{cov} \left(u(t_i, 0)^2, u(t_j, k\delta)^2 \right) + \text{cov} \left(u(t_j, 0)^2, u(t_i, k\delta)^2 \right) \right).
\end{aligned}$$

Now we can use the following well-known fact: if ξ and η are zero-mean Gaussian random variables, then $\text{cov}(\xi^2, \eta^2) = 2 \text{cov}(\xi, \eta)^2$, in particular, $\text{Var}(\xi^2) = 2 \text{Var}(\xi)^2$ (this is a corollary of the Isserlis theorem [18]). Then we get

$$\begin{aligned}
s^2 &= 2 \sum_{i=1}^n \alpha_i^2 \rho_{t_i t_i}^H(0)^2 + 4 \sum_{1 \leq i < j \leq n} \alpha_i \alpha_j \rho_{t_i t_j}^H(0)^2 + 4 \sum_{k=1}^{\infty} \sum_{i=1}^n \alpha_i^2 \rho_{t_i t_i}^H(k)^2 \\
&\quad + 4 \sum_{k=1}^{\infty} \sum_{1 \leq i < j \leq n} \alpha_i \alpha_j \left(\rho_{t_i t_j}^H(k)^2 + \rho_{t_j t_i}^H(k)^2 \right).
\end{aligned}$$

Taking into account the equality $\rho_{t_s}^H(k) = \rho_{t_s}^H(-k)$, we may rewrite this expression in the more compact form:

$$s^2 = 2 \sum_{i=1}^n \left(\alpha_i^2 \sum_{k=-\infty}^{+\infty} \rho_{t_i t_i}^H(k)^2 \right) + 4 \sum_{1 \leq i < j \leq n} \left(\alpha_i \alpha_j \sum_{k=-\infty}^{+\infty} \rho_{t_i t_j}^H(k)^2 \right).$$

Thus the equality (15) is verified. This completes the proof of Theorem 1. \square

3 Parameter estimation

Let us consider the following statistical problem. It is supposed that for fixed t_1, \dots, t_n and fixed step $\delta > 0$, the random field u given by (3) is observed at the points $x_k = k\delta$, $k = 1, \dots, N$. So the observations have the following form:

$$\{u(t_i, k\delta), i = 1, \dots, n, k = 1, \dots, N\}.$$

Our aim is to estimate the unknown parameters H , σ and κ . We shall do this in two steps. We start with construction of a strongly consistent estimator of H that does not depend on κ and σ . Also, we shall establish its asymptotic normality. Then assuming that H is known, we shall estimate the parameters σ and κ .

In what follows we assume that $H \neq \frac{1}{2}$, because otherwise the model is non-identifiable. The parameters σ and κ are assumed to be positive.

3.1 Estimation of H

In order to estimate H without knowledge of σ and κ , it suffices to observe $u(t_i, x_k)$ only at three time instants $t_1 < t_2 < t_3$. If we write (12) at these points and replace convergences with equalities, we shall get the following system of equations

$$\begin{cases} V_N(t_1) = \sigma^2 c_H t_1^{H+1} + \kappa^2 c_{\frac{1}{2}} t_1^{3/2}, \\ V_N(t_2) = \sigma^2 c_H t_2^{H+1} + \kappa^2 c_{\frac{1}{2}} t_2^{3/2}, \\ V_N(t_3) = \sigma^2 c_H t_3^{H+1} + \kappa^2 c_{\frac{1}{2}} t_3^{3/2}. \end{cases} \quad (17)$$

Excluding the unknown parameter κ from the system, we obtain

$$\begin{cases} t_2^{-3/2} V_N(t_2) - t_1^{-3/2} V_N(t_1) = \sigma^2 c_H \left(t_2^{H-\frac{1}{2}} - t_1^{H-\frac{1}{2}} \right), \\ t_3^{-3/2} V_N(t_3) - t_2^{-3/2} V_N(t_2) = \sigma^2 c_H \left(t_3^{H-\frac{1}{2}} - t_2^{H-\frac{1}{2}} \right). \end{cases} \quad (18)$$

Then excluding σ we arrive at the following estimating equation for H :

$$\frac{t_3^{H-\frac{1}{2}} - t_2^{H-\frac{1}{2}}}{t_2^{H-\frac{1}{2}} - t_1^{H-\frac{1}{2}}} = \frac{t_3^{-3/2} V_N(t_3) - t_2^{-3/2} V_N(t_2)}{t_2^{-3/2} V_N(t_2) - t_1^{-3/2} V_N(t_1)}. \quad (19)$$

The solution of (19) (if exists), can be viewed as an estimator of H .

Note that the left-hand side of (19) is indeterminate for $H = 1/2$. However, it is easy to see by l'Hôpital's rule that there exists the limit

$$\lim_{H \rightarrow \frac{1}{2}} \frac{t_3^{H-\frac{1}{2}} - t_2^{H-\frac{1}{2}}}{t_2^{H-\frac{1}{2}} - t_1^{H-\frac{1}{2}}} = \lim_{H \rightarrow \frac{1}{2}} \frac{t_3^{H-\frac{1}{2}} \log t_3 - t_2^{H-\frac{1}{2}} \log t_2}{t_2^{H-\frac{1}{2}} \log t_2 - t_1^{H-\frac{1}{2}} \log t_1} = \frac{\log t_3 - \log t_2}{\log t_2 - \log t_1}.$$

Therefore, one may define by continuity

$$f(H) := \begin{cases} \frac{t_3^{H-1/2} - t_2^{H-1/2}}{t_2^{H-1/2} - t_1^{H-1/2}}, & \text{if } H \neq \frac{1}{2}, \\ \frac{\log t_3 - \log t_2}{\log t_2 - \log t_1} & \text{if } H = \frac{1}{2}. \end{cases} \quad (20)$$

Then the estimator of H is defined as

$$\widehat{H}_N = f^{(-1)} \left(\frac{t_3^{-3/2} V_N(t_3) - t_2^{-3/2} V_N(t_2)}{t_2^{-3/2} V_N(t_2) - t_1^{-3/2} V_N(t_1)} \right), \quad (21)$$

where $f^{(-1)}$ denotes the inverse function of f . In order to prove its existence, we need to establish that $f: \mathbb{R} \rightarrow (0, \infty)$ is a one-to-one function. This is true, since f is always a strictly increasing function (see Fig. 1) as shown in the following lemma.

Lemma 1. *For any $0 < t_1 < t_2 < t_3$, the function $f: \mathbb{R} \rightarrow (0, \infty)$ defined by (20) is strictly increasing with respect to H .*

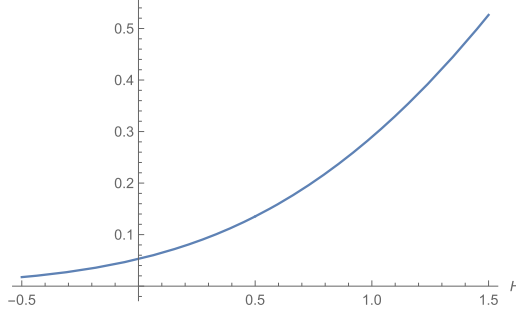


Fig. 1. The function $f(H)$ for $t_1 = 1, t_2 = 2, t_3 = 3$

Proof. We prove the statement for the case $H \in (\frac{1}{2}, \infty)$. The interval $(-\infty, \frac{1}{2})$ is considered similarly. The derivative of f with respect to H is equal to

$$f'(H) = \frac{\left(t_2^{H-\frac{1}{2}} - t_1^{H-\frac{1}{2}}\right) \left(t_3^{H-\frac{1}{2}} \log t_3 - t_2^{H-\frac{1}{2}} \log t_2\right)}{\left(t_2^{H-\frac{1}{2}} - t_1^{H-\frac{1}{2}}\right)^2} - \frac{\left(t_3^{H-\frac{1}{2}} - t_2^{H-\frac{1}{2}}\right) \left(t_2^{H-\frac{1}{2}} \log t_2 - t_1^{H-\frac{1}{2}} \log t_1\right)}{\left(t_2^{H-\frac{1}{2}} - t_1^{H-\frac{1}{2}}\right)^2}. \quad (22)$$

