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Transport equation driven by a stochastic measure

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Abstract The stochastic transport equation is considered where the randomness is given by a symmetric integral with respect to a stochastic measure. For a stochastic measure, only σ-additivity in probability and continuity of paths is assumed. Existence and uniqueness of a weak solution to the equation are proved.

Keywords Stochastic transport equation, weak solution, stochastic measure, symmetric integral

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1 Introduction

We consider the stochastic transport equation that formally can be written in the form

$$\frac{\partial u(t,x)}{\partial t}\,dt + b(t,x)\frac{\partial u(t,x)}{\partial x}\,dt + \frac{\partial u(t,x)}{\partial x}\circ d\mu(t) = 0, \qquad u(0,x) = u_0(x), \quad x\in\mathbb{R}, \quad t\in[0,T]. \tag{1}$$

Here μ is a stochastic measure (SM), see Definition 1 below. We assume that μ is defined on the Borel σ-algebra of $[0,T]$, and the process $\mu_t = \mu((0,t])$ has a continuous paths. Assumptions on b and u_0 are given in Section 3. Equation (1) is to be understood in the weak sense. The definition of a weak solution is given in (6).

We will prove existence and uniqueness of the solution. Similarly to other types of stochastic transport equation, we demonstrate that the solution is given by the formula $u(t,x) = u_0(X_t^{-1}(x))$ where $X_t(x)$ satisfies the auxiliary equation (7).

The stochastic integral with respect to μ is defined as a symmetric integral. This Stratonovich-type integral was studied in [18], we recall its definition and basic properties in Section 2.2. SMs include many important classes of processes, but we can prove existence of the integral only for integrands of the form $f(\mu_t, t)$ where $f \in \mathbb{C}^{1,1}(\mathbb{R} \times [0, T])$. Thus, we will find our solution u having this form.

For the stochastic transport equation driven by the Wiener process, the existence and uniqueness of the solution were proved under different assumptions on b and u_0, see [3, 5, 7, 12, 25]. It was shown that a stochastic term in the transport equation leads to regularization of the solution, see [1, 6, 7]. Equation in bounded domain was studied in [14]. In these papers, the stochastic term is given by the Stratonovich integral and solution is considered in the weak sense. In [26] the existence and uniqueness of stochastic strong solution are obtained, the renormalized weak solution was studied in [28].

Transport equation with other stochastic integrators is less studied. The existence and uniqueness of the solution to equation driven by the Lévy white noise was proved in [16], to equation driven by the fractional Brownian motion – in [15]. In the latter papers, techniques from white noise analysis and the Malliavin calculus approach were used.

In this paper, we consider the rather general stochastic integrator. At the same time, we assume some restrictive assumptions on b and u_0, and study the case of one-dimensional spatial variable.

The recent results for the equations driven by stochastic measures may be found in [2, 9, 10].

The rest of the paper is organized as follows. In Section 2 we recall the definitions and basic facts concerning stochastic measures and symmetric integrals. Also we prove the analogue of the Fubini theorem for our integral that we will need below. In Section 3 we give our assumptions on the equation and formulate the main result. Section 4 is devoted to the proof of the existence of the solution, and we give the explicit formula for u. In Section 5, under some additional assumptions, we obtain the uniqueness of the solution.

2 Preliminaries

2.1 Stochastic measures

Let $\mathsf{L}_0 = \mathsf{L}_0(\Omega, \mathcal{F}, \mathsf{P})$ be the set of all real-valued random variables defined on the complete probability space $(\Omega, \mathcal{F}, \mathsf{P})$ (more precisely, the set of equivalence classes). Convergence in L_0 means the convergence in probability. Let X be an arbitrary set and $\mathcal{B}$ be a σ-algebra of subsets of X.

Definition 1. A σ-additive mapping $\mu : \mathcal{B} \to \mathsf{L}_0$ is called *stochastic measure* (SM).

We do not assume existence of moments or martingale properties for the SM. In other words, μ is L_0-valued vector measure.

Important examples of SMs are orthogonal stochastic measures, α-stable random measures defined on a σ-algebra for $\alpha \in (0, 1) \cup (1, 2]$ (see [21, Chapter 3]).