Therefore, it suffices to prove the inequality

$$\begin{aligned} & \left(t_2^{H-\frac{1}{2}} - t_1^{H-\frac{1}{2}}\right) \left(t_3^{H-\frac{1}{2}} \log t_3 - t_2^{H-\frac{1}{2}} \log t_2\right) \\ & > \left(t_3^{H-\frac{1}{2}} - t_2^{H-\frac{1}{2}}\right) \left(t_2^{H-\frac{1}{2}} \log t_2 - t_1^{H-\frac{1}{2}} \log t_1\right). \end{aligned} \quad (23)$$

In order to establish (23), observe that the function $h(x) = x \log x, x > 0$, is strictly convex (indeed, its second derivative $h''(x) = 1/x > 0$). This means that for any $\alpha \in (0, 1), x > 0$ and $y > 0$,

$$\alpha h(x) + (1 - \alpha)h(y) > h(\alpha x + (1 - \alpha)y). \quad (24)$$

Let us take

$$x = t_3^{H-\frac{1}{2}} > 0, \quad y = t_1^{H-\frac{1}{2}} > 0, \quad \text{and} \quad \alpha = \frac{t_2^{H-\frac{1}{2}} - t_1^{H-\frac{1}{2}}}{t_3^{H-\frac{1}{2}} - t_1^{H-\frac{1}{2}}} \in (0, 1).$$

Then

$$1 - \alpha = \frac{t_3^{H-\frac{1}{2}} - t_2^{H-\frac{1}{2}}}{t_3^{H-\frac{1}{2}} - t_1^{H-\frac{1}{2}}}, \quad \alpha x + (1 - \alpha)y = t_2^{H-\frac{1}{2}},$$

and (24) becomes

$$\begin{aligned} & \frac{t_2^{H-\frac{1}{2}} - t_1^{H-\frac{1}{2}}}{t_3^{H-\frac{1}{2}} - t_1^{H-\frac{1}{2}}} (H - \frac{1}{2}) t_3^{H-\frac{1}{2}} \log t_3 + \frac{t_3^{H-\frac{1}{2}} - t_2^{H-\frac{1}{2}}}{t_3^{H-\frac{1}{2}} - t_1^{H-\frac{1}{2}}} (H - \frac{1}{2}) t_1^{H-\frac{1}{2}} \log t_1 \\ & > (H - \frac{1}{2}) t_2^{H-\frac{1}{2}} \log t_2, \end{aligned}$$

which is equivalent to (23). \square

The above lemma yields that the estimator \widehat{H}_N is well defined at least for sufficiently large N (when the right-hand side of estimating equation (19) becomes positive). The asymptotic properties of \widehat{H}_N are summarized in the following theorem, which is the first main result of the paper.

Theorem 2. 1. For any $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$, \widehat{H}_N is a strongly consistent estimator of the parameter H as $N \rightarrow \infty$.

2. For $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, \frac{3}{4})$, the estimator \widehat{H}_N is asymptotically normal:

$$\sqrt{N} (\widehat{H}_N - H) \xrightarrow{d} \mathcal{N}(0, \zeta^2) \quad \text{as } N \rightarrow \infty,$$

where

$$\begin{aligned} \zeta^2 &= \frac{1}{D^2 \sigma^4 c_H^2} \sum_{i,j=1}^3 r_{it_j}(H) L_i L_j, \\ L_1 &= \frac{t_3^{H-\frac{1}{2}} - t_2^{H-\frac{1}{2}}}{t_1^{3/2}}, \quad L_2 = \frac{t_1^{H-\frac{1}{2}} - t_3^{H-\frac{1}{2}}}{t_2^{3/2}}, \quad L_3 = \frac{t_2^{H-\frac{1}{2}} - t_1^{H-\frac{1}{2}}}{t_3^{3/2}}, \\ D &= \left(t_2^{H-\frac{1}{2}} - t_1^{H-\frac{1}{2}} \right) \left(t_3^{H-\frac{1}{2}} \log t_3 - t_2^{H-\frac{1}{2}} \log t_2 \right) \\ &\quad - \left(t_3^{H-\frac{1}{2}} - t_2^{H-\frac{1}{2}} \right) \left(t_2^{H-\frac{1}{2}} \log t_2 - t_1^{H-\frac{1}{2}} \log t_1 \right). \end{aligned}$$

Remark 2. The inequality (23) from the proof of Lemma 1 implies that $D > 0$ for all $H \neq 1/2$.

Proof. 1. The strong consistency follows from the construction of the estimator. Indeed, (12) implies that

$$\frac{t_3^{-3/2} V_N(t_3) - t_2^{-3/2} V_N(t_2)}{t_2^{-3/2} V_N(t_2) - t_1^{-3/2} V_N(t_1)} \rightarrow f(H) \quad \text{a. s., as } N \rightarrow \infty. \quad (25)$$

Then the convergence $\widehat{H}_N \rightarrow H$ a. s. as $N \rightarrow \infty$ follows from (21) and (25) due to the continuity of $f^{(-1)}$.

2. By taking expectations in the equalities (18), we get the following relations

$$\begin{aligned} t_2^{-3/2} \mu(t_2) - t_1^{-3/2} \mu(t_1) &= \sigma^2 c_H \left(t_2^{H-\frac{1}{2}} - t_1^{H-\frac{1}{2}} \right), \\ t_3^{-3/2} \mu(t_3) - t_2^{-3/2} \mu(t_2) &= \sigma^2 c_H \left(t_3^{H-\frac{1}{2}} - t_2^{H-\frac{1}{2}} \right), \end{aligned} \quad (26)$$

whence

$$\frac{t_3^{-3/2}\mu(t_3) - t_2^{-3/2}\mu(t_2)}{t_2^{-3/2}\mu(t_2) - t_1^{-3/2}\mu(t_1)} = \frac{t_3^{H-\frac{1}{2}} - t_2^{H-\frac{1}{2}}}{t_2^{H-\frac{1}{2}} - t_1^{H-\frac{1}{2}}} = f(H), \quad (27)$$

or

$$H = f^{(-1)}\left(\frac{t_3^{-3/2}\mu(t_3) - t_2^{-3/2}\mu(t_2)}{t_2^{-3/2}\mu(t_2) - t_1^{-3/2}\mu(t_1)}\right).$$

Therefore,

$$\sqrt{N}(\widehat{H}_N - H) = \sqrt{N}\left(g(V_N(t_1), V_N(t_2), V_N(t_3)) - g(\mu(t_1), \mu(t_2), \mu(t_3))\right),$$

where

$$g(x_1, x_2, x_3) = f^{(-1)}\left(\frac{t_3^{-3/2}x_3 - t_2^{-3/2}x_2}{t_2^{-3/2}x_2 - t_1^{-3/2}x_1}\right).$$