Many examples of the SMs on the Borel subsets of $[0, T]$ may be given by the Wiener-type integral

$$\mu(A) = \int_{[0,T]} \mathbf{1}_A(t)\, dX_t. \tag{2}$$

We note the following cases of processes X_t in (2) that generate SM.

1. X_t is any square integrable continuous martingale.
2. $X_t = W_t^H$ is the fractional Brownian motion with Hurst index $H > 1/2$, see Theorem 1.1 [11].
3. $X_t = S_t^k$ is the sub-fractional Brownian motion for $k = H - 1/2$, $1/2 < H < 1$, see Theorem 3.2 (ii) and Remark 3.3 c) in [23].
4. $X_t = Z_H^k(t)$ is the Hermite process, $1/2 < H < 1$, $k \geq 1$, see [8], [24, Section 3.1.3]. $Z_H^2(t)$ is known as the Rosenblatt process, see also [22, Section 3].

The detailed theory of stochastic measures is presented in [20].

The results of this paper will be obtained under the following assumption on μ.

Assumption A1. *μ is an SM on Borel subsets of $[0, T]$, and the process $\mu_t = \mu((0, t])$ has continuous paths on $[0, T]$.*

Processes X_t in examples 1–4 are continuous, therefore A1 holds in these cases.

2.2 Symmetric integral

The symmetric integral of random functions with respect to stochastic measures was considered in [18]. We review the basic facts and definitions concerning this integral.

Definition 2. Let ξ_t and η_t be random processes on $[0, T]$, $0 = t_0^n < t_1^n < \cdots < t_{j_n}^n = T$ be a sequence of partitions such that $\max_k |t_k^n - t_{k-1}^n| \to 0$, $n \to \infty$. We define

$$\int_{(0,T]} \xi_t \circ d\eta_t := \text{p-}\lim_{n\to\infty} \sum_{k=1}^{j_n} \frac{\xi_{t_{k-1}^n} + \xi_{t_k^n}}{2} (\eta_{t_k^n} - \eta_{t_{k-1}^n}), \tag{3}$$

provided that this limit in probability exists for any such sequence of partitions.

For a Wiener process η_t and adapted ξ_t we obtain the classical Stratonovich integral. If η_t and ξ_t are Hölder continuous with exponents γ_η and γ_ξ, $\gamma_\eta + \gamma_\xi > 1$, then value of (3) equals to the integral defined in [27].

The following theorem describes the class of processes for which the integral is well-defined.

Theorem 1 (Theorem 4.6 [18]). *Let A1 hold, $f \in \mathbb{C}^{1,1}(\mathbb{R} \times [0, T])$. Then integral (3) of $f(\mu_t, t)$ with respect to μ_t is well-defined, and*

$$\int_{(0,T]} f(\mu_t, t) \circ d\mu_t = F(\mu_T, T) - \int_{(0,T]} F'_t(\mu_t, t)\, dt, \tag{4}$$

where $F(z, t) = \int_0^z f(y, t)\, dy$.

Some other properties and equations with the symmetric integral are considered in [18–20].

2.3 Fubini theorem for symmetric integral

We will need the following auxiliary statement.

Lemma 1. *Let $f : \mathbb{R}\times[0,T]\times\mathbb{R} \to \mathbb{R}$ be measurable and has continuous derivatives $f'_y(y,t,x)$, $f'_t(y,t,x)$. Assume that*

$$|f(y,t,x)| \le g(x), \quad |f'_y(y,t,x)| \le g_1(x), \quad |f'_t(y,t,x)| \le g_2(x)$$

for some $g, g_1, g_2 \in \mathsf{L}^1(\mathbb{R}, dx)$. Then

$$\int_{\mathbb{R}} \int_{(0,T]} f(\mu_t, t, x) \circ d\mu_t \, dx = \int_{(0,T]} \int_{\mathbb{R}} f(\mu_t, t, x)\, dx \ \circ d\mu_t. \tag{5}$$

Proof. Denote

$$F(z,t,x) = \int_0^z f(y,t,x)\, dy, \quad \tilde{F}(z,t) = \int_0^z \int_{\mathbb{R}} f(y,t,x)\, dx\, dy, \quad z \in \mathbb{R}.$$

Theorem 1 and assumptions of the lemma imply that the integrals in (5) are well-defined. Applying (4), we transform left-hand side and right-hand side of (5)

$$\int_{\mathbb{R}} \int_{(0,T]} f(\mu_t, t, x) \circ d\mu_t \, dx = \int_{\mathbb{R}} \Big(F(\mu_T, T, x) - \int_{(0,T]} F'_t(\mu_t, t, x)\, dt \Big) dx,$$

$$\int_{(0,T]} \int_{\mathbb{R}} f(\mu_t, t, x)\, dx \ \circ d\mu_t = \tilde{F}(\mu_T, T) - \int_{(0,T]} \tilde{F}'_t(\mu_t, t)\, dt.$$

The equalities

$$\int_{\mathbb{R}} F(\mu_T, T, x)\, dx = \tilde{F}(\mu_T, T)$$

$$\Leftrightarrow \int_{\mathbb{R}} \int_0^{\mu_T} f(y, T, x)\, dy\, dx = \int_0^{\mu_T} \int_{\mathbb{R}} f(y, T, x)\, dx\, dy,$$

$$\int_{\mathbb{R}} \int_{(0,T]} F'_t(\mu_t, t, x)\, dt\, dx = \int_{(0,T]} \tilde{F}'_t(\mu_t, t)\, dt$$

$$\Leftrightarrow \int_{\mathbb{R}} \int_{(0,T]} \int_0^{\mu_t} f'_t(y, t, x)\, dy\, dt\, dx = \int_{(0,T]} \int_0^{\mu_t} \int_{\mathbb{R}} f'_t(y, t, x)\, dx\, dy\, dt.$$

hold by usual Fubini's theorem. □

3 The problem. Formulation of the main result

We consider equation (1) in the weak form. This means that $u : [0,T] \times \mathbb{R} \times \Omega \to \mathbb{R}$ is a measurable random function such that for each $\varphi \in \mathbb{C}_0^\infty(\mathbb{R})$ holds

$$\int_{\mathbb{R}} u(t,x)\varphi(x)\, dx = \int_{\mathbb{R}} u_0(x)\varphi(x)\, dx$$
$$+ \int_0^t \int_{\mathbb{R}} u(s,x)\Big(b(s,x)\varphi'(x) + \frac{\partial b(s,x)}{\partial x}\varphi(x) \Big)\, dx\, ds$$

$$+\int_0^t \int_{\mathbb{R}} u(s,x)\varphi'(x)\,dx\ \circ d\mu(s). \tag{6}$$

By $\mathbb{C}_0^\infty(\mathbb{R})$ we denote the class of infinitely differentiable functions with the compact support.

For our equation, we will refer to the following assumptions.

Assumption A2. *$u_0 : \mathbb{R}\times\Omega \to \mathbb{R}$ is measurable and has continuous derivative in x.*

Assumption A3. *$|u_0(x)| \le C(\omega)$ for some finite random constant $C(\omega)$.*

Assumption A4. *$b : [0,T]\times\mathbb{R} \to \mathbb{R}$ is continuous, $\frac{\partial b(t,x)}{\partial x}$ is continuous and bounded.*

Assumption A5. $\sup_{t\in[0,T]}\int_{|x|\ge r}\frac{|b(t,x)|}{1+|x|}\,dx \to 0$, $r\to\infty$.

Note that, by A4, b is globally Lipschitz continuous in x.

For each fixed $\omega\in\Omega$, we consider the following auxiliary equation

$$X_t(x) = x + \int_0^t b(r, X_r(x))\,dr + \mu_t, \quad 0\le t\le T. \tag{7}$$

Assumption A4 imply that (7) has a unique solution on $[0,T]$ for each x.

By the well known result of theory of ordinary differential equations, the solution has a continuous derivative

$$X'_t(x) = \frac{\partial}{\partial x}X_t(x).$$

We have

$$\begin{aligned} X'_t(x) &= 1 + \int_0^t \frac{\partial b(r, X_r(x))}{\partial x}X'_r(x)\,dr \\ \Rightarrow \frac{\partial}{\partial t}X'_t(x) &= \frac{\partial b(t, X_t(x))}{\partial x}X'_t(x) \\ \Rightarrow X'_t(x) &= \exp\left\{\int_0^t \frac{\partial b(s, X_s(x))}{\partial x}\,ds\right\}. \end{aligned} \tag{8}$$

Therefore, $X'_t(x) > 0$, and the function $X_t^{-1}(x)$, where the inverse is taken with respect to variable x, is well-defined.