In order to derive the desired asymptotic normality from the convergence (13), we shall apply the delta method. Denoting

$$h(x) = \frac{d}{dx}f^{(-1)}(x) = \frac{1}{f'(f^{(-1)}(x))}, \quad (28)$$

we see that the partial derivatives of g equal

$$\begin{aligned} g'_1(x_1, x_2, x_3) &= h\left(\frac{t_3^{-3/2}x_3 - t_2^{-3/2}x_2}{t_2^{-3/2}x_2 - t_1^{-3/2}x_1}\right) \cdot \frac{t_1^{-3/2}(t_3^{-3/2}x_3 - t_2^{-3/2}x_2)}{(t_2^{-3/2}x_2 - t_1^{-3/2}x_1)^2}, \\ g'_2(x_1, x_2, x_3) &= -h\left(\frac{t_3^{-3/2}x_3 - t_2^{-3/2}x_2}{t_2^{-3/2}x_2 - t_1^{-3/2}x_1}\right) \cdot \frac{t_2^{-3/2}(t_3^{-3/2}x_3 - t_1^{-3/2}x_1)}{(t_2^{-3/2}x_2 - t_1^{-3/2}x_1)^2}, \\ g'_3(x_1, x_2, x_3) &= h\left(\frac{t_3^{-3/2}x_3 - t_2^{-3/2}x_2}{t_2^{-3/2}x_2 - t_1^{-3/2}x_1}\right) \cdot \frac{t_3^{-3/2}}{t_2^{-3/2}x_2 - t_1^{-3/2}x_1}. \end{aligned} \quad (29)$$

By the delta method, we derive from (13) the convergence (2) with

$$\sigma^2 = \sum_{i,j=1}^3 r_{it_j}(H)g'_i g'_j(\mu(t_1), \mu(t_2), \mu(t_3)).$$

It remains to prove that $g'_i(\mu(t_1), \mu(t_2), \mu(t_3)) = L_i/(D\sigma^2 c_H)$, $i = 1, 2, 3$. Let us consider in detail the proof of this equality for $i = 1$, the cases $i = 2$ and $i = 3$ are considered similarly. Using successively (27), (28) and (22), we get

$$h\left(\frac{t_3^{-3/2}x_3 - t_2^{-3/2}x_2}{t_2^{-3/2}x_2 - t_1^{-3/2}x_1}\right)\Bigg|_{x_i=\mu(t_i)} = h(f(H)) = \frac{1}{f'(H)} = \frac{\left(t_2^{H-\frac{1}{2}} - t_1^{H-\frac{1}{2}}\right)^2}{D}.$$

After inserting this expression into (29) and taking into account the relations (26), we obtain

$$\begin{aligned} g'_1(\mu(t_1), \mu(t_2), \mu(t_3)) &= \frac{\left(t_2^{H-\frac{1}{2}} - t_1^{H-\frac{1}{2}}\right)^2}{D} \cdot \frac{t_1^{-3/2} \left(t_3^{H-\frac{1}{2}} - t_2^{H-\frac{1}{2}}\right)}{\sigma^2 c_H \left(t_2^{H-\frac{1}{2}} - t_1^{H-\frac{1}{2}}\right)^2} \\ &= \frac{t_1^{-3/2} \left(t_3^{H-\frac{1}{2}} - t_2^{H-\frac{1}{2}}\right)}{D \sigma^2 c_H} = \frac{L_1}{D \sigma^2 c_H}. \end{aligned}$$

Note also that the above representation yields $g'_1(x_1, x_2, x_3) \neq 0$ in the neighborhood of the point $(\mu(t_1), \mu(t_2), \mu(t_3))$, which is necessary for applying the delta method. The derivatives g'_2 and g'_3 are considered similarly. \square

The estimator of H was obtained as a solution to some exponential equation. However, it would be more convenient for applications and modeling to have the explicit form of the estimator. It turns out that in some particular cases it is possible to solve the estimating equation explicitly. Let us consider such example in more detail.

Assume that $t_1 = h > 0$, $t_2 = 2h$, $t_3 = 4h$. Substituting these values in the definition of f , we get

$$f(H) = \frac{t_3^{H-\frac{1}{2}} - t_2^{H-\frac{1}{2}}}{t_2^{H-\frac{1}{2}} - t_1^{H-\frac{1}{2}}} = \frac{4^{H-\frac{1}{2}} h^{H-\frac{1}{2}} - 2^{H-\frac{1}{2}} h^{H-\frac{1}{2}}}{2^{H-\frac{1}{2}} h^{H-\frac{1}{2}} - h^{H-\frac{1}{2}}} = 2^{H-\frac{1}{2}}.$$

Therefore, $f^{(-1)}(x) = \frac{1}{2} + \log_2 x$, $x > 0$, consequently (21) becomes¹

$$\widehat{H}_N = \frac{1}{2} + \log_2^+ \frac{t_3^{-3/2} V_N(t_3) - t_2^{-3/2} V_N(t_2)}{t_2^{-3/2} V_N(t_2) - t_1^{-3/2} V_N(t_1)}. \quad (30)$$

In this case

$$L_1 = \frac{4^{H-\frac{1}{2}} - 2^{H-\frac{1}{2}}}{h^{2-H}}, \quad L_2 = -\frac{4^{H-\frac{1}{2}} - 1}{2^{\frac{3}{2}} h^{2-H}}, \quad L_3 = \frac{2^{H-\frac{1}{2}} - 1}{8h^{2-H}},$$

and

$$\begin{aligned} D &= h^{2H-1} \left(2^{H-\frac{1}{2}} - 1\right) \left(4^{H-\frac{1}{2}} \log(4h) - 2^{H-\frac{1}{2}} \log(2h)\right) \\ &\quad - h^{2H-1} \left(4^{H-\frac{1}{2}} - 2^{H-\frac{1}{2}}\right) \left(2^{H-\frac{1}{2}} \log(2h) - \log h\right) \end{aligned}$$

¹We use \log_2^+ rather than \log_2 so that the right-hand side of (30) is always well defined. The function $\log_2^+ x$ is defined as $\log_2 x$ if $x > 0$ and 0 if $x \leq 0$. Note that the expression under \log_2^+ in (30) eventually becomes positive, since it converges to $2^{H-1/2}$.

$$\begin{aligned}
&= h^{2H-1} \left(4^{H-\frac{1}{2}} - 2^{H-\frac{1}{2}} \right) \left(2^{H-\frac{1}{2}} \log(4h) - \log(2h) - 2^{H-\frac{1}{2}} \log(2h) + \log h \right) \\
&= h^{2H-1} 2^{H-\frac{1}{2}} \left(2^{H-\frac{1}{2}} - 1 \right) \left(2^{H-\frac{1}{2}} \log 2 - \log 2 \right) \\
&= h^{2H-1} 2^{H-\frac{1}{2}} \left(2^{H-\frac{1}{2}} - 1 \right)^2 \log 2.
\end{aligned}$$

Denoting $\ell_i = D^{-1} L_i h^{1+H} \log 2$, we arrive at the following result.

Corollary 1. *Let $t_1 = h > 0$, $t_2 = 2h$, $t_3 = 4h$. Then the estimator \widehat{H}_N can be written in the explicit form (30). In this case Theorem 2 holds with*

$$\varsigma^2 = \frac{1}{\sigma^4 c_H^2 h^{2+2H} (\log 2)^2} \sum_{i,j=1}^3 r_{t_i t_j}(H) \ell_i \ell_j,$$

$$\text{where } \ell_1 = \frac{1}{2^{H-\frac{1}{2}} - 1}, \ell_2 = -\frac{2^{H-\frac{1}{2}} + 1}{2^{H+1} (2^{H-\frac{1}{2}} - 1)}, \ell_3 = \frac{1}{2^{H+\frac{5}{2}} (2^{H-\frac{1}{2}} - 1)}.$$

Remark 3. Evidently, the explicit form of the estimator can be obtained also in a slightly more general case, when $t_1 = h$, $t_2 = ah$, $t_3 = a^2 h$ with some $a > 0$. This leads to the estimator of the form (30) with \log_a^+ instead of \log_2^+ .

3.2 Estimation of σ and κ

Now we assume that the Hurst index H is known and investigate the estimation of the coefficients σ and κ . From the first two equations of (17), one can derive the following estimators:

$$\widehat{\sigma}_N^2 = \frac{t_1^{-3/2} V_N(t_1) - t_2^{-3/2} V_N(t_2)}{c_H (t_1^{H-1/2} - t_2^{H-1/2})}, \quad \widehat{\kappa}_N^2 = \frac{t_1^{-1-H} V_N(t_1) - t_2^{-1-H} V_N(t_2)}{c_{\frac{1}{2}} (t_1^{1/2-H} - t_2^{1/2-H})}. \quad (31)$$

Now we are ready to formulate and prove the second main result of the paper.

Theorem 3. *1. For any $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$, $(\widehat{\sigma}_N^2, \widehat{\kappa}_N^2)$ is a strongly consistent estimator of the parameter (σ^2, κ^2) as $N \rightarrow \infty$.*

2. For $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, \frac{3}{4})$, the estimator $(\widehat{\sigma}_N^2, \widehat{\kappa}_N^2)$ is asymptotically normal:

$$\sqrt{N} \begin{pmatrix} \widehat{\sigma}_N^2 - \sigma^2 \\ \widehat{\kappa}_N^2 - \kappa^2 \end{pmatrix} \xrightarrow{d} \mathcal{N}(0, \Sigma) \quad \text{as } N \rightarrow \infty, \quad (32)$$

where the asymptotic covariance matrix Σ consists of the following elements:

$$\begin{aligned}
\Sigma_{11} &= \frac{t_1^{-3} (r_{t_1 t_1}(H) + r_{t_1 t_2}(H)) + t_2^{-3} (r_{t_1 t_2}(H) + r_{t_2 t_2}(H))}{c_H^2 (t_1^{2H-1} - 2(t_1 t_2)^{H-\frac{1}{2}} + t_2^{2H-1})}, \\
\Sigma_{12} = \Sigma_{21} &= \frac{t_1^{-\frac{5}{2}-H} (r_{t_1 t_1}(H) + r_{t_1 t_2}(H)) + t_2^{-\frac{5}{2}-H} (r_{t_1 t_2}(H) + r_{t_2 t_2}(H))}{c_H c_{\frac{1}{2}} \left(2 - t_1^{H-\frac{1}{2}} t_2^{\frac{1}{2}-H} - t_1^{\frac{1}{2}-H} t_2^{H-\frac{1}{2}} \right)}, \\
\Sigma_{22} &= \frac{t_1^{-2-H} (r_{t_1 t_1}(H) + r_{t_1 t_2}(H)) + t_2^{-2-H} (r_{t_1 t_2}(H) + r_{t_2 t_2}(H))}{c_{\frac{1}{2}}^2 (t_1^{1-2H} - 2(t_1 t_2)^{\frac{1}{2}-H} + t_2^{1-2H})}.
\end{aligned}$$

Proof. Using the definition (31) of the estimator $\widehat{\sigma}_N^2$, we rewrite the error $\widehat{\sigma}_N^2 - \sigma$ in the following form

$$\begin{aligned}\widehat{\sigma}_N^2 - \sigma^2 &= \frac{t_1^{-3/2}\widehat{V}_N(t_1) - t_2^{-3/2}\widehat{V}_N(t_2) - \sigma^2 c_H t_1^{H-\frac{1}{2}} + \sigma^2 c_H t_2^{H-\frac{1}{2}}}{c_H \left(t_1^{H-\frac{1}{2}} - t_2^{H-\frac{1}{2}} \right)} \\ &= \frac{1}{c_H \left(t_1^{H-\frac{1}{2}} - t_2^{H-\frac{1}{2}} \right)} \left(t_1^{-3/2} \left(\widehat{V}_N(t_1) - \sigma^2 c_H t_1^{H+1} - \kappa^2 c_{\frac{1}{2}} t_1^{3/2} \right) \right. \\ &\quad \left. - t_2^{-3/2} \left(\widehat{V}_N(t_2) - \sigma^2 c_H t_2^{H+1} - \kappa^2 c_{\frac{1}{2}} t_2^{3/2} \right) \right) \\ &= \frac{t_1^{-3/2} \left(\widehat{V}_N(t_1) - \mu(t_1) \right) - t_2^{-3/2} \left(\widehat{V}_N(t_2) - \mu(t_2) \right)}{c_H \left(t_1^{H-\frac{1}{2}} - t_2^{H-\frac{1}{2}} \right)},\end{aligned}$$

where the last equality follows from (11). Similarly, one can represent $\widehat{\kappa}_N^2 - \kappa^2$ as follows:

$$\widehat{\kappa}_N^2 - \kappa^2 = \frac{t_1^{-1-H} \left(\widehat{V}_N(t_1) - \mu(t_1) \right) - t_2^{-1-H} \left(\widehat{V}_N(t_2) - \mu(t_2) \right)}{c_{\frac{1}{2}} \left(t_1^{\frac{1}{2}-H} - t_2^{\frac{1}{2}-H} \right)}.$$

Hence, we see that the random vector in the left-hand side of (32) is a linear transformation of the left-hand side of (13) (for $n = 2$), namely

$$\begin{aligned}\sqrt{N} \begin{pmatrix} \widehat{\sigma}_N^2 - \sigma^2 \\ \widehat{\kappa}_N^2 - \kappa^2 \end{pmatrix} &= \begin{pmatrix} \frac{t_1^{-3/2}}{c_H \left(t_1^{H-\frac{1}{2}} - t_2^{H-\frac{1}{2}} \right)} & - \frac{t_2^{-3/2}}{c_H \left(t_1^{H-\frac{1}{2}} - t_2^{H-\frac{1}{2}} \right)} \\ \frac{t_1^{-1-H}}{c_{\frac{1}{2}} \left(t_1^{\frac{1}{2}-H} - t_2^{\frac{1}{2}-H} \right)} & - \frac{t_2^{-1-H}}{c_{\frac{1}{2}} \left(t_1^{\frac{1}{2}-H} - t_2^{\frac{1}{2}-H} \right)} \end{pmatrix} \begin{pmatrix} \sqrt{N} \left(\widehat{V}_N(t_1) - \mu(t_1) \right) \\ \sqrt{N} \left(\widehat{V}_N(t_2) - \mu(t_2) \right) \end{pmatrix}. \quad (33)\end{aligned}$$