Note that X_t is the sum of a differentiable function of t and μ_t, X'_t is a differentiable function of t. Therefore, by Theorem 1, the integral of the form

$$\int_{(0,T]} g(X_t, X'_t, \mu_t, t)\circ d\mu_t, \quad g\in\mathbb{C}^{1,1,1,1}(\mathbb{R}^3\times[0,T]),$$

is well-defined.

The main result of the paper is the following.

Theorem 2. *1) Let Assumptions A1, A2, A4 hold, $X_t(x)$ be the solution of (7). Then the random function*

$$u(t,x) = u_0(X_t^{-1}(x)) \tag{9}$$

satisfies (6).

2) In addition, let Assumptions A3 and A5 hold. Then solution (9) is unique in the class of measurable random functions $u(t,x) = h(\mu_t, t, x)$, such that $h(\cdot,\cdot,x) \in \mathbb{C}^{1,1}(\mathbb{R} \times [0,T])$ for each $x \in \mathbb{R}$, and $|u(t,x)| \le C(\omega)$ for some finite random constant $C(\omega)$.

Remark 1. Note that $u(t,x) = u_0(X_t^{-1}(x))$ has a form $h(\mu_t, t, x)$ from the second part of the theorem. This follows from Assumption A2 and standard statements about the differentiability of inverse functions. From (7) and (8) we have that $X_t(x) = g(\mu_t, t, x)$, where $g \in \mathbb{C}^{1,1,1}(\mathbb{R} \times [0,T] \times \mathbb{R})$. For the mapping

$$(\mu, t, x) \to (\mu, t, g(\mu, t, x)),$$

the matrix of the first derivatives is nondegenerated. Therefore, the inverse mapping is well-defined and smooth (see, for example, [17, Section 7.3]).

Remark 2. Let us compare our assumptions with those made in other papers. Usually, it is supposed that u_0 is measurable and bounded (see, for example, [7, 12, 15, 25]). We additionally assume that u_0 has a continuous derivative, we need this to guarantee that the symmetric integral of $u_0(X_t^{-1}(x))$ be well-defined.

Condition of differentiability of b is standard, boundedness of $\frac{\partial b}{\partial x}$ may be assumed in some L_p norm (see [7, 25]) or uniformly ([15]). Note that in [12] the main result was obtained for arbitrary bounded measurable b.

Our integrability condition A5 is technical and is important for our method. It is similar to respective assumptions in [1, 3, 13].

4 Existence of the solution

In this section, we prove the first statement of our theorem.

By the chain rule (4), for $\varphi \in \mathbb{C}_0^\infty(\mathbb{R})$ we have

$$\begin{aligned} d_t\Big[X_t'(x)\varphi(X_t(x))\Big] &= \varphi(X_t(x))d_t\Big[X_t'(x)\Big] + X_t'(x)d_t\Big[\varphi(X_t(x))\Big] \\ &\overset{(7),(8)}{=} \varphi(X_t(x))\frac{\partial b(t,X_t(x))}{\partial x}X_t'(x)\,dt \\ &+X_t'(x)\varphi'(X_t(x))b(t,X_t(x))\,dt + X_t'(x)\varphi'(X_t(x)) \circ d\mu(t). \end{aligned} \quad (10)$$

Applying the change of variables $y = X_t(x)$, we get

$$\begin{aligned} \int_{\mathbb{R}} u_0(X_t^{-1}(y))\varphi(y)\,dy &= \int_{\mathbb{R}} u_0(x)X_t'(x)\varphi(X_t(x))\,dx \\ = \int_{\mathbb{R}} u_0(x)\Big[X_t'(x)\varphi(X_t(x))\Big|_{t=0} &+ \int_0^t d_s\big[X_s'(x)\varphi(X_s(x))\big]\Big]\,dx \\ \overset{(10)}{=} \int_{\mathbb{R}} u_0(x)\varphi(x)\,dx + \int_{\mathbb{R}} u_0(x)\int_0^t &\varphi(X_s(x))\frac{\partial b(s,X_s(x))}{\partial x}X_s'(x)\,ds\,dx \\ + \int_{\mathbb{R}} u_0(x)\int_0^t &X_s'(x)\varphi'(X_s(x))b(s,X_s(x))\,ds\,dx \\ + \int_{\mathbb{R}} u_0(x)\int_0^t &X_s'(x)\varphi'(X_s(x)) \circ d\mu_s\,dx \end{aligned}$$