Therefore, taking into account the convergence (13), we conclude that (33) weakly converges in distribution to a bivariate normal distribution with the following covariance matrix:

$$\Sigma = \begin{pmatrix} \frac{t_1^{-3/2}}{c_H \left(t_1^{H-\frac{1}{2}} - t_2^{H-\frac{1}{2}} \right)} & - \frac{t_2^{-3/2}}{c_H \left(t_1^{H-\frac{1}{2}} - t_2^{H-\frac{1}{2}} \right)} \\ \frac{t_1^{-1-H}}{c_{\frac{1}{2}} \left(t_1^{\frac{1}{2}-H} - t_2^{\frac{1}{2}-H} \right)} & - \frac{t_2^{-1-H}}{c_{\frac{1}{2}} \left(t_1^{\frac{1}{2}-H} - t_2^{\frac{1}{2}-H} \right)} \end{pmatrix} \begin{pmatrix} r_{t_1 t_1}(H) & r_{t_1 t_2}(H) \\ r_{t_1 t_2}(H) & r_{t_2 t_2}(H) \end{pmatrix}$$

$$\times \begin{pmatrix} \frac{t_1^{-3/2}}{c_H \left(t_1^{H-\frac{1}{2}} - t_2^{H-\frac{1}{2}} \right)} & \frac{t_1^{-1-H}}{c_{\frac{1}{2}} \left(t_1^{\frac{1}{2}-H} - t_2^{\frac{1}{2}-H} \right)} \\ -\frac{t_2^{-3/2}}{c_H \left(t_1^{H-\frac{1}{2}} - t_2^{H-\frac{1}{2}} \right)} & -\frac{t_2^{-1-H}}{c_{\frac{1}{2}} \left(t_1^{\frac{1}{2}-H} - t_2^{\frac{1}{2}-H} \right)} \end{pmatrix} = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12} & \Sigma_{22} \end{pmatrix}.$$

This completes the proof. \square

3.3 Maximum likelihood estimation and Fisher information

In this subsection we analyze the efficiency of the estimator $(\widehat{\sigma}_N^2, \widehat{\kappa}_N^2)$ by comparing it to the maximum likelihood estimator (MLE). Note that MLE is hard to compute, however, it is possible to identify the corresponding Fischer information matrix. We use for construction of MLE the same observations as in the previous subsection, namely let the observation vector be

$$\mathbf{X}_N = (u(t_1, \delta), u(t_1, 2\delta), \dots, u(t_1, N\delta), u(t_2, \delta), u(t_2, 2\delta), \dots, u(t_2, N\delta))^\top.$$

Obviously, \mathbf{X}_N has $2N$ -dimensional Gaussian distribution with probability density

$$f(\mathbf{X}_N, \theta) = (2\pi)^{-N} (\det \Gamma_N)^{-\frac{1}{2}} \exp \left\{ -\frac{1}{2} \mathbf{X}_N^\top \Gamma_N^{-1} \mathbf{X}_N \right\},$$

where Γ_N is the covariance matrix of \mathbf{X}_N , that is,

$$\Gamma_N = \begin{pmatrix} \rho_{t_1 t_1}^H(0) & \cdots & \rho_{t_1 t_1}^H(N-1) & \rho_{t_2 t_1}^H(0) & \cdots & \rho_{t_2 t_1}^H(N-1) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \rho_{t_1 t_1}^H(N-1) & \cdots & \rho_{t_1 t_1}^H(0) & \rho_{t_2 t_1}^H(N-1) & \cdots & \rho_{t_2 t_1}^H(0) \\ \rho_{t_2 t_1}^H(0) & \cdots & \rho_{t_2 t_1}^H(N-1) & \rho_{t_2 t_2}^H(0) & \cdots & \rho_{t_2 t_2}^H(N-1) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \rho_{t_2 t_1}^H(N-1) & \cdots & \rho_{t_2 t_1}^H(0) & \rho_{t_2 t_2}^H(N-1) & \cdots & \rho_{t_2 t_2}^H(0) \end{pmatrix}.$$

Due to (9), this matrix can be decomposed as $\Gamma_N = \sigma^2 \Gamma_N^b + \kappa^2 \Gamma_N^w$, where Γ_N^b and Γ_N^w are the covariance matrices of

$$(u_b(t_1, \delta), u_b(t_1, 2\delta), \dots, u_b(t_1, N\delta), u_b(t_2, \delta), u_b(t_2, 2\delta), \dots, u_b(t_2, N\delta))^\top$$

and

$$(u_w(t_1, \delta), u_w(t_1, 2\delta), \dots, u_w(t_1, N\delta), u_w(t_2, \delta), u_w(t_2, 2\delta), \dots, u_w(t_2, N\delta))^\top,$$

respectively. The log-likelihood function is

$$\ell(\mathbf{X}_N, \theta) = -N \log(2\pi) - \frac{1}{2} \log(\det \Gamma_N) - \frac{1}{2} \mathbf{X}_N^\top \Gamma_N^{-1} \mathbf{X}_N.$$

Then, MLE of $\theta = (\sigma^2, \kappa^2)$ is obtained as the solution to the following system of equations:

$$\frac{\partial \ell(\mathbf{X}_N, \theta)}{\partial \sigma^2} = -\frac{1}{2} \text{tr}(\Gamma_N^{-1} \Gamma_N^b) + \frac{1}{2} \mathbf{X}_N^\top \Gamma_N^{-1} \Gamma_N^b \Gamma_N^{-1} \mathbf{X}_N = 0, \quad (34)$$

$$\frac{\partial \ell(\mathbf{X}_N, \theta)}{\partial \kappa^2} = -\frac{1}{2} \text{tr}(\Gamma_N^{-1} \Gamma_N^w) + \frac{1}{2} \mathbf{X}_N^\top \Gamma_N^{-1} \Gamma_N^w \Gamma_N^{-1} \mathbf{X}_N = 0 \quad (35)$$

(here and after we use the differentiation formulas of a matrix with respect to given parameter,² see, e. g., [32] for more details).

The maximum likelihood estimator $\hat{\theta}_N^{mle}$ of θ can hardly be written in the explicit form, since the estimating equations involve the inverse matrix Γ_N^{-1} , which depends nonlinearly on σ^2 and κ^2 . However, using the general theory of maximum likelihood estimation for dependent observations [11], it is possible to establish the asymptotic normality in the form

$$(T_N(\theta))^{1/2} \left(\hat{\theta}_N^{mle} - \theta \right) \xrightarrow{d} \mathcal{N}(\mathbf{0}, I_2) \quad \text{as } n \rightarrow \infty, \quad (36)$$

where $T_N(\theta)$ is the Fisher information matrix and I_2 is the 2×2 identity matrix. The rigorous proof of (36) as well as a careful analysis of the asymptotic behavior of $T_N(\theta)$ requires an additional investigation. To the best of our knowledge, even for much simpler model of the mixed fractional Brownian motion, this problem has not been completely solved yet, see the recent paper [15] for details. Therefore, here we restrict ourselves to the identification of the matrix $T_N(\theta)$.