$$\overset{(5)}{=} \int_{\mathbb{R}} u_0(x)\varphi(x)\,dx + \int_0^t \int_{\mathbb{R}} u_0(x)\varphi(X_s(x))\frac{\partial b(s, X_s(x))}{\partial x} X_s'(x)\,dx\,ds$$
$$+ \int_0^t \int_{\mathbb{R}} u_0(x)X_s'(x)\varphi'(X_s(x))b(s, X_s(x))\,dx\,ds$$
$$+ \int_0^t \int_{\mathbb{R}} u_0(x)X_s'(x)\varphi'(X_s(x))\,dx \circ d\mu_s.$$

Lemma 1 may be applied here because φ has a compact support. Assumption A4 and (8) imply that $C_1 \le X_s' \le C_2$ for some positive constants C_1 and C_2, therefore set $\{x : \varphi'(X_s(x)) \neq 0\}$ is bounded.

Taking the inverse change of variable $x = X_t^{-1}(y)$, we obtain

$$\int_{\mathbb{R}} u_0(X_t^{-1}(y))\varphi(y)\,dy$$
$$= \int_{\mathbb{R}} u_0(x)\varphi(x)\,dx + \int_0^t \int_{\mathbb{R}} u_0(X_s^{-1}(y))\varphi(y)\frac{\partial b(s, y)}{\partial x}\,dy\,ds$$
$$+ \int_0^t \int_{\mathbb{R}} u_0(X_s^{-1}(y))\varphi'(y)b(s, y)\,dy\,ds + \int_0^t \int_{\mathbb{R}} u_0(X_s^{-1}(y))\varphi'(y)\,dy \circ d\mu_s.$$

Thus, $u(t, x) = u_0(X_t^{-1}(x))$ satisfies (6).

5 Uniqueness of the solution

In this section, we prove the second statement of our theorem. We will follow the standard approach (see, for example, proof of the uniqueness of the solution in [3, 13]).

Let $u(t, x)$ satisfy (6) with $u_0(x) = 0$. We will obtain that $u(t, x) = 0$ what implies the uniqueness of the solution.

For this case, from (6) for $\varphi \in \mathbb{C}_0^\infty(\mathbb{R})$ we get

$$\int_{\mathbb{R}} u(t, x)\varphi(x)\,dx = \int_0^t \int_{\mathbb{R}} u(s, x)\Big(b(s, x)\varphi'(x) + \frac{\partial b(s, x)}{\partial x}\varphi(x)\Big)\,dx\,ds$$
$$+ \int_0^t \int_{\mathbb{R}} u(s, x)\varphi'(x)\,dx \circ d\mu_s. \qquad (11)$$

Our solution has a form $u(t, x) = h(\mu_t, t, x)$. Denote

$$G(\mu_t, t, y) = \int_{\mathbb{R}} u(t, x)\varphi(x - y)\,dx,$$

where $G(z, t, y) \in \mathbb{C}^{1,1,\infty}(\mathbb{R} \times [0, T] \times \mathbb{R})$. We have that $G(z, 0, y) = 0$ because $u(0, x) = 0$, and

$$\frac{\partial}{\partial \mu_t} G(\mu_t, t, \mu_t) = \frac{\partial}{\partial z} G(\mu_t, t, \mu_t) + \frac{\partial}{\partial y} G(\mu_t, t, \mu_t)$$
$$= \frac{\partial}{\partial z} G(\mu_t, t, \mu_t) - \int_{\mathbb{R}} u(t, x)\varphi'(x - \mu_t)\,dx. \qquad (12)$$

We obtain

$$G(\mu_t,t,\mu_t)\overset{(4)}{=}\int_{(0,t]}\frac{\partial}{\partial s}G(\mu_s,s,\mu_s)\,ds+\int_{(0,t]}\frac{\partial}{\partial \mu_s}G(\mu_s,s,\mu_s)\circ d\mu_s$$

$$\overset{(11),(12)}{=}\int_0^t\int_{\mathbb{R}}b(s,x)u(s,x)\varphi'(x-\mu_s)\,dx\,ds$$

$$+\int_0^t\int_{\mathbb{R}}\frac{\partial b(s,x)}{\partial x}u(s,x)\varphi(x-\mu_s)\,dx\,ds$$

$$+\int_0^t\int_{\mathbb{R}}u(s,x)\varphi'(x-\mu_s)\,dx\;\circ d\mu_s-\int_0^t\int_{\mathbb{R}}u(s,x)\varphi'(x-\mu_s)\,dx\;\circ d\mu_s$$

$$=\int_0^t\int_{\mathbb{R}}b(s,x)u(s,x)\varphi'(x-\mu_s)\,dx\,ds$$

$$+\int_0^t\int_{\mathbb{R}}\frac{\partial b(s,x)}{\partial x}u(s,x)\varphi(x-\mu_s)\,dx\,ds.$$

For $V(t,z)=u(t,z+\mu_t)$, applying the change of the variable $x=z+\mu_t$, get

$$\int_{\mathbb{R}}V(t,z)\varphi(z)\,dz=\int_0^t\int_{\mathbb{R}}b(s,z+\mu_s)V(s,z)\frac{d\varphi(z)}{dz}\,dz\,ds$$
$$+\int_0^t\int_{\mathbb{R}}\frac{\partial b(s,z+\mu_s)}{\partial z}V(s,z)\varphi(z)\,dz\,ds. \tag{13}$$

Let ϕ_ε be a standard mollifier,

$$\phi_\varepsilon(x)=\frac{1}{\varepsilon}\phi\Big(\frac{x}{\varepsilon}\Big),\quad \phi\in\mathbb{C}_0^\infty(\mathbb{R}),\quad \operatorname{supp}\phi\subset[-1,1],$$
$$\phi(x)\ge 0,\quad \int_{\mathbb{R}}\phi(x)\,dx=1.$$

Denote $V_\varepsilon(t,x):=V(t,\cdot)*\phi_\varepsilon$. Substituting $\varphi(z)=\phi_\varepsilon(x-z)$ in (13), we obtain that

$$V_\varepsilon(t,x)=\int_{\mathbb{R}}V(t,z)\phi_\varepsilon(x-z)\,dz$$
$$=-\int_0^t\int_{\mathbb{R}}b(s,z+\mu_s)V(s,z)\phi_\varepsilon'(x-z)\,dz\,ds$$
$$+\int_0^t\int_{\mathbb{R}}\frac{\partial b(s,z+\mu_s)}{\partial z}V(s,z)\phi_\varepsilon(x-z)\,dz\,ds.$$

We take the derivative with respect to t, use the notation $B(t,z)=b(t,z+\mu_t)$, and get

$$\frac{\partial V_\varepsilon(t,x)}{\partial t}=-\int_{\mathbb{R}}B(t,z)V(t,z)\frac{\partial\phi_\varepsilon(x-z)}{\partial x}\,dz$$
$$+\int_{\mathbb{R}}\frac{\partial B(t,z)}{\partial z}V(t,z)\phi_\varepsilon(x-z)\,dz$$
$$=-\frac{\partial}{\partial x}\int_{\mathbb{R}}B(t,z)V(t,z)\phi_\varepsilon(x-z)\,dz$$

$$+\int_{\mathbb{R}}\Big[\frac{\partial}{\partial z}[B(t,z)V(t,z)]-B(t,z)\frac{\partial V(t,z)}{\partial z}\Big]\phi_\varepsilon(x-z)\,dz$$
$$\overset{(\diamond)}{=}-\frac{\partial}{\partial x}(BV(t,\cdot)*\phi_\varepsilon)(x)+\frac{\partial}{\partial x}(BV(t,\cdot)*\phi_\varepsilon)(x)$$
$$-\int_{\mathbb{R}}B(t,z)\frac{\partial V(t,z)}{\partial z}\phi_\varepsilon(x-z)\,dz=-\int_{\mathbb{R}}B(t,z)\frac{\partial V(t,z)}{\partial z}\phi_\varepsilon(x-z)\,dz.$$