Lemma 2. *The Fisher information matrix $T_N(\theta)$ has the form*

$$T_N(\theta) = \begin{pmatrix} \frac{1}{2} \text{tr} \left(\left(\Gamma_N^{-1} \Gamma_N^b \right)^2 \right) & \frac{1}{2} \text{tr} \left(\left(\Gamma_N^{-1} \Gamma_N^b \right) \left(\Gamma_N^{-1} \Gamma_N^w \right) \right) \\ \frac{1}{2} \text{tr} \left(\left(\Gamma_N^{-1} \Gamma_N^b \right) \left(\Gamma_N^{-1} \Gamma_N^w \right) \right) & \frac{1}{2} \text{tr} \left(\left(\Gamma_N^{-1} \Gamma_N^w \right)^2 \right) \end{pmatrix}.$$

Proof. In order to identify $T_N(\theta)$, let us calculate the second derivatives. Note that

$$\begin{aligned} \frac{\partial}{\partial \sigma^2} \left(\Gamma_N^{-1} \Gamma_N^b \right) &= \left(\frac{\partial}{\partial \sigma^2} \Gamma_N^{-1} \right) \Gamma_N^b = \left(-\Gamma_N^{-1} \Gamma_N^b \Gamma_N^{-1} \right) \Gamma_N^b = - \left(\Gamma_N^{-1} \Gamma_N^b \right)^2, \\ \frac{\partial}{\partial \sigma^2} \left(\Gamma_N^{-1} \Gamma_N^b \Gamma_N^{-1} \right) &= \left(\frac{\partial}{\partial \sigma^2} \Gamma_N^{-1} \Gamma_N^b \right) \Gamma_N^{-1} + \Gamma_N^{-1} \Gamma_N^b \left(\frac{\partial}{\partial \sigma^2} \Gamma_N^{-1} \right) \\ &= - \left(\Gamma_N^{-1} \Gamma_N^b \right)^2 \Gamma_N^{-1} - \Gamma_N^{-1} \Gamma_N^b \left(\Gamma_N^{-1} \Gamma_N^b \Gamma_N^{-1} \right) \\ &= -2 \left(\Gamma_N^{-1} \Gamma_N^b \right)^2 \Gamma_N^{-1}. \end{aligned}$$

Hence,

$$\frac{\partial^2 \ell(\mathbf{X}_N, \theta)}{\partial (\sigma^2)^2} = \frac{1}{2} \text{tr} \left(\left(\Gamma_N^{-1} \Gamma_N^b \right)^2 \right) - \mathbf{X}_N^\top \left(\Gamma_N^{-1} \Gamma_N^b \right)^2 \Gamma_N^{-1} \mathbf{X}_N.$$

Taking expectations, we obtain that the corresponding element of the Fisher information matrix equals

$$-\mathbb{E} \left[\frac{\partial^2 \ell(\mathbf{X}_N, \theta)}{\partial (\sigma^2)^2} \right] = -\frac{1}{2} \text{tr} \left(\left(\Gamma_N^{-1} \Gamma_N^b \right)^2 \right) + \mathbb{E} \left[\mathbf{X}_N^\top \left(\Gamma_N^{-1} \Gamma_N^b \right)^2 \Gamma_N^{-1} \mathbf{X}_N \right]$$

²For square matrices X and Y , $\partial(XY) = (\partial X)Y + X(\partial Y)$, $\partial(X^{-1}) = -X^{-1}(\partial X)X^{-1}$, $\partial(\ln(\det X)) = \text{tr}(X^{-1}\partial X)$, $\partial(\text{tr}(X)) = \text{tr}(\partial X)$.

$$= \frac{1}{2} \operatorname{tr} \left(\left(\Gamma_N^{-1} \Gamma_N^b \right)^2 \right),$$

since for any matrix $A = (a_{ij})_{i,j=1,\dots,2N}$, we have the equality $\mathbb{E}[\mathbf{X}_N^\top A \mathbf{X}_N] = \sum_{i,j} a_{ij} \mathbb{E}[X_i X_j] = \operatorname{tr}(A \Gamma_N)$. Arguing as above, one can write the derivatives

$$\begin{aligned} \frac{\partial^2 \ell(\mathbf{X}_N, \theta)}{\partial (\kappa^2)^2} &= \frac{1}{2} \operatorname{tr} \left(\left(\Gamma_N^{-1} \Gamma_N^w \right)^2 \right) - \mathbf{X}_N^\top \left(\Gamma_N^{-1} \Gamma_N^w \right)^2 \Gamma_N^{-1} \mathbf{X}_N \\ \frac{\partial^2 \ell(\mathbf{X}_N, \theta)}{\partial \sigma^2 \partial \kappa^2} &= \frac{1}{2} \operatorname{tr} \left(\Gamma_N^{-1} \Gamma_N^w \Gamma_N^{-1} \Gamma_N^b \right) \\ &\quad - \frac{1}{2} \mathbf{X}_N^\top \Gamma_N^{-1} \left(\Gamma_N^b \Gamma_N^{-1} \Gamma_N^w + \Gamma_N^w \Gamma_N^{-1} \Gamma_N^b \right) \Gamma_N^{-1} \mathbf{X}_N, \end{aligned}$$

and calculate their expectations, identifying other elements of $T_N(\theta)$. \square

Remark 4. 1. Similarly to the previous subsection, in the case $H = \frac{1}{2}$ it is impossible to estimate both parameters, σ^2 and κ^2 , simultaneously. Only estimation of the sum $\sigma^2 + \kappa^2$ is possible. In this case $\Gamma_N^b = \Gamma_N^w$, therefore the estimation equations (34) and (35) coincide.

2. The results of this subsection are valid for any other observations vector of the form $\mathbf{X} = (u(t_i, x_k), i = 1 \dots, M, k = 1, \dots, N)$ and its covariance matrix Γ (with decomposition $\Gamma = \sigma^2 \Gamma^b + \kappa^2 \Gamma^w$).

3. Similar approach can be applied to the case, when H is unknown, that is, to the problem of estimation of all three parameters σ^2, κ^2 and H .

4 Simulations

Let us illustrate the theoretical properties of the estimator by some numerical results. We consider the model with the coefficients $\sigma = \kappa = 1$ for various values of H . For each value of the Hurst index H , we simulate 50 sample paths of the solution $u(t, x)$ to the equation (1). The trajectories of a solution are generated by the discretization of the formula (3).

We choose $t_1 = 0.25, t_2 = 0.5, t_3 = 1$ as the moments of observations, so that the conditions of Corollary 1 are satisfied and the estimator of H can be computed by the explicit formula (30). For each t_i we observe $u(t_i, k\delta), k = 1, \dots, N$, with the step $\delta = 1$.

Table 1 reports the means and standard deviations of \widehat{H}_N for various H and N . We see that the estimates converge to the true value of the parameter H . However, the convergence is much slower compared to the estimation of H in the pure fractional case (i. e. $\kappa = 0$), which was considered in [3]. The results become poorer when H approaches $1/2$, or when H is close to zero. It's worth mentioning that the best performance of \widehat{H}_N is observed for large values of H (0.8 and 0.9), for which the asymptotic normality does not hold.

The means and standard deviations of the estimates $\widehat{\sigma}_N^2$ and $\widehat{\kappa}_N^2$ are reported in Tables 2–3. Here we also clearly see that the estimators converge to the true values of the parameters, however the results for both estimators become worse when H

Table 1. Means and standard deviations of the estimator \widehat{H}_N

N		2^8	2^9	2^{10}	2^{11}	2^{12}
$H = 0.1$	Mean	-0.0121	-0.0299	0.0353	0.0527	0.0682
	Std. dev.	0.5237	0.5552	0.2061	0.1242	0.0752
$H = 0.2$	Mean	0.1616	0.1415	0.1910	0.1961	0.1799
	Std. dev.	0.3419	0.2243	0.1540	0.0897	0.0692
$H = 0.3$	Mean	0.1543	0.2845	0.2685	0.2955	0.2999
	Std. dev.	0.4451	0.4177	0.1930	0.1314	0.0781
$H = 0.4$	Mean	0.2254	0.2725	0.3129	0.2854	0.3313
	Std. dev.	0.8089	0.8803	0.4661	0.1608	0.1299
$H = 0.6$	Mean	0.4563	0.3495	0.5266	0.5618	0.5775
	Std. dev.	0.7108	0.9010	0.2926	0.1992	0.1108
$H = 0.7$	Mean	0.6384	0.7024	0.7160	0.7151	0.6980
	Std. dev.	0.3391	0.1512	0.1065	0.0848	0.0511
$H = 0.8$	Mean	0.8160	0.8042	0.8073	0.8022	0.8074
	Std. dev.	0.1929	0.0922	0.0644	0.0467	0.0333
$H = 0.9$	Mean	0.8583	0.8722	0.8815	0.8939	0.8958
	Std. dev.	0.1216	0.0772	0.0671	0.0474	0.0334

Table 2. Means and standard deviations of the estimator $\widehat{\sigma}_N^2$ for $\sigma = 1, \kappa = 1$

N		2^8	2^9	2^{10}	2^{11}	2^{12}
$H = 0.1$	Mean	1.0084	1.0511	1.0571	1.0405	1.0440
	Std. dev.	0.3971	0.2933	0.1829	0.1355	0.0909
$H = 0.2$	Mean	1.0963	1.0478	1.0407	1.0183	1.0170
	Std. dev.	0.3742	0.2370	0.1888	0.1261	0.1014
$H = 0.3$	Mean	1.0107	1.0437	0.9616	0.9939	1.0032
	Std. dev.	0.4530	0.3618	0.2454	0.1673	0.1216
$H = 0.4$	Mean	0.8117	1.0042	1.0566	1.0964	1.0660
	Std. dev.	0.9423	0.7003	0.4789	0.3048	0.2167
$H = 0.6$	Mean	1.0905	1.1579	1.0793	1.0667	1.0673
	Std. dev.	0.6755	0.5750	0.3875	0.2967	0.2154
$H = 0.7$	Mean	0.9536	1.0545	1.0653	1.0106	0.9889
	Std. dev.	0.4650	0.3498	0.2526	0.1649	0.1202
$H = 0.8$	Mean	1.0171	1.0216	1.0287	1.0252	1.0095
	Std. dev.	0.4619	0.2805	0.2465	0.1720	0.1368
$H = 0.9$	Mean	1.1814	1.1236	1.0738	1.0391	1.0308
	Std. dev.	0.8814	0.7882	0.6851	0.4922	0.3681

is close to $1/2$. We observe that unlike \widehat{H}_N , the estimator $\widehat{\sigma}^2$ converges slowly for $H = 0.9$, demonstrating better results for small H . Note that the similar situation for the coefficient at the fractional Brownian motion takes place in the pure fractional case, see [2].

Table 3. Means and standard deviations of the estimator $\widehat{\kappa}_N^2$ for $\sigma = 1$, $\kappa = 1$

N		2^8	2^9	2^{10}	2^{11}	2^{12}
$H = 0.1$	Mean	1.0472	1.0133	1.0005	0.9953	0.9886
	Std. dev.	0.1664	0.1483	0.0963	0.0606	0.0442
$H = 0.2$	Mean	0.9608	0.9720	0.9732	0.9876	0.9859
	Std. dev.	0.2430	0.1712	0.1338	0.0834	0.0720
$H = 0.3$	Mean	0.9899	0.9638	1.0355	1.0075	1.0032
	Std. dev.	0.4197	0.3056	0.2059	0.1472	0.1017
$H = 0.4$	Mean	1.2052	1.0076	0.9417	0.9092	0.9365
	Std. dev.	0.8747	0.6671	0.4480	0.2988	0.2188
$H = 0.6$	Mean	0.8934	0.8285	0.9070	0.9258	0.9279
	Std. dev.	0.6869	0.5600	0.3733	0.2869	0.2137
$H = 0.7$	Mean	1.0483	0.9526	0.9519	0.9863	1.0040
	Std. dev.	0.4351	0.3250	0.2228	0.1448	0.1078
$H = 0.8$	Mean	0.9996	0.9938	0.9905	0.9848	0.9974
	Std. dev.	0.3113	0.1941	0.1429	0.1031	0.0795
$H = 0.9$	Mean	1.0017	0.9877	0.9767	0.9870	0.9945
	Std. dev.	0.2985	0.2311	0.2072	0.1480	0.1084

Funding

KR was supported by the Sydney Mathematical Research Institute under Ukrainian Visitors Program.

References

- [1] Arcones, M.A.: Limit theorems for nonlinear functionals of a stationary Gaussian sequence of vectors. *Ann. Probab.* **22**(4), 2242–2274 (1994). <https://doi.org/10.1214/aop/1176988503>
- [2] Avetisian, D., Ralchenko, K.: Ergodic properties of the solution to a fractional stochastic heat equation, with an application to diffusion parameter estimation. *Mod. Stoch. Theory Appl.* **7**(3), 339–356 (2020). <https://doi.org/10.15559/20-VMSTA162>
- [3] Avetisian, D.A., Ralchenko, K.V.: Estimation of the Hurst and diffusion parameters in fractional stochastic heat equation. *Theory Probab. Math. Stat.* **104**, 61–76 (2021). <https://doi.org/10.1090/tpms/1145>
- [4] Avetisian, D.A., Shevchenko, G.M.: Estimation of diffusion parameter for stochastic heat equation with white noise. *Bulletin of Taras Shevchenko National University of Kyiv. Series: Physics & Mathematics* **3**, 9–16 (2018). <https://doi.org/10.17721/1812-5409.2018/3.1>
- [5] Bibinger, M., Trabs, M.: Volatility estimation for stochastic PDEs using high-frequency observations. *Stoch. Process. Appl.* **130**(5), 3005–3052 (2020). [MR4080737. https://doi.org/10.1016/j.spa.2019.09.002](https://doi.org/10.1016/j.spa.2019.09.002)
- [6] Cai, C., Chigansky, P., Kleptsyna, M.: Mixed Gaussian processes: A filtering approach. *Ann. Probab.* **44**(4), 3032–3075 (2016). [MR3531685. https://doi.org/10.1214/15-AOP1041](https://doi.org/10.1214/15-AOP1041)