In ($\diamond$) we have used that ϕ_ε has a compact support, and, by integration by parts,

$$\int_{\mathbb{R}}\frac{\partial}{\partial z}[B(t,z)V(t,z)]\phi_\varepsilon(x-z)\,dz=\int_{\mathbb{R}}\phi_\varepsilon(x-z)\,d[B(t,z)V(t,z)]$$
$$=-\int_{\mathbb{R}}[B(t,z)V(t,z)]\,d_z\phi_\varepsilon(x-z)=\int_{\mathbb{R}}[B(t,z)V(t,z)]\frac{\partial\phi_\varepsilon(x-z)}{\partial x}\,dz$$
$$=\frac{\partial}{\partial x}(BV(t,\cdot)*\phi_\varepsilon)(x).$$

Thus,

$$\frac{\partial V_\varepsilon(t,x)}{\partial t}+\Big(B(t,z)\frac{\partial V(t,z)}{\partial z}\Big)*\phi_\varepsilon(x)=0. \tag{14}$$

Denote

$$\mathcal{R}_\varepsilon(B,V)=\frac{\partial V_\varepsilon(t,x)}{\partial t}+B(t,x)\frac{\partial V_\varepsilon(t,x)}{\partial x}\overset{(14)}{=}B\frac{\partial(\phi_\varepsilon*V)}{\partial x}-\phi_\varepsilon*\Big(B\frac{\partial V}{\partial x}\Big). \tag{15}$$

Lemma II.1 i) [4] gives that for each fixed t

$$\mathcal{R}_\varepsilon(B,V_\varepsilon)\to 0,\quad \varepsilon\to 0\quad\text{in}\quad \mathsf{L}^1_{loc}(\mathbb{R},dx), \tag{16}$$

provided that $B(t,\cdot)\in\mathsf{W}^{1,1}_{loc}(\mathbb{R})$, $V(t,\cdot)\in\mathsf{L}^\infty_{loc}(\mathbb{R},dx)$. These conditions hold due to assumptions of our theorem.

Consider $\pi_r(x)=\pi_1(x/r)$, where

$$\pi_1(x)=\begin{cases}1, & |x|<1,\\ 1-2(|x|-1)^2, & 1\le|x|\le 3/2,\\ 2(|x|-2)^2\in[0,1], & 3/2\le|x|\le 2,\\ 0, & |x|>2.\end{cases}$$

Then $\pi_r\in\mathbb{C}^{(1)}(\mathbb{R})$, and $|\pi_r'|\le\frac{C}{r}$. We have that

$$\int_{\mathbb{R}}d_x(B(V_\varepsilon)^2\pi_r)=\lim_{x\to+\infty}B(V_\varepsilon)^2\pi_r(x)-\lim_{x\to-\infty}B(V_\varepsilon)^2\pi_r(x)=0$$

because π_r has a bounded support, therefore

$$\int_{\mathbb{R}}(V_\varepsilon)^2\pi_r\,d_xB+\int_{\mathbb{R}}B\pi_r\,d_x(V_\varepsilon)^2+\int_{\mathbb{R}}B(V_\varepsilon)^2\,d\pi_r=0$$
$$\Leftrightarrow\int_{\mathbb{R}}B\pi_rV_\varepsilon\frac{\partial V_\varepsilon}{\partial x}\,dx=-\frac{1}{2}\int_{\mathbb{R}}(V_\varepsilon)^2\pi_r\frac{\partial B}{\partial x}\,dx-\frac{1}{2}\int_{\mathbb{R}}B(V_\varepsilon)^2\pi_r'\,dx.$$

We multiply (15) by $V_\varepsilon(t,x)\pi_r(x)$, take the integral over $\mathbb{R}$, and get

$$\int_{\mathbb{R}} \mathcal{R}_\varepsilon(B, V_\varepsilon)V_\varepsilon\pi_r\,dx = \frac{1}{2}\int_{\mathbb{R}} \frac{\partial V_\varepsilon^2}{\partial t}\pi_r\,dx + \int_{\mathbb{R}} BV_\varepsilon\pi_r\frac{\partial V_\varepsilon}{\partial x}\,dx$$
$$\Leftrightarrow \int_{\mathbb{R}} \mathcal{R}_\varepsilon(B, V_\varepsilon)V_\varepsilon\pi_r\,dx = \frac{1}{2}\frac{\partial}{\partial t}\int_{\mathbb{R}} V_\varepsilon^2\pi_r\,dx - \frac{1}{2}\int_{\mathbb{R}}(V_\varepsilon)^2\pi_r\frac{\partial B}{\partial x}\,dx$$
$$-\frac{1}{2}\int_{\mathbb{R}} B(V_\varepsilon)^2\pi_r'\,dx.$$

From (16) it follows that for fixed r and t

$$\int_{\mathbb{R}} \mathcal{R}_\varepsilon(B, V_\varepsilon)V_\varepsilon\pi_r\,dx \to 0, \quad \varepsilon\to 0.$$

Therefore,

$$\lim_{\varepsilon\to 0}\Big(\frac{\partial}{\partial t}\int_{\mathbb{R}} V_\varepsilon^2\pi_r\,dx - \int_{\mathbb{R}}(V_\varepsilon)^2\pi_r\frac{\partial B}{\partial x}\,dx - \int_{\mathbb{R}} B(V_\varepsilon)^2\pi_r'\,dx\Big) = 0$$
$$\Leftrightarrow \frac{\partial}{\partial t}\int_{\mathbb{R}} V^2\pi_r\,dx - \int_{\mathbb{R}} V^2\pi_r\frac{\partial B}{\partial x}\,dx = \int_{\mathbb{R}} BV^2\pi_r'\,dx. \tag{17}$$

Because $\pi_r'(x) = 0$ for $|x|\le r$ or $|x|\ge 2r$, and $|\pi_r'|\le\frac{C}{r}$, we have

$$\Big|\int_{\mathbb{R}} BV^2\pi_r'\,dx\Big| \le \|V\|^2_{\mathsf{L}_\infty}\int_{r\le|x|\le 2r}\frac{|B(t,x)|}{1+|x|}(1+|x|)|\pi_r'(x)|\,dx \to 0,$$
$$r\to\infty, \tag{18}$$

where convergence holds uniformly in t for each fixed ω.

In (18) we have used the following estimates. If $\sup_t|\mu_t| = M(\omega)$, then

$$\int_{r\le|x|\le 2r}\frac{|B(t,x)|}{1+|x|}\,dx \overset{y=x+\mu_t}{=} \int_{r\le|y-\mu_t|\le 2r}\frac{|b(t,y)|}{1+|y-\mu_t|}\,dy$$
$$\le \int_{|y|\ge r-M(\omega)}\frac{|b(t,y)|}{1+|y|-M(\omega)}\,dy \overset{\text{A5}}{\to} 0, \quad r\to\infty.$$

Integrating (17) in t and taking into account that

$$\int_{\mathbb{R}} V^2(t,x)\pi_r(x)\,dx\Big|_{t=0} = \int_{\mathbb{R}} u(0,x)^2\pi_r(x)\,dx = 0,$$

we get

$$\int_{\mathbb{R}} V^2\pi_r\,dx = \int_0^t\int_{\mathbb{R}} V^2\pi_r\frac{\partial B}{\partial x}\,dx\,ds + \int_0^t\int_{\mathbb{R}} BV^2\pi_r'\,dx\,ds. \tag{19}$$

Consider

$$g_r(t,x) = V^2(t,x)\pi_r(x).$$

By A4, we have $\left|\frac{\partial B}{\partial x}\right| \le K$ for some constant K. From (19), we get

$$\int_{\mathbb{R}} g_r(t,x)\,dx \le \int_0^t \int_{\mathbb{R}} g_r(s,x)\left|\frac{\partial B}{\partial x}\right| dx\,ds + R_r$$
$$\le K\int_0^t \int_{\mathbb{R}} g_r(s,x)\,dx\,ds + R_r,$$
$$R_r = \sup_t \left|\int_0^t \int_{\mathbb{R}} BV^2\pi_r'\,dx\,ds\right| \overset{(18)}{\to} 0, \quad r \to \infty.$$

From the Gronwall inequality for $h(t) = \int_{\mathbb{R}} g_r(t,x)\,dx$, we get

$$\int_{\mathbb{R}} g_r(t,x)\,dx \le R_r e^{Kt}.$$

Taking $r \to \infty$, we get

$$\int_{\mathbb{R}} g_r(t,x)\,dx \to \int_{\mathbb{R}} V^2\,dx, \quad R_r e^{Kt} \to 0 \Rightarrow V = 0$$

that finishes the proof of uniqueness of the solution.

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