- [7] Cai, C.H., Huang, Y.Z., Sun, L., Xiao, W.L.: Maximum likelihood estimation for mixed fractional Vasicek processes. *Fractal Fract.* **6**(1), 44 (2022). <https://doi.org/10.3390/fractalfract6010044>
- [8] Cheridito, P.: Mixed fractional Brownian motion. *Bernoulli* **7**(6), 913–934 (2001). [MR1873835. https://doi.org/10.2307/3318626](https://doi.org/10.2307/3318626)
- [9] Cialenco, I., Huang, Y.: A note on parameter estimation for discretely sampled SPDEs. *Stoch. Dyn.* **20**(3), 2050016–28 (2020). [MR4101083. https://doi.org/10.1142/S0219493720500161](https://doi.org/10.1142/S0219493720500161)
- [10] Cialenco, I., Kim, H.-J.: Parameter estimation for discretely sampled stochastic heat equation driven by space-only noise. *Stoch. Process. Appl.* **143**, 1–30 (2022). [MR4332773. https://doi.org/10.1016/j.spa.2021.09.012](https://doi.org/10.1016/j.spa.2021.09.012)
- [11] Crowder, M.J.: Maximum likelihood estimation for dependent observations. *J. R. Stat. Soc. B* **38**(1), 45–53 (1976). [MR0403035](https://doi.org/10.2307/2335335)
- [12] Da Silva, J.L., Erraoui, M.: Mixed stochastic differential equations: Existence and uniqueness result. *J. Theor. Probab.* **31**(2), 1119–1141 (2018). [MR3803926. https://doi.org/10.1007/s10959-016-0738-9](https://doi.org/10.1007/s10959-016-0738-9)
- [13] Ditlevsen, S., De Gaetano, A.: Mixed effects in stochastic differential equation models. *REVSTAT* **3**(2), 137–153 (2005). [MR2259358](https://doi.org/10.1007/s10959-016-0738-9)
- [14] Dozzi, M., Mishura, Y., Shevchenko, G.: Asymptotic behavior of mixed power variations and statistical estimation in mixed models. *Stat. Inference Stoch. Process.* **18**(2), 151–175 (2015). [MR3348583. https://doi.org/10.1007/s11203-014-9106-5](https://doi.org/10.1007/s11203-014-9106-5)
- [15] Dufitinema, J., Pynnönen, S., Sottinen, T.: Maximum likelihood estimators from discrete data modeled by mixed fractional Brownian motion with application to the Nordic stock markets. *Commun. Stat., Simul. Comput.* **51**(9), 5264–5287 (2022). [MR4491681. https://doi.org/10.1080/03610918.2020.1764581](https://doi.org/10.1080/03610918.2020.1764581)
- [16] Guerra, J., Nualart, D.: Stochastic differential equations driven by fractional Brownian motion and standard Brownian motion. *Stoch. Anal. Appl.* **26**(5), 1053–1075 (2008). [MR2440915. https://doi.org/10.1080/07362990802286483](https://doi.org/10.1080/07362990802286483)
- [17] Han, M., Xu, Y., Pei, B.: Mixed stochastic differential equations: averaging principle result. *Appl. Math. Lett.* **112**, 106705–7 (2021). [MR4145479. https://doi.org/10.1016/j.aml.2020.106705](https://doi.org/10.1016/j.aml.2020.106705)
- [18] Isserlis, L.: On a formula for the product-moment coefficient of any order of a normal frequency distribution in any number of variables. *Biometrika* **12**(1/2), 134–139 (1918). <https://doi.org/10.1093/biomet/12.1-2.134>
- [19] Kozachenko, Y., Melnikov, A., Mishura, Y.: On drift parameter estimation in models with fractional Brownian motion. *Statistics* **49**(1), 35–62 (2015). [MR3304366. 2014.907294. https://doi.org/10.1080/02331888.2014.907294](https://doi.org/10.1080/02331888.2014.907294)
- [20] Kubilius, K., Mishura, Y., Ralchenko, K.: Parameter Estimation in Fractional Diffusion Models. *Bocconi & Springer Series*, vol. 8, p. 390. Bocconi University Press, Milan Springer, Cham (2017). [MR3752152. https://doi.org/10.1007/978-3-319-71030-3](https://doi.org/10.1007/978-3-319-71030-3)
- [21] Kukush, A., Lohvinenko, S., Mishura, Y., Ralchenko, K.: Two approaches to consistent estimation of parameters of mixed fractional Brownian motion with trend. *Stat. Inference Stoch. Process.* **25**(1), 159–187 (2022). [MR4419677. https://doi.org/10.1007/s11203-021-09252-6](https://doi.org/10.1007/s11203-021-09252-6)
- [22] Markussen, B.: Likelihood inference for a discretely observed stochastic partial differential equation. *Bernoulli* **9**(5), 745–762 (2003). [MR2047684. https://doi.org/10.3150/bj/1066418876](https://doi.org/10.3150/bj/1066418876)

- [23] Mishura, Y., Ralchenko, K., Shklyar, S.: Maximum likelihood drift estimation for gaussian process with stationary increments. *Austrian Journal of Statistics* **46**(3–4), 67–78 (2017). <https://doi.org/10.17713/ajs.v46i3-4.672>
- [24] Mishura, Y., Ralchenko, K., Shevchenko, G.: Existence and uniqueness of mild solution to stochastic heat equation with white and fractional noises. *Theory Probab. Math. Stat.* **98**, 149–170 (2019). <https://doi.org/10.1090/tpms/1068>
- [25] Mishura, Y.S., Shevchenko, G.M.: Rate of convergence of Euler approximations of solution to mixed stochastic differential equation involving Brownian motion and fractional Brownian motion. *Random Oper. Stoch. Equ.* **19**(4), 387–406 (2011). [MR2871847. https://doi.org/10.1515/ROSE.2011.021](https://doi.org/10.1515/ROSE.2011.021)
- [26] Mishura, Y.S., Shevchenko, G.M.: Existence and uniqueness of the solution of stochastic differential equation involving Wiener process and fractional Brownian motion with Hurst index $H > 1/2$. *Commun. Stat., Theory Methods* **40**(19–20), 3492–3508 (2011). [MR2860753. https://doi.org/10.1080/03610926.2011.581174](https://doi.org/10.1080/03610926.2011.581174)
- [27] Mishura, Y., Shevchenko, G.: Mixed stochastic differential equations with long-range dependence: Existence, uniqueness and convergence of solutions. *Comput. Math. Appl.* **64**(10), 3217–3227 (2012). [MR2989350. https://doi.org/10.1016/j.camwa.2012.03.061](https://doi.org/10.1016/j.camwa.2012.03.061)
- [28] Mishura, Y., Zili, M.: *Stochastic Analysis of Mixed Fractional Gaussian Processes* p. 194. ISTE Press, London Elsevier Ltd, Oxford (2018). [MR3793191](https://doi.org/10.1016/j.camwa.2012.03.061)
- [29] Patrizio, C.R., Thompson, D.W.: Understanding the role of ocean dynamics in midlatitude sea surface temperature variability using a simple stochastic climate model. *J. Climate* **35**(11), 3313–3333 (2022). <https://doi.org/10.1175/JCLI-D-21-0184.1>
- [30] Piterbarg, L., Rozovskii, B.: Maximum likelihood estimators in the equations of physical oceanography. In: *Stochastic Modelling in Physical Oceanography*. *Progr. Probab.*, vol. 39, pp. 397–421. Birkhäuser Boston, Boston, MA (1996)
- [31] Prakasa Rao, B.L.S.: Maximum likelihood estimation in the mixed fractional Vasicek model. *Journal of the Indian Society for Probability and Statistics* **22** (2021). <https://doi.org/10.1007/s41096-020-00094-8>
- [32] Schott, J.R.: *Matrix Analysis for Statistics*, 2nd edn. *Wiley Series in Probability and Statistics*. Wiley-Interscience. John Wiley & Sons, Hoboken, NJ (2005)
- [33] Shevchenko, G.: Mixed stochastic delay differential equations. *Theory Probab. Math. Stat.* **89**, 181–195 (2014). [MR3235184. https://doi.org/10.1090/S0094-9000-2015-00944-3](https://doi.org/10.1090/S0094-9000-2015-00944-3)
- [34] Vergara, R.C.: *Development of geostatistical models using stochastic partial differential equations*. PhD thesis, MINES, Paris Tech (2018). http://cg.ensmp.fr/bibliotheque/public/CARRIZO_These_02513.pdf
- [35] Wiquvist, S., Golightly, A., McLean, A.T., Picchini, U.: Efficient inference for stochastic differential equation mixed-effects models using correlated particle pseudo-marginal algorithms. *Comput. Stat. Data Anal.* **157**, 107151–26 (2021). [MR4192029. https://doi.org/10.1016/j.csda.2020.107151](https://doi.org/10.1016/j.csda.2020.107151)
- [36] Zili, M.: On the mixed fractional Brownian motion. *J. Appl. Math. Stoch. Anal.*, 32435–9 (2006). [MR2253522. https://doi.org/10.1155/JAMSA/2006/32435](https://doi.org/10.1155/JAMSA/2006/32435)